Equilibrium Analysis in Cake Cutting

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Cake Cutting

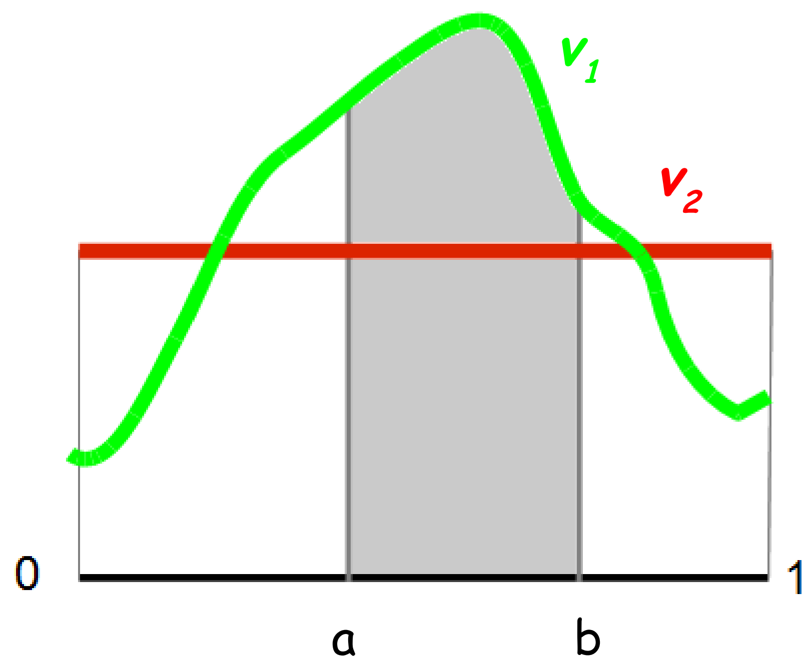
- Fundamental model in fair division; the allocation of a divisible ressource (land, time, computer memory) among agents with heterogeneous preferences

- Studied since the 1940's, starting with a Polish group of mathematicians (Banach, Knaster, Steinhaus)

- Large body of literature in mathematics, political science, economics; recent interest in computer science

Model

- The cake is the interval [0, 1]
- Set of agents $N = \{1,\ldots, n\}$
- Each agent i has a (normalized) valuation function V_i over the cake, which is the integral of a value density function v_i



Model

- A piece of cake X is a finite union of disjoint subintervals of [0,1]. A contiguous piece is a single subinterval

- The valuation of agent i for a piece X is given by the integral of their density function over the piece:

$$V_i(X) = \sum_{I \in X} \int_I v_i(x)\,dx$$

- An allocation X = (X_1, ..., X_n) is an assignment of pieces to agents such that each agent i receives piece X_i and all the X_i are disjoint.

Fairness Criteria

- **Proportionality**

  An allocation $X = (X_1, \ldots, X_n)$ is proportional if each agent gets a fair share of the cake: $V_i(X_i) \geq 1/n, \forall i \in N$

- **Envy-Freeness**

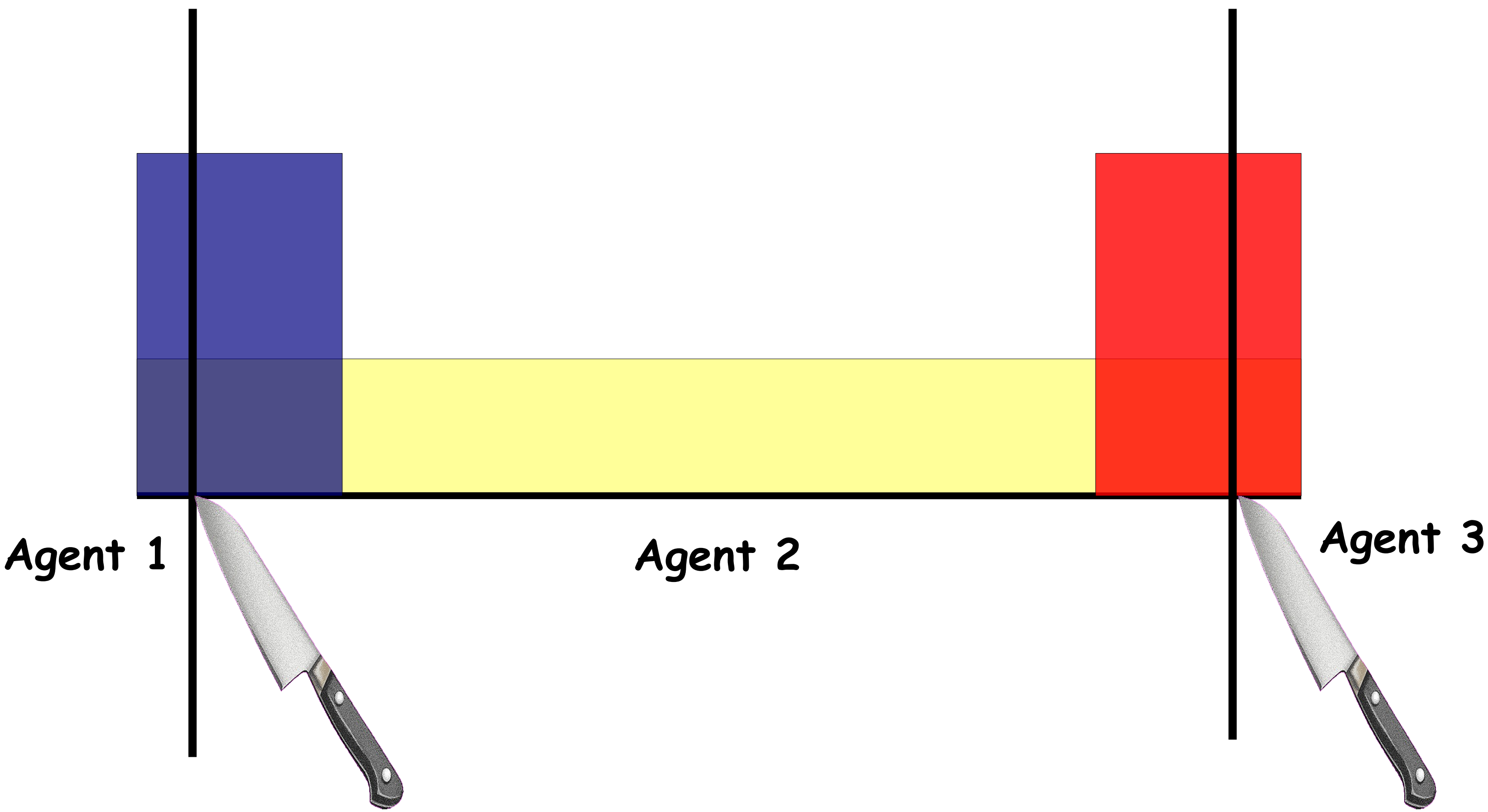
  No agent likes someone else's piece more than their own: $V_i(X_i) \geq V_i(X_j), \forall i,j \in N$

- Trivial envy-free allocation: throw away the entire cake

- Envy-freeness implies proportionality when the entire cake is allocated.

Fairness Criteria

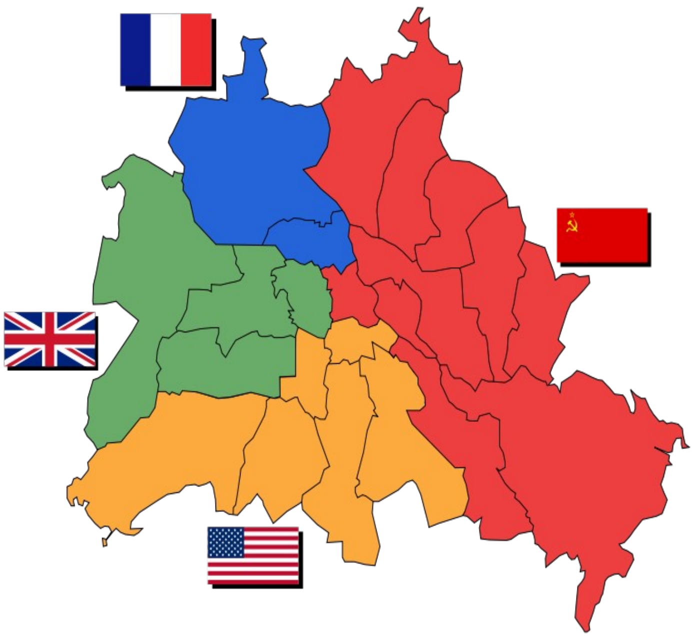
- Proportional allocation which is not envy-free:



Additional Criteria

- **Perfect**
  - Allocation X is perfect if $V_i(X_j) = 1/n$, $\forall i,j \in N$
  - Perfect allocations are guaranteed to exist (Neyman) with at most $(n-1)n^2$ cuts when the VDFs are continuous (Alon)

- **Contiguous Pieces**
  - Envy-free allocations with contiguous pieces are guaranteed to exist, but no finite algorithm can find them
  - Bound the number of cuts: some protocols can allocate countable unions of crumbles

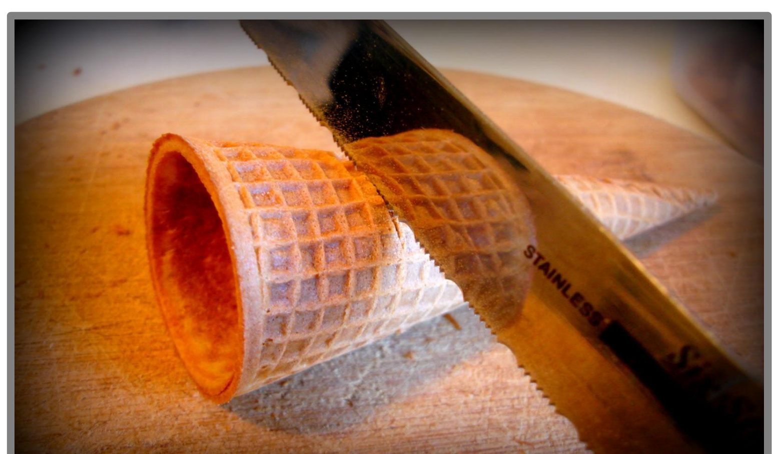


Berlin divided by the Potsdam Conference (1945)

Cut-and-Choose

Agent *1* cuts the cake in equal halves
Agent *2* chooses his favourite piece
Agent *1* takes the remainder

- The allocation is contiguous, proportional, envy-free, and Pareto optimal



Dubins-Spanier

A referee slides a knife across the cake, from left to right

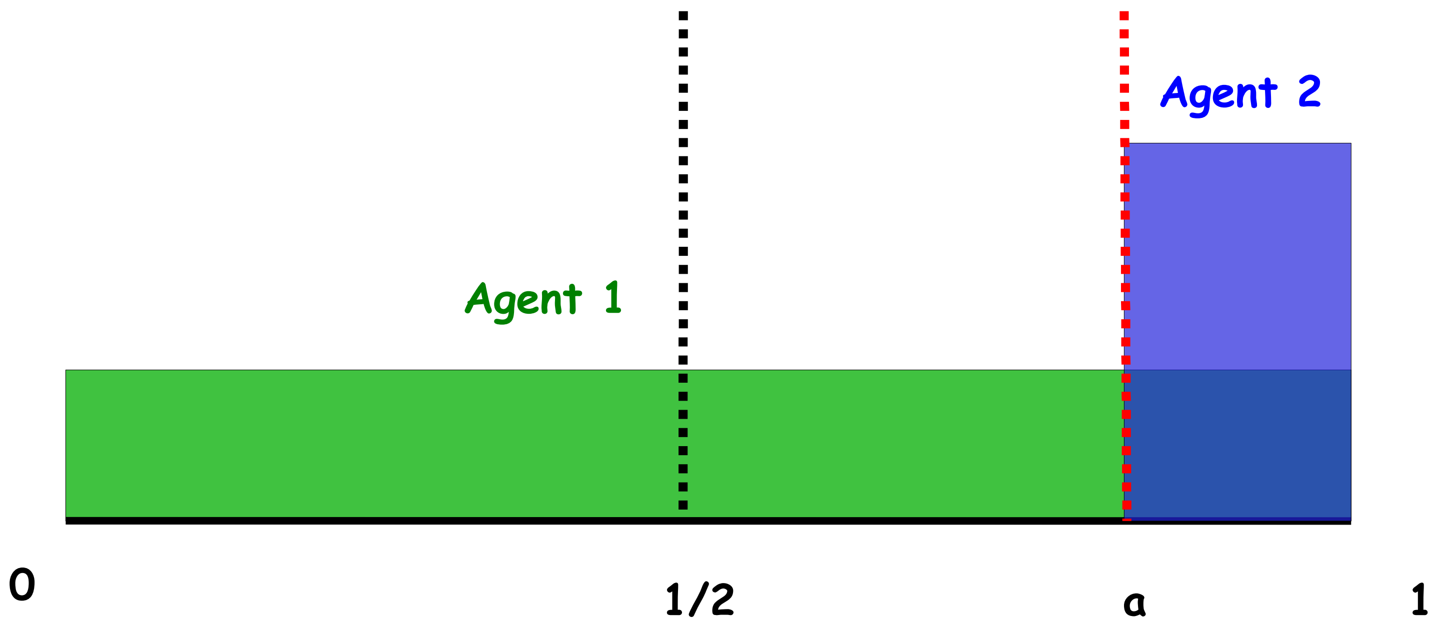
When the knife reaches a point such that one of the agents values 1/n the piece to the left of the knife, that agent shouts ***CUT!***

The first agent to call cut receives the left piece and exits

Repeat with the remaining n-1 agents on the leftover cake
(except now call cut at 1/(n-1) of the remainder)

Truthfulness

- Classical protocols assume honest players



Example: Cut-and-Choose

Truthfulness

- Recent work on mechanism design (Maya and Nisan, 2012; Chen, Lai, Parkes, and Proccacia, 2010; Mossel and Tamuz, 2010)
- General envy-free, proportional, and truthful mechanism:

> - Find perfect allocation $X = (X_1, \ldots, X_n)$
> - Draw random permutation π over N
> - For $i = 1$ to n
>   - Allocate piece X_i to agent π_i

- Unfortunately there is no algorithm to compute perfect allocations
  - Deterministic mechanisms exist only for simple valuations

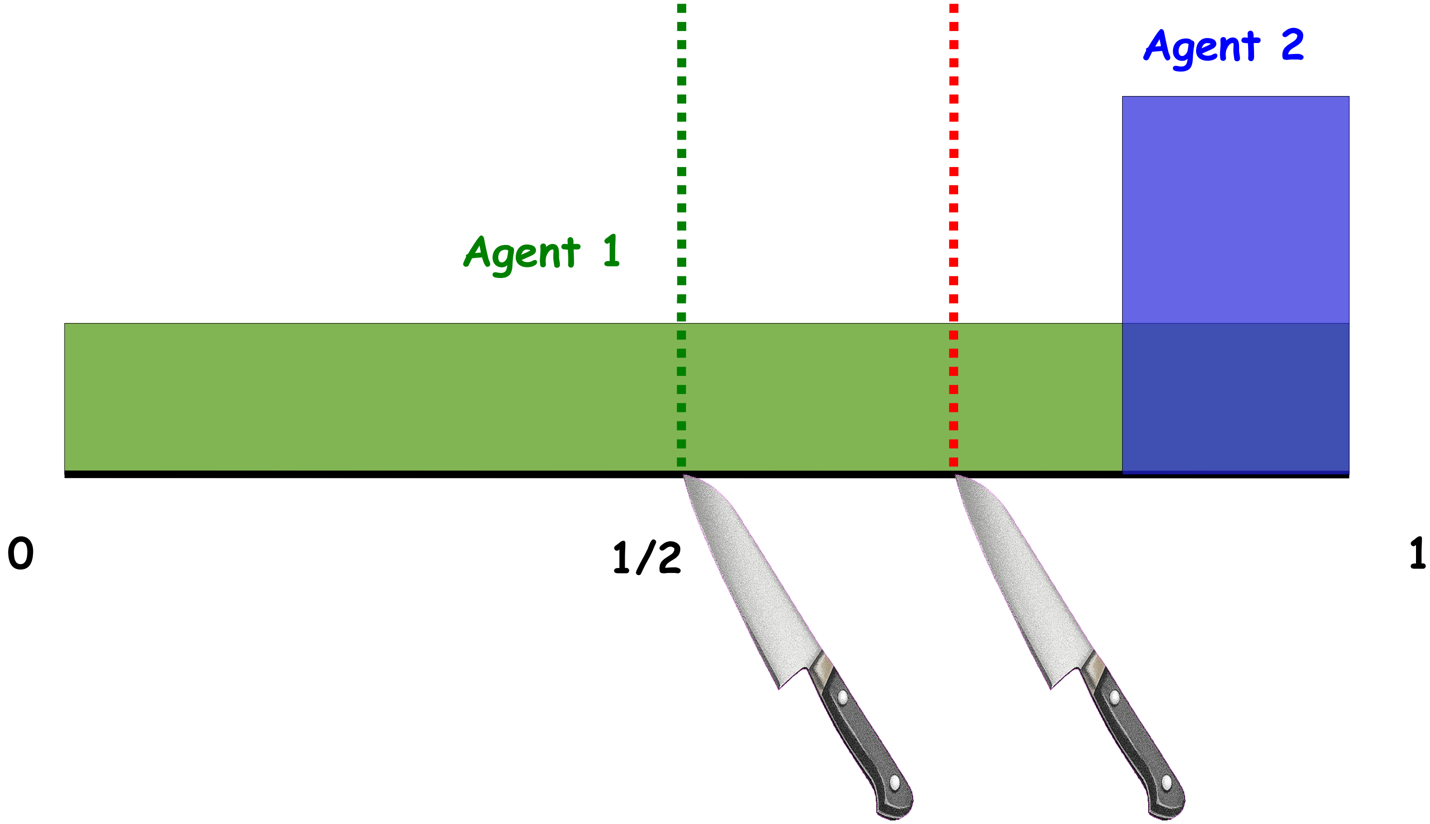
This Work : Equilibrium Analysis

Motivation :

- While the classical protocols are not truthful, they are often very simple, intuitive, and can be implemented by the agents following a sequence of natural operations
- What do the equilibria of such protocols look like?

Strategic Dubins-Spanier

The agents have no incentives to follow Dubins-Spanier even when $n = 2$



Moving Knife Game

Setup:

- Knife moves from left to right
- The agents can shout stop at any time
  - The first agent to do so receives the piece to the left of the knife and exits
  - The game continues with the remaining $n-1$ agents
- If multiple agents call stop simultaneously, ties are broken using a fixed permutation of N

Moving Knife Game

- Threshold strategies:
  - The strategy of each agent i is a vector $T_i = (t_{i,1}, \ldots, t_{i,n}) \in [0, 1]^n$
  - The agent calls cut in round j when the left piece is worth exactly t_{ij}.
- Complete information and strictly positive value density functions

Moving Knife Game

Theorem:

Consider a moving knife game with strictly positive value density functions and deterministic tie-breaking. Then every pure Nash equilibrium of the game induces an envy-free allocation that contains the entire cake.

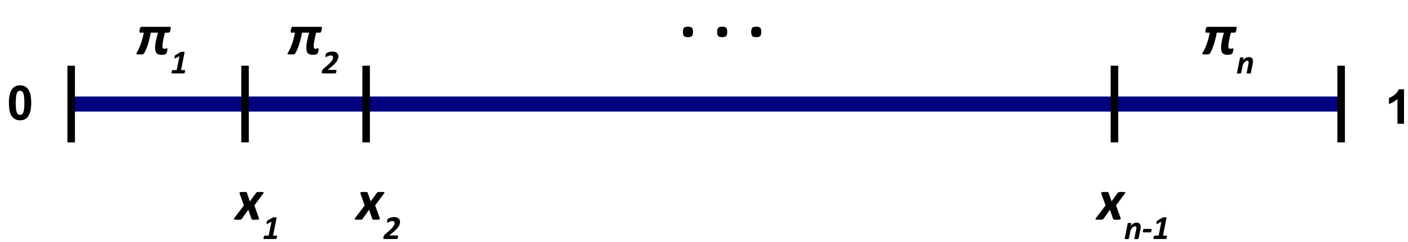
Proof (sketch):

- If an agent is envious of an earlier piece – just call cut faster in that round.
- If envious of a later piece, "skip" rounds until reaching that piece (by setting all intermediate thresholds as late as possible)

Moving Knife Game

Theorem:

Given any envy-free allocation with *n-1* cuts, there exists a deterministic tie-breaking rule π such that the game has a pure Nash equilibrium inducing this allocation.



Proof (sketch):

Set $t_{ij} = V_i([x_{j-1}, x_j])$ and tie-breaking rule π

Moving Knife Game

Theorem (characterization):

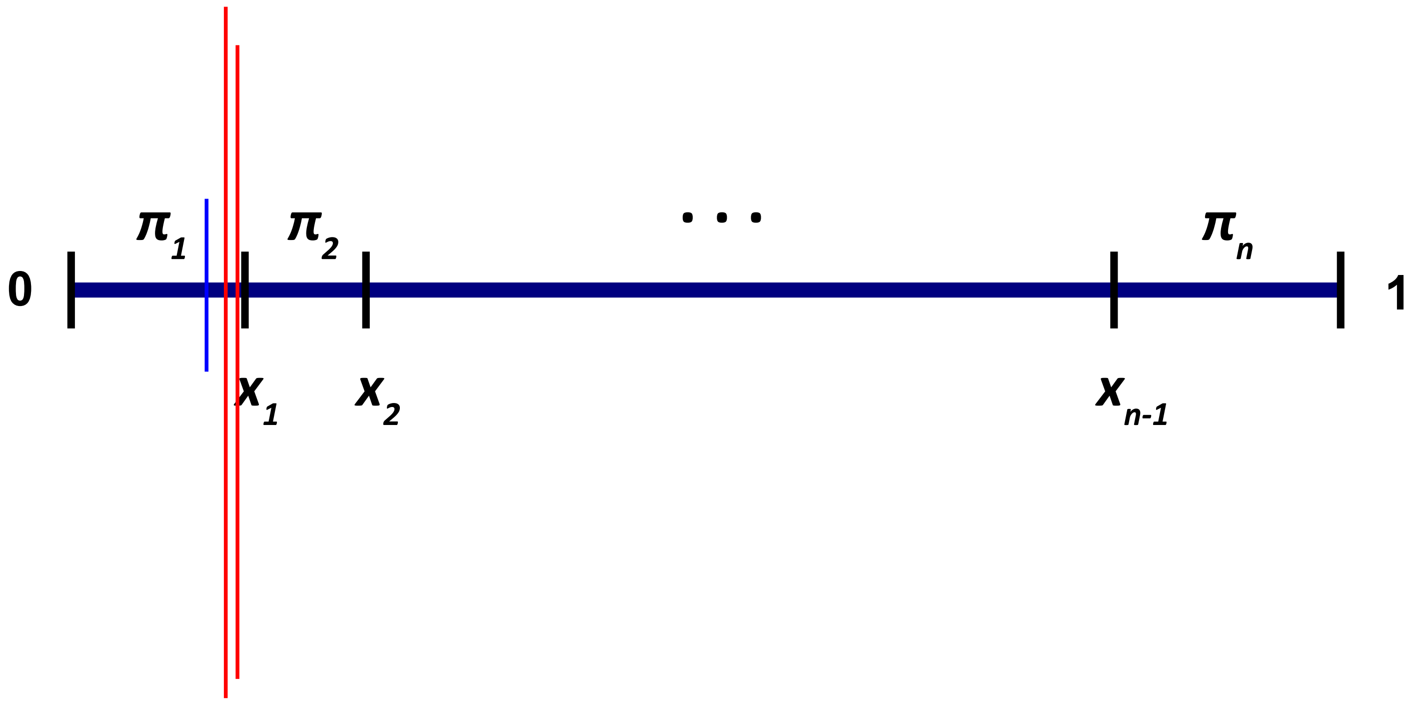
A strategy profile *T* is a Nash equilibrium under a deterministic tie-breaking rule

if and only if

i). The induced allocation is envy-free,
ii). contains the entire cake, and
iii). in every round except the last, the agent that is allocated the piece has an active competitor that calls cut simultaneously.

Moving Knife Game

Theorem: For every $\varepsilon > 0$, the game has an ε-equilibrium that is independent of tie-breaking, which induces an ε-envy-free allocation that contains the entire cake.



Open Questions

Does the moving knife game have mixed strategy equilibria for every deterministic tie-breaking rule, such that the entire cake is allocated with positive probability?

What do the equilibria of Dubins-Spanier look like under richer strategy spaces?

Open Questions

One round moving knife game:

Knife moves continuously from left to right

The agents can shout cut at any time

The first agent to call cut receives the piece to the left of the knife and the game ends

When the value density functions are strictly positive, the one round game has a unique pure Nash equilibrium at zero

- Any other interesting mixed Nash equilibria?