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Pre-readings for Remich

TOPOLOGY WITH A SUBJECT

This is a short introduction to what I am going to present at the Remich-workshop and some suggestions for pre-reading. There will be some references in my text, but not all of them are suggested. I will give a list of suggestions in the end of this paper.

The participants in the workshop are specialists in Psychology and/or Mathematics and other scientific domains as well. I will try to address them all but, also in this text, there will be sections focusing on domains which may fall outside the expertise of some participants. So just skip them! This applies of course most to some of the mathematical stuff. I will try to explain better at the workshop, using some drawings etc.

The problem in Mathematics of selecting something from something

In ordinary life and in science we are used to take something apart from some more or less infinite background or just focusing on some part of a whole as a figure on a ground. In science we select, perhaps using some criteria, perhaps using what happens to be at hand in the surroundings, often a combination, some sample for further study. In experimental science we often take “a sample of one”, as when we take some iron ball and let it roll down a slide and watch what happens when we change the slope. Much of scientific method, especially in the social sciences, deals with what influence this act of selection has on the interpretation of the results of investigating the sample. This is about scientific design and statistics, etc. The object of science has a selecting and investigating subject, and without that we would have no knowledge, as we would also not have in ordinary life.

The situation has been somewhat different in Mathematics. Without going deeply into the Philosophy of Mathematics I think we can conclude that the relation between subject and object is not the same as in the empirical sciences. Either Mathematics is considered to be a product of the mind, pure – and purified – subjectivity, not an object for science but a tool for science, or belonging to some eternal and unchangeable Platonic domain of objective ideas, “the thoughts of God” as Einstein once claimed. In both cases there has not been an explicit opposition between a selective subject and the objective domain, and the act of selection has not been conceptualized as a necessary part of Mathematics itself.

I guess we all remember from our school that the mathematical textbook said something like: “Let us take an arbitrary triangle and inscribe it in a circle” or “Let us choose one point in a given open interval”. Something was “given” and something was “taken” or “chosen”, but this was just the opening prayer before starting the real thing. In the classroom it was the teacher who gave, took and chose, and he was obviously no part of Mathematics himself, although some pupils perhaps felt it that way.

However, this “innocence” came to an end around 1900. Of course Mathematics for hundreds of years has had ways to define selections of parts from wholes. But that was not by describing an act of selection which presupposed some agency or selective activity by a subject but by referring to a rule or a criterion which was itself a part of the mathematical domain. So Mathematics made selections without any help from outside. An example: If we have some finite set of numbers Mathematics can select the biggest one, because the relation “bigger than” is part of Mathematics itself.

As said this came to an end around 1900 [10]. We can condense a long discussion with an example. Let us [opening prayer once more] think of the real axis of numbers and all its subsets, i.e. all possible parts that could be taken out of it. Does there then exist a function, i.e. a mathematical rule or criterion, which for every non-empty subset defines, singles out, one point which is a member of the subset, a so-called “choice function”? For instance the rule could be “take the biggest number”. But this would not do if the subset has no biggest number as for instance the real numbers beween 0 and 1, with 0 and 1 themselves excluded, the so-called open interval from 0 to 1. But then let us take some more sophisticated rule. Of course to each case we could tailor some ad hoc rule, but it has been proven definitely, that Mathematics cannot do itself, because there exist no rule inside Mathematics which could generate all the specific rules.

If on the other hand such a choice-function does not exist within mathematics we are really bad off. Lots of structures needed would not “exist” within Mathematics. This lead in the years around 1900 Felix Hausdorff to claim that in some way there still existed some objective “order” in all sets which could be used to select out some point which was the “maximal” one according to the order, although the order could not be made explicit. In 1904 the same was claimed in a slightly other version by Ernst Zermelo, who more directly postulated that if you had a set of different sets of elements there always existed a new set with just one element from each of the former ones, and that this was a new basic axiom in Mathematics “The Axiom of Choice”.

This was of course very controversial and let to long-standing discussions, although the axiom is generally recognized today. Many mathematicians rightly felt that some (divine or human) subject or agent was being smuggled into and polluting pure Mathematics. Perhaps somebody also thought that the “ordering” and “maximal” principles of Hausdorff reminded of the medieval “ontological” proof of God’s existence as the “maximal” step on the ladder of perfection. In any case I did.

But you could also conclude that the Axiom of Choice of course on one hand is a necessary part of Mathematics, but that on the other hand you have to give up the illusion of Mathematics as having no inherent subject or agency. Perhaps Mathematics is not only a model of objective structures in the world (and their generalizations to higher dimensions, etc.) but also of our interactions as subjects with this world. Perhaps Mathematics is also a picture of the fact that we as humans both can select something from the world by rules or criteria using universal properties or qualities of objects, and that we because of our bodily existence in one place at one time can just “pick up” what is “at hand”. Perhaps Mathematics is basically a picture of this fundamental duality in our relation to the world.

There are other "stamps" of subjectivity in modern Mathematics. Since Kurt Gödel's work in the late 1930's we know that on one hand an axiomatic system needs to be internally consistent to be "sound" but on the other hand, if it is describing a domain of Mathematics with a magnitude (cardinality) as the natural numbers, or greater, it is impossible to prove the consistency within the axiomatic system itself. Somebody has to imagine something (in mathematical terms a "model") which this somebody believes to exist and which the axiomatic system is a picture of (in mathematical terms "maps"). Mathematics cannot prove the consistency with its own means, without "help". Perhaps this is a picture of the necessity of imagination in our knowledge of the world, which also pops up in modern Mathematics.

Also in modern Mathematics we have the concept of "decidability". To give an example: When defining the now famous Mandelbrot Set we start out by defining some procedure starting in a point (x,y) in a two-dimensional co-ordinate system and "landing" in a new point (x,y). Then the procedure is repeated again and again in a so-called iterative or recursive process. If in some step in this process the point (x,y) has a greater distance from (0,0) than 2, we know that the starting point is not included in the Mandelbrot Set, but in mathematical terms in its complement. This can be done for all points in the co-ordinate system (although it will be waste of effort of course if the starting point itself is more distant from (0,0) than 2). For any starting point in the complement we will at some point in the process know, that it is in the complement. The complement is "decidable". But for starting points in the Mandelbrot Set itself (with a few exceptions) we will never know if the reason that the distance from (0,0) is less than 2 for points in the recursive process is because it will never be greater than 2, or if it is because we have not yet reached long enough in the process. The Mandelbrot Set is, in contrast to its complement, "undecidable". This can be defined in pure mathematical terms, but still has a clear connotation of somebody outside the two sets who shall make a decision in finite time. And this asymmetry of borders between a set and its complement is perhaps also a picture of an asymmetry in our figure-ground operations meeting the world outside Mathematics [5]. In topological terms the complement of the Mandelbrot Set is an open set, and the Mandelbrot Set itself is a closed set, thus defining a tie between decidability and openness. The border (in topological terms: boundary) of the two complementary sets belongs to the Mandelbrot Set, and not to its complement.

A topological model[1] for the human selection of decidable categories from the material world

At the workshop I will present a topological model (in two versions) for the human selection of categories or classes from an infinite world of objects. Perhaps our world is not really infinite, but the model is, and it gives a better picture of our interaction with the world than any finite model, which may be my excuse. It is not a model for our concepts, but for a dialectic structure "beneath"

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[1] The term "model" is here and throughout the paper used in its ordinary meaning in science as some rather simple structure picturing some important aspects of something more complex. Especially it is used here as designating an axiomatic system "modeling" some domain which can be described with a more comprehensive set of theorems. However, in a few cases "model" is used in a more special meaning from Mathematical Logic designating some imagined entity the "existence" of which is taken as proof of consistency of an axiomatic system. The different meanings of the term should be clear from the context.

concepts and language, a basic practical interaction with the physical world which opens for a "superstructure" of concepts, affections, bonds, motives etc. not confined by the Procrustean bed of mechanicism, which has been the sad fate of Psychology [9].

The above description of "subjectivity" in Mathematics has been a premise for this model, but a tacit premise. It has been a ghost in the background when I first presented the model in 1983 [4]. Instead I chose a so-called "extensional" or "objectivist" approach. i.e. only describing the "effects" of subjectivity on the organization of categories of objects in a duality of categories and in an asymmetry of categories and their complements. There were no explicit references in the model to subjects or agency whatsoever. To this end I used concepts from General Topology or Set Theoretical Topology which were already at hand. But all the reasons for choosing this model were psychological and not mathematical! The test of the model should then be that the mathematical background popped up again, now as derived from the psychological arguments. I know that the possibility of circularity and self-deception is lurking, but I think I escaped. In a later exposition [8] I turned it upside down and started with Mathematics and the foundation and history of Mathematical Logic.

And now to the model! The model operates with a world or universe, Ù, of objects[2] and two kinds of categories, sense categories and choice categories which are subsets of Ù. Sense categories are classes of objects which can be defined using our senses, and their extensions with equipment, as measuring devises towards objects' properties or qualities in universal terms, form, size, color, etc. Choice categories are classes of objects defined by pointing, taking, keeping, etc. and by relations between singular objects defined by contact, kinship, common history, etc. The objective basis for sense categories is what in Philosophy and Logic has been called qualitative identity. The objective basis for choice categories is accordingly objects' numerical identity. I shall not dwell on this distinction here but refer to the suggested readings later on, to make this introduction short [6; 7].

The axiomatic system (in its first version) has 11 axioms and a lot of derived theorems. The axioms are not necessarily interpretatively basic, as some of the theorems are more fundamental in this sense. However, the set of axioms is the most concentrated "condensation" of the totality of claims in logically independent statements. Its consistency cannot be proved formally within itself, as told above, but is has a simple model when sense categories are unions of open intervals on the real axis, and choice categories are sets of separated points. However, this model is much too special to cover what is intended with the set of axioms, and is only mentioned here as a proof of consistency.

The language used in the axioms is ordinary language and not mathematical symbols, to make them more transparent for psychologists. However, it can easily be translated to what in Mathematical Logic is called First Order Language. It is so to say First Order Language Equivalent. This has great

[2] Objects are here naturally delimitted units or every-day "things", including persons, etc., as e.g. the individuals or particulars discussed by P. F. Strawson [11]. In this context we shall not go into the more subtle discussion of what constitutes an object in relation to its parts and connections as it is done in mereotopology [14].

advantages when interpreting the consequences of axiomatic systems in relation to cardinality (the different degrees of infinity of sets).

Furthermore, this first version of the axiomatic set is not applied on the whole world of objects. The original purpose of the model was to defend the introduction of choice categories in a psychology which, as in cognitivism, was based exclusively on sense categories. It was considered as trivial, that some objects may be identical in relation to sense categories, and from that reason had to be distinguished from each others as choice categories. The existence of such doublets was used by P. F. Strawson to defend the existence of "individuals" or particulars that could not be defined with their universal properties alone [11]. And of course this was also a critique of cognitivism. As another extreme there might be "unica", i.e. objects which with a finite description in universal terms could be identified alone in universal terms, in which case their identity as choice categories was unnecessary for singling them out. But in between these extremes there is the whole domain of ordinary everyday objects which in one hand are different in the sense, that every time we have two of them we can tell the difference (otherwise they would be doublets) but where on the other hand it would be an infinite and impossible task to describe an object to be sure that no other object would fit the description (otherwise it would be a "unica"). It is this dominant part of our everyday world which is the target of this first version of the model. This has not been made sufficient explicit in earlier expositions [4; 8]. And finally I stress here that infinity in the model, which appears in some theorems as a consequence of the axioms, is not necessarily infinity of objects or descriptions in the real world in the strict sense, but just what exceeds our capacity for selecting and deciding. It is a model of what is "practicable" within finite time in the "mess" of ordinary objects.

Here are the axioms where Ù, as told, is the world of objects excluding doublets and "unica".

Ax. 1: There is more than one object in Ù

Ax. 2: The intersection of two sense categories is a sense category

Ax. 3: The union of any set of sense categories is a sense category

Ax. 4: For any two objects in Ù there are two disjunct sense categories so that one object is in the one and the other object in the other (disjunct categories have no objects in common)

Ax. 5: No sense category contains just one object

Ax. 6: No non-empty choice category is a sense category

Ax. 7: There exist a non-empty choice category

Ax. 8: Any non-empty choice category contains a choice category containing only one object

Ax. 9: The intersection of two choice categories is a choice category

Ax. 10: The union of two choice categories is a choice category

Ax. 11: The intersection of a choice category and a sense category is a choice category

From this set of axioms it can be derived, that the empty set Ø and the universe of objects Ù are sense categories and this together with Ax. 2 and 3 define the set of sense categories as so-called open sets in a topological space. Ax. 4 says in addition that it is a so-called Hausdorff-space, and Ax. 5 that it further is a perfect Hausdorff-space or just a perfect topology. This is the classical foundation of most topological spaces in Mathematics. And you can say that the choice categories as defined in the axioms form a more discrete structure in this topology. Basically the topology is asymmetric in the sense that it follows from the axioms that the complement of some sense categories are not sense categories, and that the complement of choice categories are never choice categories. This corresponds to the interpretation of the topology as a picture of the asymmetry, or asymmetric border, between figure and ground induced in the world of objects with the introduction of a subject. The subject puts its stamp on the objective world as an arrow of orientation in matter, to be a little metaphorical. This is in contrast to mereotopology which at the outset is symmetrical without reference to a subject or agency, and from that reason has problems interpreting asymmetries [13].

I will return on the surprising richness of theorems derived from the axioms and their psychological interpretations in the workshop. Here I will just point to one question in relation to the set of axioms because it is connected with my suggestion for pre-reading to the mathematicians participating in the workshop.

Completeness: An extreme case or "baseline" topology

If sense categories and choice categories are interpreted as two kinds of categories we as human subjects are able to extract or identify as subsets in the world of objects using definition by universal properties and selection by "just taking" something "at hand", respectively, then this could be combined not only as intersections mentioned in Ax. 11 but also as unions, where "pure" sense categories and choice categories are special cases. If we now define such a union as a "decidable category" we can set up a new comprehensive topology for decidable categories.

One fundamental question immediately arises in this case. It can easily be proven that there are subsets of Ù that are neither sense categories nor choice categories. But are there subsets that are not decidable (unions of the two kinds of categories), or is the set of decidable categories "complete" in the sense that there is a space organized as described with the axioms where all subsets are decidable? In this case we could claim that there need not be any other ways of selecting subsets in the world than as described here.

This question of possible completeness is not quite easy to answer using ordinary well-known concepts from set theoretical topology. However, it has been proven by professor Jørgen Hoffmann-Jørgensen [3], Institute of Mathematics, Aarhus University, that if you use the Axiom of Choice in a version close to Hausdorff's "maximal" principle, known as Zorn's Lemma, you can prove the "completeness" of the topology of decidable categories. The proof refers to some exotic structure known as Maximal Perfect Topologies [2; 12]. It is special in the sense that the topology is proved to exist but you cannot give an explicit example. In mathematical terms it is not constructive.

In addition to this proven implication it is very likely that the opposite is also true, i.e. that the completeness of the topology implies the Axiom of Choice ("Hoffmann's Conjecture"). In this case the completeness is a new version of the Axiom, and a very fundamental one! The conjecture has not been proven, however, despite attempts in Aarhus and Moscow.

So we eventually returned to fundamental Mathematics from psychological premises, and it is tempting to think that we have hit some common "ontological" basis for the two domains.

Non-completeness: More realistic cases

The complete case discussed above is, however, extreme in several aspects but at the same time an important "baseline" or departure for comparisons and interpretation of less extreme and more realistic cases in relation to psychology.

The complete case describes a situation where we "equipped" with decidable categories so to say have access to every possible subset of objects in Ù. The "price" is that we cannot construct an explicit model of the world which maps its structure. The universe of objects Ù is chaotic or in mathematical language completely "disconnected" [2]. If we on the other hand supply the 11 axioms with new ones, specifying more special cases by introducing some constraints on the range of choice categories and in the structure of sense categories, some subsets in Ù will no more be decidable or accessible. The set of decidable categories will no longer be complete, but on the other hand we can construct some model which maps its structure.

If for instance we reduce or constrain the set of sense categories to have a countable basis or even be metric the set of decidable categories will be reduced drastically, and if in addition it should be parametric, i.e. open intervals in some finite dimensional space of numbers, this will go even further. But what comes out is a well-ordered structure of subsets in Ù.

There is a complementarity between strict explicit order in our system of categories and the access to distinctions in the world. You cannot have both. I think this is a well known fact to psychologists fighting with the opposition between formalized, parametric methods and more intuitive or "qualitative" ones.

Some preliminary conclusions and perspectives

As hinted above there seem to be convergence, or even some identity, in the ontology of Mathematics and Psychology, where a duality of sense categories and choice categories, or of qualitative and numerical identity, has a common structure. In addition it seems that the duality between numerical and qualitative identity contained in the axiomatic system could also be seen as laying behind modern Physics. When Bohr could not persuade Einstein about entanglement of particles in Quantum Mechanics he did not only refer to the particles as a connected "unity" which has been interpreted as a holism, but in fact also to their "uniqueness" which was apparently not understood. Perhaps Bohr had an implicit ontology not far from the present one. Some of his references to Danish philosophy could point in that direction, but at this stage it is only a bold suggestion.

Extensions of the axiomatic model

As said above the model presented is a first version. There is a later, simpler (only 9 axioms) and more comprehensive (including "unica") which I can present at the workshop. But the question of completeness has not yet been investigated formally in this case, which is one reason why I chose to present the first version.

Furthermore the model, in both versions, can without any substantial changes be generalized to comprise objects moving and changing properties in time. Here I use the so-called compact-open topology for continuous function-spaces [1].

Suggested reading:

For everybody participating in the workshop I will recommend [5; 6; 7 and 9]

For the mathematicians already familiar with basic topology I recommend [2] which is an introduction to maximal perfect (non-constructive) topologies, and supplementary [12] for some examples of application. For a proof of the existence of a complete space of decidable categories defined as unions of sense categories and choice categories, given the validity of Zorn's Lemma, see [3] first 6 pages. The section "4. Example" on the last pages in [3] is an independent demonstration of a constructive example of a perfect Hausdorff-space on a countable point-set but with a non-countable basis.

A note (to the mathematicians): What I here have called "decidable categories" is by Hoffmann-Jørgensen called "resolvable categories" which correspond to subsets in what Hewitt calls an "irresolvable" topology, which corresponds to what I have called "complete".

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