IMPLEMENTATION OF THE KURGANOV-TADMOR HIGH
RESOLUTION SEMI-DISCRETE CENTRAL SCHEME FOR
NUMERICAL SOLUTION OF THE EVAPORATION PROCESS IN DRY
EXPANSION EVAPORATORS

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ABSTRACT

A set of partial differential equations is derived in mass flow, enthalpy, pressure, and evaporator wall temperature based on the continuity equation, Navier-Stokes equations, and the energy equation for the refrigerant in addition to the heat exchange equation for the wall (the latter accounts for convective heat exchange with the refrigerant and the ambient). The combination of pressure and enthalpy thermodynamic variables allows for complete specification of the thermodynamic state spatially and temporally in the different zones in the evaporator. Details on the implementation of the Kurganov-Tadmor scheme for the governing equations are given and results are shown for step-responses in inlet mass-flow and outlet volume-flow.

Keywords: Evaporator modeling, Discretization scheme, Two-phase flow

INTRODUCTION

The growing concern for consumption of world energy resources and related global warming emphasizes the continuing need for increasingly higher efficiencies of energy consuming systems. Modeling and simulation of these systems is an important way of addressing this optimization problem, however, it requires a certain degree of model detail depending on the specific phenomena to be investigated. As conventional refrigeration and heat pump components approach an increasing level of maturity there is limited potential in individual component optimization as compared to a total system optimization. The benefits in carrying out such efforts are described in, e.g., Refs. [1, 2].

Two main components in the aforementioned systems are the heat exchangers which are usually subject to two-phase flow and heat transfer. Traditionally, component models for condensers and evaporators are solved using NTU-ε - methods with a fixed UA (or heat transfer coefficient) for the component. The need to investigate control strategies of these systems demands realistic modeling of parameters such as superheat or subcooling hence giving rise to a widespread use of moving-boundary models [3, 4, 5]. The latter models show an advantage in demonstrating dynamical behavior of the component in relation to capacities, temperature, and pressure levels although formulated as lumped models. At the same time, the lumping also renders these models computationally fast and makes them well suited for evaluation of system control and for model predictive control. The drawback is loss of detail in the modeling and the handling of fluid-zone switching with resulting numerical obstacles.

In an effort to investigate the dynamic behavior of two-phase refrigeration-cycle systems we are seeking a mathematical formulation that provides a stable numerical solution despite a dynamically changing number of fluid zones. We further seek to capture detailed dynamics in the heat exchange between refrigerant and evaporator wall. To this end several works have been carried out on fully distributed
models solving the governing equations [6, 7, 8]. Generally, the fast dynamics of the full set of governing equations are neglected (such as pressure waves) allowing for faster computation times and numerically stable codes. Recently similar models have been used for the analysis of distribution phenomena in multi-pass evaporators [9].

In the present paper, we aim to provide a modeling framework that captures all the dynamics of the full set of governing equations and the characteristics of the components while still being numerically stable and efficient. In the first approach this includes solving the fast dynamics of pressure propagation which may later be taken out for simplification. This is accomplished by applying the Kurganov-Tadmor (KT) scheme [10] with appropriate boundary conditions. The major strengths of the KT scheme are simplicity (e.g. non-staggered and free from Riemann solvers) and stability, the latter being a major issue with standard methods. The scheme can be formulated as a system of ODEs allowing simple integration in complete system models.

MODEL EQUATIONS

In the following, a set of partial differential equations is derived describing evaporator dynamic operation. We use the continuity equation, Navier-Stokes equations, and the energy equation [4]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \mathbf{v}_i)}{\partial x_i} = 0, \quad (1)
\]

\[
\frac{\partial (\rho w_i)}{\partial t} + \frac{\partial (\rho w_i v_k)}{\partial x_k} = -\delta_{i,k} \frac{\partial P}{\partial x_k} + \frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i, \quad (2)
\]

\[
\rho \left( \frac{\partial h}{\partial t} + w_i \frac{\partial h}{\partial x_i} \right) - \left( \frac{\partial P}{\partial t} + w_i \frac{\partial P}{\partial x_i} \right) =
\sigma_{ik} \frac{\partial w_j}{\partial x_k} + \frac{\partial}{\partial x_j} \left( \frac{\kappa}{\partial x_j} \right), \quad (3)
\]

where \( \rho, w_i, P, h, T, \sigma_{ik}, g_i, \kappa, x_i, \) and \( t \) are the refrigerant mass density, refrigerant velocity, refrigerant pressure, refrigerant enthalpy, refrigerant temperature, the viscous tensor, the gravitation constant, the refrigerant thermal conductivity, the coordinate vector, and time, respectively. Einstein summation convention is employed in Eqs. (1)-(3).

To simplify the problem, we shall assume next that viscous heating in Eqn. (3) is neglected, that gravity effects are unimportant, and that variations in physical properties basically take place along one coordinate direction (the length direction of the evaporator). Then, we have

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho w)}{\partial z} = 0, \quad (4)
\]

\[
\frac{\partial (\rho w)}{\partial t} + \frac{\partial (\rho w^2)}{\partial z} = -\frac{\partial P}{\partial z} - \left( \frac{\partial P}{\partial z} \right)_{fric}, \quad (5)
\]

\[
\rho \left( \frac{\partial h}{\partial t} + w \frac{\partial h}{\partial z} \right) - \left( \frac{\partial P}{\partial t} + w \frac{\partial P}{\partial z} \right) =
\frac{4}{D_i} \alpha(T_w - T), \quad (6)
\]

where \( \frac{\partial P}{\partial z} \) is a friction factor correlation depending on local properties and conditions and \( w \) is the one-dimensional velocity along the evaporator length coordinate and the heat-transfer coefficient satisfies

\[
\alpha = \frac{D_i}{4} \frac{\partial}{\partial n} \left( \frac{k}{\rho A} \right) \frac{1}{T_w - T}, \quad (7)
\]

with \( D_i, T_w, \) and \( n \) being the evaporator inner diameter, wall temperature, and normal direction, respectively.

We now recast the above equation set in terms of new variables (pressure, mass flow, and enthalpy) \( P, \dot{m} = \rho A w, \) and \( h \)

\[
\frac{\partial P}{\partial h} \frac{\partial P}{\partial \dot{m}} + \frac{\partial P}{\partial h} \frac{\partial h}{\partial t} + \frac{1}{A} \frac{\partial \dot{m}}{\partial z} = 0, \quad (8)
\]

\[
1 \frac{\partial \dot{m}}{\partial t} + \frac{\partial}{\partial z} \left( \frac{\dot{m}^2}{\rho A} \right) = -\frac{\partial P}{\partial z} - \left( \frac{\partial P}{\partial z} \right)_{fric}, \quad (9)
\]

\[
\rho \left( \frac{\partial h}{\partial t} + \frac{\dot{m}}{A} \frac{\partial h}{\partial z} \right) - \left( \frac{\partial P}{\partial t} + \frac{\dot{m}}{\rho A} \frac{\partial P}{\partial z} \right) =
\frac{4}{D_i} \alpha(T_w - T), \quad (10)
\]

keeping in mind that \( \rho \equiv \rho(P, h), T \equiv T(P, h), \) and we have assumed that the cross-sectional area \( A \) does not depend on \( z \). We need an additional differential equation in the wall temperature to complete the model framework. This equation is

\[
(C_W \rho_A w) \frac{\partial T_w}{\partial t} = \alpha_{pi} D_i (T - T_w)
\]

\[
+ \alpha_{pi} D_{o,fj} (T_A - T_w) + \lambda A w \frac{\partial^2 T_w}{\partial z^2}, \quad (11)
\]

where \( C_W, \rho_A w, \alpha_{pi}, A_{pi}, f_{j}, T_A, \) and \( \lambda \) are the wall heat capacity, wall mass density, wall cross-sectional area, heat-transfer coefficient between
evaporator wall and ambient, outer diameter, fin factor, ambient temperature, and wall heat conductivity, respectively. The friction term is modelled as

$$\left( \frac{\partial P}{\partial z} \right)_{f r i c} = \frac{1}{2} f \dot{m}^2 \rho A^2 ,$$

with a friction factor $f$ corresponding to a simple model of Darcy–Weisbach friction.

The computational domain and subsequent discretization is indicated in Fig. 1. The above formulation is essentially a homogeneous flow model since we have formulated the equation set in terms of overall density and enthalpy neglecting the separation in gas and liquid phases as is more conventional in modelling two-phase flows [11]. In this first approach we assume a constant heat transfer coefficient independent of flow regime and conditions. In principle, the only information lacking to completely specify the dependence of $\dot{m}$, $P$, $h$, and $T_W$ as a function of $z$ and $t$ are initial conditions (all variables must be known everywhere in $z$ at $t = 0$) and appropriate boundary conditions at all times. For an evaporator tube of length $L$ we have an inlet mass-flow at $z = 0$ and an outlet volume flow $\dot{V} = \dot{m}/\rho$ at $z = L$. The pressure is then determined by the evaporator as a balance of temperatures (heat flow), mass flows and the thermodynamic relations. Pipe wall ends ($z = 0, L$) are insulated, heat-flux conditions and refrigerant enthalpy are specified at the inlet $z = 0$. The mathematical relations are then:

$$\dot{m}(0, t) = \dot{m}_{in} ,$$
$$h(0, t) = h_{in} ,$$
$$\dot{m}(L, t) = \rho(L, t) \dot{V} ,$$

The above set of equations requires dynamic access to thermodynamic routines giving $\rho$ and $T$ as a function of the (dynamic) values of $P$ and $h$. Note that $P$ and $h$ (in contrast to $\rho$ and $T$) are convenient dependent variables to solve for as they specify all thermodynamic properties including the quality $x$ (i.e., the mass percentage of vapor content in the two-phase region) unambiguously. The present model allows for determining all parameters in the evaporator spatially and dynamically.

**NUMERICAL DISCRETIZATION SCHEMES**

Straight-forward central finite differences applied to Eqs. (8)-(11) show spurious oscillations which quickly destroy the numerical solution. High-resolution schemes have been developed to avoid such oscillations and allow possible discontinuities in the solutions. We have implemented the KT scheme [10] in the second order semi-discrete form. The KT scheme aims to solve a set of equations:

$$\phi_t + H(\phi, \phi_z) = 0 ,$$

where $\phi$ is a vector of functions depending on $t$ and $z$. The second-order semi-discrete KT scheme can be written as

$$\frac{d\phi_j}{dt} = -\frac{1}{2} \left( H(\phi_j, (\phi^+_z)_j) + H(\phi_j, (\phi^-_z)_j) \right)$$
$$+ \frac{a_j}{2} \left( (\phi^+_z)_j - (\phi^-_z)_j \right) ,$$

where $z_j = j \Delta z$ with $\Delta z = L/N$ and where $j = 0, ..., N$ is the spatial discretization index.
(\phi_j(t) \approx \phi(j\Delta z,t))$, $a_j$ is the maximum local speed, and

$$
(\phi^\pm_j)_j = \frac{\Delta \phi_j^{1/2}}{\Delta z} + \frac{(\Delta \phi_j^{1/2})'}{2\Delta z},
$$

(19)

with

$$
\Delta \phi_j^{1/2} = \phi_j+1 - \phi_j,
$$

(20)

and

$$
(\Delta \phi_j^{1/2})' = \text{minmod}(\Delta \phi_j+1 - \Delta \phi_j,
\frac{1}{2}(\Delta \phi_j+1 - \Delta \phi_j-1),\Delta \phi_j - \Delta \phi_j-1),
$$

(21)

where the minmod flux limiter function is defined by

$$
\text{minmod}(a,b,c) = \begin{cases} 
\min(a,b,c) & \text{if } a,b,c > 0 \\
\max(a,b,c) & \text{if } a,b,c < 0 \\
0 & \text{otherwise}
\end{cases}
$$

(22)

More details on the KT-scheme can be found in Ref. [10]. In the following, we will highlight the specific implementation details on evaporator modelling. The KT scheme is applied only to Eqs. (8)-(16). The virtual points for the KT-scheme are constructed using second-order extrapolation, e.g.,

$$
\bar{m}_{-1} = 3\bar{m}_0 - 3\bar{m}_1 + \bar{m}_2,
$$

(29)

which assumes the functions can be approximated by a second-order Taylor series at the boundary. The thermodynamic quantities, $T(h,P)$ and $P(h,P)$, are evaluated using Ref. [12]. We generate an interpolation lookup table for fast access to these relations. The working medium is R600a. The derivatives $\frac{\partial \rho}{\partial h}|_P$ and $\frac{\partial \rho}{\partial \rho}|_P$ are evaluated numerically from values in the lookup table.

**SIMULATION RESULTS**

<table>
<thead>
<tr>
<th>Variable/Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_i$</td>
<td>$6 \cdot 10^{-3}$</td>
<td>m</td>
</tr>
<tr>
<td>$D_o$</td>
<td>$8 \cdot 10^{-3}$</td>
<td>m</td>
</tr>
<tr>
<td>$f_{in}$</td>
<td>10</td>
<td>[-]</td>
</tr>
<tr>
<td>$A$</td>
<td>$\frac{\pi}{4}D_i^2$</td>
<td>m²</td>
</tr>
<tr>
<td>$C_W$</td>
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<td>J/K</td>
</tr>
<tr>
<td>$\rho_W$</td>
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<td>kg/m³</td>
</tr>
<tr>
<td>$A_w$</td>
<td>$\frac{\pi}{4}(D_o^2 - D_i^2)$</td>
<td>m²</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>386</td>
<td>W/(mK)</td>
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</tr>
<tr>
<td>$\alpha_o$</td>
<td>100</td>
<td>W/(m²K)</td>
</tr>
<tr>
<td>$T_A$</td>
<td>300</td>
<td>K</td>
</tr>
<tr>
<td>$f$</td>
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<td>[-]</td>
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<tr>
<td>$\bar{m}_{in}$</td>
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<td>kg/s</td>
</tr>
<tr>
<td>$\bar{h}_{in}$</td>
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<td>J/kg</td>
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<tr>
<td>$\bar{V}$</td>
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<td>m³/s</td>
</tr>
<tr>
<td>$P(t=0,z)$</td>
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<td>Pa</td>
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<tr>
<td>$T(t=0,z)$</td>
<td>263</td>
<td>K</td>
</tr>
<tr>
<td>$T_W(t=0,z)$</td>
<td>273</td>
<td>K</td>
</tr>
<tr>
<td>$h(t=0,z)$</td>
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<tr>
<td>$\bar{m}(t=0,z)$</td>
<td>$5.31 \cdot 10^{-3}$</td>
<td>kg/s</td>
</tr>
</tbody>
</table>

Table 1: Parameters, boundary values, and initial data used in the simulations.

In the following, we use the parameters, boundary values, and initial data shown in Table 1. First, we find a steady-state by evolving the initial data in Table 1 until the temporal derivatives become zero (takes approximately 50 s). The resulting steady-state values of $\bar{m}$, $P$, $h$ and $T_W$ as function of position $z$ are shown in Fig. 2. The mass-flow is constant over the tube while the pressure decreases and
Figure 2: Steady-state solution found after letting the initial data from Table 1 evolve for 50 s. Units are according to Table 1.
Figure 3: $\dot{m}$, $T_W$, and $T$ at $z = 0$ (in) and $z = 10$ m (out) as a function of time when the inlet massflow $\dot{m}_m$ is reduced by a factor of two at $t = 0.5$ s and increased again to the initial value ($5.3 \cdot 10^{-3}$ kg/s) at $t = 25$ s. Time axis is broken in upper and middle plot to exclude the non-varying part of the functions. Units refer to Table 1.

Figure 4: $\dot{m}$, $T_W$, and $T$ at $z = 0$ (in) and $z = 10$ m (out) as a function of time when the outlet volume flow $\dot{V}$ is reduced by a factor of two at $t = 0.5$ s and increased again to the initial value ($1.75 \cdot 10^{-3}$ m$^3$/s) at $t = 25$ s. Time axis is broken to exclude the non-varying part of the functions. Units are according to Table 1.
the enthalpy rises. The wall temperature $T_W$ drops slightly until dryout at $z \sim 6$ m whereafter the temperature rises quickly and becomes almost equal to the ambient temperature at $z = 10$ m.

To test the dynamic response and numerical stability of the model we next investigate the model response to steps in inlet massflow and outlet volume flow. In the remaining results, all simulations are started from the steady-state condition in Fig. 2.

First we reduce the inlet mass flow $\dot{m}_{in}$ by a factor of two at $t = 0.5$ s and increase it again by a factor of two at $t = 25$ s. Figure 3 shows simulation results of $\dot{m}$, $T$, and $T_W$ at $z = 0$ (in) and $z = 10$ m (out). The model behaves as expected and is numerically stable. The under- and over-shoot of $T_{out}$ at $t = 0.5$ and 25 s reflects the effects of the dynamic pressure wave propagating in the system as discussed later.

In Fig. 4 we show simulation results of $\dot{m}$, $T$, and $T_W$ at $z = 0$ (in) and $z = 10$ m (out) for the case where the outlet volume flow $\dot{V}$ is reduced by a factor of two at $t = 0.5$ s and increased again to its original value at $t = 25$ s. The inlet massflow is kept constant and the outlet massflow returns to its original value while pressure and temperatures change so as to ensure continuity. Note that in this case the superheated zone vanishes ($T_{out} < T_{in}$) and liquid is ejected from the tube during the low volume-flow period. Again, the model behaves as expected and is numerically stable. The over- and under-shoot of $T_{out}$ at $t = 0.5$ s and 25 s reflects again the instantaneous change in pressure related to the dynamic pressure wave traveling in the systems.

When the inlet or outlet flows are changed instantaneously as in Figs. 3 and 4, a pressure wave is formed shuttling back and forth a few times in the evaporator tube. Figure 5 show a detailed spatio-temporal view of $\dot{m}$ near $t = 0.5$ s when the outlet volume flow is reduced as in the case from Fig. 4. The wave is formed at $z = 10$ m and moves against the flow towards the inlet. At $t \sim 0.57$ s it reaches the inlet position at $z = 0$ and reflects back into the system. The speed of the wave is not constant over the evaporator tube due to the changing speed of sound through the passage of mainly the two-phase region.

**CONCLUSION**

We have implemented the second order semidiscrete Kurganov-Tadmor scheme for an evaporator formulated in terms of enthalpy, pressure, mass-
flow, and wall temperature. The scheme works well while being stable against even large steps in the inlet and outlet flows and changing number of fluid zones. Since the present model addresses the full set of partial differential equations, including the spatio-temporal details of pressure-wave effects, it is much more computational demanding than ordinary moving-boundary models or distributed models with simplified momentum equations. Thus its main purpose is to model evaporators as a benchmark for simpler and faster models ensuring similar behavior at long timescales.

REFERENCES