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A Local Stable Bootstrap for Power Variations of Pure-Jump Semimartingales and Activity Index Estimation*

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Abstract

We provide a new resampling procedure - the local stable bootstrap - that is able to mimic the dependence properties of realized power variations for pure-jump semimartingales observed at different frequencies. This allows us to propose a bootstrap estimator and inference procedure for the activity index of the underlying process, β, as well as a bootstrap test for whether it obeys a jump-diffusion or a pure-jump process, that is, of the null hypothesis $\mathcal{H}_0 : \beta = 2$ against the alternative $\mathcal{H}_1 : \beta < 2$. We establish first-order asymptotic validity of the resulting bootstrap power variations, activity index estimator, and diffusion test for $\mathcal{H}_0$. Moreover, the finite sample size and power properties of the proposed diffusion test are compared to those of benchmark tests using Monte Carlo simulations. Unlike existing procedures, our bootstrap test is correctly sized in general settings. Finally, we illustrate use and properties of the new bootstrap diffusion test using high-frequency data on three FX series, the S&P 500, and the VIX.

Keywords: Activity index, Bootstrap, Blumenthal-Getoor index, Confidence Intervals, High-frequency Data, Hypothesis Testing, Realized Power Variation, Stable Processes.

JEL classification: C12, C14, C15, G1

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1 Introduction

Itô semimartingales comprise an important class of continuous time processes that are widely used to model financial asset prices, asset return volatility, volume of trades, among others. Hence, power variations of discrete increments of such processes have similarly become imperative risk measures in asset- and derivatives pricing, risk management, and portfolio selection.[1] Defined as the sum of absolute (log-)innovations raised to a given power, p, these risk measures contain the realized variance as a special case with $p = 2$. When the leading term of the Itô semimartingale is a Brownian motion, it is well known that, under weak conditions, the realized variance converges to the quadratic variation of the process as the interval between successive observations progressively shrinks, see, e.g., Jacod & Shiryaev (2003). If, however, the leading term is a pure-jump process with activity index $\beta \in (0, 2)$, cf. Aït-Sahalia & Jacod (2009), the realized variance diverges since power variation statistics with $\beta < p$ are not asymptotically bounded (formal definitions are given below). Hence, it is important to estimate and make inference on β to determine whether the underlying process is a jump-diffusion or a pure-jump semimartingale, that is, to distinguish between the two hypotheses $\mathcal{H}_0 : \beta = 2$ and $\mathcal{H}_1 : \beta < 2$, when specifying financial models and making inference on financial risk measures.

The limiting behavior of realized power variations has been extensively studied in the continuous Brownian semimartingale case, see, e.g., Barndorff-Nielsen & Shephard (2002, 2003), and later extended to allow for jumps that may affect the limiting properties of the statistics in Jacod (2008). However, while the Brownian semimartingale model is commonly adopted in financial economics, e.g. Andersen & Benzoni (2012) and references therein, recent empirical evidence suggests that pure-jump semimartingales with infinite activity jumps may provide a better approximation for logarithmic innovations in equities, equity indices and exchange rates, see, e.g., Wu (2008), Aït-Sahalia & Jacod (2009, 2012), and Jing, Kong & Liu (2012); for option pricing, e.g., Carr & Wu (2003) and Wu (2006); and for volatility modeling, see, e.g., Carr, Geman, Madan & Yor (2003), Todorov & Tauchen (2011*b*), Andersen, Bondarenko, Todorov & Tauchen (2014), and Todorov, Tauchen & Grynkiv (2014). Either in the process of establishing this evidence, motivated by it, or independently, the study of the limiting behavior of realized power variations has been extended to the case where the underlying innovation process obeys a pure-jump semimartingale, see Aït-Sahalia & Jacod (2009), Todorov & Tauchen (2011*a*), Todorov (2013) along with earlier work by Woerner (2003, 2007). In particular, consistency and asymptotic central limit theory for realized power variation estimates of the stochastic scale have been established and used to construct estimators of the underlying activity index, β.

However, while asymptotically valid, central limit theory based coverage for both the stochastic scale and β may have poor finite sample properties, especially when applied to a moderate number of intra-daily observations. This is exemplified by the Monte Carlo study in Barndorff-Nielsen & Shephard (2005), who show that the feasible asymptotic theory for realized variance in a Brownian semimartingale setting may provide a poor guide to the finite sample distribution, and by Jing, Kong,

[1] See, e.g., Andersen, Bollerslev & Diebold (2010) and Barndorff-Nielsen & Shephard (2007) for reviews and references.

Liu & Mykland (2012), who show that the activity index estimator of Aït-Sahalia & Jacod (2009) suffer from a non-negligible bias and large mean-squared errors in finite samples. As a result, existing tests of the null hypothesis $\mathcal{H}_0$ against $\mathcal{H}_1$ may suffer from similar distortions. To improve finite sample inference on the realized variance estimator for the continuous Brownian semimartingale case, Gonçalves & Meddahi (2009) propose a wild bootstrap and show that it achieves second-order refinements. Hounyo (2014) provides further improvements with a local Gaussian bootstrap, which achieves third-order refinements. However, none of the two procedures accommodate power variations at different frequencies, nor allow the process to be a pure-jump semimartingale.

In this paper, we propose a new resampling procedure - the local stable bootstrap - that is able to mimic the dependence properties of power variations for pure-jump semimartingales observed at different frequencies. As a result, we use the bootstrap to propose a new estimator and inference procedure for the activity index, β, along with a new test for whether the underlying process is a jump-diffusion or a pure-jump semimartingale, that is, of $\mathcal{H}_0$ against $\mathcal{H}_1$. We establish first-order asymptotic validity of the local stable bootstrap as well as the resulting estimator of β. Moreover, and as a by-product of the analysis, we establish consistency of a bipower variation estimator for the stochastic scale when the underlying process obeys a pure-jump semimartingale. We design the local stable bootstrap test for $\mathcal{H}_0$ to have good size properties by targeting the resampling towards a specific null hypothesis, similar to the recommendations in Davidson & MacKinnon (1999). Hence, when assessing the finite sample size and power of our test in a Monte Carlo study, it is not surprising that we find our test to be correctly sized in a variety of settings, in contrast to existing tests for $\mathcal{H}_0$ that are oversized for all sample sizes and settings considered. Finally, we illustrate the practical use of the local stable bootstrap procedure by testing $\mathcal{H}_0$ using high-frequency data on three exchange rate series, the S&P 500 index and the VIX from 2011. We find that the (null) presence of a diffusion is rarely rejected for the S&P 500, rejected 60-87% of the days for the VIX, whereas the rejection rates for the exchange rate series falls between the two. Moreover, we show that existing tests uniformly reject more often than our bootstrap test, verifying the patterns from the simulation study.

The outline of the paper is as follows. Section 2 lays out the setup, assumptions, and review existing results. Section 3 introduces the local stable bootstrap procedure and establishes its asymptotic properties. Section 4 contains the simulation study, and Section 5 provides the empirical analysis. Finally, Section 6 concludes. An appendix contains additional assumptions, proofs, and implementation details. The following notation is used throughout: $\mathbb{P}^*$, $\mathbb{E}^*$ and $\mathbb{V}^*$ denotes the probability measure, expected value and variance, respectively, induced by the bootstrap resampling and is, thus, conditional on a realization of the original time series. In addition, for a generic sequence of bootstrap statistics X_n^*, we write $X_n^* = o_{p^*}(1)$ or $X_n^* \xrightarrow{\mathbb{P}^*} 0$ as $n \to \infty$, in probability-$\mathbb{P}$, if for any $\varepsilon > 0$ and any $\delta > 0$, $\lim_{n\to\infty} \mathbb{P}\left[\mathbb{P}^*\left(|X_n^*| > \delta\right) > \varepsilon\right] = 0$. Similarly, we write $X_n^* = O_{p^*}(1)$ as $n \to \infty$, in probability-$\mathbb{P}$, if for all $\varepsilon > 0$ there exists a $M_\varepsilon < \infty$ such that $\lim_{n\to\infty} \mathbb{P}\left[\mathbb{P}^*\left(|X_n^*| > M_\varepsilon\right) > \varepsilon\right] = 0$. Finally, we write $X_n^* \xrightarrow{d^*} X$ as $n \to \infty$, in probability-$\mathbb{P}$, if conditional on the original sample, X_n^* converges weakly to some X under $\mathbb{P}^*$, for all samples contained in a set with probability-$\mathbb{P}$ approaching one.

2 Setup, Assumptions, and Existing Results

This section introduces a general Itô semimartingale framework and provides the necessary assumptions to perform the theoretical analysis. Moreover, it defines the realized power variations statistics of interest, the activity index of the underlying process and its estimator. Finally, we review some asymptotic results, which are relevant for designing our local stable bootstrap.

2.1 The Framework

Let Z denote the logarithmic asset price process defined on a filtered probability space, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where the information filtration $(\mathcal{F}_t) \subseteq \mathcal{F}$ is an increasing family of σ-fields satisfying $\mathbb{P}$-completeness and right continuity. In particular, we assume that Z obeys an Itô semimartingale of the form

$$dZ_t = \alpha_t dt + \sigma_{t-} dL_t + dY_t, \quad 0 \leq t \leq 1, \tag{1}$$

where α_t and σ_t are some processes with càdlàg paths; L_t is a Lévy process, which is a martingale, if of infinite variation, and a sum of jumps, if of finite variation; and, finally, Y_t is some "residual" jump process, which is dominated by L_t over small time scales. As will be formalized below, we assume throughout that L_t is "locally" a stable process with activity parameter β, that is, L_t behaves like a stable process over small time increments.[2] In particular, we will restrict attention to the empirically relevant case $1 < \beta \leq 2$, i.e., where the high-frequency movements in Z_t behave locally like a stable process of infinite activity with stochastic scale, σ_t. For the boundary case $\beta = 2$, L_t is a Brownian motion at high frequencies and for $1 < \beta < 2$, L_t behaves like a pure-jump process, but continues to dominate the drift term, α_t. Hence, separating the cases $\beta = 2$ and $\beta < 2$ is of a central importance for, e.g., volatility measurement, see, e.g., Todorov & Tauchen (2011*a*) and Todorov (2013).

The process in (1) covers most widely applied models of financial asset prices, for example, the affine class of jump-diffusions in Duffie, Pan & Singleton (2000), and the time-changed Lévy models of Carr et al. (2003). Furthermore, note that the "residual" jump process, Y_t, need not be independent of L_t (nor α_t and σ_t). This implies that Z_t need not share tail behavior of L_t, which may be driven, for example, by tempered stable process, whose tail behavior may be very different from that of a stable process. Exactly such flexible modeling of the tails has been emphasized, e.g., for characterizing investor equity, variance, and jump risk premia, see Bollerslev & Todorov (2011), and by Andersen, Fusari & Todorov (2014) in the context of index option pricing.

2.2 The Objective

To set the stage, we assume to have a discrete set of observations Z_{t_i}, $i = 0, 1, \ldots, n$, available from an equidistant sampling grid, that is, where $t_i = i/n$ $\forall i$, and define a general class of power variation

[2] Details on stable processes may be found in, e.g., Sato (1999).

statistics as

$$V_n(p,Z,\upsilon) \equiv \sum_{i=\upsilon}^{n} |\Delta_i^{n,\upsilon} Z|^p, \qquad \Delta_i^{n,\upsilon} Z = Z_{t_i} - Z_{t_{i-\upsilon}} = \sum_{j=1}^{\upsilon} \Delta_{i+1-j}^{n,1} Z, \tag{2}$$

which are indexed by the frequency υ. In particular, we seek to make bootstrap inference on the quantities,

$$V_n(p,Z,1) = \sum_{i=1}^{n} \left|\Delta_i^{n,1} Z\right|^p, \quad V_n(p,Z,2) = \sum_{i=2}^{n} \left|\Delta_{i-1}^{n,1} Z + \Delta_i^{n,1} Z\right|^p, \tag{3}$$

whose asymptotic central limit theory have already been studied in Todorov & Tauchen (2011*a*) and Todorov (2013) under similar assumptions, see also Woerner (2003, 2007). To improve the finite sample inference in the special case where Z_t is a Brownian semimartingale without jumps, Gonçalves & Meddahi (2009) propose a wild bootstrap procedure for $V_n(2,Z,1)$, showing that it achieves second-order refinements, and Hounyo (2014) provides further improvements by proposing a local Gaussian bootstrap, which achieves third-order refinements. The present problem, however, is much more demanding as we seek to make bootstrap inference on $V_n(p,Z,\upsilon)$, that is, for some power p, difference orders $\upsilon = (1,2)$, and, perhaps most importantly, for the general class of processes (1).

The added challenge is readily illustrated by the definition of the generalized Blumenthal-Getoor index, cf. Aït-Sahalia & Jacod (2009), which, under certain regularity conditions that will be stated below, suggests that

$$\beta = \inf\left\{p > 0 : \operatorname*{plim}_{n\to\infty} V_n(p,Z,1) < \infty\right\}, \tag{4}$$

that is, power variations diverge for powers greater than the "local" activity index, β.[3] This implies that unless $\beta = 2$, the widely applied realized variance quantity, $V_n(2,Z,1)$, diverges. Hence, we also seek to utilize our bootstrap procedure for (3) to make inference on the activity estimator,

$$\hat{\beta}(p) = \frac{p\ln(2)}{\ln\left(V_n(p,Z,2)\right) - \ln\left(V_n(p,Z,1)\right)} \mathbf{1}\{V_n(p,Z,2) \neq V_n(p,Z,1)\}, \tag{5}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. Note that (5) combines the estimators from Todorov & Tauchen (2011*a*) and Todorov (2013), and it is recently used empirically by Andersen, Bondarenko, Todorov & Tauchen (2014) to study the high-frequency dynamics of S&P 500 equity-index options.

Remark 1. *The analysis is performed without consideration of market microstructure noise, which is known to contaminate observed prices at tick-by-tick frequencies. Several ways of correcting for noise-induced effects have been proposed in the context of quadratic variation estimation for Brownian semimartingales. However, for pure-jump semimartingales, common practice is to use moderately sampled data to alleviate the impact of noise. Hence, the use of a bootstrap inference procedure is particularly warranted in this settings since the feasible asymptotic theory may deviate substantially from finite sample distributions, see, e.g., the remarks on Barndorff-Nielsen & Shephard (2005) and*

[3] The index in (4) is a generalization of the original Blumenthal-Getoor index, proposed by Blumenthal & Getoor (1961), which is only defined for pure-jump Lévy processes.

Jing, Kong, Liu & Mykland (2012) in the introduction above.

2.3 Assumptions

First, let us recall the definition of a Lévy process on Jacod & Shiryaev (2003, p. 75), which states that L_t is a Lévy process with characteristic triplet (b, c, ν) if its (logarithmic) characteristic function is given by

$$\ln \mathbb{E}\left[e^{\mathrm{i}uL_t}\right] = \mathrm{i}tub - tcu^2/2 + t\int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - iu\kappa(x)\right)\nu(dx) \tag{6}$$

where $\kappa(\cdot)$ is a continuous truncation function, which behaves like $\kappa(x) = x$ in a neighborhood of the origin, and $\nu(\cdot)$ is the Lévy measure, whose density controls the activity of the process. Following, e.g., Todorov (2013), we will for simplicity assume throughout that $\kappa(-x) = -\kappa(x)$ and, furthermore, let $\kappa'(x) = x - \kappa(x)$. Intuitively, the truncation function assists in quantifying the asymptotic behavior of L_t depending on the activity index, β. For example, when $\beta \in (1, 2)$, we need both $\kappa'(x)$ and $\kappa(x)$ to decompose the infinity activity Lévy process into the martingale component of the small jumps and large jumps, see, e.g., the discussion in Aït-Sahalia & Jacod (2012).

Assumption 1. *Let the constants A_1 and A_2 satisfy $A_1 > 0$ and $A_2 \geq 0$, respectively, then*

(a) *L_t is a Lévy process with characteristic triplet $(0, 0, \nu)$ where the Lévy measure ν has density defined by $\nu(x) = \nu_1(x) + \nu_2(x)$ with*

$$\nu_1(x) = A_1|x|^{-(\beta+1)} \quad \text{and} \quad |\nu_2(x)| = A_2|x|^{-(\beta'+1)} \quad \text{when } |x| \leq x_0$$

for some $x_0 > 0$, $\beta' < 1$, $\beta \in (1, 2]$, $\int_{\mathbb{R}} |x|\nu(x)dx < \infty$ and, finally, where

$$A_1 = \left(\frac{4\Gamma(2-\beta)|\cos(\beta\pi/2)|}{\beta(\beta-1)}\right)^{-1}, \quad \text{when } \beta \in (1, 2).$$

(b) *Y_t is an Itô semimartingale with a characteristic triplet (B, C, ν^Y) given by*

$$\left(B, C, \nu^Y\right) = \left(\int_0^t \int_{\mathbb{R}} \kappa(x)\nu_s^Y(dx)ds, 0, dt \otimes \nu_t^Y(dx)\right)$$

with $\int_{\mathbb{R}}(|x|^{\beta'+\epsilon} \wedge 1)\nu_t^Y(dx)$ being locally bounded and predictable, and where β' satisfies the conditions in (**a**) *and $\epsilon > 0$ is arbitrarily small. A formal definition of Itô semimartingales, including the characteristic triplet, is provided in Jacod & Shiryaev (2003, pp. 75-76).*

Assumption 1 imposes conditions similar to those in Aït-Sahalia & Jacod (2009), Todorov & Tauchen (2011, 2012), and Todorov (2013). In particular, it formalizes the notion of local stable behavior over small time increments, that is, we have $h^{-1/\beta}L_{ht} \xrightarrow{d} S_t$ for $h \to 0$ where convergence holds under the Skorokhod topology on the space of càdlàg functions, and S_t is a strictly stable process with characteristic function

$$\ln \mathbb{E}\left[e^{iuS_t}\right] = -t|u|^{\beta}/2, \tag{7}$$

see, e.g., Todorov & Tauchen (2012, Lemma 1). Intuitively, the result follows as $\beta' < \beta$ such that the behavior from the Lévy density of a stable, $\nu_1(x)$, dominates the contribution from the other jump measure $\nu_2(x)$, which is not necessarily Lévy, when $h \to 0$. As in Todorov & Tauchen (2012), we conveniently normalize the constant A_1 since it ensures that when $\beta \to 2$, the jump process converges finite-dimensionally to a Brownian motion. This normalization seems innocuous from a modeling perspective since we observe Z_t, whose leading small time increment behavior is determined by an integral of $\sigma_{t-}dL_t$, and not the components σ_{t-} and dL_t separately.

Similar to the residual jump component $\nu_2(x)$ in L_t, the activity of the high-frequency "residual", Y_t, is also restricted by the index β'. However, unlike the former, whose time variation is also determined by the stochastic scale, σ_t, the latter is almost unrestricted in its time variation. This allows Z_t to exhibit different, and more general, variation from that implied by the stable measure, $\nu_1(x)$, at larger time increments. For example, and as thoroughly discussed in Todorov & Tauchen (2012) and Todorov (2013), having two high-frequency jump "residual" components makes Z_t general enough to nest time-changed Lévy processes and Lévy-driven CARMA models.

Finally, we need to impose conditions on the drift, a_t, and the stochastic scale, σ_t. Intuitively, these are required to obey Itô semimartingales, which may be arbitrarily driven by Brownian motions and random Poisson measures with locally bounded coefficients. Since the technical details are identical to those in Todorov & Tauchen (2011*a*, (3.10)-(3.11)) and Todorov (2013, Assumption B), they are deferred to Appendix A for ease of exposition. It is important to note, however, that the conditions allow for dependence between the innovations in α_t, σ_t, and the driving Lévy process, L_t, which is important for financial applications, see, among others, Klüppelberg, Lindner & Maller (2004), Bollerslev & Todorov (2011), Andersen, Fusari & Todorov (2014), and Todorov et al. (2014).

Remark 2. *The assumption of symmetry for A_1 when $x > 0$ and $x < 0$ as well as the conditions $\beta' < 1$ and $\beta \in (1, 2]$ (though $\beta' < \beta$ remains) may be relaxed following the work of Todorov (2013). In particular, this involves replacing the pair $(V_n(p, Z, 1), V_n(p, Z, 2))$ with $(V_n(p, Z, 2), V_n(p, Z, 4))$, and perform an analysis similar to the one below. Whereas the latter combination is more robust to drift and asymmetric jumps in L_t, it is at the expense of a somewhat larger asymptotic variance when estimating the integrated power variation of the stochastic scale and, as a result, the activity index.*

2.4 Review of Relevant Asymptotic Results

Before stating asymptotic results that are relevant for designing our local stable bootstrap, we need to impose a few additional, yet mild restrictions on the activity indices β, β', and the power p.

Assumption 2. *In addition to the restrictions implied by Assumption 1, the activity indices β and β', along with the power p, are assumed to satisfy one of the following conditions:*

(a) *$p \in (0, \beta)$;*

(b) *$\beta' < \beta/2$, $\beta > \sqrt{2}$ as well as $p \in \left(\frac{|\beta-1|}{2} \vee \frac{2-\beta}{2(\beta-1)} \vee \frac{\beta\beta'}{2(\beta-\beta')}, \beta/2\right)$;*

While Assumption 2 (a) provides a mild condition for sub-additivity of functionals of the form $|x|^p$, Assumption 2 (b) gives sufficient conditions on β, β', and p to invoke a central limit theorem for power variation statistics below. The lower bound on p is here determined by the drift and the less active jump components of Z. In particular, $p > \frac{2-\beta}{2(\beta-1)}$ and $p > \frac{|\beta-1|}{2}$ are induced by the presence of a drift term, leading to the restriction $\beta > \sqrt{2}$. The remaining lower and upper bounds $p > \frac{\beta\beta'}{2(\beta-\beta')}$ and $p < \beta/2$, respectively, are required to eliminate the contribution from less active residual jump components at high frequencies, see also Todorov & Tauchen (2011*a*, Remark 3.7).

Finally, let us define $\mu_p(\beta) = \mathbb{E}[|S_i|^p]$ where $S_0, S_1, \ldots$ are i.i.d. strictly β-stable random variables whose characteristic function satisfies (7) for $t = 1$, and, moreover, let $\boldsymbol{\Sigma}(p, \beta, k) = \mathbb{E}[\boldsymbol{S}_1 \boldsymbol{S}'_{1+k}]$ for $k = 0, 1$ where $\boldsymbol{S}_i = (|S_i|^p - \mu_p(\beta), |S_i + S_{i+1}|^p - 2^{p/\beta}\mu_p(\beta))'$, then we may state the following lemma, which combines results from Todorov & Tauchen (2011*a*) and Todorov (2013).

Lemma 1. *Under Assumption 1 and Assumption 3 of Appendix A, then if additionally*

(a) *Assumption 2 (a) holds,*

$$n^{p/\beta-1} V_n(p, Z, 1) \xrightarrow{\mathbb{P}} \mu_p(\beta) \int_0^1 |\sigma_s|^p ds, \qquad n^{p/\beta-1} V_n(p, Z, 2) \xrightarrow{\mathbb{P}} 2^{p/\beta}\mu_p(\beta) \int_0^1 |\sigma_s|^p ds,$$

(b) *Assumption 2 (b) holds,*

$$\sqrt{n} \begin{pmatrix} n^{p/\beta-1} V_n(p, Z, 1) - \mu_p(\beta) \int_0^1 |\sigma_s|^p ds \\ n^{p/\beta-1} V_n(p, Z, 2) - 2^{p/\beta}\mu_p(\beta) \int_0^1 |\sigma_s|^p ds \end{pmatrix} \xrightarrow{d_s} \boldsymbol{\Omega}(p, Z) \times \boldsymbol{\mathcal{N}}$$

where $\boldsymbol{\mathcal{N}}$ is a two-dimensional standard normal random variable defined on an extension of the original probability space and orthogonal to $\mathcal{F}$, and $\boldsymbol{\Omega}(p, Z) = \int_0^1 |\sigma_s|^{2p} ds \times \boldsymbol{\Xi}$ where the 2×2 matrix $\boldsymbol{\Xi} \equiv (\Xi_{i,j})_{1 \le i,j \le 2}$ is defined as $\boldsymbol{\Xi} = \boldsymbol{\Sigma}(p, \beta, 0) + \boldsymbol{\Sigma}(p, \beta, 1) + \boldsymbol{\Sigma}(p, \beta, 1)'$.

Proof. Under the stated assumptions, the two consistency results in (a) follow by applying Todorov & Tauchen (2012, Lemma 1) in conjunction with Todorov & Tauchen (2011*a*, Theorem 3.2 (b)) and Todorov (2013, Theorem 2 (a)) for $V_n(p, Z, 1)$ and $V_n(p, Z, 2)$, respectively. Similarly, the joint central limit theorem in (b) follows by Todorov & Tauchen (2012, Lemma 1) in conjunction with Todorov & Tauchen (2011*a*, Theorem 3.4 (b)), Todorov (2013, Theorem 2 (b)) and a stable Cramer-Wold theorem, see Varneskov (2014, Lemma C.1 (d)). □

Corollary 1. *Under the conditions for Lemma 1 (b),*

$$\sqrt{n}\left(\hat{\beta}(p) - \beta\right) \xrightarrow{d_s} \Omega_\beta(p, Z) \times \mathcal{N}, \qquad \Omega_\beta(p, Z) = \frac{\int_0^1 |\sigma_s|^{2p} ds}{\left(\int_0^1 |\sigma_s|^p ds\right)^2} \times \frac{\beta^4}{\mu_p^2(\beta) p^2 (\ln(2))^2} \times \tilde{\Xi}$$

where $\mathcal{N}$ is a univariate standard normal random variable defined on an extension of the original probability space and orthogonal to $\mathcal{F}$, and $\tilde{\Xi} = \Xi_{1,1} - 2^{1-p/\beta}\Xi_{1,2} + 2^{-2p/\beta}\Xi_{2,2}$ with the 2×2 covariance matrix $\boldsymbol{\Xi} = (\Xi_{i,j})_{1 \leq i,j \leq 2}$ defined as in Lemma 1.

Lemma 1 formalizes the asymptotic behavior of $V_n(p, Z, 1)$ and $V_n(p, Z, 2)$, whose joint law is described using the notion of stable convergence, see, e.g., Jacod & Shiryaev (2003, pp. 512-518) and Podolskij & Vetter (2010) for details. The specific combination of estimators, $V_n(p, Z, 1)$ and $V_n(p, Z, 2)$, and their use in defining $\hat{\beta}(p)$, is inspired by the approach in Andersen, Bondarenko, Todorov & Tauchen (2014), who, however, do not state any formal asymptotic results for $\hat{\beta}(p)$. Furthermore, they define L_t (in our notation) to be a standard, strictly stable process, whereas we only require it to be locally stable, as described by Assumption 1. Most importantly for our purposes, however, the central limit theory in Lemma 1 and Corollary 1 highlight the dependence patterns, we seek to replicate with our proposed bootstrap procedure in order to perform inference on power variation statistics and the activity index, β, respectively. This is described in detail next.

3 The Local Stable Bootstrap

In this section, we propose a novel bootstrap procedure to perform inference on the 2-dimensional vector of power variation statistics $(V_n(p, Z, 1), V_n(p, Z, 2))'$ and, in conjunction with the delta method, the activity index estimator, $\hat{\beta}(p)$. Specifically, we suggest to resample the (possibly) higher-order increments $\Delta_i^{n,v} Z$ for each $i = v, \ldots, n$ such as to mimic their dependence properties. In order to motivate our bootstrap procedure, let us highlight two features of the locally stable process Z. From Todorov & Tauchen (2012, Lemma 1), we already know $h^{-1/\beta} L_{ht} \xrightarrow{d} S_t$ for $h \to 0$. Then, as the remaining terms in (1) are of strictly lower order under Assumptions 1 and 2, it is straightforward to deduce that

$$h^{-1/\beta} \frac{Z_{t+sh} - Z_t}{\sigma_t} \xrightarrow{d} S'_{t+s} - S'_t \qquad \text{as} \quad h \to 0, \tag{8}$$

similarly, with convergence under the Skorokhod topology on the space of càdlàg functions where S'_t has a distribution identical to that of the strictly stable process S_t, which is described in (7). Furthermore, it follows by self-similarity of strictly stable processes that

$$S_t - S_s \stackrel{d}{=} |t - s|^{1/\beta} S_1, \quad 0 \leq s < t. \tag{9}$$

Intuitively, the result in (8) suggests that each high frequency increment of Z behaves locally like a stable process with a constant scale σ_t, which is "known" at the onset of the increment. Hence, if the stochastic process σ_t was directly observable at each discrete time point t_i, $i = 0, \ldots, n$, we could scale the increments of Z accordingly, and its resulting (infill asymptotic) behavior will be similar to that of a sequence of i.i.d. stable random variables, which suggests that a wild bootstrap-type procedure will be appropriate in this setting. Hence, we introduce a particular wild bootstrap – the local stable bootstrap – that may be summarized as the following 3-step algorithm.

Algorithm 1.

Step 1. Estimate the activity index, β, of the process Z using the estimator $\hat{\beta}(p)$ defined in (5).

Step 2. Generate an $n+1$ sequence of identically and independently distributed $\hat{\beta}(p)$-stable random variables $S_1^*, S_2^*, \ldots, S_{n+1}^*$, whose characteristic function are defined as

$$\ln \mathbb{E}\left[e^{\mathrm{i}uS_i^*}\right] = -|u|^{\hat{\beta}(p)}/2, \qquad \forall i = 1, \ldots, n+1. \tag{10}$$

Step 3. The local stable bootstrap generates observations according to

$$\Delta_i^{n,\upsilon} Z^* = \Delta_i^{n,1} Z \cdot \Big(\sum_{t=1}^{\upsilon} S_{i+t-1}^*\Big), \qquad i = \upsilon, \ldots, n, \tag{11}$$

and redefines the power variation statistics $V_n(p, Z, 1)$ and $V_n(p, Z, 2)$ as follows

$$V_n^*(p, Z, 1) = \sum_{i=1}^{n} |\Delta_i^{n,1} Z^*|^p, \qquad V_n^*(p, Z, 2) = \sum_{i=1}^{n} |\Delta_i^{n,2} Z^*|^p.$$

The three steps of the bootstrap algorithm deserves a few comments. First, to fully appreciate the careful design of Step 3, let us explicate the bootstrap power variation statistics as

$$V_n^*(p, Z, 1) = \sum_{i=1}^{n} |\Delta_i^{n,1} Z|^p |S_i^*|^p, \qquad V_n^*(p, Z, 2) = \sum_{i=1}^{n} |\Delta_i^{n,1} Z|^p |S_i^* + S_{i+1}^*|^p. \tag{12}$$

This decomposition highlights the respective contributions of the two components in $\Delta_i^{n,\upsilon} Z^*$ to the power variation statistics. Heuristically, the first component in each statistic, $\Delta_i^{n,1} Z$, contains information about the "scale" of the process Z, that is, about $n^{-1/\beta}\sigma_{t_{i-1}}$, and the second component, $\sum_{t=1}^{\upsilon} S_{i+t-1}^*$, is included to mimic the local asymptotic dependence, which arises as a result of using (possibly) higher-order increments in the construction of the power variation statistics.

Second, we stress that a direct generalization of the two bootstrap procedures in Gonçalves & Meddahi (2009) and Hounyo (2014), respectively, to power variation statistics for pure-jump semimartingales will *not* work for those based on high-order increments, that is, when $\upsilon > 1$. To clarify this point, note that we may write the resulting generalization of the procedure in Gonçalves & Meddahi (2009) as

$$\tilde{V}_n^*(p, Z, 2) = \sum_{i=2}^{n} |\Delta_{i-1}^{n,1} Z \cdot S_i^* + \Delta_i^{n,1} Z \cdot S_{i+1}^*|^p$$

when $\upsilon = 2$. A similar generalization of the procedure in Hounyo (2014) will also share this generic form, albeit it will be defined in blocks of contiguous observations. The main problem with a direct application of both bootstraps is the lack of separation between the additive components inside of the power functional $|x|^p$, which prevents the procedures, in combination with $V_n^*(p, Z, 1)$, from replicating the moments of the joint central limit theorem in Lemma 1 (b).

Third, the simple form of the characteristic function in Step 2 results from normalization of the constant A_1 in Assumption 1, which implies that the local asymptotic behavior of $h^{-1/\beta}L_t$ is like that of a strictly stable process with characteristic function (7). In general, however, the validity of our bootstrap algorithm pertains to the case where $A_1 > 0$ is arbitrary. The only two changes are that the characteristic function, we simulate from, becomes increasingly complicated, being of the general form (6), and that the characteristic parameters $\mu_p(\beta)$ and $\boldsymbol{\Sigma}(p, \beta, k)$ will have to be redefined, see, e.g., the corresponding definitions in Todorov & Tauchen (2011*a*) and Todorov (2013).

3.1 Moments of Bootstrap Power Variation Statistics

We start examining the properties of the local stable bootstrap by establishing asymptotic results for the first two moments of the bootstrap power variation statistics in Step 3. Before proceeding, however, let us define the analogous characteristic parameters of $S_1^*, S_2^*, \ldots, S_{n+1}^*$, the sequence of i.i.d. $\hat{\beta}(p)$-stable random variables generated in Step 2 of the bootstrap, as $\mathbb{E}^*[|S_i^*|^p] = \mu_p(\hat{\beta}(p))$ and $\mathbb{E}^*[\boldsymbol{S}_1^* \boldsymbol{S}_{1+k}^{*\prime}] = \boldsymbol{\Sigma}(p, \hat{\beta}(p), k)$ for $k = 0, 1$ where $\boldsymbol{S}_i^* = (|S_i^*|^p - \mu_p(\hat{\beta}(p)), |S_i^* + S_{i+1}^*|^p - 2^{p/\hat{\beta}(p)}\mu_p(\hat{\beta}(p)))'$. We, then, specifically seek to describe

$$\boldsymbol{\mathcal{E}}_n^*(p, Z) \equiv \mathbb{E}^* \left[n^{p/\beta - 1} \begin{pmatrix} V_n^*(p, Z, 1) \\ V_n^*(p, Z, 2) \end{pmatrix} \right], \quad \boldsymbol{\Omega}_n^*(p, Z) \equiv \mathbb{V}^* \left[\sqrt{n} n^{p/\beta - 1} \begin{pmatrix} V_n^*(p, Z, 1) \\ V_n^*(p, Z, 2) \end{pmatrix} \right] \tag{13}$$

and their probability limits $\boldsymbol{\mathcal{E}}^*(p, Z) = \text{plim}_{n\to\infty} \boldsymbol{\mathcal{E}}_n^*(p, Z)$, respectively, $\boldsymbol{\Omega}^*(p, Z) = \text{plim}_{n\to\infty} \boldsymbol{\Omega}_n^*(p, Z)$.

Lemma 2. *Suppose S_i^*, $i = 1, \ldots, n+1$, are i.i.d. strictly $\hat{\beta}(p)$-stable random variables, defined as described in Step 2 of the local stable bootstrap algorithm, then*

$$\boldsymbol{\mathcal{E}}_n^*(p, Z) = \begin{pmatrix} 1 \\ 2^{p/\hat{\beta}(p)} \end{pmatrix} \mu_p(\hat{\beta}(p)) n^{p/\beta - 1} V_n(p, Z, 1), \qquad \text{and}$$

$$\boldsymbol{\Omega}_n^*(p, Z) = n^{2p/\beta - 1} \boldsymbol{\Sigma}(p, \hat{\beta}(p), 0) \sum_{i=1}^{n} |\Delta_i^{n,1} Z|^{2p} + n^{2p/\beta - 1} \sum_{i=1}^{n-1} |\Delta_i^{n,1} Z|^p |\Delta_{i+1}^{n,1} Z|^p \left(\boldsymbol{\Sigma}(p, \hat{\beta}(p), 1) + \boldsymbol{\Sigma}(p, \hat{\beta}(p), 1)' \right).$$

Lemma 2 shows that the moments of the bootstrap power variation statistics depend on the characteristic parameters of the $\hat{\beta}(p)$-stable random variables generated in Step 2 as well as the properties of $V_n(p, Z, 1)$, $V_n(2p, Z, 1)$, and the bipower variation statistic, $BV_n(2p, Z, 1) = \sum_{i=1}^{n-1} |\Delta_i^{n,1} Z|^p |\Delta_{i+1}^{n,1} Z|^p$. Under Assumptions 1, 3, and $p \in (0, \beta/2)$, we may invoke Lemma 1 (a) to show

$$n^{p/\beta - 1} V_n(p, Z, 1) \xrightarrow{\mathbb{P}} \mu_p(\beta) \int_0^1 |\sigma_s|^p ds \qquad \text{and} \qquad n^{2p/\beta - 1} V_n(2p, Z, 1) \xrightarrow{\mathbb{P}} \mu_{2p}(\beta) \int_0^1 |\sigma_s|^{2p} ds.$$

However, before being able to characterize the probability limit of the whole asymptotic covariance matrix, $\boldsymbol{\Omega}^*(p, Z)$, a similar convergence result needs to be established for $BV_n(2p, Z, 1)$.

Theorem 1. *Under Assumption 1, 3, and $p \in (0, \beta/2)$, then*

$$n^{2p/\beta-1} BV_n(2p, Z, 1) \xrightarrow{\mathbb{P}} \mu_p^2(\beta) \int_0^1 |\sigma_s|^{2p} ds.$$

Theorem 1 extends previous consistency results for the bipower variation statistic, see Barndorff-Nielsen & Shephard (2004, Theorem 2) for the original result and Barndorff-Nielsen, Graversen, Jacod & Shephard (2006, Theorem 1) for a generalization, by allowing Z to obey the general class of locally stable processes (1) instead of Brownian semimartingale with finite activity jumps.[4] Moreover, the result allows us to state the following corollary:

Corollary 2. *Suppose that the conditions for Lemmas 1 (b) and 2 along with $p < \hat{\beta}(p)/2$, then*

$$\boldsymbol{\Omega}^*(p, Z) = \int_0^1 |\sigma_s|^{2p} ds \big(\mu_{2p}(\beta) \boldsymbol{\Sigma}(p, \beta, 0) + \mu_p^2(\beta) \left(\boldsymbol{\Sigma}(p, \beta, 1) + \boldsymbol{\Sigma}(p, \beta, 1)'\right)\big).$$

Proof. The power variation results follow from Lemma 1 (a) and Theorem 1. Consistency of the characteristic parameters $\boldsymbol{\Sigma}(p, \beta, k)$ follows since $p < \hat{\beta}(p)/2$, guaranteeing the existence of $2p$ moments for S_i^*, $i = 1, \dots n+1$, $\hat{\beta}(p)$ is consistent for β by Corollary 1, and since continuity of all non-degenerate stable distributions allows us to invoke the continuous mapping theorem. □

Corollary 2 shows that the bootstrap variance, $\boldsymbol{\Omega}_n^*(p, Z)$, will only be a consistent estimator of the asymptotic variance $\boldsymbol{\Omega}(p, Z)$ if $\mu_{2p}(\beta) = \mu_p^2(\beta) = 1$, which is not possible as it would contradictory imply that $\Sigma(p, \beta, 0)_{1,1} = 0$, that is, the "variablility" of the strictly β-stable process is 0. However, despite $\boldsymbol{\Omega}_n^*(p, Z)$ not being consistent for $\boldsymbol{\Omega}(p, Z)$, an asymptotically valid bootstrap can still be achieved for the *studentized* distribution. In particular, let us define

$$Q(2p) = \int_0^1 |\sigma_s|^{2p} ds, \qquad \boldsymbol{M}(p, \beta) = \mu_{2p}(\beta) \boldsymbol{\Sigma}(p, \beta, 0) + \mu_p^2(\beta) \left(\boldsymbol{\Sigma}(p, \beta, 1) + \boldsymbol{\Sigma}(p, \beta, 1)'\right)$$

such that $\boldsymbol{\Omega}^*(p, Z) = Q(2p) \boldsymbol{M}(p, \beta)$, then we consider

$$\boldsymbol{T}_n^* \equiv \left(\widehat{\boldsymbol{\Omega}}_n^*(p, Z)\right)^{-1/2} \sqrt{n} n^{p/\beta-1} \begin{pmatrix} V_n^*(p, Z, 1) - \mathbb{E}^*[V_n^*(p, Z, 1)] \\ V_n^*(p, Z, 2) - \mathbb{E}^*[V_n^*(p, Z, 2)] \end{pmatrix} \tag{14}$$

where

$$\widehat{\boldsymbol{\Omega}}_n^*(p, Z) = \widehat{Q}_n^*(2p, \hat{\beta}(p)) \boldsymbol{M}(p, \hat{\beta}(p)), \qquad \widehat{Q}_n^*(2p, \hat{\beta}(p)) = \mu_{2p}^{-2}(\hat{\beta}(p)) n^{2p/\hat{\beta}(p)-1} V_n^*(2p, Z, 1) \tag{15}$$

[4]Note that both Barndorff-Nielsen & Shephard (2004) and Barndorff-Nielsen et al. (2006) develop central limit theory for the bipower variation statistic. However, as this is not necessary for our further analysis of the properties of the proposed local stable bootstrap, we leave this for further research.

and $\boldsymbol{M}(p, \hat{\beta}(p))$ is the feasible analogue of $\boldsymbol{M}(p, \beta)$. The key aspect for the validity of the bootstrap procedure is that we use a consistent estimator $\widehat{\boldsymbol{\Omega}}_n^*(p, Z)$ for $\boldsymbol{\Omega}^*(p, Z)$ when constructing the studentized bootstrap t-statistic, $\boldsymbol{T}_n^*$, such that its asymptotic variance is a 2-dimensional identity matrix $\boldsymbol{I}_2$.

Remark 3. *An implication of Lemma 2 is that the ratio*

$$\mathbb{E}^*[n^{p/\beta-1}V_n^*(p,Z,2)]/\mathbb{E}^*[n^{p/\beta-1}V_n^*(p,Z,1)] = 2^{p/\hat{\beta}(p)} \xrightarrow{\mathbb{P}} 2^{p/\beta}$$

under the conditions of Corollary 1. Hence, in addition to using the local stable bootstrap to make inference on power variation statistics, we may also utilize the resampling procedure to mimic the asymptotic behavior of the ratio $(n^{p/\beta-1}V_n(p,Z,2))/(n^{p/\beta-1}V_n(p,Z,1)) \xrightarrow{\mathbb{P}} 2^{p/\beta}$*, under the conditions of Lemma 1 (a), and, as a result, to make inference on the activity index,* β*.*

3.2 A Bootstrap CLT for Power Variation Statistics

In this section, we proceed by establishing asymptotic central limit theory for the studentized bootstrap t-statistic, $\boldsymbol{T}_n^*$, in (14) along with its first-order asymptotic validity for corresponding studentized statistic from the asymptotic distribution,

$$\boldsymbol{T}_n \equiv \left(\widehat{\boldsymbol{\Omega}}_n(p,Z)\right)^{-1/2} \sqrt{n} \begin{pmatrix} n^{p/\beta-1}V_n(p,Z,1) - \mu_p(\beta)\int_0^1 |\sigma_s|^p ds \\ n^{p/\beta-1}V_n(p,Z,2) - 2^{p/\beta}\mu_p(\beta)\int_0^1 |\sigma_s|^p ds \end{pmatrix} \tag{16}$$

where $\widehat{\boldsymbol{\Omega}}_n(p, Z)$ is a consistent estimator of the asymptotic covariance matrix $\boldsymbol{\Omega}(p, Z)$ in Lemma 1. In particular, we let

$$\widehat{\boldsymbol{\Omega}}_n(p,Z) = \frac{n^{p/\hat{\beta}(p)-1}}{\mu_{2p}(\hat{\beta}(p))} V_n(2p,Z,1) \times \widehat{\boldsymbol{\Xi}}$$

where $\widehat{\boldsymbol{\Xi}}$ is a consistent estimator of the matrix $\boldsymbol{\Xi}$ that is written out explicitly in Appendix C.

Theorem 2. *Let Assumptions 1, 2 (b), and 3 of Appendix A hold and suppose* S_i^*, $i = 1, \ldots, n+1$, *are i.i.d. strictly* $\hat{\beta}(p)$*-stable random variables, defined as described in Step 2 of the local stable bootstrap algorithm, then as* $n \to \infty$,

(a) $\boldsymbol{T}_n^* \xrightarrow{d^*} N(\boldsymbol{0}, \boldsymbol{I}_2)$ *in probability-*$\mathbb{P}$,

(b) $\sup_{\boldsymbol{x}\in\mathbb{R}^2} |\mathbb{P}^*(\boldsymbol{T}_n^* \le \boldsymbol{x}) - \mathbb{P}(\boldsymbol{T}_n \le \boldsymbol{x})| \xrightarrow{\mathbb{P}} 0$.

Theorem 2 is the main asymptotic result in the paper. It provides the theoretical justification for using the local stable bootstrap to consistently estimate the distribution of $\boldsymbol{T}_n$. Moreover, it allows us to use the bootstrap to construct percentile-t (bootstrap studentized statistic) intervals for the stochastic scale of pure-jump semimartingales, $\mu_p(\beta)\int_0^1 |\sigma_s|^p ds$, along with non-linear transformations thereof, an application of which is shown below. As mentioned above, Theorem 2 generalizes existing bootstrap results for power variation statistics, cf. Gonçalves & Meddahi (2009) and Hounyo (2014), by allowing the power, p, to take other values than 2, by accommodating difference orders $v = (1, 2)$, and, most importantly, by allowing Z_t to obey the general class of processes (1).

Remark 4. *Theorem 2 may straightforwardly be adapted to perform feasible inference on the stochastic scale* $\mu_p(\beta) \int_0^1 |\sigma_s|^p$ *by replacing* β *in* T_n^* *with a consistent estimator* $\hat{\beta}(p)$*. To see this, let*

$$\bar{\boldsymbol{\mathcal{E}}}_n^*(p, Z) \equiv \mathbb{E}^* \left[n^{p/\hat{\beta}(p)-1} \begin{pmatrix} V_n^*(p, Z, 1) \\ V_n^*(p, Z, 2) \end{pmatrix} \right], \quad \bar{\boldsymbol{\Omega}}_n^*(p, Z) \equiv \mathbb{V}^* \left[\sqrt{n} n^{p/\hat{\beta}(p)-1} \begin{pmatrix} V_n^*(p, Z, 1) \\ V_n^*(p, Z, 2) \end{pmatrix} \right]$$

then, under the conditions for Theorem 2, it follows that

$$\begin{aligned} \bar{\mathcal{E}}_n^*(p, Z) &= \left(\mu_p(\hat{\beta}(p)), 2^{2/\hat{\beta}(p)} \mu_p(\hat{\beta}(p)) \right)' n^{p/\hat{\beta}(p)-1} V_n(p, Z, 1), \qquad \text{and} \\ \bar{\Omega}_n^*(p, Z) &= n^{2p/\hat{\beta}(p)-1} \Sigma(p, \hat{\beta}(p), 0) \sum_{i=1}^{n} |\Delta_i^{n,1} Z|^{2p} \\ &\quad + n^{2p/\hat{\beta}(p)-1} \sum_{i=1}^{n-1} |\Delta_i^{n,1} Z|^p |\Delta_{i+1}^{n,1} Z|^p \left(\Sigma(p, \hat{\beta}(p), 1) + \Sigma(p, \hat{\beta}(p), 1)' \right). \end{aligned}$$

by Lemma 2 such that $\text{plim}_{n\to\infty} \bar{\boldsymbol{\mathcal{E}}}_n^*(p, Z) = \boldsymbol{\mathcal{E}}^*(p, Z)$ *and* $\text{plim}_{n\to\infty} \bar{\boldsymbol{\Omega}}_n^*(p, Z) = \boldsymbol{\Omega}^*(p, Z)$ *by combining Lemma 1, Corollaries 1-2, and the continuous mapping theorem. In addition,*

$$\left(\bar{\boldsymbol{\Omega}}_n^*(p, Z) \right)^{-1/2} \sqrt{n} n^{p/\hat{\beta}(p)-1} \begin{pmatrix} V_n(p, Z, 1) - \mathbb{E}^*[V_n(p, Z, 1)] \\ V_n(p, Z, 2) - \mathbb{E}^*[V_n(p, Z, 2)] \end{pmatrix} \xrightarrow{d^*} N(\mathbf{0}, \boldsymbol{I}_2)$$

in probability-$\mathbb{P}$ *follows from Theorem 2. These results are immediate since* $\hat{\beta}(p)$ *is not random under the bootstrap probability measure* $\mathbb{P}^*$ *and* $\hat{\beta}(p) \xrightarrow{\mathbb{P}} \beta$ *by Corollary 1. Unlike the feasible inference result for the stochastic scale in, e.g., Todorov (2013, Theorem 3), this demonstrates that the local stable bootstrap procedure may be implemented without additional bias corrections.*

3.3 A Bootstrap CLT for Activity Index Estimation

An important implication of Theorem 2 is that we may deduce a consistency result as well as a central limit theorem for a bootstrap activity index estimator, denoted by $\hat{\beta}^*(p)$. In particular, and as explicated in Lemma 2 and Remark 2, the logarithmic ratio of bootstrap power variations at frequencies $v = (1, 2)$ may be studied to learn about the activity index estimator $\hat{\beta}(p)$ and, consequently, about the underlying β, similar to the formulation of the former in (5). Hence, we propose the estimator

$$\hat{\beta}^*(p) = \frac{p \ln(2)}{\ln\left(V_n^*(p, Z, 2)\right) - \ln\left(V_n^*(p, Z, 1)\right)} \mathbf{1}\{V_n^*(p, Z, 2) \neq V_n^*(p, Z, 1)\}, \tag{17}$$

whose consistency for β follows by combining results from Lemma 2 and Corollaries 1-2. Furthermore, we may apply the delta method in conjunction with the central limit theorem in Theorem 2 to characterize the asymptotic distribution of $\hat{\beta}^*(p)$. This is summarized in the following theorem:

Theorem 3. *Suppose the conditions of Theorem 2, then it follows that*

$$\tau_n^* \equiv \sqrt{n}\left(\widehat{\Omega}_\beta^*(p,Z)\right)^{-1/2}\left(\hat{\beta}^*(p)-\hat{\beta}(p)\right) \xrightarrow{d^*} N(0,1), \text{ in probability-}\mathbb{P},$$

where the estimator of the asymptotic variance $\widehat{\Omega}_\beta^*(p,Z)$ *is defined as*

$$\widehat{\Omega}_\beta^*(p,Z) = (\hat{\beta}^*(p))^4 \cdot (p\ln(2))^{-2} \cdot \widehat{Q}_n^*(2p,\hat{\beta}(p)) \cdot \widehat{\zeta}(p,Z,\hat{\beta}(p)),$$

with $\widehat{Q}_n^*(2p,\hat{\beta}(p))$ *given in* (15), $\widehat{\zeta}(p,Z,\hat{\beta}(p))$ *is defined through*

$$\frac{\widehat{\zeta}(p,Z,\hat{\beta}(p))}{n^{2-2p/\hat{\beta}(p)}} = \frac{M(p,\hat{\beta}(p))_{1,1}}{(V_n^*(p,Z,1))^2} - \frac{M(p,\hat{\beta}(p))_{1,2}+M(p,\hat{\beta}(p))_{2,1}}{V_n^*(p,Z,1)V_n^*(p,Z,2)} + \frac{M(p,\hat{\beta}(p))_{2,2}}{(V_n^*(p,Z,2))^2}$$

and $M(p,\hat{\beta}(p))_{i,j}$ *is the* (i,j)*-th element of the matrix* $\boldsymbol{M}(p,\hat{\beta}(p))$ *in* (15).

Theorem 3 justifies using the local stable bootstrap to make inference on the activity index, β, for pure-jump semimartingales via bootstrap percentile-t intervals. Moreover, it allows us to propose a bootstrap test of whether Z_t is a pure-jump semimartingale or a jump diffusion. However, if the bootstrap is used blindly to construct such a test, the resulting procedure may have poor finite sample properties, see, e.g., Hall & Wilson (1991). One way to avoid this problem is to allow the bootstrap test to differ from the bootstrap confidence intervals by generating the bootstrap distribution for the former under a specific and pre-specified null hypothesis, as discussed in Davidson (2007). Not only will tests based on a *null hypothesis resampling procedure* differ from interval-based tests, they often have superior size properties. Indeed, Davidson & MacKinnon (1999) show that in order to minimize the error in rejection probability under the null hypothesis of a bootstrap test (i.e., its Type I error), we should generate the bootstrap data as efficiently as possible, see also MacKinnon (2009). For specificity, this entails generating the bootstrap data under the restriction specified by the null hypothesis $\mathcal{H}_0 : \beta = 2$. A simple and natural way to accommodate this restriction in resampling procedure is to implement the bootstrap Algorithm 1 with $\hat{\beta}(p) = 2$ as follows:

Algorithm 2.

Step 1. Under the restriction specified by $\mathcal{H}_0 : \beta = 2$, use $\hat{\beta}(p) = 2$.

Step 2. Generate an $n+1$ sequence of identically and independently distributed 2-stable random variables $S_1^*, S_2^*, \ldots, S_{n+1}^*$, whose characteristic function are defined as

$$\ln \mathbb{E}\left[e^{\mathrm{i}uS_i^*}\right] = -|u|^2/2, \qquad \forall i = 1,\ldots,n+1,$$

and which are independent of the original sample, Z_{t_i}, $i = 0,\ldots,n$.

Step 3. Same as Step 3 of Algorithm 1 using the 2-stable random variables $S_1^*, S_2^*, \ldots, S_{n+1}^*$.

Corollary 3. *Suppose the conditions of Theorem 2, then for $\hat{\beta}(p) = 2$ under $\mathcal{H}_0$, it follows that*

$$\tau_n^*(2) \equiv \sqrt{n}\left(\widehat{\Omega}_\beta^*(p,Z)\right)^{-1/2}\left(\hat{\beta}^*(p) - 2\right) \xrightarrow{d^*} N(0,1), \text{ in probability-}\mathbb{P},$$

where the estimator of the asymptotic variance $\widehat{\Omega}_\beta^*(p,Z)$ is defined as

$$\widehat{\Omega}_\beta^*(p,Z) = (\hat{\beta}^*(p))^4 \cdot (p\ln(2))^{-2} \cdot \widehat{Q}_n^*(2p,2) \cdot \widehat{\zeta}(p,Z,2).$$

Proof. Follows directly from Theorem 3. □

The bootstrap testing procedure in Algorithm 2 and Corollary 3 are targeted against a specific null hypothesis $\mathcal{H}_0 : \beta = 2$. Of course, one may be interested in different null hypotheses, for example, a pre-specified null $\tilde{\mathcal{H}}_0 : \beta \in (\sqrt{2}, 2)$ or $\bar{\mathcal{H}}_0 : \beta = \hat{\beta}(p)$. In either case, the bootstrap data generating process in Algorithm 2 can easily be adapted to satisfy the requisite null. Nevertheless $\mathcal{H}_0$ against the alternative $\mathcal{H}_1 : \beta < 2$ seems to be the most interesting hypothesis for many problems in finance and econometrics such as, e.g., option pricing, risk premia characterization, and volatility modeling.

4 Simulation Study

In this section, we assess the finite sample properties of the proposed bootstrap test in Algorithm 2 for $\mathcal{H}_0 : \beta = 2$ against $\mathcal{H}_1 : \beta < 2$ using Monte Carlo simulations. For comparison, we also include a feasible test based on the limiting result in Corollary 1, which is inspired by the approach in Andersen, Bondarenko, Todorov & Tauchen (2014) and combines the methods in Todorov & Tauchen (2011*a*) and Todorov (2013). Since the latter is a state-of-the-art benchmark, their relative properties will speak directly to the usefulness of our bootstrap test. We detail how to implement both testing procedures in Appendix C. Finally, we include a test based on Algorithm 1 where $\hat{\beta}(p)$ is used to generate the bootstrap sample to gauge the benefits of targeting the bootstrap test against a specific $\mathcal{H}_0$.

4.1 Simulation Setup

We simulate the data to match a standard 6.5-hour trading day and normalize the trading window to the unit interval, $t \in [0,1]$, such that 1 second corresponds to an increment of size 1/23400. At each increment, we generate observations of Z_t according to the general process

$$dZ_t = adt + \sigma_t dL_t + dY_t, \qquad dY_t = \int_{\mathbb{R}} k_2 x \mu\,(dt, dx) \tag{18}$$

where the drift, a, is assumed to be constant, the locally stable process, L_t is either modeled as a standard Brownian motion under the null hypothesis $\mathcal{H}_0 : \beta = 2$ or as a symmetric tempered stable process, e.g. Rosinski (2007), with compensator $\nu_t(dx) = dt \otimes \nu(dx)$,

$$\nu(dx) = c_1 \exp(-\lambda_1 |x|)|x|^{-(\beta+1)} dx \quad \text{where } c_1 > 0,\ \lambda_1 > 0 \text{ and } \beta \in \left(\sqrt{2}, 2\right) \tag{19}$$

under the alternative, consistent with Assumptions 1-2. We will use the notation $L_t = W_t$ for the locally stable process under $\mathcal{H}_0$ to avoid confusion. The tempered stable process under the alternative is simulated using the series representation in Rosinski (2001), see also Todorov (2009). The stochastic scale, σ_t, is assumed to follow a two-factor model,

$$\sigma_t = \text{s-exp}(b_0 + b_1 \tau_{1,t} + b_2 \tau_{2,t}) \quad \text{where} \quad d\tau_{1,t} = a_1 \tau_{1,t} dt + dB_{1,t},$$
$$d\tau_{2,t} = a_2 \tau_{2,t} dt + (1 + \phi \tau_{2,t}) dB_{2,t}, \quad \text{Corr}(B_{1,t}, W_t) = \rho_1, \quad \text{Corr}(B_{2,t}, W_t) = \rho_2,$$

and both $B_{1,t}$ and $B_{2,t}$ are standard Brownian motions, following, e.g., Chernov, Gallant, Ghysels & Tauchen (2003) and Huang & Tauchen (2005). The function s-exp is an exponential with a linear growth function splined in at high values of its argument: s-exp$(x) = \exp(x)$ if $x \leq x_0$ and s-exp$(x) = \frac{\exp(x_0)}{\sqrt{x_0}} \sqrt{x_0 - x_0^2 + x^2}$ if $x > x_0$ with $x_0 = \ln(1.5)$. Note that the stochastic scale is driven by two standard Brownian motions, which are correlated with the dominant term in (18) under the null hypothesis, thus allowing for leverage effects. Since such correlation statistics are not well-defined under the alternative when L_t is pure-jump process, we will here assume $L_t \perp\!\!\!\perp (B_{1,s}, B_{2,s})\ \forall t, s$.

The residual jump process, Y_t, is assumed to obey a symmetric tempered stable process with either of the following two compensators $\nu_t^Y(dx) = dt \otimes \nu^Y(dx)$,

$$\nu^Y(dx) = c_2 \frac{\exp(-x^2/(2\sigma_2^2))}{\sqrt{2\pi}\sigma_2} dx \quad \text{or} \quad \nu^Y(dx) = c_2 \exp(-\lambda_2|x|)|x|^{\beta'+1} dx \tag{20}$$

where $\sigma_2 > 0$, $c_2 > 0$, $\lambda_2 > 0$ and $\beta' \in [0, 1)$. Whereas the first compensator in (20) captures a compound Poisson process with normally distributed mean-zero jumps, the second specification models the residual jumps as a symmetric tempered stable process of finite activity. The compound Poisson process has activity index zero, while it is β' for the tempered stable process.

We consider ten different specifications within this general setting. For all cases, we fix some parameters according to Huang & Tauchen (2005): $\alpha = 0.03$, $b_0 = -1.2$, $b_1 = 0.04$, $b_2 = 1.5$, $a_1 = -0.00137$, $a_2 = -1.386$, $\phi = 0.25$, and $\rho_1 = \rho_2 = -0.3$. Moreover, we initialize the two factors at the beginning of each "trading day" by drawing the most persistent factor from its unconditional distribution, $\tau_{1,0} \sim N(0, 1/(2a_1)))$, and by letting the strongly mean-reverting factor, $\tau_{2,t}$, start at zero. For the remaining parameters, the ten cases are described in Table 1. Out of the ten cases, the first six, that is, DGP's A-F, model Z_t under the null hypothesis, $\mathcal{H}_0$, whereas DGP's G-J specify Z_t as pure-jump semimartingales under the alternative, $\mathcal{H}_1$. In particular, DGP's A-B use the same parameters as Todorov (2009) to calibrate to contribution of Y_t, specified as a tempered stable process, to the total variation of the series. These reflect the empirical results in Huang & Tauchen (2005) and set the variation of Y_t to be 0.1 on average, which is 10% of the average variation in the dominant component under the null hypothesis. DGP's C and D are variants where the activity of Y_t have been increased. In DGP's E and F, on the other hand, Y_t is modeled as a compound Poisson processes with relatively infrequent jumps (e.g., once per trading day) of "moderate size", which, for example, may capture discontinuous movements surrounding news announcements. Whereas Y_t is specified similarly for DGP's G-J, L_t is implemented as in (19) with activity indices $\beta = \{1.51, 1.91\}$.

Once Z_t has been simulated, we construct equidistant samples $t_i = i/n$ for $i = 0, \ldots n$ and generate returns $\Delta_i^{n,1} Z = Z_{t_i} - Z_{t_{i-1}}$. Here, we primarily consider $n = \{39, 78, 195, 390\}$, which corresponds to sampling observations every $\{10, 5, 2, 1\}$ minutes, respectively. Note that the impact of market microstructure noise is greatly alleviated at these relatively sparse frequencies, in particular for very liquid assets such as those we consider in the empirical analysis below. Moreover, we implement the tests with $p = \{0.7, 0.9\}$ and assign significance at a 5% nominal level. Finally, the simulation study is carried out using 999 bootstrap samples for each of the 1000 Monte Carlo replications. The rejection rates of $\mathcal{H}_0$ are reported in Table 2 for DGP's A-F (size) and in Table 3 for DGP's G-J (power).

4.2 Simulation Results

There are several interesting observations from Table 2. First, the feasible test based on the central limit result in Corollary 1, labelled CLT, is oversized for all DGP's considered, showing rejection rates in the 8-35% range, often much larger than the nominal 5% level. Furthermore, the size distortions remain when the sample size is increased from $n = 39$ to $n = 390$. Second, the local stable bootstrap test based on Algorithm 1, labelled LSB 1, has better size properties than the CLT test, but with rejection rates in the 4-11% range, it is still slightly liberal with respective to size. Third, the proposed bootstrap test based on Algorithm 2, labelled LSB 2, is conservative for small samples, but as the sample size approaches $n = 195$, the size of the tests are very close to the 5% nominal level. Hence, this shows the value of generating the bootstrap sample more efficiently by targeting the test towards a specific null hypothesis, consistent with the results in Davidson & MacKinnon (1999). Fourth, the tests based on $p = 0.7$ generally exhibit slightly lower rejection rates than those for $p = 0.9$. Finally, we wish to highlight the results for DGP F. When the residual jump process, Y_t, is modeled as large infrequent jumps, the size distortions of the CLT test are particularly pronounced. However, the LSB 2 test is *not* affected by such jumps and has a slightly conservative size around 3-3.5%.

The rejection rates in Table 3 for DGP's G-J illustrate that all tests have power against the alternative hypothesis $\mathcal{H}_1 : \beta < 2$. Not surprisingly, we find that the LSB 2 test has low power for small sample sizes. This is the price we pay for having a correctly sized test. However, its power increases dramatically when the sample size is increased to $n = \{195, 390\}$ observations. The corresponding rejection rates for both the CLT and LSB 1 tests are larger. However, as emphasized by Horowitz & Savin (2000) and Davidson & MacKinnon (2006), such power results are misleading since the sizes of the respective tests are liberal for all sampling frequencies considered, especially for the CLT test. Interestingly, despite the fact that all tests using power $p = 0.9$ violate the condition $p < \beta/2$ for the two alternative DGP's with $\beta = 1.51$, their relative finite sample rejection rates resembles the two cases with $\beta = 1.91$ where the condition holds. Last, when the sample size is increased to $n = 780$ observations, all tests display rejection rates of approximately 100%.

In general, the proposed local stable bootstrap tests of the null hypothesis $\mathcal{H}_0 : \beta = 2$ provide useful alternatives to existing central limit theory based tests that have (much) better size properties. Whereas the bootstrap test based on Algorithm 2 displays the best size properties, the test based on Algorithm 1 may have a slight edge in terms of power, in particular for smaller samples.

5 Empirical Analysis

We analyze the null hypothesis, $\mathcal{H}_0 : \beta = 2$, using high-frequency data on three exchange rate series, the S&P 500 index and the VIX from 2011, which presents an interesting and diverse period with calm markets in the beginning of the year followed by a turbulent month of August where stock prices dropped sharply in fear of contagion of the European sovereign debt crises to Italy and Spain. In particular, we use observations on the Euro (EUR), Japanese Yen (JPY), and the Swiss Franc (CHF) against the U.S. Dollar (USD) from Tick Data. These are collected from both pit and electronic trading and cover *whole* trading days. Moreover, we use futures contracts on the S&P 500, that are traded on the Chicago Mercantile Exchange (CME) Group during regular trading hours from 8.30-15.15 CT. The high-frequency VIX observations cover the same trading window. In general, all three markets are very liquid and the use of futures contracts for the S&P 500 eliminates the need for adjustments due to dividend payments. To strike a compromise between the liquidity of the series and concerns about market microstructure noise, we construct series of one- and two-minute logarithmic returns on each *full* trading day. For the FX series, these are of length $n = 1439$ and $n = 719$. Similarly, for the S&P 500 and the VIX, they contain $n = 404$ and $n = 201$ observations, respectively. For all five assets, we compute the mean and median estimates of β using (5) across the trading days as well as the rejection rates of $\mathcal{H}_0$ for the CLT, LSB 1, and LSB 2 tests. The estimator and the tests are implemented with powers $p = \{0.7, 0.9\}$ and using a 5% nominal level. The results are reported in Table 4.

From Table 4, we see that $\mathcal{H}_0$ is rarely rejected for the S&P 500 series, the rejection rates being approximately 4-7% for both the CLT and LSB 1 tests and 1-4% for the LSB 2 test. For the VIX, on the other hand, the average and median activity index estimates are around 1.35-1.6 and the rejection rates of $\mathcal{H}_0$ are much higher, being in the 60-87% range. These results for the CLT test corroborate the findings in Andersen, Bondarenko, Todorov & Tauchen (2014) by showing that the S&P 500 and the VIX are (usually) best described as a jump-diffusion and a pure-jump semimartingale, respectively. The corresponding estimates for the three FX series fall between the two extremes. Furthermore, there are differences between the one- and two-minute results. Whereas the β estimates using two-minute sampling fall in the 1.90-2.00 interval and the rejection rates are between 8-25%, the comparable ranges for series sampled every minute are 1.80-1.95 and 20-56%. The frequent rejection of $\mathcal{H}_0$ contradicts the findings in Todorov & Tauchen (2010) and Cont & Mancini (2011), who, using five-minute log-returns on the DM-USD exchange rate from the 1990's, argue that exchange rates are best described as jump-diffusions. Instead, on many trading days, we find that all three tests provide support for the use of a pure-jump semimartingale model. As shown by Carr & Wu (2003), these results may have important implications for the *daily* pricing of exchange rate derivatives. Notice, however, that there are striking differences between the rejection rates from the CLT and LSB 2 tests. For example, when considering the EUR-USD exchange rate and $p = 0.9$, the CLT test rejects $\mathcal{H}_0$ on 55 out of the 207 full trading days in the sample (26.57%), whereas the LSB 2 test only rejects on 42 days (20.29%). Given the liberal size of the CLT test, this suggest that it may wrongfully reject $\mathcal{H}_0$ on 13 trading days out of a year, which, again, may lead to a *daily* misspecification of the exchange rate model. In fact, we find that the LSB 2 test rejects uniformly less than the CLT test for *all* series, which is consistent with the size properties of the two procedures, as illustrated by the simulation study. This clearly highlights the usefulness of the proposed local stable bootstrap procedure, which may be used to construct a correctly sized and more conservative test of $\mathcal{H}_0$ than existing methods.

Remark 5. *Last, note that the lower β estimates and resulting higher rejection rates of $\mathcal{H}_0$ for one-minute compared to two-minute sampled exchange rate series are not easily explained by market microstructure noise. To see this, suppose $X_{t_i} = Z_{t_i} + U_{t_i}$ where $U_{t_i} \sim i.i.d.N(0, \sigma_u^2)$. Then, since $\Delta_i^{n,1} Z = O_p(n^{-1/\beta})$ for $\beta \in (\sqrt{2}, 2]$ and $\Delta_i^{n,1} U = O_p(1)$, we have*

$$V_n(p, X, 1) \approx V_n(p, U, 1), \quad V_n(p, X, 2) \approx V_n(p, U, 2) \approx V_n(p, U, 1), \qquad \text{as} \quad n \to \infty$$

and, consequently, it follows that $\hat{\beta}(p) \xrightarrow{\mathbb{P}} \infty$. In other words, adding a noise component will inflate the activity index estimates, not result in more frequent rejection of $\mathcal{H}_0$.

6 Conclusion

We provide a new resampling procedure - the local stable bootstrap - that is able to mimic the dependence properties of power variations for pure-jump semimartingales observed at different frequencies. This allows us to propose a bootstrap estimator and inference procedure for the activity index of the underlying process as well as a test for whether it is a jump-diffusion or a pure-jump semimartingale. We establish first-order asymptotic validity of the resulting bootstrap power variations, activity index estimator, and diffusion test. Moreover, we examine the finite sample size and power of the proposed diffusion test using Monte Carlo simulations and show that, unlike existing tests, it is correctly sized in general settings. Finally, we test for the (null) presence of a diffusive component using high-frequency data on three exchange rate series, the S&P 500 index and the VIX from 2011. We find that the null hypothesis is rarely rejected for the S&P 500 series, rejected 60-87% of the days for the VIX, whereas the rejection rates for the exchange rate series falls in between the two. Importantly, we show that existing tests uniformly reject more often than our bootstrap test, verifying the results from the simulation study and illustrating the usefulness of our correctly sized bootstrap test.

Finally, we note that the proposed resampling procedure is generally applicable to processes, which behave locally like an infinite activity stable process. Hence, it may possibly be adapted to - and has potential to improve upon the finite sample inference for - alternative activity index estimators in the literature that rely on a similar approximation over small time scales such as those in Aït-Sahalia & Jacod (2009), Zhao & Wu (2009), Jing, Kong, Liu & Mykland (2012), and Todorov (2015). Another interesting direction for further research is to extend the local stable bootstrap to make it robust against market microstructure noise, possibly in combination with an existing noise-robust activity index estimator such as the one proposed by Jing, Kong & Liu (2011) based on pre-averaging. However, both extensions are beyond the scope of the paper.

Parameter Configurations for the Simulation Study		
DGP	Specification of L_t	Specification of Y_t
A	Brownian Motion	TS with $(\beta', k_2, c_2, \lambda_2) = (0.1, 0.0119, 0.125, 0.015)$
B	Brownian Motion	TS with $(\beta', k_2, c_2, \lambda_2) = (0.5, 0.0161, 0.4, 0.015)$
C	Brownian Motion	TS with $(\beta', k_2, c_2, \lambda_2) = (0.8, 0.0106, 0.1, 0.015)$
D	Brownian Motion	TS with $(\beta', k_2, c_2, \lambda_2) = (0.9, 0.0161, 0.1, 0.015)$
E	Brownian Motion	CP with $(k_2, c_2, \sigma_2) = (1, 0.1, 1)$
F	Brownian Motion	CP with $(k_2, c_2, \sigma_2) = (1, 1, 1.5)$
G	TS with $(\beta, k_1, c_1, \lambda_1) = (1.51, 1, 1, 0.25)$	$Y_t = 0$
H	TS with $(\beta, k_1, c_1, \lambda_1) = (1.91, 1, 1, 0.25)$	CP with $(k_2, c_2, \sigma_2) = (1, 0.1, 1)$
I	TS with $(\beta, k_1, c_1, \lambda_1) = (1.51, 1, 1, 0.25)$	TS with $(\beta', k_2, c_2, \lambda_2) = (0.05, 0.0106, 0.1, 0.015)$
J	TS with $(\beta, k_1, c_1, \lambda_1) = (1.91, 1, 1, 0.25)$	TS with $(\beta', k_2, c_2, \lambda_2) = (0.05, 0.0106, 0.1, 0.015)$

Table 1: Parameter configurations. This table provides an overview of the parameter configurations for the dominant component, L_t, and the residual jump process, Y_t, of the general price process (18) for the simulation study. Here, "TS" and "CP" abbreviate tempered stable and compound Poisson processes, respectively, which are defined using the compensators in (20). Hence, DGP's A-F capture the null hypothesis $\mathcal{H}_0 : \beta = 2$ whereas DGP's G-J capture the one-sided alternative $\mathcal{H}_1 : \beta < 2$.

Rejection Rates under $\mathcal{H}_0$										
	$p = 0.7$					$p = 0.9$				
	$\hat{\beta}$-Mean	$\hat{\beta}$-Med	CLT	LSB 1	LSB 2	$\hat{\beta}$-Mean	$\hat{\beta}$-Med	CLT	LSB 1	LSB 2
DGP A										
$n = 39$	2.36	2.09	10.00	5.00	1.20	2.29	2.07	13.20	7.40	3.70
$n = 78$	2.23	2.07	10.10	4.90	1.70	2.17	2.05	13.60	7.50	3.50
$n = 195$	2.04	2.00	10.40	6.10	3.50	2.03	1.99	14.80	9.80	5.00
$n = 390$	2.01	2.00	10.60	5.40	3.60	1.99	2.00	13.10	8.30	4.70
DGP B										
$n = 39$	2.60	2.10	10.10	4.60	1.20	2.47	2.07	12.70	6.60	3.20
$n = 78$	2.23	2.07	10.80	4.90	1.70	2.17	2.04	13.90	7.60	3.30
$n = 195$	2.04	1.99	10.60	6.60	3.90	2.03	1.98	14.30	10.00	5.40
$n = 390$	2.01	1.99	9.70	5.30	3.60	1.99	1.99	12.30	7.80	4.40
DGP C										
$n = 39$	2.73	2.14	8.00	4.40	1.50	2.58	2.09	11.10	6.40	3.40
$n = 78$	2.26	2.08	9.30	5.10	2.20	2.20	2.05	12.60	7.70	4.00
$n = 156$	2.05	2.00	10.00	6.70	3.80	2.04	1.99	13.40	10.40	5.40
$n = 390$	2.02	2.00	9.10	5.40	3.70	2.01	1.99	11.40	8.50	4.30
DGP D										
$n = 39$	2.68	2.10	9.00	4.30	1.50	2.21	2.06	11.90	6.30	3.30
$n = 78$	2.21	2.05	10.20	5.90	2.10	2.16	2.03	13.60	8.60	4.20
$n = 195$	2.03	1.99	11.50	7.10	4.30	2.02	1.98	15.20	11.10	5.90
$n = 390$	2.00	1.98	10.30	6.40	3.70	1.99	1.98	12.60	8.40	4.90
DGP E										
$n = 39$	2.59	2.12	8.50	5.60	1.20	2.32	2.08	12.90	8.20	3.70
$n = 78$	2.29	2.08	8.30	4.80	1.50	2.25	2.03	12.20	7.70	2.60
$n = 195$	2.07	2.01	8.20	5.40	3.70	2.05	2.01	11.20	8.80	4.90
$n = 390$	2.04	2.00	8.20	5.90	3.90	2.02	2.00	9.80	7.80	4.90
DGP F										
$n = 39$	1.51	1.71	25.20	8.40	1.30	2.02	1.70	34.40	8.90	3.20
$n = 78$	1.88	1.74	27.80	10.50	2.10	1.84	1.71	35.00	11.00	3.60
$n = 195$	1.86	1.81	25.10	9.90	3.00	1.81	1.79	31.60	8.20	3.40
$n = 390$	1.87	1.86	22.40	8.70	3.50	1.83	1.84	28.60	7.40	3.10

Table 2: Size results. This table provides rejection frequencies of the null hypothesis $\mathcal{H}_0 : \beta = 2$ for DGP's A-F in Table 1, sample sizes $n = \{39, 78, 156, 390\}$, powers $p = \{0.7, 0.9\}$ along with three different tests CLT, LSB 1, and LSB 2. In particular, CLT is the feasible test based on Corollary 1, see also Andersen, Bondarenko, Todorov & Tauchen (2014), LSB 1 is the local stable bootstrap test based on Algorithm 1, and LSB 2 is the local stable bootstrap test based on Algorithm 2, which is targeted against $\mathcal{H}_0$. The nominal level of the tests is 5%. $\hat{\beta}$-Mean and $\hat{\beta}$-Med denote the mean and median, respectively, of the activity index estimator $\hat{\beta}(p)$ in (5). Finally, the exercise is performed for 999 bootstrap samples for every one of the 1000 Monte Carlo replications.

Rejection Rates under $\mathcal{H}_1$										
	$p = 0.7$					$p = 0.9$				
	$\hat{\beta}$-Mean	$\hat{\beta}$-Med	CLT	LSB 1	LSB 2	$\hat{\beta}$-Mean	$\hat{\beta}$-Med	CLT	LSB 1	LSB 2
DGP G										
$n = 39$	1.99	1.82	22.40	14.20	4.00	1.98	1.83	27.80	15.40	7.20
$n = 78$	1.78	1.71	39.90	28.80	12.70	1.79	1.72	44.40	27.80	14.30
$n = 195$	1.64	1.59	79.20	69.70	52.90	1.66	1.62	77.50	65.10	43.70
$n = 390$	1.60	1.59	99.60	98.90	97.50	1.63	1.61	98.20	96.00	88.70
$n = 780$	1.56	1.55	100.00	100.00	100.00	1.58	1.57	100.00	100.00	99.50
DGP H										
$n = 39$	2.56	2.05	12.80	8.50	2.80	2.71	2.02	18.40	10.70	5.70
$n = 78$	2.08	1.96	18.40	12.90	4.90	2.10	1.94	23.60	15.70	7.50
$n = 195$	1.95	1.92	43.90	34.90	22.80	1.92	1.86	44.90	36.70	21.90
$n = 390$	1.92	1.91	91.40	89.20	84.60	1.89	1.90	84.90	78.70	68.20
$n = 780$	1.90	1.88	100.00	100.00	100.00	1.87	1.88	99.80	99.60	99.40
DGP I										
$n = 39$	3.06	1.87	27.70	15.50	4.10	2.29	1.90	32.30	15.80	7.20
$n = 78$	1.81	1.66	42.70	28.90	11.00	1.82	1.70	47.00	27.00	12.50
$n = 195$	1.62	1.57	77.30	65.50	47.40	1.64	1.60	76.00	59.10	38.00
$n = 390$	1.58	1.57	99.60	97.70	94.70	1.59	1.59	98.50	91.90	82.50
$n = 780$	1.54	1.54	100.00	99.70	98.50	1.56	1.56	99.60	97.80	95.80
DGP J										
$n = 39$	2.68	2.11	17.70	9.30	2.70	2.36	2.03	22.60	12.00	5.40
$n = 78$	2.23	1.95	24.30	14.00	5.40	2.12	1.92	29.20	16.10	7.00
$n = 195$	1.94	1.93	45.30	33.30	21.60	1.91	1.86	47.20	33.50	19.70
$n = 390$	1.92	1.90	94.30	88.20	81.10	1.89	1.86	86.90	76.00	64.80
$n = 780$	1.89	1.88	100.00	99.50	98.70	1.85	1.82	100.00	97.90	96.30

Table 3: Power results. This table provides rejection frequencies of the null hypothesis $\mathcal{H}_0 : \beta = 2$ for DGP's G-J in Table 1, sample sizes $n = \{39, 78, 156, 390, 780\}$, powers $p = \{0.7, 0.9\}$ along with three different tests CLT, LSB 1, and LSB 2. In particular, CLT is the feasible test based on Corollary 1, see also Andersen, Bondarenko, Todorov & Tauchen (2014), LSB 1 is the local stable bootstrap test based on Algorithm 1, and LSB 2 is the local stable bootstrap test based on Algorithm 2, which is targeted against $\mathcal{H}_0$. The nominal level of the tests is 5%. $\hat{\beta}$-Mean and $\hat{\beta}$-Med denote the mean and median, respectively, of the activity index estimator $\hat{\beta}(p)$ in (5). Finally, the exercise is performed for 999 bootstrap samples for every one of the 1000 Monte Carlo replications.

Activity Index Estimates and Diffusion Tests Based On Empirical Data										
	$p = 0.7$					$p = 0.9$				
	$\hat{\beta}$-Mean	$\hat{\beta}$-Med	CLT	LSB 1	LSB 2	$\hat{\beta}$-Mean	$\hat{\beta}$-Med	CLT	LSB 1	LSB 2
EUR-USD										
1-min	1.85	1.84	49.28	45.89	43.96	1.95	1.94	26.57	26.09	20.29
2-min	1.92	1.91	24.64	20.29	16.43	1.99	1.97	14.49	14.01	10.14
USD-CHF										
1-min	1.88	1.86	42.51	38.16	34.78	1.94	1.93	25.60	23.67	18.84
2-min	1.98	1.98	14.49	12.56	9.18	2.02	2.02	13.04	12.08	7.73
USD-JPY										
1-min	1.82	1.81	56.04	53.62	50.72	1.90	1.88	42.51	40.58	34.30
2-min	1.99	1.98	13.53	13.04	9.18	1.99	1.98	13.53	13.04	9.18
S&P 500										
1-min	2.20	2.14	5.95	3.97	3.97	2.29	2.24	3.57	2.78	2.38
2-min	2.14	2.07	6.75	5.16	1.98	2.20	2.13	3.97	4.37	1.19
VIX										
1-min	1.41	1.35	86.64	85.78	83.19	1.51	1.46	82.76	80.60	77.59
2-min	1.51	1.42	71.98	70.26	65.95	1.58	1.49	69.83	68.53	61.21

Table 4: Summary statistics. This table provides activity index estimates as well as rejection rates of null hypothesis $\mathcal{H}_0 : \beta = 2$, using 1-minute and 2- minute return series, powers $p = \{0.7, 0.9\}$ along with the three different tests; CLT, LSB 1, and LSB 2. The activity indices are estimated and tested using regular exchange days in 2011. The three FX series are constructed using FX observations, which are collected from both pit and electronic trading and cover *whole* trading days. Hence, the 1-minute and 2-minute series have $n = 1439$ and $n = 719$ observations, respectively. The S&P 500 series are constructed using futures contracts during regular trading hours at the CME from 8.30-15.15 CT. Hence, the 1-minute and 2-minute series have $n = 404$ and $n = 201$ observations, respectively. The high-frequency VIX series are of similar length. CLT is the feasible test based on Corollary 1, see also Andersen, Bondarenko, Todorov & Tauchen (2014), LSB 1 is the local stable bootstrap test based on Algorithm 1, and LSB 2 is the local stable bootstrap test based on Algorithm 2, which is targeted against $\mathcal{H}_0$. The nominal level of the tests is 5%. $\hat{\beta}$-Mean and $\hat{\beta}$-Med denote the mean and median, respectively, of the activity index estimator $\hat{\beta}(p)$ in (5). Finally, we used 999 replications for the bootstrap resampling.

References

Aït-Sahalia, Y. & Jacod, J. (2009), 'Estimating the degree of activity of jumps in high frequency data', *Annals of Statistics* **37**, 2202–2244.

Aït-Sahalia, Y. & Jacod, J. (2012), 'Analyzing the spectrum of asset returns: Jump and volatility components in high frequency data', *Journal of Economic Literature* **50**, 1007–1050.

Andersen, T., Bollerslev, T. & Diebold, F. X. (2010), Parametric and nonparametric measurements of volatility, *in* Y. Ait-Sahalia & L. P. Hansen, eds, 'Handbook of Financial Econometrics', Elsevier, North Holland.

Andersen, T. G. & Benzoni, L. (2012), Stochastic volatility, *in* R. A. Meyers, ed., 'Encyclopedia of Complexity and Systems Science', Springer-Verlag. forthcoming.

Andersen, T. G., Bondarenko, O., Todorov, V. & Tauchen, G. (2014), 'The fine structure of equity-index option dynamics', *Journal of Econometrics* . forthcoming.

Andersen, T. G., Fusari, N. & Todorov, V. (2014), 'Parametric inference and dynamic state state recovery from option panels', *Econometrica* . forthcoming.

Barndorff-Nielsen, O. E., Graversen, S. E., Jacod, J. & Shephard, N. (2006), 'Limit theorems for bipolar variation in financial econometrics', *Econometric Theory* **22**, 677–719.

Barndorff-Nielsen, O. E. & Shephard, N. (2002), 'Econometric analysis of realized volatility and its use in estimating stochastic volatility models', *Journal of the Royal Statistical Society Series B* **64**, 253–280.

Barndorff-Nielsen, O. E. & Shephard, N. (2003), 'Realized power variation and stochastic volatility models', *Bernoulli* **9**, 243–265.

Barndorff-Nielsen, O. E. & Shephard, N. (2004), 'Power and Bipower Variation with Stochastic Volatility and Jumps', *Journal of Financial Econometrics* **2**, 1–37.

Barndorff-Nielsen, O. E. & Shephard, N. (2005), How accurate is the asymptotic approximation to the distribution of realised volatility?, *in* D. Andrews, J. Powell, P. Ruud & J. Stock, eds, 'Identification and Inference for Econometric Models. A Festschrift in Honour of T.J. Rothenberg', Cambridge University Press, Cambridge, UK, pp. 306–372.

Barndorff-Nielsen, O. E. & Shephard, N. (2007), Variation, jumps, market frictions and high frequency data in financial econometrics, *in* R. Blundell, T. Persson & W. K. Newey, eds, 'Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress', Cambridge University Press, Cambridge, UK, pp. 328–372.

Bhattacharya, R. N. & Rao, R. R. (1986), *Normal Approximation and Asymptotic Expansions*, Siam: Classics in Applied Mathematics.

Blumenthal, R. & Getoor, R. (1961), 'Sample functions of stochastic processes with independent increments', *Journal of Math. Mech.* **10**, 493–516.

Bollerslev, T. & Todorov, V. (2011), 'Tails, fears, and risk premia', *Journal of Finance* **66**, 2165–2211.

Carr, P., Geman, H., Madan, D. & Yor, M. (2003), 'Stochastic volatility for Lévy processes', *Mathematical Finance* **58**, 345–382.

Carr, P. & Wu, L. (2003), 'The finite moment log stable process and option pricing', *The Journal of Finance* **LVIII**, 753–778.

Chernov, M., Gallant, A. R., Ghysels, E. & Tauchen, G. (2003), 'Alternative models for stock price dynamics', *Journal of Econometrics* **116**, 225–257.

Cont, R. & Mancini, C. (2011), 'Nonparametric tests for pathwise properties of semimartingales', *Bernoulli* **17**, 781–813.

Davidson, R. (2007), Bootstrapping econometric models. Working paper, McGill University.

Davidson, R. & MacKinnon, J. G. (1999), 'The size distortion of bootstrap tests', *Econometric Theory* **15**, 361–376.

Davidson, R. & MacKinnon, J. G. (2006), 'The power of bootstrap and asymptotic tests', *Journal of Econometrics* **133**, 421–441.

Duffie, D., Pan, J. & Singleton, K. (2000), 'Transform analysis and asset pricing for affine jump-diffusions', *Econometrica* **68**, 1343–1376.

Gonçalves, S. & Meddahi, N. (2009), 'Bootstrapping realized volatility', *Econometrica* **1**, 283–306.

Hall, P. & Wilson, S. R. (1991), 'Two guidelines for bootstrap hypothesis testing', *Biometrics* **47**, 757–762.

Horowitz, J. L. & Savin, N. E. (2000), 'Empirically relevant critical values for hypothesis tests: A bootstrap approach', *Journal of Econometrics* **95**, 375–389.

Hounyo, U. (2014), Bootstrapping realized volatility and realized beta under a local gaussianity assumption. Unpublished manuscript, University of Oxford.

Huang, X. & Tauchen, G. (2005), 'The relative contribution of jumps to total price variance', *Journal of Financial Econometrics* **3**, 456–499.

Jacod, J. (2008), 'Asymptotic properties of realized power variations and related functionals of semimartingales', *Stochastic Processes and their Applications* **118**, 517–559.

Jacod, J. & Protter, P. (2012), *Discretization of Processes*, Springer-Verlag: Berlin.

Jacod, J. & Shiryaev, A. N. (2003), *Limit Theorems for Stochastic Processes*, 2nd Edition, Springer-Verlag: New-York.

Jing, B.-Y., Kong, X.-B. & Liu, Z. (2011), 'Estimating the jump activity index under noisy observations using high-frequency data', *Journal of the American Statistical Association* **106**, 558–568.

Jing, B.-Y., Kong, X.-B. & Liu, Z. (2012), 'Modeling high-frequency financial data by pure jump processes', *The Annals of Statistics* **40**, 759–784.

Jing, B.-Y., Kong, X.-B., Liu, Z. & Mykland, P. (2012), 'On the jump activity index for semimartingales', *Journal of Econometrics* **166**, 213–223.

Klüppelberg, C., Lindner, A. & Maller, R. (2004), 'A continuous-time GARCH process driven by a Lévy process: Stationarity and second-order behavior', *Journal of Applied Probability* **41**, 601–622.

MacKinnon, J. G. (2009), Bootstrap hypothesis testing, *in* D. A. Belsley & J. Kontoghiorghes, eds, 'Handbook of Computational Econometrics', Chichester, Wiley.

Pauly, M. (2011), 'Weighted resampling of martingale difference arrays with applications', *Electronic Journal of Statistics* **5**, 41–52.

Podolskij, M. & Vetter, M. (2010), 'Understanding limit theorems for semimartingales: A short survey', *Statistica Nederlandica* **64**, 329–351.

Rosinski, J. (2001), Series representations of Lévy processes from the perspective of point processes, *in* O. E. Barndorff-Nielsen, T. Mikosch & S. Resnick, eds, 'Lévy Processes-Theory and Applications.', Boston: Birkhäser.

Rosinski, J. (2007), 'Tempering stable processes', *Stochastic Processes and Their Applications* **117**, 677–707.

Sato, K. (1999), *Lévy processes and Infinitely Divisible Distributions*, Cambridge University Press, United Kingdom.

Shao, X. (2010), 'The dependent wild bootstrap', *Journal of the American Statistical Association* **105**, 218–235.

Todorov, V. (2009), 'Estimation of continuous-time stochastic volatility models with jumps using high-frequency data', *Journal of Econometrics* **148**, 131–148.

Todorov, V. (2013), 'Power variation from second order differences for pure jump semimartingales', *Stochastic Processes and their Applications* **123**, 2819–2850.

Todorov, V. (2015), 'Jump activity estimation for pure-jump semimartingales via self-normalized statistics', *Annals of Statistics* . forthcoming.

Todorov, V. & Tauchen, G. (2010), 'Activity signature functions for high-frequency data analysis', *Journal of Econometrics* **154**, 125–138.

Todorov, V. & Tauchen, G. (2011*a*), 'Limit theorems for power variations of pure-jump processes with application to activity estimation', *The Annals of Applied Probability* **21**, 546–588.

Todorov, V. & Tauchen, G. (2011*b*), 'Volatility jumps', *Journal of Business and Economic Statistics* **219**, 356–371.

Todorov, V. & Tauchen, G. (2012), 'Realized laplace transforms for pure-jump semimartingales', *Annals of Statistics* **40**, 1233–1262.

Todorov, V., Tauchen, G. & Grynkiv, I. (2014), 'Volatility activity: Specification and estimation', *Journal of Econometrics* **178**, 180–193.

Varneskov, R. T. (2014), Estimating the quadratic variation spectrum of noisy asset prices using generalized flat-top realized kernels. Unpublished Manuscript, Aarhus University.

Woerner, J. (2003), 'Variational sums and power variation: A unifying approach to model selection and estimation in semimartingale models', *Statistics and Decisions* **21**, 47–68.

Woerner, J. (2007), 'Inference in Lévy-type stochastic volatility models', *Advances in Applied Probability* **39**, 531–549.

Wu, L. (2006), 'Dampened power law: Reconciling the tail behavior of financial security returns', *The Journal of Business* **79**, 1445–1473.

Wu, L. (2008), Modeling financial security returns using Lévy processes, *in* J. Birge & L. Linetsky, eds, 'Handbooks in Operations Research and Management Science, Volume 15: Financial Engineering', Elsevier, North Holland.

Zhao, Z. & Wu, W. B. (2009), 'Nonparametric inference of discretely sampled stable Lévy processes', *Journal of Econometrics* **153**, 83–92.

A Additional Assumptions

Before proceeding to the remaining assumptions for the theoretical analysis, let us fix some notation. In particular, let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ define the non-negative real line, and let $(E, \mathcal{E})$ denote an auxiliary measurable space on the original filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Then,

Assumption 3. *The drift, α_t, and the stochastic scale, σ_t, are Itô semimartingales of the form*

$$\begin{aligned} \alpha_t &= \alpha_0 + \int_0^t b_s^\alpha ds + \int_0^t \int_E \kappa\left(\delta^\alpha(s,x)\right) \underline{\tilde{\mu}}(ds,dx) + \int_E \kappa\left(\delta^\alpha(s,x)\right) \underline{\mu}(ds,dx) \\ \sigma_t &= \sigma_0 + \int_0^t b_s^\sigma ds + \int_0^t \int_E \kappa\left(\delta^\sigma(s,x)\right) \underline{\tilde{\mu}}(ds,dx) + \int_E \kappa\left(\delta^\sigma(s,x)\right) \underline{\mu}(ds,dx) \end{aligned} \tag{A.1}$$

where the different components satisfy the following:

(a) $|\sigma_t|^{-1}$ *and* $|\sigma_{t-}|^{-1}$ *are strictly positive;*

(b) $\underline{\mu}$ *is a homogenous Poison random measure on* $\mathbb{R}_+ \times E$ *with compensator (Lévy measure)* $dt \otimes \lambda(dx)$*. Furthermore,* $\underline{\mu}$ *may have arbitrary dependence with* L_t*;*

(c) *the processes* $\delta^\alpha(t,x)$ *and* $\delta^\sigma(t,x)$ *are predictable, left-continuous with right limits in* t*. In addition, let* T_k *denote a sequence of stopping times increasing to* $+\infty$*, then* $\delta^\alpha(t,x)$ *and* $\delta^\sigma(t,x)$ *are assumed to satisfy*

$$|\delta^\alpha(t,x)| + |\delta^\sigma(t,x)| \leq \gamma_k(x), \quad \forall t \leq T_k,$$

where $\gamma_k(x)$ *is a deterministic function on* $\mathbb{R}$ *satisfying* $\int_{\mathbb{R}} (|\gamma_k(x)|^{\beta+\epsilon} \wedge 1) dx < \infty$*,* β *being the activity index defined in Assumption 1 and* $\epsilon > 0$ *is arbitrarily small;*

(d) *the processes* b_t^α *and* b_t^σ *are both Itô semimartingales of the form* (A.1) *with components satisfying restrictions equivalent to* (**b**) *and* (**c**).

The regularity conditions on α_t and σ_t implied by Assumption 3 are identical to the corresponding conditions in Todorov & Tauchen (2011*a*) and Todorov (2013). This is not surprising as our bootstrap procedure carefully seeks to mimic the dependence in the original series, $\Delta_i^{n,\nu} Z$, such that we may obtain a bootstrap central limit theorem, which is similar to their results (presented in Section 2.4). Similar to the proofs in Todorov (2013), we will rely on the following stronger assumption when establishing some of the asymptotic results below, in particular Theorem 1, and then use a standard localization argument to extend them to the weaker case in Assumption 3, see, for example, the discussion in Jacod & Protter (2012, Section 4.4.1).

Assumption 3'. *In addition to Assumption 3, the following holds*

(a) *the processes* b_t^α*,* b_t^σ*,* $|\sigma_t|$ *and* $|\sigma_t|^{-1}$ *are uniformly bounded;*

(b) *the processes* $|\delta^\alpha(t,x)| + |\delta^\sigma(t,x)| \leq \gamma(x)$ *for all* t *where* $\gamma(x)$ *is a deterministic function on* $\mathbb{R}$ *satifying* $\int_{\mathbb{R}} |\gamma(x)|^{\beta+\epsilon}$*,* β *being the activity index defined in Assumption 1 and* $\epsilon \in [\beta, 2]$*;*

(c) *the coefficients of the Itô semimartingales b_t^α and b_t^σ satisfy conditions, which are analogous to the conditions* (**a**) *and* (**b**) *above;*

(d) *the process $\int_{\mathbb{R}}(|x|^{\beta'+\epsilon}\wedge 1)\nu_t^Y(dx)$ is bounded and so are the jumps of L and Y.*

B Proofs of Theoretical Results

In the following proofs, we will use the notation $\mathbb{E}_i^n[\cdot]\equiv\mathbb{E}[\cdot|\mathcal{F}_{t_i}]$. Furthermore, K denotes a constant, which may change from line to line and from (in)equality to (in)equality. Moreover, for a given $d\times d$ matrix $\boldsymbol{A}$, $\|\boldsymbol{A}\|$ denotes the Euclidean matrix norm and $\varpi_i(\boldsymbol{A})$ denotes its i-th eigenvalue.

B.1 Proof of Lemma 2

We first establish the result for $\boldsymbol{\mathcal{E}}_n^*(p,Z)$ by utilizing the properties of the bootstrap expectation operator to rewrite the vector as

$$\boldsymbol{\mathcal{E}}_n^*(p,Z)n^{1-p/\beta}=\begin{pmatrix}\sum_{i=1}^n|\Delta_i^{n,1}Z|^p\mathbb{E}^*[|S_i^*|^p]\\ \sum_{i=1}^n|\Delta_i^{n,1}Z|^p\mathbb{E}^*[|S_i^*+S_{i+1}^*|^p]\end{pmatrix}=\begin{pmatrix}\mu_p(\hat\beta(p))\sum_{i=1}^n|\Delta_i^{n,1}Z|^p\\ 2^{p/\hat\beta(p)}\mu_p(\hat\beta(p))\sum_{i=1}^n|\Delta_i^{n,1}Z|^p\end{pmatrix}$$

and, then, by using the definition of $V_n(p,Z,1)$. For the asymptotic covariance matrix $\boldsymbol{\Omega}_n^*(p,Z)$, we establish the result element-by-element. First, for the two diagonal terms, it follows that

$$\mathbb{V}^*\left[\sqrt{n}n^{p/\beta-1}V_n^*(p,Z,1)\right]=\Sigma(p,\hat\beta(p),0)_{1,1}n^{2p/\beta-1}\sum_{i=1}^n|\Delta_i^{n,1}Z|^{2p},$$

where $\boldsymbol{\Sigma}(p,\hat\beta(p),k)=(\Sigma(p,\hat\beta(p),k)_{i,j})_{1\le i,j\le 2}$, and

$$\begin{aligned}\mathbb{V}^*\left[\sqrt{n}n^{p/\beta-1}V_n^*(p,Z,2)\right]&=\Sigma(p,\hat\beta(p),0)_{2,2}n^{2p/\beta-1}\sum_{i=1}^n|\Delta_i^{n,1}Z|^{2p}\\&\quad+2\Sigma(p,\hat\beta(p),1)_{2,2}n^{2p/\beta-1}\sum_{i=1}^{n-1}|\Delta_i^{n,1}Z|^p|\Delta_{i+1}^{n,1}Z|^p,\end{aligned}$$

respectively, using the properties of the bootstrap variance operator. For the cross-product terms, we have $\mathrm{Cov}^*[n^{p/\beta-1/2}V_n^*(p,Z,1),n^{p/\beta-1/2}V_n^*(p,Z,2)]=n^{2p/\beta-1}\mathrm{Cov}^*[V_n^*(p,Z,1),V_n^*(p,Z,2)]$ where

$$\begin{aligned}\mathrm{Cov}^*\left[V_n^*(p,Z,1),V_n^*(p,Z,2)\right]&=\sum_{i=1}^n\sum_{j=1}^n|\Delta_i^{n,1}Z|^p|\Delta_j^{n,1}Z|^p\mathrm{Cov}^*\left[|S_i^*|^p,|S_j^*+S_{j+1}^*|^p\right]\\&=\sum_{i=1}^n|\Delta_i^{n,1}Z|^{2p}\mathrm{Cov}^*\left[|S_i^*|^p,|S_i^*+S_{i+1}^*|^p\right]\\&\quad+\sum_{i=1}^{n-1}|\Delta_{i+1}^{n,1}Z|^p|\Delta_i^{n,1}Z|^p\mathrm{Cov}^*\left[|S_{i+1}^*|^p,|S_i^*+S_{i+1}^*|^p\right]\end{aligned}$$

with $\text{Cov}^*[|S_i^*|^p, |S_j^* + S_{j+1}^*|^p] = \Sigma(p, \hat{\beta}(p), 0)_{1,2}$, $\text{Cov}^*\left[|S_{i+1}^*|^p, |S_i^* + S_{i+1}^*|^p\right] = \Sigma(p, \widehat{\beta}(p), 1)_{1,2}$, and, similarly, for $n^{2p/\beta-1}\text{Cov}^*[V_n^*(p, Z, 2), V_n^*(p, Z, 1)]$. Finally, collecting terms as

$$\mathbf{\Omega}_n^*(p, Z) = n^{2p/\beta-1}\mathbf{\Sigma}(p, \hat{\beta}(p), 0) \sum_{i=1}^{n} |\Delta_i^{n,1} Z|^{2p}$$
$$+ n^{2p/\beta-1} \sum_{i=1}^{n-1} |\Delta_i^{n,1} Z|^p |\Delta_{i+1}^{n,1} Z|^p \left(\mathbf{\Sigma}(p, \hat{\beta}(p), 1) + \mathbf{\Sigma}(p, \hat{\beta}(p), 1)'\right),$$

provides the final result. □

B.2 Proof of Theorem 1

We collect two approximation error bounds, based on the results in Todorov (2013, Section 5.2.2), and highlight them as an auxiliary lemma since they will be useful later in the proof.

Lemma 3. *Under Assumptions 1, 3, and $p \in (0, \beta/2)$, it holds that*

$$\mathbb{E}\left[\left||\sigma_{t_{i-}}|^p - |\sigma_s|^p\right|\right] \leq K n^{-1/(\beta+\epsilon)\wedge 1}, \qquad \mathbb{E}\left[\left||\sigma_s|^p - |\sigma_{t_{(i-1)-}}|^p\right|\right] \leq K n^{-1/(\beta+\epsilon)\wedge 1},$$

for some $s \in [t_{i-1}, t_i]$ and $\epsilon \in [0, 2-\beta]$

Proof. By using the same arguments as for Todorov (2013, Equations (29) and (30)). □

Next, we make the decomposition

$$n^{2p/\beta-1} BV_n(2p, Z, 1) - \mu_p^2(\beta) \int_0^1 |\sigma_s|^{2p} ds = E_1 + E_2 + E_3 \tag{B.1}$$

where the three terms may be written as

$$E_1 = \frac{1}{n}\sum_{i=1}^{n} |\sigma_{t_{(i-1)-}}|^p |\sigma_{t_{i-}}|^p \left(n^{p/\beta}|\Delta_i^{n,1} S|^p n^{p/\beta}|\Delta_{i+1}^{n,1} S|^p - \mu_p^2(\beta)\right),$$
$$E_2 = \mu_p^2(\beta) \sum_{i=1}^{n-1} \left(\frac{1}{n}|\sigma_{t_{(i-1)-}}|^p |\sigma_{t_{i-}}|^p - \int_{t_{i-1}}^{t_i} |\sigma_s|^{2p} ds\right),$$
$$E_3 = n^{2p/\beta-1} \sum_{i=1}^{n-1} \left(|\Delta_i^{n,1} Z|^p |\Delta_{i+1}^{n,1} Z|^p - |\sigma_{t_{(i-1)-}}|^p |\sigma_{t_{i-}}|^p |\Delta_i^{n,1} S|^p |\Delta_{i+1}^{n,1} S|^p\right),$$

and analyze each of the three terms separately. *First*, for E_1, write

$$E_1 = \frac{1}{n}\sum_{i=1}^{n-1} |\sigma_{t_{(i-1)-}}|^{2p} \left(n^{p/\beta}|\Delta_i^{n,1} S|^p n^{p/\beta}|\Delta_{i+1}^{n,1} S|^p - \mu_p^2(\beta)\right)$$
$$+ \frac{1}{n}\sum_{i=1}^{n-1} |\sigma_{t_{(i-1)-}}|^p \left(|\sigma_{t_{i-}}|^p - |\sigma_{t_{(i-1)-}}|^p\right) \left(n^{p/\beta}|\Delta_i^{n,1} S|^p n^{p/\beta}|\Delta_{i+1}^{n,1} S|^p - \mu_p^2(\beta)\right) \equiv E_{1,1} + E_{1,2}$$

for which we may bound the second term as

$$|E_{1,2}| \leq \frac{K}{n}\sum_{i=1}^{n-1}\left||\sigma_{t_{i-}}|^p - |\sigma_{t_{(i-1)-}}|^p\right| \leq O_p(n^{-1/(\beta+\epsilon)\wedge 1}) \quad \text{for } \epsilon \in [0, 2-\beta] \tag{B.2}$$

using boundedness of $|\sigma_{t_{(i-1)-}}|$ from above an below along with the existence of the p-th absolute moment of a strictly stable process for $p < \beta$. For the second inequality, we use Lemma 3. Next, define $\chi_i = n^{1/\beta}\Delta_i^{n,1}S$ and rewrite $E_{1,1}$ as $E_{1,1} = n^{-1}\sum_{i=1}^{n-1}|\sigma_{t_{(i-1)-}}|^{2p}(|\chi_i|^p|\chi_{i+1}|^p - \mu_p^2(\beta))$. Then, by Todorov & Tauchen (2012, Lemma 1), it follows that $\chi_i \xrightarrow{d} S_i$ with S_i, $i = 1, \dots, n$, being the self-similar, strictly stable process defined via the characteristic function (7). Then, as

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}_{i-1}^n\left[|\sigma_{t_{(i-1)-}}|^{2p}\left(|\chi_i|^p|\chi_{i+1}|^p - \mu_p^2(\beta)\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n-1}|\sigma_{t_{(i-1)-}}|^{2p}\left(\mathbb{E}_{i-1}^n\left[|\chi_i|^p\right]\mathbb{E}_{i-1}^n\left[|\chi_{i+1}|^p\right] - \mu_p^2(\beta)\right) = 0,
\end{aligned}$$

since $\mathbb{E}_{i-1}^n\left[|\chi_i|^p\right] = \mu_p(\beta)$, and

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n-1}\mathbb{E}_{i-1}^n\left[|\sigma_{t_{(i-1)-}}|^{4p}\left(|\chi_i|^p|\chi_{i+1}|^p - \mu_p^2(\beta)\right)^2\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n-1}|\sigma_{t_{(i-1)-}}|^{4p}\mathbb{E}_{i-1}^n\left[\left(|\chi_i|^p|\chi_{i+1}|^p - \mu_p^2(\beta)\right)^2\right] \leq \frac{K}{n^2}\sum_{i=1}^{n-1}\mathbb{E}_{i-1}^n\left[\left(|\chi_i|^p|\chi_{i+1}|^p - \mu_p^2(\beta)\right)^2\right] \leq \frac{K}{n}
\end{aligned}$$

using, again, boundedness of $|\sigma_{t_{(i-1)-}}|$ from above an below along with existence of the p-th absolute moment of a strictly stable process for $p < \beta/2$. This implies $E_{1,1} = o_p(n^{-1/2})$.

Second, for E_2, write

$$E_2 = \mu_p^2(\beta)\frac{1}{n}\sum_{i=1}^{n-1}|\sigma_{t_{i-}}|^p\left(|\sigma_{t_{(i-1)-}}|^p - |\sigma_{t_{i-}}|^p\right) + \mu_p^2(\beta)\sum_{i=1}^{n-1}\left(\frac{1}{n}|\sigma_{t_{i-}}|^{2p} - \int_{t_i}^{t_{i+1}}|\sigma_s|^{2p}ds\right)ds \xrightarrow{\mathbb{P}} 0 \tag{B.3}$$

using Lemma 3 for the first term, as in (B.2), and Riemann integrability for the second term.

Third, to show $E_3 \xrightarrow{\mathbb{P}} 0$, it suffices to establish a bound for

$$\begin{aligned}
&n^{p/\beta}\left(|\Delta_i^{n,1}Z|^p - |\sigma_{t_{(i-1)-}}|^p|\Delta_i^{n,1}S|^p\right) \\
&= n^{p/\beta}\left(|\Delta_i^{n,1}Z|^p - \left|\int_{t_{i-1}}^{t_i}\sigma_{s-}dL_t\right|^p\right) + n^{p/\beta}\left(\left|\int_{t_{i-1}}^{t_i}\sigma_{s-}dL_t\right|^p - |\sigma_{t_{(i-1)-}}|^p|\Delta_i^{n,1}S|^p\right) \equiv E_{3,1} + E_{3,2}
\end{aligned}$$

and use it in conjunction with an addition and subtraction argument to bound the whole sum. Before proceeding, we highlight the following two algebraic inequalities

$$\big||a+b|^p - |a|^p\big| \le |b|^p \qquad \text{and} \qquad \big|\sum_i a_i\big|^p \le \sum_i |a_i|^p \tag{B.4}$$

for $a, b \in \mathbb{R}$, $a_i \in \mathbb{R}\ \forall i$, and $p \in (0,1)$. Using these, we may decompose

$$|E_{3,1}| \le \big|n^{1/\beta}\int_{t_{i-1}}^{t_i} \alpha_s ds + n^{1/\beta}\int_{t_{i-1}}^{t_i} dY_s\big|^p \le \big|n^{1/\beta}\int_{t_{i-1}}^{t_i} \alpha_s ds\big|^p + \big|n^{1/\beta}\int_{t_{i-1}}^{t_i} dY_s\big|^p = O_p\left(n^{(1/\beta-1)p}\right)$$

where the last inequality follows by Assumptions 1 and 3' since $\beta' < 1$. Next, rewrite $E_{3,2}$ as

$$\begin{aligned} E_{3,2} &= n^{p/\beta}\big|\sigma_{t_{(i-1)-}}\Delta_i^{n,1}L - \sigma_{t_{(i-1)-}}\Delta_i^{n,1}L + \int_{t_{i-1}}^{t_i}\sigma_{s-}dL_t\big|^p - n^{p/\beta}|\sigma_{t_{(i-1)-}}|^p|\Delta_i^{n,1}S|^p \\ &\le n^{p/\beta}\big|\int_{t-i}^{t_i}(\sigma_s - \sigma_{t_{(i-1)-}})dL_s\big|^p + n^{p/\beta}|\sigma_{t_{(i-1)-}}|^p\left(|\Delta_i^{n,1}L|^p - |\Delta_i^{n,1}S|^p\right) \equiv E_{3,2,1} + E_{3,2,2}. \end{aligned}$$

Then, we may bound $E_{3,2,1}$ as

$$\begin{aligned} E_{3,2,1} &\le \int_{t_{i-1}}^{t_i} |\sigma_{s-} - \sigma_{t_{(i-1)-}}|^p |n^{1/\beta}dL_s|^p \\ &\le \sup_{s\in[t_{i-1},t_i]} |\sigma_{s-} - \sigma_{t_{(i-1)-}}|^p \int_{t_{i-1}}^{t_i} |n^{1/\beta}dL_s|^p \le O_p(n^{-(p/\beta\wedge 1-\iota)}) \times O_p(n^{-1}), \quad \forall \iota > 0 \end{aligned}$$

using that $\mathbb{E}[|\sigma_s - \sigma_t|] \le K|t-s|^{p/\beta\wedge 1-\iota}$ for $s,t>0$ in conjunction with $n^{1/\beta}dL_s \overset{d}{=} S_i$, cf. Todorov & Tauchen (2012) and (9), as $n \to \infty$ and $\beta' < 1$ and the existence of absolute moments for the self-similar, strictly stable process S_i when $p < \beta$. Last, before establishing a bound for $E_{3,2,1}$, we note that L_t may be decomposed as

$$L_t = S_t + \tilde{S}_t - \hat{S}_t,$$

see Todorov (2013, Equation (22)), where $\tilde{S}_t$ and $\hat{S}_t$ are pure-jump Lévy processes with the first two characteristics zero with respect to the truncation function $\kappa(\cdot)$, and Lévy densities $2|\nu_2(x)|\mathbf{1}\{\nu_2(x) < 0\}$ and $|\nu_2(x)|$, respectively, see the supplementary appendix for Todorov & Tauchen (2012) for details on this decomposition. Then, using boundedness of $|\sigma_{t_{(i-1)-}}|$ and the inequalities (B.4), we may write

$$|E_{3,2,2}| \le K\big||n^{1/\beta}\Delta_i^{n,1}L|^p - |n^{1/\beta}\Delta_i^{n,1}S|^p\big| \le K\left(|n^{1/\beta}\Delta_i^{n,1}\hat{S}|^p + |n^{1/\beta}\Delta_i^{n,1}\tilde{S}|^p\right) = O_p\left(n^{(1/\beta-1/\beta')p}\right)$$

since the activity indices of $\hat{S}$ and $\tilde{S}$ are determined by $|\nu_2(x)|$, i.e., they are β'. The results for $E_{3,1}$ and $E_{3,2}$ may be combined to show $E_3 \overset{\mathbb{P}}{\to} 0$, concluding the proof. □

B.3 Proof of Theorem 2

The proof of the main theorem proceeds in two steps:

Step 1. Show the desired result for $\widetilde{\boldsymbol{T}}_n^*$ where

$$\widetilde{\boldsymbol{T}}_n^* \equiv (\boldsymbol{\Omega}_n^*(p,Z))^{-1/2}\sqrt{n}n^{p/\beta-1}\begin{pmatrix} V_n^*(p,Z,1)-\mathbb{E}^*[V_n^*(p,Z,1)] \\ V_n^*(p,Z,2)-\mathbb{E}^*[V_n^*(p,Z,2)] \end{pmatrix}.$$

Step 2. Show $\widehat{\boldsymbol{\Omega}}_n^*(p,Z)-\boldsymbol{\Omega}_n^*(p,Z)\xrightarrow{\mathbb{P}}\mathbf{0}$ using Corollary 2.

First, for **Step 1**, let $\boldsymbol{\Phi}(\boldsymbol{x};\boldsymbol{V})$ be the multivariate distribution function of $N(\mathbf{0},\boldsymbol{V})$ on $\mathbb{R}^2$. Then, we will first show that

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2}\left|\mathbb{P}^*\left(\widetilde{\boldsymbol{T}}_n^*\leq\boldsymbol{x}\right)-\mathbb{P}\left(\widetilde{\boldsymbol{T}}_n\leq\boldsymbol{x}\right)\right|\xrightarrow{\mathbb{P}}\mathbf{0}, \tag{B.5}$$

with

$$\widetilde{\boldsymbol{T}}_n \equiv (\boldsymbol{\Omega}_n(p,Z))^{-1/2}\sqrt{n}\begin{pmatrix} n^{p/\beta-1}V_n(p,Z,1)-\mu_p(\beta)\int_0^1|\sigma_s|^p\,ds \\ n^{p/\beta-1}V_n(p,Z,2)-2^{p/\beta}\mu_p(\beta)\int_0^1|\sigma_s|^p\,ds \end{pmatrix},$$

where

$$\boldsymbol{\Omega}_n(p,Z)=\mathbb{V}\left[\sqrt{n}n^{p/\beta-1}\begin{pmatrix} V_n(p,Z,1) \\ V_n(p,Z,2)\end{pmatrix}\right].$$

Under Assumptions 1, 2 (b), and 3 of Appendix A, $\widetilde{\boldsymbol{T}}_n\xrightarrow{d_s}N(\mathbf{0},\boldsymbol{I}_2)$ follows from Lemma 1 (b). Hence, we may invoke a multivariate version of Polya's Theorem, see, e.g., Bhattacharya & Rao (1986), to establish

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2}\left|\mathbb{P}\left(\widetilde{\boldsymbol{T}}_n\leq\boldsymbol{x}\right)-\boldsymbol{\Phi}(\boldsymbol{x};\boldsymbol{I}_2)\right|\xrightarrow{\mathbb{P}}\mathbf{0}.$$

Hence, if we can prove that

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2}\left|\mathbb{P}^*\left(\widetilde{\boldsymbol{T}}_n^*\leq\boldsymbol{x}\right)-\boldsymbol{\Phi}(\boldsymbol{x};\boldsymbol{I}_2)\right|\xrightarrow{\mathbb{P}}\mathbf{0}, \tag{B.6}$$

then (B.5) follows by the triangle inequality. To show (B.6), rewrite $\widetilde{\boldsymbol{T}}_n^*$ as

$$\widetilde{\boldsymbol{T}}_n^*=(\boldsymbol{\Omega}_n^*(p,Z))^{-1/2}\sqrt{n}\sum_{i=1}^n\boldsymbol{D}_i\boldsymbol{Z}_i^*=\sqrt{n}\sum_{i=1}^n\boldsymbol{z}_i^*,$$

with $\boldsymbol{z}_i^*\equiv(\boldsymbol{\Omega}_n^*(p,Z))^{-1/2}\boldsymbol{D}_i\boldsymbol{Z}_i^*$ where

$$\boldsymbol{D}_i=n^{p/\beta-1}\left|\Delta_i^{n,1}Z\right|^p\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad\text{and}\quad\boldsymbol{Z}_i^*=\begin{pmatrix}|S_i^*|^p-\mathbb{E}^*[|S_i^*|^p] \\ \left|S_i^*+S_{i+1}^*\right|^p-\mathbb{E}^*[\left|S_i^*+S_{i+1}^*\right|^p]\end{pmatrix}.$$

Note that $\boldsymbol{Z}_i^*$ may be written as $\boldsymbol{Z}_i^*=(|S_i^*|^p-\mu_p(\hat\beta(p)),|S_i^*+S_{i+1}^*|^p-2^{p/\hat\beta(p)}\mu_p(\hat\beta(p))'$, and, furthermore, that $\boldsymbol{Z}_i^*$ is a mean-zero and one-dependent vector.

Next, we follow Pauly (2011) and rely on a modified Cramer-Wold device to establish the bootstrap central limit theorem. Let $\mathcal{D}=\{\boldsymbol{\lambda}_k:k\in\mathbb{N}\}$ be a countable dense subset of the unit circle of $\mathbb{R}^2$, then this implies that for any $\boldsymbol{\lambda}\in\mathcal{D}$, we need to show $\boldsymbol{\lambda}'\widetilde{\boldsymbol{T}}_n^*\xrightarrow{d^*}N(0,1)$, in probability-$\mathbb{P}$, as $n\to\infty$. By

Lemma 2, $\mathbb{V}^*[\boldsymbol{\lambda}'\widetilde{\boldsymbol{T}}_n^*] = 1$ for all n. Hence, we are left with showing that $\boldsymbol{\lambda}'\widetilde{\boldsymbol{T}}_n^*$ is asymptotically normally distributed conditional on the original sample and with probability-$\mathbb{P}$ approaching one. To account for the vectors $\boldsymbol{z}_i^*$ being one-dependent, we prove this using a large-block-small-block argument.[5] In particular, we rely on large blocks of L_n successive observations followed by a small block consisting of a single element. Formally, let L_n be an integer such that $L_n \propto n^\alpha$ for $0 < \alpha < \frac{\delta}{2(1+\delta)}$ some arbitrarily small $\delta > 0$, and let $k_n = \lfloor \frac{n}{L_n+1} \rfloor$. Then, define the (large) blocks

$$\mathcal{L}_j = \{i \in \mathbb{N} : (j-1)(L_n+1) + 1 \leq i \leq j\,(L_n+1) - 1\} \quad \text{where} \quad 1 \leq j \leq k_n,$$

and $\mathcal{L}_{k_n+1} = \{i \in \mathbb{N} : k_n(L_n+1) + 1 \leq i \leq n\}$. Moreover, let $U_j^* = \sum_{i \in \mathcal{L}_j} \boldsymbol{\lambda}'\boldsymbol{z}_i^*$ for $j = 1, \ldots, k_n+1$, such that we can make the decomposition

$$\boldsymbol{\lambda}'\widetilde{\boldsymbol{T}}_n^* = \sqrt{n}\sum_{j=1}^{k_n+1} U_j^* + \sqrt{n}\sum_{j=1}^{k_n} \boldsymbol{\lambda}'\boldsymbol{z}_{j(L_n+1)}^*.$$

Next, we need to show that

(a) $\sqrt{n}\sum_{j=1}^{k_n} \boldsymbol{\lambda}'\boldsymbol{z}_{j(L_n+1)}^* = o_{p^*}(1)$ in probability-$\mathbb{P}$, and

(b) for some $\delta > 0$, $\sum_{j=1}^{k_n+1} \mathbb{E}^*|\sqrt{n}U_j^*|^{2+\delta} \xrightarrow{\mathbb{P}} 0$,

since conditions **(a)**-**(b)**, in conjunction with $\{U_j^*\}_{1 \leq j \leq k_n+1}$ being an independent array, conditionally on the original sample, suffices to show that $\sqrt{n}\sum_{j=1}^{k_n+1} U_j^* \xrightarrow{d^*} N(0,1)$, in probability-$\mathbb{P}$.

For **(a)**. Since $\mathbb{E}^*[\boldsymbol{z}_i^*] = \boldsymbol{0}$ for all i, it suffices to show that $\mathbb{V}^*[\sqrt{n}\sum_{j=1}^{k_n} \boldsymbol{\lambda}'\boldsymbol{z}_{j(L_n+1)}^*] = o_p(1)$. This simplifies since, for $L_n \geq 1$ with n sufficiently large, the elements $z_{j(L_n+1)}^*$ are independent along j conditional on the original sample such that

$$\mathbb{V}^*\left[\sqrt{n}\sum_{j=1}^{k_n} \boldsymbol{\lambda}'\boldsymbol{z}_{j(L_n+1)}^*\right] = \boldsymbol{\lambda}'\left(\boldsymbol{\Omega}_n^*(p,Z)\right)^{-1/2}\boldsymbol{\Lambda}_n^*\left(\boldsymbol{\Omega}_n^*(p,Z)\right)^{-1/2}\boldsymbol{\lambda}$$

where $\boldsymbol{\Lambda}_n^* = \mathbb{V}^*[\sqrt{n}\sum_{j=1}^{k_n} \boldsymbol{D}_{j(L_{n+1})}\boldsymbol{Z}_{j(L_n+1)}^*]$. By the Cauchy-Schwarz inequality, it follows that

$$\left\|\mathbb{V}^*\left[\sqrt{n}\sum_{j=1}^{k_n} \boldsymbol{\lambda}'\boldsymbol{z}_{j(L_n+1)}^*\right]\right\| \leq \|\boldsymbol{\lambda}\|^2\left\|\left(\boldsymbol{\Omega}_n^*(p,Z)\right)^{-1/2}\right\|^2\|\boldsymbol{\Lambda}_n^*\|.$$

Next, since $\boldsymbol{\Omega}_n^*(p,Z) \xrightarrow{\mathbb{P}} \boldsymbol{\Omega}^*(p,Z)$ by Corollary 2,

$$\left\|\left(\boldsymbol{\Omega}_n^*(p,Z)\right)^{-1/2}\right\|^2 = \operatorname{Tr}\left(\left(\boldsymbol{\Omega}_n^*(p,Z)\right)^{-1}\right) \xrightarrow{\mathbb{P}} \operatorname{Tr}\left(\left(\boldsymbol{\Omega}^*(p,Z)\right)^{-1}\right) = \left\|\left(\boldsymbol{\Omega}^*(p,Z)\right)^{-1/2}\right\|^2$$

[5] See, e.g., the proof of Shao (2010, Theorem 1) for a similar approach.

by the continuous mapping theorem where

$$\left\|(\mathbf{\Omega}^*(p,Z))^{-1/2}\right\|^2 = \varpi_1\left((\mathbf{\Omega}^*(p,Z))^{-1}\right) + \varpi_2\left((\mathbf{\Omega}^*(p,Z))^{-1}\right) = \varpi_1^{-1}\left(\mathbf{\Omega}^*(p,Z)\right) + \varpi_2^{-1}\left(\mathbf{\Omega}^*(p,Z)\right)$$

with $\varpi_1^{-1}\left(\mathbf{\Omega}^*(p,Z)\right) + \varpi_2^{-1}\left(\mathbf{\Omega}^*(p,Z)\right) = O_p(1)$ since $\int_0^1 |\sigma_s|^{2p}\, ds > 0$ by Assumption 3 (a). For $\|\mathbf{\Lambda}_n^*\|$, we have

$$\mathbf{\Lambda}_n^* = n\sum_{j=1}^{k_n} \boldsymbol{D}_{j(L_{n+1})}\mathbb{E}^*\left[\boldsymbol{Z}^*_{j(L_n+1)}\boldsymbol{Z}^{*\prime}_{j(L_n+1)}\right]\boldsymbol{D}'_{j(L_{n+1})},$$

which implies

$$\begin{aligned}\|\mathbf{\Lambda}_n^*\| &= n\sum_{j=1}^{k_n}\left\|\boldsymbol{D}_{j(L_{n+1})}\right\|^2\left\|\mathbb{E}^*\left[\boldsymbol{Z}^*_{j(L_n+1)}\boldsymbol{Z}^{*\prime}_{j(L_n+1)}\right]\right\| \\ &= n\sum_{j=1}^{k_n}\left\|\boldsymbol{D}_{j(L_{n+1})}\right\|^2\left\|\mathbf{\Sigma}(p,\hat{\beta}(p),0)\right\| \le K\sum_{j=1}^{k_n} n\left\|\boldsymbol{D}_{j(L_{n+1})}\right\|^2\end{aligned}$$

Since for any i, $\|\boldsymbol{D}_i\|^2 = 2n^{2(p/\beta-1)}|\Delta_i^{n,1}Z|^{2p}$, it follows that

$$\|\mathbf{\Lambda}_n^*\| \le Kn^{2p/\beta-1}\sum_{j=1}^{k_n}\left|\Delta^{n,1}_{j(L_n+1)}Z\right|^{2p} = O_p(k_n/n),$$

which, using $k_n = \lfloor \frac{n}{L_n+1}\rfloor \le \frac{n}{L_n} = n^{1-\alpha}$, is $O_p(L_n^{-1}) = O_p(n^{-\alpha})$. Combining the asymptotic bounds results for $\|\mathbf{\Lambda}_n^*\|$ and $\|\left(\mathbf{\Omega}_n^*(p,Z)\right)^{-1/2}\|^2$ with $\alpha > 0$ provides **(a)**.

Next, we verify **(b)**. For any $1 \le j \le k_n+1$, it follows by the c_r-inequality,

$$\left|U_j^*\right|^{2+\delta} = \left|\sum_{i\in\mathcal{L}_j}\boldsymbol{\lambda}'\boldsymbol{z}_i^*\right|^{2+\delta} \le L_n^{2+\delta-1}\|\boldsymbol{\lambda}\|^{2+\delta}\left\|(\mathbf{\Omega}_n^*(p,Z))^{-1/2}\right\|^{2+\delta}\sum_{i\in\mathcal{L}_j}\|\boldsymbol{D}_i\|^{2+\delta}\|\boldsymbol{Z}_i^*\|^{2+\delta}.$$

Hence, we have

$$\begin{aligned}\mathbb{E}^*\left[\left|U_j^*\right|^{2+\delta}\right] &\le L_n^{2+\delta-1}\|\boldsymbol{\lambda}\|^{2+\delta}\left\|(\mathbf{\Omega}_n^*(p,Z))^{-1/2}\right\|^{2+\delta}\sum_{i\in\mathcal{L}_j}\|\boldsymbol{D}_i\|^{2+\delta}\,\mathbb{E}^*\left[\|\boldsymbol{Z}_i^*\|^{2+\delta}\right] \\ &\le KL_n^{1+\delta}\left\|(\mathbf{\Omega}_n^*(p,Z))^{-1/2}\right\|^{2+\delta}\sum_{i\in\mathcal{L}_j}\|\boldsymbol{D}_i\|^{2+\delta}\end{aligned}$$

since we may select $\delta > 0$ arbitrarily small. Then, by arguments similar to those for **(a)**,

$$\sum_{j=1}^{k_n+1}\mathbb{E}^*\left[\left|\sqrt{n}U_j^*\right|^{2+\delta}\right] \le Kn^{1+\delta/2}L_n^{1+\delta}\left\|(\mathbf{\Omega}_n^*(p,Z))^{-1/2}\right\|^{2+\delta}\sum_{j=1}^{k_n+1}\sum_{i\in\mathcal{L}_j}\|\boldsymbol{D}_i\|^{2+\delta}$$

$$= Kn^{1+\delta/2}L_n^{1+\delta}\left\|(\mathbf{\Omega}_n^*(p,Z))^{-1/2}\right\|^{2+\delta}\sum_{j=1}^{k_n+1}\sum_{i\in\mathcal{L}_j}\left(2n^{2(p/\beta-1)}\left|\Delta_i^{n,1}Z\right|^{2p}\right)^{\frac{2+\delta}{2}}$$

$$= KL_n^{1+\delta}n^{-\delta/2}\left\|(\mathbf{\Omega}_n^*(p,Z))^{-1/2}\right\|^{2+\delta}\left(n^{p(2+\delta)/\beta-1}\sum_{i=1}^{n}\left|\Delta_i^{n,1}Z\right|^{p(2+\delta)}\right),$$

where $\|(\mathbf{\Omega}_n^*(p,Z))^{-1/2}\|^{2+\delta}=O_p(1)$ since $(\varpi_1^{-1}(\mathbf{\Omega}^*(p,Z))+\varpi_2^{-1}(\mathbf{\Omega}^*(p,Z)))^{(2+\delta)/2}=O_p(1)$, as in the proof of **(a)**, and

$$n^{p(2+\delta)/\beta-1}\sum_{i=1}^{n}\left|\Delta_i^{n,1}Z\right|^{p(2+\delta)}=O_p(1)$$

by Lemma 1 (b) since $\delta>0$ is arbitrarily small. This implies that the whole term,

$$\sum_{j=1}^{k_n+1}\mathbb{E}^*\left[\left|U_j^*\right|^{2+\delta}\right]\leq O_p(L_n^{1+\delta}n^{-\delta/2})=O_p(n^{\alpha(1+\delta)-\delta/2})\xrightarrow{\mathbb{P}}0$$

by $\alpha(1+\delta)-\delta/2<0$ or, equivalently, $\alpha<\frac{\delta}{2(1+\delta)}$, providing condition **(b)**. Hence, $\widetilde{\boldsymbol{T}}_n^*\xrightarrow{d^*}N(\mathbf{0},\boldsymbol{I}_2)$ with probability-$\mathbb{P}$ approaching one, which, in conjunction with a multivariate version of Polya's Theorem gives (B.6) and concludes the proof of **Step 1**.

Second, for **Step 2**, we may use the definitions of $\widehat{\mathbf{\Omega}}_n^*(p,Z)$, $\mathbf{\Omega}_n^*(p,Z)$, and (14) to write

$$\boldsymbol{T}_n^*=\left(\widehat{\mathbf{\Omega}}_n^*(p,Z)\right)^{-1/2}(\mathbf{\Omega}_n^*(p,Z))^{1/2}\,\widetilde{\boldsymbol{T}}_n^*.$$

Hence, to obtain the desired central limit theory for $\boldsymbol{T}_n^*$, using the result for $\widetilde{\boldsymbol{T}}_n^*$ in **Step 1**, it suffices to show that

$$\left(\widehat{\mathbf{\Omega}}_n^*(p,Z)\right)^{-1}\times\mathbf{\Omega}_n^*(p,Z)=\left((\mathbf{\Omega}_n^*(p,Z))^{-1}\times\widehat{\mathbf{\Omega}}_n^*(p,Z)\right)^{-1}\xrightarrow{\mathbb{P}^*}\boldsymbol{I}_2. \tag{B.7}$$

Corollary 2 directly implies that $\mathbf{\Omega}_n^*(p,Z)\xrightarrow{\mathbb{P}^*}\mathbf{\Omega}^*(p,Z)=Q(2p)\boldsymbol{M}(p,\beta)$ since convergence in probability follows conditional on the original sample. Moreover, we have that the bootstrap variance estimator decomposes $\widehat{\mathbf{\Omega}}_n^*(p,Z)=\widehat{Q}_n^*(2p,\hat{\beta}(p))\boldsymbol{M}(p,\hat{\beta}(p))$ where $\boldsymbol{M}(p,\hat{\beta}(p))\xrightarrow{\mathbb{P}^*}\boldsymbol{M}(p,\beta)$, similar to Corollary 2. Consequently, (B.7) follows by the continuous mapping theorem if we can show that

$$\widehat{Q}_n^*(2p,\hat{\beta}(p))=\mu_{2p}^{-2}(\hat{\beta}(p))n^{2p/\hat{\beta}(p)-1}V_n^*(2p,Z,1)\xrightarrow{\mathbb{P}^*}Q(2p)=\int_0^1|\sigma_s|^{2p}ds. \tag{B.8}$$

Here, we utilize the fact that convergence in L_1 implies convergence in probability and that all elements of the sum in the bootstrap power variation $V_n^*(2p,Z,1)$ are non-negative. In particular, we have

$$\mathbb{E}^*\left[|V_n^*(2p,Z,1)|\right]=\sum_{i=1}^{n}|\Delta_i^{n,1}Z_i|^{2p}\times\mathbb{E}^*\left[|S_i^*|^{2p}\right]=\mu_{2p}(\hat{\beta}(p))\sum_{i=1}^{n}|\Delta_i^{n,1}Z_i|^{2p}$$

and, as a result,

$$\mathbb{E}^*\left[\left|\widehat{Q}_n^*(2p,\hat{\beta}(p))\right|\right] = \mu_{2p}^{-1}(\hat{\beta}(p))n^{2p/\hat{\beta}(p)-1}\sum_{i=1}^{n}|\Delta_i^{n,1}Z_i|^{2p} \xrightarrow{\mathbb{P}} \int_0^1 |\sigma_s|^{2p}ds,$$

which follows using Lemma 1 (a), Corollary 1, and the continuous mapping theorem. This verifies condition (B.8) and concludes the proof of **Step 2**. □

B.4 Proof of Theorem 3

By Theorem 2 and the delta method, we have

$$\sqrt{n}\left(\widehat{\Omega}_\beta^*(p,Z)\right)^{-1/2}\left(\hat{\beta}^*(p)-\hat{\beta}(p)\right) \xrightarrow{d^*} N(0,1), \text{ in probability-}\mathbb{P},$$

where $\hat{\beta}^*(p)$ is defined in (17), $\hat{\beta}\,(p)$ may be written as

$$\hat{\beta}\,(p) = \frac{p\ln(2)}{\ln\left(\mathbb{E}^*[V_n^*\,(p,Z,2)]\right)-\ln\left(\mathbb{E}^*[V_n^*\,(p,Z,1)]\right)}\mathbf{1}\{\mathbb{E}^*[V_n^*\,(p,Z,2)]\neq\mathbb{E}^*[V_n^*\,(p,Z,1)]\},$$

and where the variance $\widehat{\Omega}_\beta^*(p,Z)$ follows as

$$\begin{aligned}\frac{\widehat{\Omega}_\beta^*(p,Z)}{n^{2-2p/\hat{\beta}(p)}} &= \begin{pmatrix}\frac{(\hat{\beta}^*(p))^2}{(p\ln(2))V_n^*(p,Z,1)} & \frac{-(\hat{\beta}^*(p))^2}{(p\ln(2))V_n^*(p,Z,2)}\end{pmatrix}\widehat{\boldsymbol{\Omega}}_n^*(p,Z)\begin{pmatrix}\frac{(\hat{\beta}^*(p))^2}{(p\ln(2))V_n^*(p,Z,1)}\\ \frac{-(\hat{\beta}^*(p))^2}{(p\ln(2))V_n^*(p,Z,2)}\end{pmatrix}\\ &= \frac{(\hat{\beta}^*(p))^4}{(p\ln(2))^2}\begin{pmatrix}\frac{1}{V_n^*(p,Z,1)} & \frac{-1}{V_n^*(p,Z,2)}\end{pmatrix}\widehat{\boldsymbol{\Omega}}_n^*(p,Z)\begin{pmatrix}\frac{1}{V_n^*(p,Z,1)}\\ \frac{-1}{V_n^*(p,Z,2)}\end{pmatrix}\\ &= \widehat{Q}_n^*\left(2p,\hat{\beta}(p)\right)\frac{(\hat{\beta}^*(p))^4}{(p\ln(2))^2}\begin{pmatrix}\frac{1}{V_n^*(p,Z,1)} & \frac{-1}{V_n^*(p,Z,2)}\end{pmatrix}\boldsymbol{M}\left(p,\hat{\beta}(p)\right)\begin{pmatrix}\frac{1}{V_n^*(p,Z,1)}\\ \frac{-1}{V_n^*(p,Z,2)}\end{pmatrix}\end{aligned}$$

using the definition of $\widehat{\boldsymbol{\Omega}}_n^*(p,Z)$ in (15). Then, as $\boldsymbol{M}(p,\hat{\beta}(p)) = (M(p,\hat{\beta}(p))_{i,j})_{1\leq i,j\leq 2}$ with elements defined as in given in (15), see also Corollary 2, we may write

$$\widehat{\Omega}_\beta^*(p,Z) = (\hat{\beta}^*(p))^4\cdot(p\ln(2))^{-2}\cdot\widehat{Q}_n^*(2p,\hat{\beta}(p))\cdot\widehat{\zeta}(p,Z,\hat{\beta}(p)),$$

where $\widehat{\zeta}(p,Z,\hat{\beta}(p))$ is defined through

$$\frac{\widehat{\zeta}(p,Z,\hat{\beta}(p))}{n^{2-2p/\hat{\beta}(p)}} = \frac{M(p,\hat{\beta}(p))_{1,1}}{(V_n^*(p,Z,1))^2} - \frac{M(p,\hat{\beta}(p))_{1,2}+M(p,\hat{\beta}(p))_{2,1}}{V_n^*(p,Z,1)V_n^*(p,Z,2)} + \frac{M(p,\hat{\beta}(p))_{2,2}}{(V_n^*(p,Z,2))^2},$$

concluding the proof. □

C Algorithm for Numerical Implementation

We detail how the proposed local stable bootstrap procedure may be used to test whether Z_t is a jump diffusion or a pure-jump semimartingale. In particular, we test the null hypothesis $\mathcal{H}_0 : \beta = 2$ against a one-sided alternative $\mathcal{H}_1 : \beta < 2$. In the following, B denotes the number of bootstrap replications for each of the M Monte Carlo replications. Then, for a given equidistant partition of the normalized time window $[0, 1]$ with step length $1/n$ do the following:

Algorithm 3: The Local Stable Bootstrap Simulation for hypothesis testing

Step 1. Simulate $n+1 \in \mathbb{N}$ points of the process Z_t under investigation (a pure-jump semimartingale or a jump diffusion). For details on how to simulate tempered stable processes, see, e.g., Todorov et al. (2014, Section 10) or, alternatively, the methodology by Rosinski (2007) based on a shot-noise decomposition of the Lévy measure.

Step 2. Estimate the activity index β of the process Z_t using the estimator $\hat{\beta}(p)$ in (5).

Step 3. Compute the studentized statistic

$$\tau_n(2) \equiv \sqrt{n}\left(\widehat{\Omega}_\beta(p,Z)\right)^{-1/2}\left(\hat{\beta}(p)-2\right)$$

where $\widehat{\Omega}_\beta\,(p,Z)$ is an consistent estimator of the asymptotic variance of $\widehat{\beta}\,(p)$. In particular,

$$\begin{aligned}\widehat{\Omega}_\beta(p,Z) &= \frac{\mu_p^2(\hat{\beta}(p))}{\mu_{2p}(\hat{\beta}(p))}\frac{n^{2p/\hat{\beta}(p)-1}V_n(2p,Z,1)}{n^{2p/\hat{\beta}(p)-2}\,(V_n(p,Z,1))^2}\times\frac{(\hat{\beta}(p))^4}{\mu_p^2(\hat{\beta}(p))p^2(\ln(2))^2}\times\widetilde{\Xi}\\ &= \frac{n}{\mu_{2p}(\hat{\beta}(p))}\times\frac{V_n(2p,Z,1)}{(V_n(p,Z,1))^2}\times\frac{(\hat{\beta}(p))^4}{p^2(\ln(2))^2}\times\widetilde{\Xi},\end{aligned} \tag{C.1}$$

with $\widetilde{\Xi} = \widehat{\Xi}_{1,1} - 2^{1-p/\beta}\widehat{\Xi}_{1,2} + 2^{-2p/\beta}\widehat{\Xi}_{2,2}$ and the matrix $\widehat{\boldsymbol{\Xi}} = (\widehat{\Xi}_{i,j})_{1\le i,j\le 2}$ is given as

$$\widehat{\boldsymbol{\Xi}} = \boldsymbol{\Sigma}(p,\hat{\beta}(p),0) + \boldsymbol{\Sigma}(p,\hat{\beta}(p),1) + \boldsymbol{\Sigma}(p,\hat{\beta}(p),1)' \tag{C.2}$$

where $\boldsymbol{\Sigma}(p,\hat{\beta}(p),k)$ for $k = 0, 1$ are defined as in Sections 2.4 and 3.1.

Step 4. Generate an $n+1$ sequence of identically and independently distributed 2-stable random variables $S_1^*, S_2^*, \ldots, S_{n+1}^*$, whose characteristic function are defined as

$$\ln\mathbb{E}\left[e^{iuS_i^*}\right] = -|u|^2/2, \qquad \forall i = 1,\ldots,n+1. \tag{C.3}$$

The observations $S_1^*, S_2^*, \ldots, S_{n+1}^*$ should be independent of observations generated in Step 1.

Step 5. Generate the local stable bootstrap observations under the restriction specified by $\mathcal{H}_0$ as follows,

$$\Delta_i^{n,v} Z^* = \Delta_i^{n,1} Z \cdot \Big(\sum_{t=1}^{v} S_{i+t-1}^* \Big), \qquad i = v, \dots, n,$$

and compute the bootstrap activity index estimator,

$$\hat{\beta}^*(p) = \frac{p \ln(2)}{\ln(V_n^*(p, Z, 2)) - \ln(V_n^*(p, Z, 1))} \mathbf{1}\{V_n^*(p, Z, 2) \neq V_n^*(p, Z, 1)\},$$

where $V_n^*(p, Z, 1)$ and $V_n^*(p, Z, 2)$ are defined in (12).

Step 6. Compute the studentized bootstrap statistic $\tau_n^*(2)$ from Corollary 3.

Step 7. Repeat Steps 4-6 B times and keep the values of $\tau_n^*(2, j)$, $j = 1, \dots, B$, where $\tau_n^*(2, j)$ is given as in Step 6. Then, sort $\tau_n^*(2, 1), \dots, \tau_n^*(2, B)$ ascendingly from the smallest to the largest as $\bar{\tau}_n^*(2, 1), \dots, \bar{\tau}_n^*(2, B)$ such that $\bar{\tau}_n^*(2, i) < \bar{\tau}_n^*(2, j)$ for all $1 \leq i < j \leq B$.

Step 8. Reject $\mathcal{H}_0$ when $\tau_n(2) < q_\alpha^*$ where q_α^* is the α quantile of the bootstrap distribution of $\tau_n^*(2)$. For example, if we let $B = 999$, then the 0.05-th quantile of $\tau_n^*(2)$ is estimated by $\tau_n^*(2, a)$ with $a = 0.05 \times (999 + 1) = 50$.

Step 9. Repeat Steps 1-8 M times to get the size or power of the bootstrap test. In particular, if Z_t is simulated as a jump diffusion, then the size is given by $M^{-1}(\#\{\tau_n(2) < q_\alpha^*\})$.

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