What is the opposite of cat?
A gentle introduction to group theory

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Abstract: This paper has originated from our interest in approaching mathematical concepts starting from people's common sense intuitions and building up from there. This goal is challenging both in designing the didactical transposition and sequencing of the mathematical subject matter, and in adopting the open and interactive teaching approach needed to achieve students' active participation and reflection. To demonstrate these challenges, and our experience of trying to cope with them, we have chosen the concept of "inverses" as used in group theory, and its common sense precursor "opposites". We present our approach via a series of workshop iterations, which summarizes our experience in the many actual workshops we ran in Israel and in Denmark.

Keywords: Intuition; group theory; inverses; opposites; bridging; abstraction hurdles.

1. Introduction

This paper has initially originated from our interest in approaching mathematical concepts starting from people's common sense intuitions and building up from there. This turns out to be a challenging goal, especially when applied to advanced mathematical topics such as abstract algebra. It is challenging both in designing the didactical transposition (Chevallard, 1985) and sequencing of the mathematical subject matter, and in adopting the open and interactive teaching approach needed to achieve students' active participation and reflection. In the theoretical background lies the extensive research in cognitive and social psychologists on *dual process theory* (See e.g., Evans & Frankish, 2009; Kahneman, 2002, 2011. For applications in education see Leron & Hazzan, 2006, 2009), and the related educational notion of *bridging* between the intuitive and analytical modes of thinking (Abrahamson, 2009; Clements, 1993; Ejersbo & Leron, 2014; Ejersbo et al, submitted.). Also in the background lies the research on intuitive thinking by Educational researchers such as Fischbein (1987), Smith et al (1993), and Stavy & Tirosh (2000).

To demonstrate these challenges, and our experience of trying to cope with them, we have chosen the concept of "inverses" as used in group theory. Basically, a group is a set with a binary operation (usually called multiplication and denoted $\cdot$) which satisfies certain properties. An important property of groups is that elements have *inverses* (more details below), so we became interested in people's intuitive conception of "opposites", and how we can move smoothly from there to the formal group inverse.

We present our approach as a series of workshop iterations[1], which summarizes our experience in the many actual workshops we ran in Israel and in Denmark. Many of these iterations are suitable for almost any age, except where functions are involved, which will require certain knowledge of algebra. Each of the iterations is followed by mathematical and didactical reflective comments.

In the classroom interactions, and in writing this article, we are being pulled by two conflicting forces: On the one hand, the wish to stay as much as possible with the students' "natural thinking" and, on the other hand, the wish to move towards the sophisticated and perhaps not-so-natural idea of group. We need to walk the fine line between these conflicting forces, and the relative weight unavoidably shifts (less natural, more sophisticated) as we progress along the iterations.

Two caveats – both stemming from the need in this article to respect focus and space constraints – are in place before embarking on our story. One, the scenarios as presented here are a highly idealized and abbreviated version of the real ones we have experienced. For example, each of our written scenarios, if played out verbatim, would probably last no more than 5 to 10 minutes, while the realistic version might take a full hour and involve more active teacher and classroom participation, a lot more guesswork and blind allies, and a much more meandering path. The second caveat has to do with the didactic approach. In our realistic workshops, we have practiced a much more interactive dialogical approach, encouraging many participants to express diverse responses, with the explicit goal of encouraging a rich classroom interaction as a way to promote students' reflection. In both these respects, the scenarios as depicted here are mere sketches of the real experience. However, they do contain the main ideas of the method, and we trust that the experienced reader will be able to flesh out the missing details.

2. From folk *opposites* to group *inverses*: Three iterations

Iteration 1: Everyday objects that come in pairs

Teacher: What's the opposite of cat?

Student (immediately): Dog.

T: What's the opposite of dog?

S: (smiling) Cat, of course.

T: What's the opposite of chair?

S: Table.

[1] The word *iterations* hints that we are visiting *the same idea* on successively higher levels.

T: What's the opposite of Long?

S: Short.

T: What's the opposite of night?

S: Day.

T: What's the opposite of Rock?

S: (hesitating)

T: What's the opposite of mouse?

S: Cat.

T: What’s the opposite of Cat?

S: Mou… Oops! (realizing suddenly that previously it was dog.)

Reflective Comments:

1. The above virtual interaction is based on a game we have played in our classes many times, and these reactions are quite typical. We suggest that there are several levels to distinguish, when investigating people's spontaneous conception of "opposite", and this iteration expresses the first (most basic) level. The "student" in this interaction should be thought of as standing for a whole classroom discussion.

2. In this first iteration we have met objects that occur naturally in pairs (in people's daily experience or in language), such as chair-table or dog-cat. If you ask people for "the opposite" of one member of the pair, most people will automatically and immediately answer with the second member of the same pair. In such cases, the folk meaning of "opposite" is almost like "partner": you can ask people what is the partner of an object and you will get the same answers. Or you can run an association test: Ask people to quickly complete sentences like "dogs and ___ ", and you will get the same answers. This relation is symmetrical in the sense that if A is the opposite of B then B is the opposite of A. But the case of cat already poses an interesting challenge. Since the opposite of both dog and mouse is cat, we get stuck when asked for the opposite of cat (is it a dog or a mouse?) Pedagogically, we see this as an opportunity to discuss in a natural way the issue of *uniqueness* of the opposite, which will come up again in the discussion of groups (where uniqueness of the inverse can in fact be proved). A sub-category consists of the natural opposite pairs – many of them adjectives – such as long-short, young-old, day-night, etc.

3. Some students like to out-smart the teacher by giving unusual answers, especially. If asked what's the opposite of cat they might for example answer flower, house, devil, grandma, or even "minus-cat". And who could say they were wrong? Here is a trick we have found useful in such situations: Instead of asking the direct question

("what's the opposite…") we ask the following indirect one: What do you think would most people answer when asked "what's the opposite of cat?" If they answer (as they invariably do) "dog", this means that in a way this is what *they* had thought before they decided to act smart!

4. The example of rock demonstrates that some (in fact many) objects do not naturally come in pairs, and so do not have an easy and automatic opposite.

5. As introduction to groups, this iteration has the advantage that it is easy, natural, and fun. It has the disadvantage that these objects do not have an operation defined on them (as required for group elements) in any natural way. Thus, how do you multiply (or add) a cat by a dog, or a cat by a chair, or even a cat by itself? In other words, there is no way that these examples could be refined and formalized to become a group. In our classes, we still like to open with these examples, despite this shortcoming, and later move on to more sophisticated ones, as demonstrated in the coming iterations.

Iteration 2: Mathematical objects

T: What's the opposite of 3?

S1: -3

S2: 1/3

T: Can 3 have two different opposites?

S3: We saw before that cat had two opposites: dog and mouse.

T: Indeed. What about numbers?

(The students take some minutes to discuss the matter between them.)

Students (summarizing the discussion): The opposite of 3 is -3 if you think of plus and it's 1/3 if you think of times.

T: Ok, let's look at it from a little different angle. What's the opposite of "adding 3"?

S1: It's "taking away 3".

T: Can you explain?

S1: If you add 3 and then take away 3, it's as if you did nothing.

T: What do you mean did nothing?

S1: It's like adding 0, the number stays the same.

T: S2, what's the opposite of "multiplying by 3"?

S2: It's "dividing by 3".

T: Why?

S2: It's like S1 said: If you multiply by 3 and then divide by 3, you come back to the same number, and it's as if you did nothing.

T: What do you mean did nothing?

S2: It's the same as multiplying by 1, and the number stays the same.

T (summarizing): As you noticed in this discussion, when we ask for the opposite of something, it depends on the operation we have in mind. Also, each operation we have met has a "neutral element" (also called "identity element") which in effect "does nothing". As you mentioned, for addition the identity element is 0 (adding 0 "does nothing") and for multiplication it is 1 (multiplying by 1 "does nothing"). In this sense, the opposite of an action is another action that *cancels out* the effect of the first: Doing first the action and then its opposite comes out as if we did nothing.

Reflective comment:

1. This iteration fixes the problem (mathematically speaking) with our first iteration, that is, that dog and cat, or table and chair, cannot even be multiplied (or added), and in particular they cannot cancel each other out.

2. In this iteration we seem to meet the first occurrence of a serious "abstraction hurdle": The combined scheme of an operation, identity element, and the definition of inverse as "cancelling out" in the sense that the product (or sum) comes out the identity. Students do know that adding 0 or multiplying by 1 leave the number unchanged, but it is a big jump from here to thinking of this not as a property of 0 or 1 but a property of the operation, and moreover, abstracting this to see the general pattern relative to an arbitrary operation. It is indeed a big jump, and we don't expect students to grasp it in one session, but conducting a lively classroom activity and conversation, can get them started and traveling part of the way. In our abbreviated scenario we glossed over these difficulties, pretending the kids already come out with the correct answers. In realistic situations, the teacher and the students will have to work much harder and longer to get there, but in our experience they do get there in the end. We will be meeting more of these "abstraction hurdles" as we proceed, and these unfortunately may be unavoidable points of discontinuity on the progression from intuitive to analytical understanding of the group definition.

Iteration 3: Real-world actions

T: What's the opposite of putting on your shoes?

S (immediately): Taking them off.

T: What's the opposite of putting on your socks?

S: Taking them off.

T: Let's look at the combined action "putting on one's socks and then putting on one's shoes". What's the opposite of *that*?

S: Taking off the shoes and then taking off the socks.

T: What happened to the *order* of the operations?

S (thinking): It's the opposite!

T: Yes, it's the opposite operations taken in reverse order.

T (under the influence of the Logo turtle): What's the opposite of turning right?

S: Turning left.

T: What's the opposite of moving forward 100 steps?

S: Moving back 100 steps.

T: What's the opposite of the combined motion forward 100 steps then turning right then forward 50 steps?

S (having taken the time to draw a picture and experiment with a paper "turtle"): It's moving back 50 steps, turning left and moving back 100 steps.

T: The same as we have seen with the shoes and socks: The opposite actions taken in reverse order.

T: What's the opposite of flipping a coin?

S: (Silence)

T: Suppose we start with head up and tail down. What happens after we flip?

S: We get tail up and head down.

T: How do you get it back to head up and tail down?

S: Flip again.

T: So, what is the opposite of flipping a coin?

S: Flipping again. It's the same operation.

T: Yes, the operation of flipping a coin is its own opposite!

Reflective Comments:

1. As we have seen, discussing the opposites of daily objects like cats and chairs is simple and intuitive, but in the end it cannot be refined in a natural way to discussing group elements, since there is no sensible way to "multiply" dogs and cats and chairs. In this respect, daily operations such as putting on one's shoes or turning right is a step forward, since these operations can be thought of as functions (but see below), and thus can be "multiplied" by the composition of functions (performing them one after the other), which is what typically happens in many groups. To recall, in composing two functions, you take the output of the first to be the input of the second. This activity also brings out more explicitly the unifying idea behind all our iterations (except the first), namely, that an element and its opposite "cancel out" each other. More formally, if you perform in succession some action a (e.g., putting on your shoes) and its opposite a' (taking them off), you come back to the initial state, which is the same as applying the identity action I. Thus $a \cdot a' = I$, and we are moving closer to the definition of inverses in groups. In the last interaction we have also seen a nice intuitive demonstration of a well-known property of inverses in a non-commutative group: The inverse of a product is the product of the inverses *in reverse order*: $(a \cdot b)' = b' \cdot a'$.

2. In moving towards the definition of a group, this iteration still has the advantage that it is easy and natural (though a bit less so than the first one) and can still be a lot of fun. It has an important additional advantage that its elements can be "multiplied", and that they more typically resemble group elements. Indeed, many typical groups are made of functions (or operations, or transformation, or permutations, as they are sometimes called), where the group operation (or "multiplication") is composition of functions.

3. As precursors to groups, some of the actions discussed in this iteration still hide a serious disadvantage: The "multiplication" (i.e., function decomposition) is not defined on *all* the elements. For example, while we can compose putting on one's shoe with taking it off (getting as a result the identity action, i.e. doing nothing), or putting on ones socks with putting on one's shoes, we cannot sensibly compose putting on one's shoe with itself (the so-called putting-on-one's-shoe squared). For example, you might try to define putting-on-one's-shoe squared, i.e., putting one's shoes on and then doing it again, to be the same as doing it only once. That is, if we denote this action p and we try to define its square as equal to itself: $p^2 = p$. But this is impossible in a group because this equality can only happen if p is the identity element. (To see this multiply each side of the equality $p^2 = p$ by the inverse of p and you get $p = e$, the identity element).

4. In contrast, there is no problem formalizing the (rigid) motions in the plane (right left, forward, back, etc.) into group elements. Each has an inverse and any two can naturally be composed. It is interesting to reflect on the reason for this difference: For a bunch of actions to qualify as a group, they all need to be one-to-one functions from a single set onto itself; only then can they be always multiplied and inverted. All the operations on physical objects we have looked at (putting on one's shoe, turning left,

moving forward) do not seem to be of this type, but some of them can be suitably formalized this way and some of them can't. For example let's look at the set of all turtle motions in Logo (turning right and left any number of degrees and moving forward and back any number of steps, as well as all chains of such operations). They can all be formalized as functions on the set of all *turtle states,* i.e., the set of all triples (x,y,h), where (x,y) represents the turtle's position and h its heading. (See Leron, 1982, and Leron & Zazkis, 1992, for details.)

5. We have postponed the discussion of "flipping a coin" to the end of this iteration, since the idea that an operation (or a function) can be its own inverse is not easy for students. In a way, "flipping" can be seen as prototypical of all the operations such that when performed twice, the second time undoes the first, bring us back to the original state (which is the same as saying that they are their own inverse). We will be working with one more such example, namely, the function $y = \frac{1}{x}$, in the next iteration.

Iteration 4: A sophisticated high-school-level activity

In what follows, we will be assuming that these students had learned about functions and about composition of functions in previous lessons, and had practiced with several examples and with various notations for functions. They have also practiced the view of a function as input-output machine, and the corresponding interpretation of composition of two functions (i.e. the two machines are serially connected so that the output of the first serves as an input to the second).

T: What's the opposite of x?

S1: -x

S2: $\frac{1}{x}$

T: Can you explain?

S1 and S2 (after some discussion): x is a number, so it is the same as we did before with 3. It's -x if we think of addition (with 0 as the identity element) and it's $\frac{1}{x}$ if we think of multiplication (with 1 as the identity element).

T: Ok, let's move on. What's the opposite of 3x?

S1: -3x

S2: $\frac{1}{3x}$

T: Can you explain?

S1 and S2: 3x is a number too, so it's the same as we did with x.

T: What if we think of 3x as a function, y = 3x ? What is the opposite of *that*?

S3: The opposite is the function y = $\frac{x}{3}$.

T: Can you explain?

S3: It's the same as the function "multiplying by 3" which we did before, whose opposite was "divide x by 3". If we multiply x by 3 and then divide the result by 3, we get back to x, as if we did nothing. So whatever the function "multiply x by 3" does, gets undone by the function "divide x by 3".

T: What about the function y = 3x +2 ? What is the opposite of *that*?

S: (silence)

T: When we talked about the function y = 3x, it helped us to say in words what this function does. Can you repeat what we said?

S4: Yes, in the function y = 3x we multiply x by 3.

T: Can you do the same for the new function y = 3x + 2 ?

S4: Yes, we multiply x by 3 and then add 2.

T: So what's the opposite of that? What do we need to do if we want "to undo" what this function does?

S4: We take away 2 and then divide by 3.

T: Yes, can you give us an example?

S4: If x = 4 then to get y we first multiply 4 by 3 (it's 12) and then add 2, so we get y = 14. To get back to 4 we first take away 2 and we get 12, and then we divide the 12 by 3 and we come back to 4.

T (summarizing): We now have *three different answers* for the opposite of 3x. In the first two (-3x and $\frac{1}{3x}$) we were thinking of 3x as a number and the operations were first addition and then multiplication. In the third answer we were thinking of 3x as a function "multiply x by 3" and its opposite function "divide x by 3" because, as S3 said, these two cancel each other out.

To summarize, we now have three "opposites" for 3x: -3x , $\frac{1}{3x}$, and $\frac{x}{3}$, corresponding to the operations of number addition (with 0 as the identity element), number multiplication (with 1 as the identity element) and function composition (with the identity function as the identity element). It seems to be a good idea, when we talk about opposites, to mention what kind of operation and of identity element we have in mind.

T: Here is a new one: What's the opposite of $\frac{1}{x}$?

S1 and S2: - $\frac{1}{x}$ or x.

T: Yes, the same as before.

S4: Maybe opposite of $\frac{1}{x}$ is $\frac{1}{x}$ if we think about it as a function? I am not sure.

T: Can you explain why you think so?

S4: Because if we think of $\frac{1}{x}$ as a function, and if we do it twice one after the other we get back to x.

T: Why is that?

S4: because 1 over 1 over x is again x.

T (Writes on the board): $\frac{1}{\frac{1}{x}} = x$.

T: Can you give us an example?

S4: If x = 5, then is $\frac{1}{x} = \frac{1}{5}$, and if you do it again, it will go back to 5 because $\frac{1}{\frac{1}{5}} = 5$.

T (summarizing): Very interesting, this shows us what may happen when we take the opposite of functions, and when the operation is composition of functions. Let us call this function f so that f(x) = y → 1/x; and let us call the identity function I so that I: x → x; and let us use · for composition of functions. Then we can express the fact that f is its own inverse by writing f · f = I, or more concisely: $f^2 = I$. Can you see that this notation says the same as what we did before? (The students engage in some discussion.)

T (continues): Can you now say in what way the following pairs are similar: addition and 0; multiplication and 1; composition (of functions) and I?

S5: Each operation is mentioned with its identity element.

T: Remember we also called the identity element the "neutral element" relative to the operation under discussion? Can you explain this name?

S6: It is neutral because it doesn't change the elements it operates on: for any number x, 0 + x = x and 1 · x = x, and for every function f, I · f = f · I = f.

T: If we now have any operation · and an identity element *e*, how can we say in general what the opposite of any element is? (Let's use the notation *a* for this element and *a'* for its opposite.)

S7: a and a' cancel out each other, so we can write $a \cdot a' = e$.

T: (summarizing): What is common to all these examples is that the product (in the generalized group sense) of an element and its "opposite" (now called *inverse*) is the identity element *e*.

Reflective comments

1. Depending on the time available and the classroom level and priorities, the teacher may now choose to engage the class in further explorations and examples, leading to the full group concept.

2. The case of the function y = $\frac{1}{x}$ being its own inverse, is relatively hard to explain and to understand. (Our student S4 would be an exceptional student indeed to come up with this idea on her own.) There are several ways for the teacher to approach this challenge. Our teacher has chosen a technical approach, utilizing the students' knowledge of fraction arithmetic ($\frac{1}{\frac{1}{x}}$ = x) which is perhaps the shortest and easiest to explain, but is it also the most enlightening? We are not sure. Next we mention two alternative approaches; presumably different readers (teachers, students) might have different tastes as to which are "best".

One alternative approach is based on an analogy that can be established between the function x → $\frac{1}{x}$ and the operation of flipping a coin (see Iteration 3 above). If we represent x as a fraction, i.e., x = $\frac{x}{1}$, then we can draw an analogy between this fraction and a coin using the following pairings: numerator = up, denominator = down, x = head, 1 = tail. The effect of our function can then be seen as "flipping" this fraction from $\frac{x}{1}$ to $\frac{1}{x}$ ("head goes down, tail goes up"). Since we have seen that the operation of flipping a coin is its own inverse, we can use the same reasoning (and the same image) here: One more flip will bring the fraction back to its original state. In our experience, people differ quite a bit in their reaction to this explanation: some find it intuitive and convincing, others do not.

A second alternative approach makes use of the following characterization of inverse functions. Suppose we have two functions f, g: A → A. Then f and g are inverse functions if and only if the following relation holds:

For all x,y in A, y = f(x) if and only if x = g(y)

The task of finding the inverse of our function is now turned into simply solving the equation y = $\frac{1}{x}$ for x, which leads to x = $\frac{1}{y}$, namely the same function (since changing the names of the variables doesn't change the function).

This approach has one advantage and one drawback. The advantage is that, unlike the other approaches, it enables students to actually *compute* the inverse function (by solving an equation) on their own. The drawback is that it is not immediately obvious why this characterization of inverse functions is equivalent to the one we have been promoting in this workshop, namely that f and g are inverse functions if g undoes what f does (and vice versa). This equivalence could of course be proved if the time available and the level of the class permitted it, which in this case would be a highly worthwhile activity in its own right.

C. Conclusion

The goal of this article has been to demonstrate the possibility of a (relatively) smooth transition from common sense intuitions to sophisticated mathematical concepts; specifically, we have chosen for our demonstration the transition from the common sense "opposites" to concept of *inverse* as used in group theory.

While many teachers and math educators view intuition (or common sense) as a threat, because it often leads to errors, we believe in contrast that intuition (of the universal recurring kind) is a precious resource, a collection of skills of which all people are experts, and that we should as much as possible ground even sophisticated mathematical concepts in their common sense precursors. True, raw intuitions often lead to non-normative responses, but we suggest (along with Smith et al, 1985, and others) to view these intuitions as a precious resource, which can serve as foundation for gradually refining and building more advanced mathematical concepts. An analogy with computer programming may be useful here. When you find an error (a "bug") in computer program, you don't discard the program – you "debug" it. Following Papert (1980), we view "bugs" in our students' thinking not as something to avoid or eliminate, but rather as opportunities for new and refined understanding, by engaging students in a process of "debugging" their intuition.

In practice we do this via classroom activities and reflective discussions, which in the article, due to space and focus considerations, have been condensed into several iterations of idealized classroom scenarios.

In the first iteration the class played with the common sense notion of "opposites", which has served as a fun way to enter the topic, though it could not be continued much further because there is no reasonable way to define "multiplication" on everyday objects such as dogs and cats and tables and chairs.

In the second iteration we entered the real work via numbers and operation on numbers. The two arithmetical operations of addition and multiplication, each with its identity element, enabled us to start gearing the discussion towards the deeper meaning of inverse as "cancelling out", or "undoing".

In the third and fourth iteration the elements to be inverted were functions with the binary operation of composition (which is typical of many groups), where the identity element was the identity function, and the meaning of inverse as undoing became even more explicit.

Not all the steps were equally smooth and we did encounter some "abstraction hurdles", notably the introduction of the identity element and the idea that "the function $\frac{1}{x}$ " turned out to be its own inverse. Though in our brief scenarios these steps might have appeared a bit like sorcery, and the students a bit too bright, our experience confirms that in reality this can be achieved within more elaborate sessions of activities and reflective discussions.

The theoretical foundation for both the gap between intuitive and analytical thinking, and for the idea of bridging this gap by "debugging intuition" can be found (beside Papert's *Mindstorms* cited above) in Dual Process Theory (e.g., Evans & Frankish, 2009; Kahneman, 2002, 2011; Leron & Hazzan, 2006, 2009), and in Kahneman's (2002, 2011) description of how System 1 (the intuitive system) operates. Briefly stated, System 1 pays attention only to the immediately salient features of the situation at hand (hence the "bugs"), and the debugging process should help the student notice and take into account the missing, hidden features.

Finally, we mention that there are other well-known intuitive ways to approach group theory, or more generally abstract algebra, such as using interactive digital environments or playing and calculating with the symmetries of the square (See e.g., Maycock & Hibbard, 2002; Cuoco & Rotman, 2013; Larsen, 2002; Leron & Dubinsky, 1995). Depending on the situation, these other approaches could be combined in various ways with the approach demonstrated here.

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