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Vortices and Jacobian Varieties

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Abstract

We investigate the geometry of the moduli space of N-vortices on line bundles over a closed Riemann surface Σ of genus $g > 1$, in the little explored situation where $1 \leq N < g$. In the regime where the area of the surface is just large enough to accommodate N vortices (which we call the dissolving limit), we describe the relation between the geometry of the moduli space and the complex geometry of the Jacobian variety of Σ. For $N = 1$, we show that the metric on the moduli space converges to a natural Bergman metric on Σ. When $N > 1$, the vortex metric typically degenerates as the dissolving limit is approached, the degeneration occurring precisely on the critical locus of the Abel–Jacobi map of Σ at degree N. We describe consequences of this phenomenon from the point of view of multivortex dynamics.

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1 Introduction

Inspired at first by the Ginzburg–Landau theory of superconductivity, several models for the dynamics of magnetic flux vortices [16, 9, 30, 24] have been studied as gauge theories in $2+1$ dimensions. The surface supporting a vortex is usually considered as a cross section of a "true" vortex tube in three spatial dimensions, but the models often make sense purely in two spatial dimensions, on surfaces like the euclidean and the hyperbolic planes, with boundaries at infinity, and also on closed surfaces such as the 2-sphere. In the present paper, we will be focusing on what is known as the abelian Higgs model for vortex dynamics, on a closed surface Σ of genus $g > 1$. In this particular setting, several interesting results have already been gathered in the literature, some of them in analogy with the most extensively studied case where Σ is taken to be the euclidean plane.

The time-independent model consists of a system of coupled partial differential equations for a connection d_a and a section ϕ of a complex line bundle $\mathcal{L} \to \Sigma$. On the surface Σ, a Kähler structure $(\Sigma, j_\Sigma, \omega_\Sigma)$, where j_Σ is the complex structure and ω_Σ the Kähler 2-form (here just the area form) needs to be specified, and the bundle $\mathcal{L}$, supposed to be nontrivial, is equipped with a hermitian metric. A real parameter τ appears in the energy and in the field equations. This determines the size of a vortex, which is significant when the geometry of Σ is fixed. Another parameter, the coupling constant λ, will be fixed at $\lambda = 1$. This models the boundary between type I and type II superconductors, and turns out to be most interesting mathematically, since the field configurations of minimal energy then satisfy the first-order vortex equations [5]. Here there are no static forces between vortices; for generic allowed values of the parameter τ, a unique static multivortex solution, up to gauge transformations, has been proved to exist for any distribution of N points on Σ [7, 13]. These points specify isolated zeroes of the section ϕ. The positive integer N, the vortex number, is also the total magnetic flux through the surface in units of the "classically quantised" flux of a single vortex. Although the vortices should be regarded as extended objects over the surface, these points on Σ (which can be superposed) can be seen as precise locations of N vortex centres, around which the energy density of the fields typically concentrates. In this regime, the moduli space of N-vortex solutions is thus the Nth symmetric power of the surface Σ,

$$\mathcal{M}_N \cong \mathrm{Sym}^N \Sigma := \Sigma^N / \mathfrak{S}_N \tag{1.1}$$

where $\mathfrak{S}_N$ is the symmetric group, and is a smooth complex manifold. In particular $\mathcal{M}_1$, as a complex manifold, is just a copy of Σ.

The dynamical model extends the time-independent model by including a kinetic energy term in the lagrangian. This is quadratic in the time derivatives of the fields. Also the connection needs to be extended to include a time-component. Since there is no static potential interaction between vortices when $\lambda = 1$, the non-relativistic motion of N vortices on Σ reduces to a geodesic motion on the moduli space $\mathcal{M}_N$, with respect to a Riemannian metric (for each value of τ) that arises from the kinetic part of the lagrangian [21, 29].

This metric can alternatively be thought of as being induced from the natural L^2-metric on the space of solutions of the vortex equations.

Quite a lot is known about this family of L^2-metrics on the moduli space $\mathcal{M}_N$ parametrised by τ. A very interesting result is a formula due to Samols expressing the metric in terms of local data around the vortex centres [28]. Each metric in the family is Kähler with respect to the natural complex structure on $\mathrm{Sym}^N\Sigma$ induced from the original one on Σ, which is independent of τ. From Samols' formula, one can derive an explicit expression for the total volume of $\mathcal{M}_N$ [23]. This volume depends only on the topological data (the genus g of Σ and the vortex number N), on the total area $A = \int_\Sigma \omega_\Sigma$ of Σ, and on τ. It does not depend on more detailed metric information.

The allowed range of the parameter τ for solutions of the vortex equations to exist is the interval $[\frac{4\pi N}{A}, +\infty[$, for a given $N > 0$. In the present work, we will be interested in the situation where τ approaches the lower end of this interval. Specifically, at $\tau = \frac{4\pi N}{A}$ the vortex equations have solutions, but ϕ vanishes everywhere — so, strictly speaking, there are no vortices. There remains a magnetic field, which is a multiple of the area form on the surface. There also remain moduli, because the magnetic field does not completely fix the holonomies (the Wilson loop variables) of the connection around noncontractible 1-cycles on Σ. The moduli space of connections in this limit of "dissolved vortices" is in fact a translate of the moduli space of flat U(1)-connections on Σ. This is the dual of the Jacobian variety $\mathrm{Jac}(\Sigma)$ associated to Σ, a torus of complex dimension g, for any value of N. Importantly, if the holonomies change with time, there will be electric fields on Σ and hence a positive field kinetic energy. The moduli space in this regime of dissolved vortices, which we can identify with the Jacobian itself by duality, therefore acquires a metric. This metric has been shown by Nasir to be flat [26], and it only depends on the complex structure of Σ, not on the detailed form of the conformal factor Ω. We rederive these results below.

Close to this critical situation, in the case where τ is arbitrarily close to but greater than $\frac{4\pi N}{A}$, the moduli space $\mathcal{M}_N$ has complex dimension N. The picture is that the section ϕ is close to zero everywhere (as its L^2-norm equals $\tau A - 4\pi N$ for static fields), but its zero locus consists of isolated points. In this paper, we shall describe an approximation to the vortex equations modelling this situation, and will call it the regime of "dissolv*ing* vortices" (as opposed to dissolv*ed*, which we defined in the preceding paragraph). Specifically, we will be mainly dealing with the case where $N < g$. (Dissolving vortices in the cases where $g = 1$ and $N = 1$, and where $g = 0$ with N arbitrary, were investigated previously in [22] and [3], respectively.) The moduli space $\mathcal{M}_N$ can be identified with the space of effective divisors of order N on the algebraic curve Σ, i.e. sets of N (not necessarily distinct) points on Σ — the zeros of ϕ with multiplicities counted algebraically. In algebraic geometry, any divisor on Σ gives rise to a holomorphic line bundle, but as ϕ has no poles, we need to restrict to the effective divisors, which give rise to line bundles with nontrivial holomorphic sections. An effective divisor is converted to a point in the Jacobian by the Abel–Jacobi map [14, 17]. At generic points of the image, the Abel–Jacobi map embeds $\mathcal{M}_N$ smoothly (and holomorphically) in $\mathrm{Jac}(\Sigma)$, but there can be singular points. However, the manifold $\mathcal{M}_1$ of effective divisors of order 1 is simply Σ, and the Abel–Jacobi map embeds $\mathcal{M}_1$ smoothly in $\mathrm{Jac}(\Sigma)$.

Our main result for the $N = 1$ case is the following

Theorem 1: *The metric describing the motion of one dissolving vortex is a natural Bergman metric on $\Sigma \cong \mathcal{M}_1$. It coincides with the Kähler metric obtained by pulling back the flat metric dual to the metric of dissolved vortices (which is determined by the polarisation of the Jacobian), via the Abel–Jacobi embedding* $\mathrm{AJ}_1 : \Sigma \hookrightarrow \mathrm{Jac}(\Sigma)$.

The proof of this Theorem is contained in Section 5. (The notion of Bergman metric we are using is explained in Appendix A, and most of the other ingredients in Section 4.) As τ grows away from $\frac{4\pi}{A}$, the metric on $\mathcal{M}_1$ is a deformation of this Bergman metric defined from the complex structure alone, and this deformation will incorporate details of the metric data on Σ (Riemannian metric and hermitian structure) needed to write down the lagrangian of the model. Unfortunately, the metric on $\mathcal{M}_1$ is not known explicitly except in limiting cases. However, $\mathcal{M}_1$ is always in the same conformal class as Σ, and its total area is an affine function of both τA and g. The opposite to the dissolving limit is the regime where $\tau \to +\infty$, which has been investigated previously [22]. Here the local geometry near the vortex is crucial. The metric at a point of $\mathcal{M}_1$ is a constant multiple of the metric on Σ with a leading correction that depends on the curvature of Σ at that point. The interpretation is that the vortex behaves almost exactly like a localised particle on Σ. So the flow of the one-vortex moduli space metric (as τ increases) turns an effective divisor, which is not a localised object when the associated holomorphic bundle and a specific holomorphic section solving the vortex equations are considered, into a localised particle.

Broadly speaking, when $N > 1$, the slow motion of a vortex is determined by two factors: on the one hand, the nature of the metric on Σ close to the vortex and the hermitian structure on the bundle $\mathcal{L} \to \Sigma$, which are essentially introduced by hand in the model, and on the other hand, the effect of other nearby vortices, which in turn is controlled by the parameter τ scaling the area $\frac{4\pi}{\tau}$ of an "effective disc" occupied by each vortex. The most salient feature of the latter effect is that the head-on symmetric scattering of two vortices along some curve of Σ occurs in finite time, with the vortices emerging along a perpendicular curve on Σ through the point of collision. This conclusion follows essentially from the fact that the directions of geodesics for a Kähler metric are determined by the complex structure alone; the simplest version of this local argument wams first spelled out by Hitchin in [15]. For dissolving vortices on a round sphere, the behaviour of vortex scattering geodesics more general than head-on was described by Baptista and Manton [3] — however, on a round sphere the metric has positive curvature, and the general scattering behaviour of dissolving vortices on a surface Σ of higher genus will be quite different.

In Section 7, we discuss the geometry of the slow motion of dissolving multi-vortices, in the case that $1 < N < g$. It is natural to expect that the Abel–Jacobi map

$$\mathrm{AJ}_N : \mathrm{Sym}^N \Sigma \longrightarrow \mathrm{Jac}(\Sigma) \tag{1.2}$$

still relates the geometry of dissolving vortices with that of dissolved N-vortices, but it turns out that this map typically has singular points (more precisely, there are singular points unless $N = 2$, $g = 3$ and Σ is not hyperelliptic). So the pull-back of the dual of the metric of dissolved vortices degenerates at the critical locus of AJ_N:

Theorem 2: *For $1 < N < g$, the slow motion of dissolving vortices is described generically by the pull-back of the dual of the flat metric of dissolved vortices via the Abel–Jacobi map (1.2). This pull-back is a Kähler metric in an open subset of $\mathcal{M}_N$, degenerating at the locus of special effective divisors of degree N.*

Geodesics of these metrics of dissolving multivortices follow the expected pattern of 90^o vortex scattering, but they become rather exotic at the degeneration locus, which consists of the union of exceptional fibres of the Abel–Jacobi map. The complement of the set of special divisors consists of the points around which the Abel–Jacobi map is a one-to-one cover. To understand what happens at the boundary of this regularity locus, we study the simplest case in Section 8, namely, the behaviour of two dissolving vortices on a hyperelliptic Riemann surface of genus 3 near the singularity. Then the image of the Abel–Jacobi map is a surface $W_2 \subset \mathrm{Jac}(\Sigma)$ with a double point, and the moduli space $\mathcal{M}_2 = \mathrm{Sym}^2\Sigma$ can be recovered from it by blowing up this singularity. We will show that the geodesic motion will be suppressed along the directions of the exceptional fibre introduced in the blow-up. As we shall explain in Section 8, an interesting implication of this observation is that the motion of two dissolving vortices on the surface is able to detect Weierstraß points.

2 Vortex dynamics and the vortex equations

Let Σ be a closed Riemann surface of genus g with a compatible metric g_Σ. We will be working on the fixed space-time $\mathbb{R} \times \Sigma$ with lorentzian metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - g_\Sigma\,, \tag{2.1}$$

where t is a time coordinate on $\mathbb{R}$. In terms of a local complex coordinate z on Σ, $g_\Sigma = \Omega\,\mathrm{d}z\mathrm{d}\bar{z}$, where Ω is a positive real function called the conformal factor, and the associated Kähler 2-form is $\omega_\Sigma = \frac{\mathrm{i}}{2}\Omega\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$.

Over Σ, we also fix a hermitian line bundle $\mathcal{L} \to \Sigma$, i.e. for each $P \in \Sigma$ the fibre $\mathcal{L}_P$ is a copy of $\mathbb{C}$ endowed with a hermitian inner product $\langle\cdot,\cdot\rangle_P$ that varies smoothly over Σ. The hermitian inner product can be used to normalise local trivialisations, and this amounts to reducing the structure group of $\mathcal{L} \to \Sigma$ from $\mathbb{C}^*$ to the subgroup $\mathrm{U}(1)$. We can use the projection pr_Σ onto Σ to pull $\mathcal{L}$ back to our spacetime $\mathbb{R} \times \Sigma$, and the resulting line bundle $\mathrm{pr}_\Sigma^*\mathcal{L} \to \mathbb{R} \times \Sigma$ can be described by time-independent transition functions.

The dynamics we want to discuss involves configurations of two fields on $\mathbb{R} \times \Sigma$: a complex field ϕ which is a smooth section of $\mathrm{pr}_\Sigma^* \mathcal{L}$, and a U(1)-connection $\mathrm{D}_{\hat{a}}$ on $\mathrm{pr}_\Sigma^* \mathcal{L}$. On a local trivialisation, ϕ is equivalent to a complex function, whereas the connection can be expressed as the covariant derivative

$$\mathrm{D}_{\hat{a}} = \mathrm{d} - \mathrm{i}\hat{a}\,, \tag{2.2}$$

where $\hat{a} = a_t\,\mathrm{d}t + a_z\,\mathrm{d}z + a_{\bar{z}}\,\mathrm{d}\bar{z}$ is a real 1-form, with z a local complex coordinate on the trivialising open set $U \subset \Sigma$ and $a_{\bar{z}} = \overline{a_z}$. Gauge transformations act on the fields as

$$\begin{aligned} \phi &\mapsto \mathrm{e}^{\mathrm{i}\chi}\phi & (2.3)\\ \mathrm{D}_{\hat{a}} &\mapsto \mathrm{D}_{\hat{a}} - \mathrm{i}\,\mathrm{d}\chi\,, & (2.4) \end{aligned}$$

with χ a real function.

The connection splits into a time and space part, $\hat{a} = a_t\,\mathrm{d}t + a$, where, locally, a_t is a real function and a is a real 1-form on Σ, both time-dependent in general. Then the covariant derivative of ϕ splits as

$$\mathrm{D}_{\hat{a}}\phi = \mathrm{D}_{\mathrm{t}}\phi\,\mathrm{d}t + \mathrm{d}_a\phi \tag{2.5}$$

where $\mathrm{D}_{\mathrm{t}}\phi = \partial_t\phi - \mathrm{i}a_t\phi$ is a section of $\mathrm{pr}_\Sigma^*\mathcal{L} \to \mathbb{R}\times\Sigma$ and $\mathrm{d}_a\phi = \mathrm{d}\phi - \mathrm{i}a\phi$ is a path of 1-forms on Σ with values in $\mathcal{L}$ (locally, these two quantities are just a complex function on space-time and a time-dependent 1-form on Σ, respectively). The (space-time) Maxwell 2-form is the curvature of the connection, $F_{\hat{a}} = \mathrm{d}\hat{a}$, which can be written as

$$F_{\hat{a}} = \mathrm{d}t \wedge e_a + b_a\,, \tag{2.6}$$

where $e_a = \partial_t a - \mathrm{d}a_t$ (the electric field) is a time-dependent 1-form and $b_a = \mathrm{d}a$ (the magnetic field) is a time-dependent 2-form, both globally defined on Σ and both invariant under gauge transformations. Complex line bundles $\mathcal{L} \to \Sigma$ are topologically classified by their Chern number (or degree) $N \in \mathbb{Z}$, which is given by the integral

$$N = \frac{1}{2\pi}\int_\Sigma b_a \quad \in \mathbb{Z} \tag{2.7}$$

and can be interpreted as a "quantised" total magnetic flux. Throughout this paper, we shall assume that N is positive.

The lagrangian of the abelian Higgs model at coupling $\lambda = 1$ is [24]

$$L = \frac{1}{2}\int_\Sigma \left\{ e_a \wedge *e_a + \mathrm{D}_{\mathrm{t}}\phi \wedge *\overline{\mathrm{D}_{\mathrm{t}}\phi} - b_a \wedge *b_a - \mathrm{d}_a\phi \wedge *\overline{\mathrm{d}_a\phi} - \frac{1}{4} * (\langle\phi,\phi\rangle - \tau)^2 \right\} \tag{2.8}$$

where τ is a fixed positive parameter. We are using the Hodge $*$-operator associated to g_Σ, which in terms of the coordinate z acts on forms as $*\mathrm{d}z = -\mathrm{i}\,\mathrm{d}z$, $*\mathrm{d}\bar{z} = \mathrm{i}\,\mathrm{d}\bar{z}$, $*1 = \omega_\Sigma$ and $*\omega_\Sigma = 1$. The contribution of the first two terms, involving the time derivatives of the fields and a_t, defines the kinetic energy of the theory, T, and the remaining terms give (minus) the potential energy,

V. Note that the kinetic energy is independent of the parameter τ. Also note that the first and fourth terms, involving 1-forms and their Hodge duals, are conformally invariant — they are unchanged if one deforms g_Σ within the same conformal class (in other words, keeps the complex structure on Σ fixed).

Static vortices, associated to fields with no time dependence and vanishing a_t, minimise the potential energy

$$V = \frac{1}{2}\int_\Sigma \left\{ b_a \wedge * b_a + \mathrm{d}_a\phi \wedge *\overline{\mathrm{d}_a\phi} + \frac{1}{4} * (\langle\phi,\phi\rangle - \tau)^2 \right\} . \tag{2.9}$$

By a standard reorganisation of the integral, it can be shown that the minimal value of V is $\pi\tau N$, and is attained precisely when the fields satisfy the first-order vortex equations on Σ [5]

$$\bar\partial_a \phi = 0 \,, \tag{2.10}$$

$$*b_a + \tfrac{1}{2}(\langle\phi,\phi\rangle - \tau) = 0 \,. \tag{2.11}$$

The first equation expresses that the section ϕ of $\mathcal{L} \to \Sigma$ is holomorphic, i.e. annihilated by the operator $\bar\partial_a : \Omega^0(\Sigma,\mathcal{L}) \to \Omega^1(\Sigma,\mathcal{L})$ (locally, $\bar\partial_a = \bar\partial - \mathrm{i}a_{\bar z}\mathrm{d}\bar z$) defined from the unitary connection d_a and the complex structure on Σ [8]. The second equation relates the curvature $b_a = \mathrm{d}a$ of the connection to the moment map of the holomorphic, hamiltonian action of U(1) on the fibres of $\mathcal{L} \to \Sigma$ (with Kähler structure induced from the hermitian metric) evaluated after ϕ. The presence of the constant $\tau \in \mathbb{R}$ relates to the ambiguity in the choice of a moment map for this action.

By integrating (2.11) over Σ and using (2.7), one finds

$$||\phi||^2_{L^2} = \tau A - 4\pi N \tag{2.12}$$

where $A := \int_\Sigma \omega_\Sigma$ is the total area of the surface and $||\phi||^2_{L^2} := \int_\Sigma \langle\phi,\phi\rangle\omega_\Sigma$. Since this squared L^2-norm is non-negative, the vortex equations can only have solutions if $\tau \geq \frac{4\pi N}{A}$. If we take $\tau > \frac{4\pi N}{A}$, there is a unique solution, up to gauge equivalence, for any choice of N unordered (not necessarily distinct) points on Σ where ϕ is required to vanish [7, 13]. This is what is called an N-vortex solution. The moduli space $\mathcal{M}_N$ of N-vortex solutions is therefore $\mathrm{Sym}^N\Sigma = \Sigma^N/\mathfrak{S}_N$, a smooth complex manifold with complex dimension N.

We shall refer to the critical situation where $\tau = \frac{4\pi N}{A}$ as the limit of *dissolved* vortices. Here, solutions of the vortex equations also exist. Since $||\phi||^2_{L^2}$ must vanish, and ϕ is smooth, $\phi = 0$ everywhere, so there are no true (localised) vortices. The first vortex equation is now trivially satisfied and the second vortex equation reduces to

$$b_a = \frac{\tau}{2}\omega_\Sigma \,, \tag{2.13}$$

which means that the magnetic flux per unit area has the constant value $\frac{2\pi N}{A}$. Notice that the value of this constant is determined by the topology and the total area A of Σ. A connection d_a satisfying equation (2.13) is called a projectively flat (or constant curvature) connection, and it always exists, but is not completely determined by the magnetic field b_a if $g \geq 1$ (which implies that Σ is not simply connected). We shall see below that the space of such connections, up to gauge equivalence, is a flat torus of real dimension $2g$, irrespective of the value of N.

To investigate vortices and their moduli space in what we shall call the *dissolving limit*, where τ slightly exceeds $\frac{4\pi N}{A}$, it is sufficient to make the approximation that ϕ is small, and to neglect the term $\langle \phi, \phi \rangle$ in the second vortex equation. In this regime, the connection is therefore taken to satisfy (2.13), as discussed above; there will be corrections to this, which in principle could be studied in perturbation theory in the parameter $\varepsilon = \tau - \frac{4\pi N}{A}$, but we shall not pursue this here. In addition, the connection must be such that the first vortex equation has a nontrivial solution. We shall see that this prescription (which substitutes the first vortex equation *per se*) picks out a subset of the connections that occur in the situation of dissolved vortices.

3 Dissolved vortices

In this section, we shall discuss the critical situation of dissolved vortices, where

$$\tau = \frac{4\pi N}{A} \,. \tag{3.1}$$

Here, the field ϕ vanishes everywhere on Σ, so the first vortex equation (2.10) is trivially satisfied. The second vortex equation (2.11) fixes the magnetic field on Σ and reduces to the equation for a projectively flat connection

$$b_a = \mathrm{d}a = \frac{\tau}{2}\,\omega_\Sigma \,. \tag{3.2}$$

The metric on Σ is Kähler, so locally there is a real Kähler potential $\mathcal{K}$ such that

$$\omega_\Sigma = \mathrm{i}\partial\bar{\partial}\mathcal{K} \,. \tag{3.3}$$

Therefore, a choice for the connection 1-form is, locally,

$$a = \frac{\mathrm{i}\tau}{4}(\bar{\partial}\mathcal{K} - \partial\mathcal{K}) \,, \tag{3.4}$$

which is real. The ambiguity in the local Kähler potential corresponds to an ambiguity in the choice of gauge.

Globally, there is not a unique unitary connection for this magnetic field. The general such connection can be expressed as $\mathrm{d}_a - \mathrm{i}\alpha$, where a is fixed as above, and α is a global, real, closed 1-form, satisfying $\mathrm{d}\alpha = 0$. If α is globally an exact form, then $a \mapsto a + \alpha$ is simply a gauge transformation of a. To project out gauge transformations, it is natural to impose the further condition $\mathrm{d} * \alpha = 0$. With this prescription, α (satisfying $\mathrm{d}\alpha = \mathrm{d} * \alpha = 0$) is a real, harmonic 1-form on Σ.

Now let us consider a time-varying connection, with the same unchanging magnetic field. This is described by a connection $\mathrm{d}_{a+\alpha} = \mathrm{d}_a - \mathrm{i}\alpha$, where d_a is fixed in time and as above, together with a time-varying, harmonic 1-form α. In addition, there is the real function a_t, also varying in time. The 1-form electric field is $e = e_{a+\alpha} = \partial_t \alpha - \mathrm{d}a_t$, and should satisfy Gauß's law, which is $\mathrm{d} * e = 0$ when ϕ vanishes. (Gauß's law is one of the Euler–Lagrange equations arising in the abelian Higgs model, and expresses the constraint that time variations of a connection should be projected orthogonally to the orbits of the group of gauge transformations in the space of infinitesimal connections, with respect to the L^2-norm.) With α harmonic, Gauß's law is satisfied by setting $a_t = 0$, and the electric field is simply $e = \partial_t \alpha$. The kinetic energy is then

$$T = \frac{1}{2} \int_\Sigma e \wedge *e = \frac{1}{2} \int_\Sigma (\partial_t \alpha) \wedge *(\partial_t \alpha) \,. \tag{3.5}$$

The space of real, harmonic 1-forms on Σ has real dimension $2g$, and is isomorphic to the space of holomorphic 1-forms on Σ, with complex dimension g. This is because each harmonic form α can be uniquely expressed in terms of a holomorphic form ω as

$$\alpha = 2\,\mathrm{Re}\,\omega = \omega + \bar{\omega} \,. \tag{3.6}$$

Then

$$*\,\alpha = -\mathrm{i}\,\omega + \mathrm{i}\,\bar{\omega} \,, \tag{3.7}$$

and it follows that $\mathrm{d}\alpha = \mathrm{d}{*\alpha} = 0$, since locally $\omega = \omega(z)\mathrm{d}z$ for some holomorphic function $\omega(z)$, with $\bar{\omega} = \overline{\omega(z)}\mathrm{d}\bar{z}$ and $\partial_{\bar{z}}\omega(z) = \partial_z \overline{\omega(z)} = 0$.

It is convenient to introduce a canonical basis of the space of holomorphic 1-forms $H^0(\Sigma, K_\Sigma)$ [11]. (K_Σ denotes the canonical sheaf of Σ.) First, we represent Σ as a $4g$-sided polygon Σ_{poly} with sides identified, as in Fig. 1. (We depict the $g = 2$ case only, for simplicity.) The labelled edges $\{a_j, b_j : 1 \le j \le g\}$ are representatives of a canonical (or symplectic) basis of 1-cycles, generating the first homology group $H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2g}$. In this context, the words canonical/symplectic refer to the fact that, with respect to this basis, the symplectic (i.e. skew-symmetric and nondegenerate) pairing $\sharp(\cdot,\cdot)$ between homology 1-cycles given by counting signed intersections with multiplicity has canonical form, namely:

$$\sharp(a_j, b_k) = \delta_{jk}, \quad \sharp(a_j, a_k) = 0 = \sharp(b_j, b_k), \qquad j, k = 1, \ldots, g \,. \tag{3.8}$$

Any other canonical basis of $H_1(\Sigma; \mathbb{Z})$ is related to this one by a linear transformation in $\mathrm{Sp}_{2g}(\mathbb{Z})$. The elements of the canonical basis of holomorphic 1-forms relative to this basis of 1-homology are denoted by $\zeta_j, 1 \le j \le g$ (locally, $\zeta_j = \zeta_j(z)\,\mathrm{d}z$, where $\zeta_j(z)$ are holomorphic functions of a complex coordinate z in the polygon). By definition, they are uniquely determined by the normalisation of a-periods

$$\oint_{a_j} \zeta_k = \delta_{jk} \,, \tag{3.9}$$

and their b-periods are denoted as

$$\oint_{b_j} \zeta_k = \Pi_{jk} \,. \tag{3.10}$$

The $g \times g$ matrix Π of b-periods has the properties, established by Riemann, that it is symmetric and its imaginary part $\mathrm{Im}\,\Pi$ is positive definite. Hence $\mathrm{Im}\,\Pi$ has an inverse (which we will use below). Notice that when we change the basis of $H_1(\Sigma;\mathbb{Z})$ using some matrix

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathrm{Sp}_{2g}\mathbb{Z}\,, \tag{3.11}$$

where B_{11}, B_{12}, B_{21} and B_{22} are $g \times g$ blocks, then the matrix of b-periods Π changes as

$$\Pi \mapsto (\Pi\, B_{12} + B_{11})^{-1}(\Pi\, B_{22} + B_{21})\,, \tag{3.12}$$

where the left factor accounts for a change of basis of $H^0(\Sigma, K_\Sigma)$ that is necessary to maintain the normalisation of a-periods.

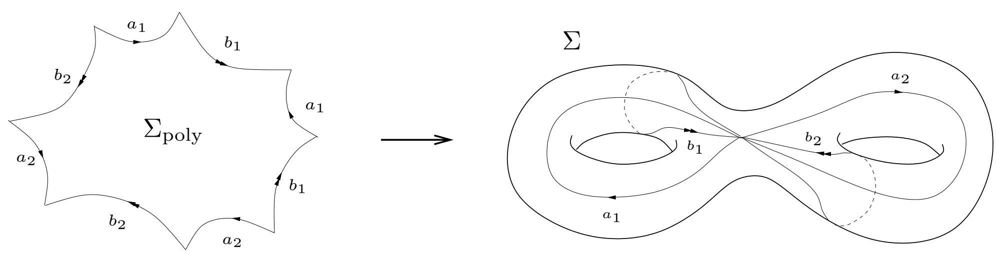

Figure 1: Constructing a closed Riemann surface from a polygon

We can now write the time-varying part of the connection, α, as

$$\alpha = \sum_{j=1}^{g} (\overline{c_j}\, \zeta_j + c_j\, \overline{\zeta_j})\,, \tag{3.13}$$

where the coefficients $c_j = c_j(t)$ are complex functions of t. Then

$$e = \sum_{j=1}^{g} \left(\frac{\mathrm{d}\overline{c_j}}{\mathrm{d}t}\, \zeta_j + \frac{\mathrm{d}c_j}{\mathrm{d}t} \overline{\zeta_j} \right) \tag{3.14}$$

and

$$*\, e = \sum_{j=1}^{g} \left(-\mathrm{i} \frac{\mathrm{d}\overline{c_j}}{\mathrm{d}t}\, \zeta_j + \mathrm{i} \frac{\mathrm{d}c_j}{\mathrm{d}t} \overline{\zeta_j} \right)\,, \tag{3.15}$$

so the kinetic energy of the connection is

$$T = \frac{1}{2}\int_\Sigma e \wedge *e \tag{3.16}$$

$$= \mathrm{i}\sum_{j,k=1}^{g} \frac{\mathrm{d}\overline{c_j}}{\mathrm{d}t}\frac{\mathrm{d}c_k}{\mathrm{d}t}\int_\Sigma \zeta_j \wedge \overline{\zeta_k} \tag{3.17}$$

$$= 2\sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)_{jk}\frac{\mathrm{d}c_j}{\mathrm{d}t}\frac{\mathrm{d}\overline{c_k}}{\mathrm{d}t}\,, \tag{3.18}$$

where, to obtain the last line, we have used the result ([11], p. 65)

$$\int_\Sigma \zeta_j \wedge \overline{\zeta_k} = -2\mathrm{i}(\mathrm{Im}\,\Pi)_{jk}\,, \tag{3.19}$$

and the symmetry of $\mathrm{Im}\,\Pi$.

The expression (3.18) can be interpreted as the kinetic energy of dissolved vortices, and arises purely from the electric field of the time-varying connection. It only depends on the complex structure of Σ. Exactly where any vortices are, and how they are moving, will be clarified below. The metric on the moduli space of dissolved vortices is obtained by dropping a factor $\frac{1}{2}$, and is therefore [26]

$$\mathrm{d}s^2 = 4\sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)_{jk}\,\mathrm{d}c_j\,\mathrm{d}\overline{c_k}\,, \tag{3.20}$$

a flat metric on a space of complex dimension g.

The final point is that the range of the coordinates c_j is not the whole of $\mathbb{C}^g$: two connections with the same curvature are (globally) gauge equivalent if and only if they have the same holonomies around all 1-cycles. The connections d_a and $\mathrm{d}_{a+\alpha}$, where α is a harmonic 1-form, have the same holonomies if the integral of α around any 1-cycle on Σ is an integer multiple of 2π, that is, if d_α has trivial holonomy. This condition is equivalent to requiring the integrals around the cycles a_j and b_j to be integral multiples of 2π. For $\alpha = \sum_{j=1}^{g}(\overline{c_j}\,\zeta_j + c_j\,\overline{\zeta_j})$, these integrals are

$$\oint_{a_j} \alpha = \overline{c_j} + c_j\,, \tag{3.21}$$

$$\oint_{b_j} \alpha = \sum_{k=1}^{g}(\Pi_{jk}\,\overline{c_k} + \overline{\Pi_{jk}}\,c_k)\,. \tag{3.22}$$

The 1-forms α representing connections with trivial holonomy therefore form a lattice of rank $2g$ inside the space of global harmonic 1-forms $\mathcal{H}_1(\Sigma;\mathbb{R}) = \mathbb{R}^{2g} \cong \mathbb{C}^g$, defined by the $2g$ real conditions,

$$\frac{1}{\pi}\mathrm{Re}\,c_j \in \mathbb{Z}\,, \quad \frac{1}{\pi}\mathrm{Re}\sum_{k=1}^{g}(\Pi_{jk}\,\overline{c_k}) \in \mathbb{Z}\,. \tag{3.23}$$

So the moduli space of connections describing dissolved vortices is $\mathbb{C}^g$ modulo this lattice, which is a complex torus [14] with the flat metric (3.20). Its volume is $(2\pi)^{2g}$ [26].

This torus is essentially the Jacobian of Σ, as the lattice here is related to the period lattice in $\mathbb{C}^g$ generated by the column vectors of the unit matrix and the column vectors of the matrix Π, as we shall show next. More precisely, this torus is dual to the Jacobian, but will be naturally identified with the Jacobian once a normalisation constant is chosen. Notice that all these results, and the metric (3.20) in particular, are independent of the vortex number N (which only appeared via the reference connection d_a).

4 Geometry of the Jacobian variety

Recall that the vector space $H^0(\Sigma, K_\Sigma)$ of holomorphic 1-forms on Σ is of complex dimension g. One can embed the group $H_1(\Sigma;\mathbb{Z})$ of homology 1-cycles into the dual space $H^0(\Sigma, K_\Sigma)^*$ as follows: for each 1-cycle γ, we consider the linear functional $\{\gamma\} \in H^0(\Sigma, K_\Sigma)^*$ defined by

$$\omega \mapsto \oint_\gamma \omega\,, \tag{4.1}$$

sending a holomorphic 1-form ω to its line integral around γ. Let Λ denote the image of this embedding, which is a lattice of rank $2g$ in $H^0(\Sigma, K_\Sigma)^*$ generated by the elements $\{a_j\}$, $\{b_j\}$,

$$\omega \mapsto \oint_{a_j} \omega\,, \quad \omega \mapsto \oint_{b_j} \omega \tag{4.2}$$

with $\{a_j, b_j : 1 \leq j \leq g\}$ the basis of homology 1-cycles introduced earlier. By definition, the Jacobian of Σ is the quotient

$$\mathrm{Jac}(\Sigma) = H^0(\Sigma, K_\Sigma)^*/\Lambda\,, \qquad \Lambda = \{H_1(\Sigma;\mathbb{Z})\} \tag{4.3}$$

and is a complex torus of real dimension $2g$. This manifold comes equipped with a complex structure, induced from multiplication by i in $H^0(\Sigma, K_\Sigma)$, which in turn comes from the complex structure j_Σ on the Riemann surface Σ. Clearly, $\mathrm{Jac}(\Sigma)$ is also an analytic abelian Lie group, with operation induced from the addition in the vector space $H^0(\Sigma, K_\Sigma)^*$.

Once we fix the canonical basis of holomorphic 1-forms $\{\zeta_k : 1 \leq k \leq g\}$, the vector space $H^0(\Sigma, K_\Sigma)^*$ acquires natural complex coordinates $\chi_k : H^0(\Sigma, K_\Sigma)^* \to \mathbb{C}$ (for $1 \leq k \leq g$), which are defined by extending linearly (over $\mathbb{R}$) the functionals

$$\{\gamma\} \mapsto \oint_\gamma \zeta_k =: \chi_k(\{\gamma\})\,, \tag{4.4}$$

defined on the image $\Lambda \cong \mathbb{Z}^{2g}$ of $H_1(\Sigma;\mathbb{Z})$, to the whole of $H^0(\Sigma, K_\Sigma)^* \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$; in particular,

$$\chi_k(\{a_j\}) = \oint_{a_j} \zeta_k = \delta_{jk}\,, \tag{4.5}$$

$$\chi_k(\{b_j\}) = \oint_{b_j} \zeta_k = \Pi_{jk}\,. \tag{4.6}$$

In these coordinates, the lattice Λ is generated over $\mathbb{Z}$ by the $2g$ vectors in $\mathbb{C}^g$ which are the column vectors of the unit matrix and the column vectors of the matrix Π, and $\mathrm{Jac}(\Sigma)$ has the explicit form $\mathbb{C}^g/\Lambda$.

The projectively flat connections $\mathrm{d}_{a+\alpha}$ that occur in the limit of dissolved vortices, discussed in the previous section, also define elements of $H^0(\Sigma, K_\Sigma)^*$, via wedge product and integration over Σ. (This is a variant of a construction usually applied to flat connections.) Specifically, once we fix a projectively flat connection d_a, we can identify another projectively flat connection $\mathrm{d}_{a+\alpha}$ with the harmonic 1-form α, which in turn defines an element of $H^0(\Sigma, K_\Sigma)^*$ via

$$\omega \mapsto \frac{1}{2\pi}\int_\Sigma \alpha \wedge \omega\,, \tag{4.7}$$

whose coordinates are

$$\chi_k = \frac{1}{2\pi}\int_\Sigma \alpha \wedge \zeta_k\,. \tag{4.8}$$

In (4.7) we are introducing a normalisation factor of 2π for convenience. Using the expansion (3.13), and the integral (3.19), and noting that $\zeta_j \wedge \zeta_k$ vanishes, we deduce that

$$\chi_k = \frac{\mathrm{i}}{\pi}\sum_{l=1}^{g} (\mathrm{Im}\,\Pi)_{kl} c_l\,. \tag{4.9}$$

Since $\mathrm{Im}\,\Pi$ is invertible, the coordinates χ_k or c_k are equally good at parametrising the element of $H^0(\Sigma, K_\Sigma)^*$ corresponding to α.

Now recall that the coordinates c_k are defined only modulo the lattice specified by the conditions (3.23). These conditions are equivalent to

$$\overline{c_j} + c_j = 2\pi m_j\,, \quad \sum_{k=1}^{g} (\Pi_{jk}\,\overline{c_k} + \overline{\Pi_{jk}}\,c_k) = 2\pi n_j\,, \tag{4.10}$$

where $\{m_j, n_j : 1 \le j \le g\}$ are integers. Eliminating $\overline{c_k}$, they become

$$\frac{\mathrm{i}}{\pi}\sum_{k=1}^{g} (\mathrm{Im}\,\Pi)_{jk} c_k = \sum_{k=1}^{g} (\Pi_{jk} m_k - \delta_{jk} n_k)\,, \tag{4.11}$$

and in terms of the coordinates on $H^0(\Sigma, K_\Sigma)^*$ defined by (4.9) they take the form

$$\chi_j = \sum_{k=1}^{g} (\Pi_{jk} m_k - \delta_{jk} n_k)\,, \tag{4.12}$$

a vector of coordinates that is an integer combination of the column vectors of the unit matrix and the column vectors of the matrix Π. But this is precisely a vector in the lattice Λ. So, for a connection $\mathrm{d}_{a+\alpha}$, equation (4.7) defines unambiguously an element in the quotient space $H^0(\Sigma, K_\Sigma)^*/\Lambda$, which is the Jacobian.

We had already pointed out that the moduli space of projectively flat connections is a (real) $2g$-torus, hence diffeomorphic to $\mathrm{Jac}(\Sigma)$. We can interpret equation (4.9) as giving an explicit diffeomorphism between the two tori (the Jacobian and its dual) expressed in local coordinates and use it to pull back the Riemannian metric (3.20) on the moduli space of projectively flat connections to obtain a metric G_J on $\mathrm{Jac}(\Sigma)$: in the coordinates above,

$$G_\mathrm{J} = 4\pi^2 \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \mathrm{d}\chi_j \mathrm{d}\overline{\chi_k}\,. \tag{4.13}$$

It is important to note that this metric is intrinsic to $\mathrm{Jac}(\Sigma)$; since it is invariant under translations, it does not depend on the choice of reference projectively flat connection d_a. Now there is another intrinsic geometric structure on $\mathrm{Jac}(\Sigma)$, namely, a symplectic form that descends from the translation-invariant symplectic form on $H^0(\Sigma, K_\Sigma)^*$ defined by linearly extending (over $\mathbb{R}$) the intersection pairing (3.8). We denote it by $\Omega_\mathrm{J}(\cdot,\cdot)$. From G_J and Ω_J, we can define a hermitian metric H_J on $\mathrm{Jac}(\Sigma)$ by setting

$$H_\mathrm{J}(\cdot,\cdot) = G_\mathrm{J}(\cdot,\cdot) + \mathrm{i}\Omega_\mathrm{J}(\cdot,\cdot)\,. \tag{4.14}$$

Since Ω_J is closed, there is an underlying Kähler structure, and one can recover any of G_J, Ω_J and H_J from just one of these structures and the complex structure (multiplication by i). Another way to see that this Kähler structure is intrinsic is to note that its pullback to $H^0(\Sigma, K_\Sigma)^* \cong \mathbb{C}^g$ coincides with the positive definite hermitian form induced from the nondegenerate pairing on $H^0(\Sigma, K_\Sigma)$ given by

$$(\zeta, \omega) \mapsto \int_\Sigma \zeta \wedge \bar{\omega}\,. \tag{4.15}$$

This bilinear form on $H^0(\Sigma, K_\Sigma)^*$ has the property of being $\mathbb{Z}$-valued on elements of the lattice Λ (this follows from (3.8) and Poincaré duality), and is called a polarisation of the torus $\mathrm{Jac}(\Sigma) = H^0(\Sigma, K_\Sigma)^*/\Lambda$. Since the matrix obtained by evaluating the imaginary part on a basis of Λ has unit determinant, one speaks of a principal polarisation. Notice that polarisations are defined in terms of the matrix of b-periods Π only up to transformations of the form (3.12) in terms of the blocks in (3.11), that is, they are parametrised by the orbits of $\mathrm{Sp}_{2g}(\mathbb{Z})$ acting on the space of symmetric, complex $g \times g$ matrices with positive definite imaginary part. Such a transformation relates hermitian forms of the form (4.14) that are isometric (i.e. pull-backs of each other under linear transformations).

The Jacobian plays another important role, that of classifying holomorphic line bundles over Σ. This classification is finer than that of (smooth) line bundles without a holomorphic structure specified. For fixed N, all line bundles are topologically equivalent, but, as we now recall [14, 17], the moduli space of holomorphic line bundles on Σ is a copy of $\mathrm{Jac}(\Sigma)$.

A holomorphic line bundle $\mathcal{L} \to \Sigma$ of first Chern class (or degree) N is determined by a divisor class of degree N. Such a class is represented by divisors on Σ, consisting of formal sums of points of Σ written as

$$\sum_{j=1}^{n+N} P_j - \sum_{k=1}^{n} Q_k \tag{4.16}$$

for some $n \in \mathbb{N}_0$. The points P_j with positive coefficients are locations of zeros of a meromorphic section of the bundle, whereas the points with negative coefficients Q_k are locations of the poles of the same section. (Note that the points appearing in a divisor need not be distinct, to account for poles or zeros of higher multiplicity; if they are distinct, the divisor is said to be reduced.) Two divisors of degree N belong to the same (linear) class, i.e. represent the same holomorphic line bundle of degree N, if their difference is a divisor of degree zero describing the locus of zeros and poles of some meromorphic function on Σ.

The relation of divisors on Σ with the Jacobian is achieved by the Abel–Jacobi map. To define this, one needs to choose a base point P_0 on Σ. Then the Abel–Jacobi image of a divisor (4.16) representing $\mathcal{L}$ is the element of $\mathrm{Jac}(\Sigma)$ defined through the divisor's action on holomorphic 1-forms via abelian integrals,

$$\omega \mapsto \sum_{j=1}^{n+N} \int_{P_0}^{P_j} \omega - \sum_{k=1}^{n} \int_{P_0}^{Q_k} \omega \,. \tag{4.17}$$

Equivalently, its coordinates on the Jacobian are

$$\chi_l = \sum_{j=1}^{n+N} \int_{P_0}^{P_j} \zeta_l - \sum_{k=1}^{n} \int_{P_0}^{Q_k} \zeta_l \,. \tag{4.18}$$

These are well defined in $\mathrm{Jac}(\Sigma)$ despite the non-uniqueness of the paths from P_0 to P_i and Q_i, since the ambiguities are contained in Λ. Also, divisors which map to the same point in the Jacobian are in one divisor class, as by Abel's theorem they correspond to the same line bundle. The Jacobi inversion theorem implies that to each point in the Jacobian there is a non-trivial divisor class, and hence a holomorphic line bundle. Notice that choosing a different base point $P_0 \in \Sigma$ amounts to a global translation in $\mathrm{Jac}(\Sigma)$. In summary, the moduli space of holomorphic line bundles on Σ of degree N (i.e. with fixed but arbitrary vortex number N) is a copy of the Jacobian of Σ.

In the dissolving limit, described at the end of Section 2, vortices are obtained from line bundles of degree N with holomorphic sections, and these are not generic. They are the bundles that arise from effective divisors, i.e., those with a divisor of zeros $P_1 + P_2 + \cdots + P_N$ but vanishing divisor of poles. The coordinates on the Jacobian corresponding such a divisor are

$$\chi_l = \sum_{j=1}^{N} \int_{P_0}^{P_j} \zeta_l \,. \tag{4.19}$$

The space of effective divisors of degree N is clearly $\mathrm{Sym}^N\Sigma$. If $N < g$, its image in $\mathrm{Jac}(\Sigma)$ is denoted W_N, which is a complex subvariety of $\mathrm{Jac}(\Sigma)$ of codimension $g - N$, whereas for $N \geq g$ it is the whole of the Jacobian. Of most interest to us is the set of effective divisors of degree 1, representable by single points P. This is a copy of Σ itself, and the Abel–Jacobi map embeds $\Sigma = \mathrm{Sym}^1(\Sigma)$ in $\mathrm{Jac}(\Sigma)$ as a smooth complex curve W_1. The coordinates of the image in $\mathrm{Jac}(\Sigma)$ of the degree one divisor P are

$$\chi_j = \int_{P_0}^{P} \zeta_j \,. \tag{4.20}$$

In the next section we show that the Abel–Jacobi image of Σ captures the geometry of the one-vortex moduli space in the dissolving limit. The generic case when $1 < N < g$ is that W_N has singularities at points of $\mathrm{Jac}(\Sigma)$ (i.e. line bundles) that admit linearly independent holomorphic sections. We shall discuss their significance further in Sections 7 and 8.

5 One vortex in the dissolving limit

In the regime that we call the limit of dissolving vortices, which models the situation where $\tau A - 4\pi N$ is small and positive, we shall adopt the following prescription to approximate solutions to the vortex equations. From the integrated second vortex equation (2.12) we see that ϕ is small in the L^2-norm, and we therefore make the approximation that the magnetic field is the same as for the situation of dissolved vortices discussed in Section 3. However, the need to solve the first vortex equation puts constraints on the connection. To investigate this it is helpful to transform from a unitary to a holomorphic gauge, where holomorphic sections of $\mathcal{L} \to \Sigma$ are locally given by holomorphic functions.

To make possible a transition to holomorphic gauge, one extends the class of gauge transformations, so that they are valued in the complexification $\mathrm{U}(1)^c = \mathbb{C}^*$ rather than $\mathrm{U}(1)$. Complex gauge transformations are still given by the formulae (2.3) and (2.4), but χ can now take complex values. The first vortex equation is invariant under complexified gauge transformations, but the second vortex equation generally is not, since it involves the hermitian structure of $\mathcal{L} \to \Sigma$, which should be multiplied by the quantity $|\mathrm{e}^{\mathrm{i}\chi}|^2$ under a complex gauge transformation $\mathrm{e}^{\mathrm{i}\chi} : \Sigma \to \mathbb{C}^*$, so that covariance is preserved. However, we are replacing the second vortex equation by the equation (3.2) of projectively flat connections, and this is invariant under the complexified gauge group.

Let us start with a $\mathbb{C}^*$-connection d_a, locally given by a connection 1-form $a = a_z \mathrm{d}z + a_{\bar z} \mathrm{d}\bar z$ for which the reality condition $a_{\bar z} = \overline{a_z}$ need not hold. Using a complex gauge transformation, it is always possible to go to a holomorphic gauge, where $a_{\bar{z}} = 0$ (in each trivialising chart). In this gauge, the transition functions for the line bundle $\mathcal{L} \to \Sigma$ are holomorphic functions from overlaps of trivialising patches to $\mathbb{C}^*$. For the connection 1-form (3.4), we can be more explicit. After making the gauge transformation

$$\chi = -\frac{\mathrm{i}\tau}{4}\mathcal{K}\,, \tag{5.1}$$

we obtain

$$a = -\frac{\mathrm{i}\tau}{2}\,\partial\mathcal{K} = -\frac{\mathrm{i}\tau}{2}\,\partial_z\mathcal{K}\,\mathrm{d}z\,, \tag{5.2}$$

so that $a_{\bar{z}} = 0$.

In holomorphic gauge, the first vortex equation reduces to

$$\bar{\partial}\phi = 0\,, \tag{5.3}$$

as the holomorphic structure operator $\bar{\partial}_a$ coincides with the $\bar{\partial}$-operator defined from the complex structure of Σ alone; that is, on a trivialisation ϕ is a holomorphic function. If $\mathcal{L}$ has no nontrivial holomorphic section then ϕ must vanish everywhere, and there is no vortex, so we exclude such bundles from further consideration. For $\mathcal{L}$ to have a holomorphic section, its divisor class must contain an effective divisor, $P_1 + P_2 + \cdots + P_N$ say. Then ϕ vanishes at these isolated points (with higher multiplicity, whenever some of the points P_j coincide), and they are the vortex centres.

Let us now focus on the one-vortex case, with $N = 1$. The divisor class of $\mathcal{L}$ must be represented by an effective divisor P, which is just any point of Σ, and ϕ has a simple zero at P. The moduli space $\mathcal{M}_1$ of one-vortex solutions is therefore Σ.

Our present task is to understand these bundles more concretely, so as to calculate the metric on $\mathcal{M}_1$. To do this, it is convenient to first fix the holomorphic line bundle $\mathcal{L} = \mathcal{L}_0$, with divisor P_0, associated with having one vortex with centre at the base point $P_0 \in \Sigma$. We fix the connection 1-form a on a trivialising patch of $\mathcal{L}_0$ to be of the form (5.2). Next we consider moving the vortex to the point P. This will require changing the holomorphic bundle to $\mathcal{L} = \mathcal{L}_P$, associated with the divisor P and different transition functions. Alternatively, it will correspond to keeping the bundle and transition functions fixed but replacing the connection by $\mathrm{d}_a - \mathrm{i}\alpha$, with α a particular harmonic 1-form that depends on P and P_0. By relating these two points of view, we will be able to calculate how α depends on P, and from this find the kinetic energy of the moving vortex.

For the first point of view, we assume that Σ is represented by the polygon Σ_{poly} in Fig. 2 (with P_0 and P not on the edges), and that our line bundles are trivialised over this (open) polygon. Introduce a complex coordinate $z : \Sigma_{\mathrm{poly}} \to \mathbb{C}$ (with $z = 0$ at some interior point distinct from P_0 and P), and let $Z_0 = z(P_0)$ and $Z = z(P)$. $\mathcal{L}_0$ is defined by certain (holomorphic) transition functions connecting (neighbourhoods of) the paired edges a_j, a_j^{-1} and b_j, b_j^{-1}.

To obtain the bundle $\mathcal{L}_P$ it is sufficient to change the transition functions by constant factors on each edge. This is a choice of gauge, and is satisfactory because we only want to change the holonomy of the connection, without changing the magnetic field. Let us call these factors e^{μ_j} (the additional factor connecting a_j to a_j^{-1}) and e^{ν_j} (the additional factor connecting b_j to b_j^{-1}).

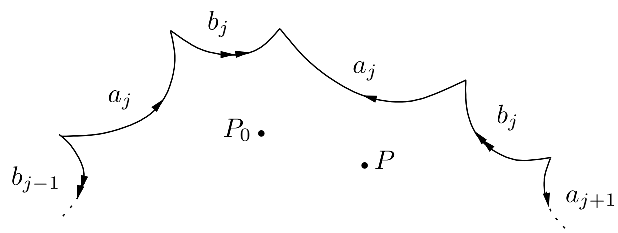

Figure 2: The polygonal trivialising patch Σ_{poly}

The dependence of these factors on P_0 and P can be determined using abelian differentials. Let $\xi_P^{P_0}$ be an abelian differential of the third kind (a meromorphic 1-form on Σ with only simple poles [11]), whose poles are located at P_0 and P with residues -1 and 1, respectively. Such a differential exists and is unique up to addition of a holomorphic 1-form (an abelian differential of the first kind). Define the locally holomorphic function

$$f(z) = -\int_0^z \xi_P^{P_0} \,. \tag{5.4}$$

f has singularities at P_0 and P and its value also depends on the homology class of the integration path in $\Sigma_{\text{poly}} \setminus \{P_0, P\}$. The singularities are of type $-\log(z - Z_0)$ and $\log(z - Z)$, so f branches at these points and is multivalued on $\Sigma_{\text{poly}} \setminus \{P_0, P\}$, the differences of values at each point being integer multiples of $2\pi\mathrm{i}$. If we now set $H_P^{P_0}(z) := \mathrm{e}^{f(z)}$, we obtain a single-valued holomorphic function on Σ_{poly} which has a simple pole at P_0 and a simple zero at P. Let ϕ_0 be the holomorphic section of $\mathcal{L}_0$ with its zero at P_0. Then $H_P^{P_0}\phi_0|_{\Sigma_{\text{poly}}}$ has a zero at P but no longer has a zero at P_0. So $H_P^{P_0}\phi_0$ is a good candidate for the holomorphic section of $\mathcal{L}_P$.

At this point, we note that ϕ_0 and $H_P^{P_0}\phi_0$ are indeed sections of different line bundles, because $H_P^{P_0}$ does not extend to a meromorphic function on Σ when the edges of Σ_{poly} are identified (if it did extend to a meromorphic function with a single zero and a single pole, Σ would necessarily be rational with $g = 0$). By comparing $H_P^{P_0}$ at equivalent points of the paired edges $\{a_j, a_j^{-1}\}$ and $\{b_j, b_j^{-1}\}$, we see that the action of $H_P^{P_0}$ on ϕ_0 amounts to including additional transition factors e^{μ_j} and e^{ν_j} in the transition functions of the bundle $\mathcal{L}_0 \to \Sigma$, namely

$$e^{\mu_j} = e^{\oint_{b_j} \xi_P^{P_0}} \,, \quad e^{\nu_j} = e^{-\oint_{a_j} \xi_P^{P_0}} \,, \tag{5.5}$$

because a path in $\Sigma_{\rm poly}$ connecting equivalent points on edges $\{a_j, a_j^{-1}\}$ is homologous to the edge b_j, and similarly a path connecting equivalent points on edges $\{b_j, b_j^{-1}\}$ is homologous to the edge a_j^{-1}. These factors are constant, because they are independent of the choice of equivalent points on the paired edges, by Cauchy's theorem. They also have no dependence on the homology classes of the paths of integration, provided the paths avoid P_0 and P, since the residues at the poles of $\xi_P^{P_0}$ are ± 1. So we see that multiplication by this global object $H_P^{P_0}$ transforms holomorphic sections of $\mathcal{L}_0 \to \Sigma$ into holomorphic sections of $\mathcal{L}_P \to \Sigma$. In the language of the theory of holomorphic vector bundles over curves, $H_P^{P_0}$ describes the action (on holomorphic sections) of a double Hecke modification [12] that shifts the twisting of a holomorphic line bundle of degree one from P_0 to P.

Now recall that $\xi_P^{P_0}$ is not unique, because one may add to it any holomorphic 1-form without changing its poles and residues. Let us choose $\xi_P^{P_0}$ to be normalised by the condition that its integrals along all the edges a_j of $\Sigma_{\rm poly}$ vanish. In bundle terms, this corresponds to fixing some of the remaining freedom to perform holomorphic gauge transformations. For this normalised $\xi_P^{P_0}$,

$$e^{\mu_j} = e^{\oint_{b_j} \xi_P^{P_0}} \tag{5.6}$$

and $\nu_j \equiv 0 \,(\text{mod}\, 2\pi\text{i})$. A Riemann bilinear relation gives [11]

$$\oint_{b_j} \xi_P^{P_0} = -2\pi\text{i} \int_{P_0}^{P} \zeta_j \,, \tag{5.7}$$

where ζ_j is the jth canonical holomorphic 1-form. (This is derived by applying the residue theorem to a product of $\xi_P^{P_0}$ and the indefinite integral of ζ_j, integrated around the boundary of $\Sigma_{\rm poly}$.) Therefore

$$\text{e}^{\mu_j} = \text{e}^{-2\pi\text{i} \int_{P_0}^{P} \zeta_j} \,, \tag{5.8}$$

or, as an equation relating the exponents,

$$\mu_j = -2\pi\text{i} \int_{P_0}^{P} \zeta_j \quad (\text{mod } 2\pi\text{i}) \,. \tag{5.9}$$

Recalling (4.20), we see that, rather remarkably, $\frac{\text{i}}{2\pi}\mu_j$ are the coordinates χ_j of the image of P in $\text{Jac}(\Sigma)$ by the Abel–Jacobi map with base-point P_0. So, the holomorphic line bundle $\mathcal{L}_P$ with effective divisor P is obtained from the holomorphic line bundle $\mathcal{L}_0$ with effective divisor P_0 using a change in transition functions which is directly related to the Abel–Jacobi image of P. (This gives yet another characterisation of the Jacobian, as the moduli space of bundle transition functions — the so-called theta characteristics.)

For the second point of view, we return to the bundle $\mathcal{L}_0$ (trivialised over $\Sigma_{\rm poly}$), with connection 1-form a and holomorphic section ϕ_0 describing a vortex centred at P_0, and now look for the connection 1-form $a + \alpha$, where α is the restriction of a global, real harmonic 1-form, that gives a vortex centred at P. We first transform to the gauge discussed above, where $a = -\frac{i\tau}{2}\partial_z\mathcal{K}\,\mathrm{d}z$. In this gauge, the first vortex equation in our trivialisation is

$$\partial_{\bar z}\phi - \mathrm{i}\alpha_{\bar z}\phi = 0\,, \tag{5.10}$$

so the function $\Sigma_{\mathrm{poly}} \to \mathbb{C}$ representing the section ϕ is not holomorphic. Now recall that $\alpha = \omega + \overline{\omega}$, where $\omega = \omega(z)\,\mathrm{d}z$ extends to a global holomorphic 1-form. Let

$$\chi(\bar z) = -\overline{\int_0^z \omega}\,, \tag{5.11}$$

which is an anti-holomorphic function on Σ_{poly}. The complex gauge transformation $e^{\mathrm{i}\chi}$ makes $\alpha_{\bar z}$ vanish on Σ_{poly}. Its effect is therefore to transform the complete connection $\mathrm{d}_a - \mathrm{i}\alpha$ to holomorphic gauge, but at the expense of changing the bundle transition functions, as $e^{\mathrm{i}\chi}$ is not single-valued on Σ. The additional factors depend on the difference between the values of χ at equivalent points on the paired edges. Before writing these down, we need to allow for the possibility of a further holomorphic gauge transformation, leaving us in holomorphic gauge. We therefore modify (5.11) to

$$\chi(z,\bar z) = -\overline{\int_0^z \omega} + \int_0^z \theta \tag{5.12}$$

where θ extends to a global holomorphic 1-form. The additional factors defining the new bundle transition functions are e^{μ_j} and e^{ν_j}, where

$$\mu_j \;=\; \mathrm{i}\overline{\oint_{b_j} \omega} - \mathrm{i}\oint_{b_j}\theta\,, \tag{5.13}$$

$$\nu_j \;=\; -\mathrm{i}\overline{\oint_{a_j} \omega} + \mathrm{i}\oint_{a_j}\theta\,. \tag{5.14}$$

As before, these factors are constants, by Cauchy's theorem.

Now expand the 1-forms ω and θ in terms of the canonical basis of holomorphic 1-forms as

$$\omega = \sum_{k=1}^{g} \overline{c_k}\,\zeta_k\,, \quad \theta = \sum_{k=1}^{g} d_k\,\zeta_k\,. \tag{5.15}$$

Then, in terms of the periods (3.9) and (3.10),

$$\mu_j \;=\; \mathrm{i}\sum_{k=1}^{g}\overline{\Pi_{jk}}\,c_k - \mathrm{i}\sum_{k=1}^{g}\Pi_{jk}d_k\,, \tag{5.16}$$

$$\nu_j \;=\; -\mathrm{i}\,c_j + \mathrm{i}\,d_j\,. \tag{5.17}$$

Choose θ so that ν_j vanishes, i.e. $d_j = c_j$. Then

$$\mu_j = \mathrm{i}\sum_{k=1}^{g}(\overline{\Pi_{jk}} - \Pi_{jk})c_k = 2\sum_{k=1}^{g}(\mathrm{Im}\,\Pi)_{jk}\,c_k\,. \tag{5.18}$$

Since the matrix $\mathrm{Im}\,\Pi$ is invertible,

$$c_k = \frac{1}{2}\sum_{j=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\,\mu_j\,. \tag{5.19}$$

Now we can use our earlier result (5.9), determining the factors μ_j which produce a holomorphic section vanishing at P. This leads to our final expression for the coefficients,

$$c_k = -\pi\mathrm{i}\sum_{j=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\int_{P_0}^{P}\zeta_j\,. \tag{5.20}$$

The harmonic 1-form α that shifts a vortex from P_0 to P, in the unitary gauge, $\alpha = \sum_{k=1}^{g}(\overline{c_k}\,\zeta_k + c_k\,\overline{\zeta_k})$, is therefore

$$\alpha = \pi\mathrm{i}\sum_{j,k=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\left[\overline{\left(\int_{P_0}^{P}\zeta_j\right)}\,\zeta_k - \left(\int_{P_0}^{P}\zeta_j\right)\overline{\zeta_k}\right]. \tag{5.21}$$

This harmonic 1-form is exactly what one would expect from the discussion of the Jacobian in the previous section. The degree one divisor P corresponds to the point on $\mathrm{Jac}(\Sigma)$ with coordinates $\chi_j = \int_{P_0}^{P}\zeta_j$, by (4.20). The related coordinates c_j are therefore

$$c_j = -\pi\mathrm{i}\sum_{k=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\int_{P_0}^{P}\zeta_k\,, \tag{5.22}$$

by the inverse of (4.9). This agrees with (5.20).

Using the expression for α, we can find the electric field associated with a moving vortex. Let us first note the infinitesimal change in α that occurs when the vortex is shifted from P to a neighbouring point which we call $P+\delta P$. Let the coordinates of P and $P+\delta P$ be Z and $Z+\delta Z$. Then the change in α is

$$\delta\alpha = \pi\mathrm{i}\sum_{j,k=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\left[\overline{\zeta_j(Z)}\,\delta\bar{Z}\,\zeta_k - \zeta_j(Z)\,\delta Z\,\overline{\zeta_k}\right]. \tag{5.23}$$

Therefore, the 1-form electric field of a moving vortex is

$$e = \pi\mathrm{i}\sum_{j,k=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\left[\overline{\zeta_j(Z)}\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t}\zeta_k - \zeta_j(Z)\frac{\mathrm{d}Z}{\mathrm{d}t}\overline{\zeta_k}\right], \tag{5.24}$$

with Hodge dual

$$*\,e = \pi\sum_{j,k=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\left[\overline{\zeta_j(Z)}\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t}\zeta_k + \zeta_j(Z)\frac{\mathrm{d}Z}{\mathrm{d}t}\overline{\zeta_k}\right], \tag{5.25}$$

so the kinetic energy of the moving vortex is

$$
\begin{aligned}
T &= \frac{1}{2}\int_\Sigma e \wedge *e && (5.26)\\
&= \frac{1}{2}\pi^2 \mathrm{i} \sum_{j,k,m,n=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk} (\mathrm{Im}\,\Pi)^{-1}_{mn} \Big[\overline{\zeta_j(Z)}\zeta_m(Z) \int_\Sigma \zeta_k \wedge \overline{\zeta_n} \\
&\qquad - \zeta_j(Z)\overline{\zeta_m(Z)} \int_\Sigma \overline{\zeta_k} \wedge \zeta_n \Big] \frac{\mathrm{d}Z}{\mathrm{d}t}\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} && (5.27)\\
&= 2\pi^2 \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \zeta_j(Z)\overline{\zeta_k(Z)}\, \frac{\mathrm{d}Z}{\mathrm{d}t}\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} && (5.28)
\end{aligned}
$$

using the integral (3.19) and the symmetry of $\mathrm{Im}\,\Pi$. By dropping a factor $\frac{1}{2}$, we obtain the metric on the one-vortex moduli space $\mathcal{M}_1$ (with complex coordinate Z)

$$\mathrm{d}s^2 = 4\pi^2 \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \zeta_j(Z)\overline{\zeta_k(Z)}\, \mathrm{d}Z\mathrm{d}\bar{Z}\,. \qquad (5.29)$$

The metric (5.29) is called a Bergman metric on $\mathcal{M}_1 \cong \Sigma$ [20, 17], as we explain in Appendix A. Its curvature is negative except, possibly, at a finite number of points where it is zero. If we use an orthonormal basis η_j, rather than the canonical basis ζ_j, for the holomorphic 1-forms, satisfying

$$\int_\Sigma \eta_j \wedge \overline{\eta_k} = -2i\delta_{jk}\,, \qquad (5.30)$$

then the metric becomes

$$\mathrm{d}s^2 = 4\pi^2 \sum_{j=1}^{g} \eta_j(Z)\overline{\eta_j(Z)}\, \mathrm{d}Z\mathrm{d}\bar{Z}\,. \qquad (5.31)$$

Although this formula looks simpler, it describes the same Bergman metric as before, since it is related to (5.29) by pull-back (an isometry).

The Bergman metric (5.29) is also simply the metric (4.13) on the Jacobian, restricted to $\mathcal{M}_1$. In fact, the coordinates of the image of P in $\mathrm{Jac}(\Sigma)$ under the Abel–Jacobi map are

$$\chi_j(Z) = \int_{Z_0}^{Z} \zeta_j\,, \qquad (5.32)$$

so their differentials are $\mathrm{d}\chi_j = \zeta_j(Z)\,\mathrm{d}Z$, and the restriction of (4.13) to $\mathcal{M}_1$ is

$$\mathrm{d}s^2 = 4\pi^2 \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \zeta_j(Z)\overline{\zeta_k(Z)}\, \mathrm{d}Z\mathrm{d}\bar{Z}\,. \qquad (5.33)$$

Equivalently, the Bergman metric is obtained from the flat metric (3.20) by restricting to $\mathcal{M}_1$ using (5.22).

One of the most interesting properties of $\mathcal{M}_1$ is its total volume (area in this case). This is

$$\mathrm{Vol}(\mathcal{M}_1) = 2\pi^2 \mathrm{i} \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk} \int_\Sigma \zeta_j \wedge \overline{\zeta_k} = 4\pi^2 \sum_{j=1}^{g} I_{jj} = 4\pi^2 g\,, \tag{5.34}$$

where I is the $g \times g$ unit matrix, with trace g. Recall that the ambient space $\mathrm{Jac}(\Sigma)$ has volume $(2\pi)^{2g}$.

6 Comparison with Samols' formula

A general formula for the metric on the N-vortex moduli space $\mathcal{M}_N$ was established by Samols [28, 24]. Samols showed, working with the linearised vortex equations and the kinetic part of the lagrangian (2.8), that the metric, which is defined as an integral over Σ, can be reduced to a local form, depending on data at the vortex centres. We shall now show that our formula for the metric of one dissolving vortex is consistent with the $\tau \to \frac{4\pi}{A}$ limit of Samols' formula for the metric on $\mathcal{M}_1 \cong \Sigma$.

Let ϕ represent the section of $\mathcal{L} \to \Sigma$ in a trivialising patch which, together with the connection, solves the vortex equations in unitary gauge for one vortex centred at Z; this ϕ has a simple zero at Z. The gauge invariant quantity $\log|\phi|^2 - \log|z - Z|^2$ has a Taylor expansion around Z [16]

$$\log|\phi|^2 - \log|z-Z|^2 = a + \frac{1}{2}\bar{b}(z-Z) + \frac{1}{2}b(\bar{z}-\bar{Z}) + \cdots \tag{6.1}$$

where $a, b, \cdots$ depend on Z. [N.B. this a is a real function of Z; it is not the connection.] Then Samols' formula for the metric on $\mathcal{M}_1$ (not dividing out by the mass of the vortex) is

$$\mathrm{d}s^2 = \pi\left(\tau\Omega + 2\frac{\partial b}{\partial Z}\right) \mathrm{d}Z\mathrm{d}\bar{Z}\,, \tag{6.2}$$

where Ω and $\frac{\partial b}{\partial Z}$ (which is real) are evaluated at Z. This formula shows that the metric on $\mathcal{M}_1$ is in the same conformal class as the original metric g_Σ on Σ.

We now show that in the dissolving limit, (6.2) reduces to the Bergman metric (5.29). The key is to look at the complex gauge transformation that converts ϕ to holomorphic gauge. By combining (5.1) and (5.11), we see that

$$\mathrm{e}^{\left\{\frac{\tau}{4}\mathcal{K} - \mathrm{i}\overline{\int_0^z \omega}\right\}}\,\phi =: \phi_{\mathrm{hol}} \tag{6.3}$$

is a holomorphic function of z vanishing at $z = Z$. Apart from its normalisation factor, it depends on Z holomorphically. So it has an expansion about Z,

$$\phi_{\mathrm{hol}}(z) = A(Z,\bar{Z})\big(z - Z + B(Z)(z-Z)^2 + \cdots\big)\,. \tag{6.4}$$

Also, using (5.20),

$$\omega = \pi\mathrm{i} \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk} \left(\overline{\int_{Z_0}^{Z} \zeta_j}\right) \zeta_k \,, \tag{6.5}$$

so, combining these formulae with (6.3), we find

$$\phi = A(Z,\bar{Z})\big(z - Z + B(Z)(z-Z)^2 + \cdots\big)\mathrm{e}^{\left\{-\frac{\tau}{4}\mathcal{K} + \pi\sum_{j,k=1}^{g}(\mathrm{Im}\,\Pi)^{-1}_{jk}\left(\int_{Z_0}^{Z}\zeta_j\right)\overline{\int_0^z \zeta_k}\right\}} \tag{6.6}$$

and therefore

$$\begin{aligned}\log|\phi|^2 \quad - \quad & \log|z-Z|^2 \\ & = -\frac{\tau}{2}\mathcal{K} + \pi \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk} \left[\left(\int_{Z_0}^{Z} \zeta_j\right) \overline{\int_0^z \zeta_k} + \left(\overline{\int_{Z_0}^{Z} \zeta_j}\right) \int_0^z \zeta_k \right] \\ & + \log|A(Z,\bar{Z})|^2 + B(Z)(z-Z) + \overline{B(Z)}(\bar{z}-\bar{Z}) + \cdots \,. \end{aligned} \tag{6.7}$$

The Taylor coefficient $\frac{1}{2}b$ is the $\bar{z}$-derivative of this, evaluated at Z,

$$\frac{1}{2}b = -\frac{\tau}{2}\partial_{\bar{Z}}\mathcal{K} + \pi \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk} \left(\int_{Z_0}^{Z} \zeta_j\right) \overline{\zeta_k(Z)} + \overline{B(Z)} \,. \tag{6.8}$$

Here all quantities are functions of Z and $\bar{Z}$. Taking the Z-derivative, multiplying by 4, and evaluating again at Z, gives

$$2\frac{\partial b}{\partial Z} = -2\tau\partial_Z\partial_{\bar{Z}}\mathcal{K} + 4\pi \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk} \zeta_j(Z)\overline{\zeta_k(Z)} \tag{6.9}$$

$$= -\tau\Omega(Z,\bar{Z}) + 4\pi \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk} \zeta_j(Z)\overline{\zeta_k(Z)} \,, \tag{6.10}$$

and hence Samols' formula (6.2) reduces to

$$\mathrm{d}s^2 = 4\pi^2 \sum_{j,k=1}^{g} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \zeta_j(Z)\overline{\zeta_k(Z)}\, \mathrm{d}Z\mathrm{d}\overline{Z} \,, \tag{6.11}$$

the Bergman metric. The conformal factor Ω cancels in this limit.

In general, Samols' formula cannot be used to calculate the vortex metric away from the dissolving limit, because the coefficient b is not known explicitly. However, the total volume of the moduli space, $\mathrm{Vol}(\mathcal{M}_N)$, is known exactly [23, 24]. For N vortices on a compact surface of genus g, with $N \leq g$,

$$\mathrm{Vol}(\mathcal{M}_N) = \pi^N \sum_{n=0}^{N} \frac{(4\pi)^n\, (\tau A - 4\pi N)^{N-n}\, g!}{n!(N-n)!(g-n)!} \,. \tag{6.12}$$

In the dissolving limit, where $\tau \to \frac{4\pi N}{A}$, the leading term is that with $n = N$, giving

$$\mathrm{Vol}(\mathcal{M}_N) = (4\pi^2)^N \frac{g!}{N!(g-N)!}\,, \tag{6.13}$$

with subleading corrections of order $\tau A - 4\pi N$. For $N = 1$ (and $g \geq 1$) this volume is $4\pi^2 g$, agreeing with (5.34). For $N = g$, the volume in the dissolving limit is $(4\pi^2)^g$, which is the volume of $\mathrm{Jac}(\Sigma)$.

Note that for $N = 1$, the exact result is

$$\mathrm{Vol}(\mathcal{M}_1) = 4\pi^2 g + \pi(\tau A - 4\pi)\,. \tag{6.14}$$

This shows that the calculation leading to (5.34) only gives the leading term in $\mathrm{Vol}(\mathcal{M}_1)$ in the dissolving limit.

7 Geometry of dissolving multivortices

In this section, we address the problem of understanding the dissolving limit $\tau \to \frac{4\pi N}{A}$ for multivortices, assuming that the two inequalities

$$1 < N < g \tag{7.1}$$

hold. As we have already stated, the image of the moduli space of vortices $\mathcal{M}_N \cong \mathrm{Sym}^N\Sigma$ under the Abel–Jacobi map (well defined after a base point $P_0 \in \Sigma$ is chosen) is a subvariety W_N of the Jacobian, but this map is in general not an embedding, in contrast to the $N = 1$ case. In analogy with our results in Section 5, one would expect the relevant metric (in the dissolving limit) to be a pull-back of the flat metric on $\mathrm{Jac}(\Sigma)$ coming from the polarisation. However, at critical points of the Abel–Jacobi map the pull-back degenerates, as we shall explain.

As before, we define the dissolving limit as the situation where the connection d_a is projectively flat on a line bundle $\mathcal{L} \to \Sigma$ of degree N, thus approximating the second vortex equation (2.11) by equation (3.2) with τ close to $\frac{4\pi N}{A}$. The first vortex equation (2.10) in turn is replaced by the constraint that we only consider holomorphic line bundles of degree N on Σ that are represented by some effective divisor — in other words, line bundles which admit at least one (nontrivial) global holomorphic section. This picks out precisely the locus W_N inside $\mathrm{Jac}(\Sigma)$.

To understand the limiting vortex metric on $\mathcal{M}_N$, we can follow essentially the same steps as in the calculation of Section 5. Suppose that we have two effective divisors of degree N,

$$D_0 = \sum_{l=1}^{N} P_l \qquad \text{and} \qquad D = \sum_{l=1}^{N} Q_l\,, \tag{7.2}$$

where the points P_l and Q_l are not necessarily distinct. One can pair points in D_0 with points in D in some arbitrary order, construct a differential of the third kind on Σ with poles of residues ± 1, respectively, on the points of each pair, and then add all these N differentials to obtain a differential of the third kind $\xi_D^{D_0}$, uniquely defined up to the addition of a differential of the first kind. Setting as before $f(z) := -\int_0^z \xi_D^{D_0}$, where the integration is along a path inside $\Sigma_{\rm poly}$ avoiding the points of both D_0 and D, we obtain a multivalued function on $\Sigma_{\rm poly} \setminus \{P_1, \ldots, P_N, Q_1, \ldots, Q_N\}$ which branches logarithmically over the points of the divisors; but the object $H_D^{D_0}(z) := {\rm e}^{f(z)}$ is single-valued on $\Sigma_{\rm poly}$, with divisor of zeroes minus divisor of poles $(H_D^{D_0}) = D - D_0$. We can use the freedom of adding a holomorphic 1-form to $\xi_D^{D_0}$ to make its a-periods vanish, and then $H_D^{D_0}$ is defined uniquely from the divisors D_0 and D.

All of this mimics the construction in Section 5 except for one issue: over Σ, it may turn out that $H_D^{D_0}$ is globally defined as a meromorphic function (and hence describes a trivial Hecke modification, relating two holomorphic sections of the same bundle). This occurs precisely when D_0 and D are effective divisors belonging to the same linear equivalence class. In this case, the two divisors sit on the same fibre of the Abel–Jacobi map, that is, they represent zeroes of two linearly independent, nontrivial holomorphic sections of the same holomorphic line bundle over Σ, and the extra transition factors ${\rm e}^{\mu_j}$ associated to $\xi_D^{D_0}$ via (5.5) are trivial. (The factors ${\rm e}^{\nu_j}$ are already trivial from the normalisation of a-periods.) Using the bilinear relation

$$\oint_{b_j} \xi_D^{D_0} = -2\pi{\rm i} \int_{D_0}^{D} \zeta_j := 2\pi{\rm i} \sum_{l=1}^{N} \left(\int_{P_0}^{P_l} \zeta_j - \int_{P_0}^{Q_l} \zeta_j \right) , \tag{7.3}$$

where the paths of integration are contained in $\Sigma_{\rm poly}$, we can express the transition factors across a-edges as

$$\mu_j = -2\pi{\rm i} \int_{D_0}^{D} \zeta_j \pmod{2\pi{\rm i}} . \tag{7.4}$$

Our second point of view of the divisor shift also extends to the multivortex case, and the harmonic 1-form that describes the change of the connection corresponding to the divisor shift can be readily computed:

$$\alpha = \pi{\rm i} \sum_{j,k=1}^{g} ({\rm Im}\,\Pi)^{-1}_{jk} \left[\overline{\left(\int_{D_0}^{D} \zeta_j \right)} \zeta_k - \left(\int_{D_0}^{D} \zeta_j \right) \overline{\zeta_k} \right] . \tag{7.5}$$

We now regard the divisor D_0 as fixed, and the divisor D as giving the centres of an N-vortex solution, i.e. a point of $\mathcal{M}_N$. We assign each point Q_l in the divisor D the coordinate $z(Q_l) =: Z_l$. Then, to shift the vortices to neighbouring points $Q_l + \delta Q_l$, with a corresponding shift in coordinates $z(Q_l + \delta Q_l) - z(Q_l) =: \delta Z_l$, we calculate the infinitesimal change of the harmonic 1-form to be

$$\delta\alpha = \pi{\rm i} \sum_{j,k=1}^{g} \sum_{l=1}^{N} ({\rm Im}\,\Pi)^{-1}_{jk} \left[\overline{\zeta_j(Z_l)}\, \delta\overline{Z_l}\, \zeta_k - \zeta_j(Z_l)\, \delta Z_l\, \overline{\zeta_k} \right] , \tag{7.6}$$

and this gives rise, when the vortex motion is dynamical, to the electric field

$$e = \pi \mathrm{i} \sum_{j,k=1}^{g} \sum_{l=1}^{N} (\mathrm{Im}\,\Pi)^{-1}_{jk} \left[\overline{\zeta_j(Z_l)} \frac{\mathrm{d}\overline{Z_l}}{\mathrm{d}t} \zeta_k - \zeta_j(Z_l) \frac{\mathrm{d}Z_l}{\mathrm{d}t} \overline{\zeta_k} \right] . \tag{7.7}$$

(We are making the simplifying assumption that the divisor D is reduced, so that the coordinates Z_l are well defined, but this is not essential.) The kinetic energy associated with the multivortex motion is therefore

$$T = \frac{1}{2} \int_{\Sigma} e \wedge *e = 2\pi^2 \sum_{j,k=1}^{g} \sum_{l,m=1}^{N} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \zeta_j(Z_l) \overline{\zeta_k(Z_m)}\, \frac{\mathrm{d}Z_l}{\mathrm{d}t} \frac{\mathrm{d}\overline{Z_m}}{\mathrm{d}t} . \tag{7.8}$$

As for $N = 1$, this resembles the kinetic energy for a particle moving on the complex manifold $\mathcal{M}_N = \mathrm{Sym}^N \Sigma$ with local complex coordinates $Z_1, \ldots, Z_N$ (which degenerate at points where the vortex centres coincide), and whose metric is the restriction of the flat Kähler metric on $\mathrm{Jac}(\Sigma)$. The Kähler $(1,1)$-form on $\mathcal{M}_N$ is given locally by

$$\omega_{\mathrm{diss}} = 2\pi^2 \mathrm{i} \sum_{j,k=1}^{g} \sum_{l,m=1}^{N} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \zeta_j(Z_l) \overline{\zeta_k(Z_m)}\, \mathrm{d}Z_l \wedge \mathrm{d}\overline{Z_m} \tag{7.9}$$

and is the pull-back, by the Abel–Jacobi map on degree N divisors, of the symplectic structure Ω_{J} on $\mathrm{Jac}(\Sigma)$ associated to the polarisation.

A crucial difference, however, from the $N = 1$ case is that for $N > 1$ this $(1,1)$-form is typically degenerate on some locus where the rank drops down. This is because the Abel–Jacobi map typically has critical points when $N > 1$, and one is left with a degenerating metric, for which the existence and uniqueness of geodesics associated to any point and direction may not hold on this locus. (The global 2-form ω_{diss} is still closed, as it is the pull-back of the closed 2-form Ω_{J}.) So ω_{diss} only defines a Riemannian structure over the set of regular points, which is an open subset of $\mathrm{Sym}^N \Sigma$. Effective divisors on this subset represent line bundles that do not admit independent holomorphic sections (with different divisors of zeroes). In contrast, in the language of algebraic geometry [1], ω_{diss} is degenerate over "special" effective divisors, which run or move in nontrivial linear systems. The directions of degeneracy on $\mathrm{Sym}^N \Sigma$ are precisely those along the complete linear system associated with a special divisor D, that is, motions from D into effective divisors linearly equivalent to D. Physically, in the dissolving limit of vortices, motion at finite speeds in these directions occurs with no electric field and hence no kinetic energy. More usefully, one can say that for vortex motion with positive kinetic energy, motion along these linear systems could occur infinitely fast in the dissolving limit, so that (metrically) these linear systems collapse to points.

The sets of special divisors D, sitting on exceptional fibres of the Abel–Jacobi map, are complex projective spaces whose dimension ℓ can be related to sheaf cohomology via the Riemann–Roch theorem [14]:

$$\begin{aligned} \ell &= \dim_{\mathbb{C}} \mathbb{P}(H^0(\Sigma, \mathcal{O}(D))) && (7.10)\\ &= \dim_{\mathbb{C}} H^1(\Sigma, \mathcal{O}(D)) + \deg D - g + 1 - 1 && (7.11)\\ &= \dim_{\mathbb{C}} H^1(\Sigma, \mathcal{O}(D)) + N - g\,. && (7.12) \end{aligned}$$

The divisor D is special precisely when the following strict inequality holds:

$$\dim_{\mathbb{C}} H^1(\Sigma, \mathcal{O}(D)) = \dim_{\mathbb{C}} H^0(\Sigma, \mathcal{O}(K_\Sigma - D))^* > g - N\,. \quad (7.13)$$

The relations among the geometry of linear systems on Σ, exceptional fibres of the Abel–Jacobi map, and singularities of the subvariety $W_N \subset \mathrm{Jac}(\Sigma)$ are summarised in the beautiful Riemann–Kempf theorem, which essentially says that a point $w \in W_N$ is a singularity of multiplicity $\binom{g-N+\ell}{\ell}$, its tangent cone being the union of images of the tangent spaces $\mathrm{T}_D\mathrm{Sym}^N\Sigma$ by the differential of the Abel–Jacobi map, where the effective divisor D runs over the complete linear system associated with (i.e. is in the fibre over) w. The subvarieties $W_N \subset \mathrm{Jac}(\Sigma)$ are locally given by determinantal equations, and their structure is an important topic in the modern algebraic geometry of curves [1].

To illustrate more concretely the behaviour of the Abel–Jacobi map for $N > 1$ and the structure of its image W_N as a complex N-fold inside the Jacobian, we briefly describe the possible behaviours at low vortex number. The qualitative behaviour at a given genus depends crucially on the complex structure of Σ, e.g. on whether Σ is hyperelliptic, and on what kind of linear systems the geometry of Σ allows. For more information, the reader is referred to the textbooks [1, 25].

Example 1: For $N = 2$, the lowest-genus case where (7.1) is satisfied is $g = 3$. In this situation there are two subcases. If Σ is a nonhyperelliptic curve (the generic situation), the image $W_2 \subset \mathrm{Jac}(\Sigma)$ of the Abel–Jacobi map is smooth, and just a copy of the moduli space $\mathcal{M}_2 = \mathrm{Sym}^2\Sigma$ inside the Jacobian. In fact, this is the only case with $N > 1$ where the 2-form (7.9) is globally nondegenerate, and the dissolving limit metric is regular everywhere. If $g = 3$ but Σ is hyperelliptic, then W_2 already has a singularity. W_2 is the singular complex surface got from the smooth surface $\mathrm{Sym}^2\Sigma$ by blowing down a copy of $\mathbb{CP}^1$ to a point, which is a double point in W_2 [27]. The exceptional $\mathbb{CP}^1$ fibre that is blown down is the pencil of degree two divisors that are orbits of the hyperelliptic involution (a g^1_2); the space of orbits is the quotient of Σ by the hyperelliptic involution, which is a $\mathbb{CP}^1$ that embeds in $\mathrm{Sym}^2\Sigma$ holomorphically with noncontractible image. This exceptional fibre has an analogue for any moduli space of 2-vortices on a hyperelliptic curve Σ [6].

Example 2: If $N = 3$, the simplest situation requires $g = 4$. Then there are three subcases. If Σ is not hyperelliptic, one can show that it can be obtained as an intersection of a quadric Q and a cubic C in $\mathbb{CP}^3$. The first subcase is when Q is smooth, hence biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then C meets each projective line of the form $\{P_1\} \times \mathbb{CP}^1$ or $\mathbb{CP}^1 \times \{P_2\}$ in Q at three points, so $\Sigma = Q \cap C$ projects to either of the two $\mathbb{CP}^1$ factors of Q as a 3-cover. The pre-images of points in $\mathbb{CP}^1$ by the two projections form effective divisors of degree 3 moving in two pencils (i.e. parametrised by two projective lines), and describe two copies F_1, F_2 of $\mathbb{CP}^1$ inside $\mathrm{Sym}^3\Sigma$, which are g_3^1's on Σ . These are the exceptional fibres of the Abel–Jacobi map. Physically, there is a metric singularity in the dissolving limit if the three vortices are centred at the points of these particular divisors. The image W_3 can be obtained by blowing down these rational curves F_1, F_2 to two points, which are ordinary double points of the 3-fold. The second subcase is when Σ is not hyperelliptic, hence $\Sigma = Q \cap C$ as before, but now Q is singular (a quadric cone); then Q can be described as a family of projective lines parametrised by a $\mathbb{CP}^1$ and all meeting at the singular point. Each line in the family again meets C at three points, and so Σ inherits one pencil of degree 3 effective divisors (a g_3^1), which is the only exceptional fibre of the Abel–Jacobi map. The image W_3 in this case is again got by blowing down this $\mathbb{CP}^1$ fibre, and this results in a double point in the 3-fold which has higher multiplicity. The third and last subcase occurs when Σ is hyperelliptic. The exceptional fibres here form a complex surface inside $\mathrm{Sym}^3\Sigma$, namely, the locus of effective divisors on Σ consisting of adding any point of Σ to the $\mathbb{CP}^1$ of hyperelliptic orbits described in the previous example; this can be described as a family of pencils (i.e. g_3^1's) parametrised by Σ. Then W_3 is obtained from $\mathrm{Sym}^3\Sigma$ by blowing down this surface to a curve isomorphic to Σ.

8 Dissolving multivortices near a singularity

To understand the behaviour of the geodesic flow close to a singularity, we shall now analyse in detail the simplest situation, which occurs in the scattering of two vortices on a hyperelliptic Riemann surface of genus three.

We start by recalling that the image W_2 of the Abel–Jacobi map for degree two effective divisors

$$\mathrm{AJ}_2 : \mathrm{Sym}^2\Sigma \longrightarrow \mathrm{Jac}(\Sigma) \tag{8.1}$$

on a hyperelliptic curve Σ with $g = 3$ has a double point, whose blow-up is the exceptional fibre in $\mathrm{Sym}^2\Sigma$, which is a projective line (see Example 1 in Section 7). Since we are only interested in the leading local behaviour near this critical locus, we will not need to use theta-functions, and will instead take the standard algebraic model

$$t_3^2 = t_1 t_2 \tag{8.2}$$

for the double point, using local coordinates $t_i : U \to \mathbb{C}$ centred at the singularity; so (8.2) gives a local equation for the image of $W_2 \cap U \subset \mathrm{Jac}(\Sigma)$ under the coordinate system, which we may regard as a hypersurface W_2' in an open neighbourhood U' of the origin of $\mathbb{C}^3$. Now we blow up $(0,0,0) \in U'$, to obtain a 3-fold $\widetilde{U}'$ together with a holomorphic map $\pi : \widetilde{U'} \to U'$ which has $\pi^{-1}(0,0,0) = \mathbb{P}(\mathrm{T}_{(0,0,0)}U') \cong \mathbb{CP}^2$ but is one-to-one everywhere else. We recall how this is constructed [2]. The manifold $\widetilde{U}'$ can be regarded as the subset of $U' \times \mathbb{CP}^2$ defined by the incidence relation

$$t_i v_j = t_j v_i \qquad \text{for all} \quad i, j \in \{1, 2, 3\} \tag{8.3}$$

where v_j are homogeneous coordinates on the projectivisation $\mathbb{CP}^2$ of the tangent space at the origin, and the map π is simply the projection $\mathrm{pr}_{U'}$ onto the first factor. In the open set of $U' \times \mathbb{CP}^2$ where $v_3 \neq 0$, for example, $\widetilde{U}'$ is described by the system of equations

$$t_1 = \frac{v_1}{v_3} t_3, \quad t_2 = \frac{v_2}{v_3} t_3 \tag{8.4}$$

which has constant rank 2, and this determines a 3-dimensional submanifold. Since the incidence relation (8.3) is trivially satisfied for $(t_1, t_2, t_3) = (0, 0, 0)$, we get indeed the whole of the $\mathbb{CP}^2$ factor as exceptional fibre.

Imposing the equation (8.2), we obtain a surface $\widetilde{W_2'} \cap \widetilde{U}'$ which is smooth; the singularity is replaced by the conic $v_3^2 = v_1 v_2$ in the exceptional fibre $\mathbb{CP}^2$, which is itself a projective line $\mathbb{CP}^1$, and the restriction

$$\pi|_{\widetilde{W_2'} \cap \widetilde{U}'} : \widetilde{W_2'} \cap \widetilde{U}' \to W_2' \cap U' \tag{8.5}$$

provides a local resolution of the double point on the surface. To find the resolution map explicitly, we should use a system of two local coordinates where a dense subset of the exceptional fibre is visible; for example, an affine coordinate on the $\mathbb{CP}^1$ factor, say $q = \frac{v_3}{v_1}$, together with one of the coordinates on the first factor, say $p = t_1$. In these coordinates, the projection is given by

$$(p, q) \mapsto (t_1, t_2, t_3) = (p, pq^2, pq) \in U' \,. \tag{8.6}$$

Working on such local patches, it is not hard to see that the projection of $\widetilde{W_2'} \cap \widetilde{U}'$ onto the second factor of $\widetilde{U}' \times \mathbb{CP}^2$ can be understood as a restriction of the standard projection

$$\mathrm{T}^{*(1,0)}\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \tag{8.7}$$

to a neighbourhood of the (image of the) zero section, which gives a very concrete picture of the resolution. The exceptional fibre of AJ_2 is identified with the zero section, parametrised by q, and our complex coordinate p parametrises the cotangent fibres.

We want to understand the effect of pulling back a Kähler metric on U' to the blow-up $\widetilde{U}'$, and in particular the behaviour of the geodesic flow near the exceptional fibre where the metric becomes degenerate. The Kähler metric we consider is the standard euclidean metric on U', $g_0 = |\mathrm{d}t_1|^2 + |\mathrm{d}t_2|^2 + |\mathrm{d}t_3|^2$, as the qualitative behaviour of the flow will not depend on anisotropy factors. Pulling back to $\widetilde{U}'$ we obtain

$$\begin{aligned} \widetilde{g} = \pi^* g_0 \quad &= \quad (1 + |q|^2 + |q|^4)\mathrm{d}p\,\mathrm{d}\bar{p} + |p|^2(1 + 4|q|^2)\mathrm{d}q\,\mathrm{d}\bar{q} \\ &\quad + \bar{p}q(1 + 2|q|^2)\mathrm{d}p\,\mathrm{d}\bar{q} + p\bar{q}(1 + 2|q|^2)\mathrm{d}q\,\mathrm{d}\bar{p} \,. \end{aligned} \tag{8.8}$$

As expected, this tensor defines a Kähler metric in the complement of the complex line with equation $p = 0$, but its rank (over $\mathbb{R}$) drops from 4 to 2 on this line, which corresponds to an affine piece of the exceptional $\mathbb{CP}^1$ fibre of the Abel–Jacobi map. To understand the geodesic flow, we should first compute the Christoffel symbols. For a Kähler metric this calculation simplifies, and moreover Christoffel symbols mixing holomorphic and anti-holomorphic directions automatically vanish [2]. We find:

$$\widetilde{\Gamma}^q_{pq} = \widetilde{\Gamma}^q_{qp} = \frac{1}{p}, \qquad \widetilde{\Gamma}^p_{pq} = \widetilde{\Gamma}^p_{qp} = \widetilde{\Gamma}^q_{pp} = \widetilde{\Gamma}^p_{pp} = 0\,, \tag{8.9}$$

$$\widetilde{\Gamma}^p_{qq} = -\frac{2p\bar{q}^2}{1+4|q|^2+|q|^4}\,, \qquad \widetilde{\Gamma}^q_{qq} = \frac{2\bar{q}(2+|q|^2)}{1+4|q|^2+|q|^4}\,. \tag{8.10}$$

These lead to the following geodesic equations:

$$\ddot{p} - \frac{2p\bar{q}^2\dot{q}^2}{1+4|q|^2+|q|^4} = 0\,, \tag{8.11}$$

$$\ddot{q} + \frac{2\dot{p}\dot{q}}{p} + \frac{2\bar{q}(2+|q|^2)\dot{q}^2}{1+4|q|^2+|q|^4} = 0\,, \tag{8.12}$$

where the derivatives are with respect to a parameter s, say.

An obvious integral of motion is the kinetic energy of the geodesic flow (up to a constant factor),

$$(1+|q|^2+|q|^4)|\dot{p}|^2 + |p|^2(1+4|q|^2)|\dot{q}|^2 + (1+2|q|^2)(\bar{p}q\dot{p}\dot{\bar{q}} + p\bar{q}\dot{\bar{p}}\dot{q})\,. \tag{8.13}$$

The conservation of this quantity already implies that the motion on the exceptional fibre $\mathbb{CP}^1$ (parametrised by the coordinate q) is suppressed in its tangent directions: as $p \to 0$, all the kinetic energy must be transferred to motion along the transverse directions parametrised by the complex coordinate p. In particular, any geodesic intersecting the exceptional fibre must do so at isolated points of the fibre.

To demonstrate that there are indeed geodesics crossing the exceptional fibre, we note that the geodesic equations above are satisfied by the rays of the tangent cone to W_2', i.e. paths of the form $s \mapsto (p,q) = (c_1 s, c_2)$ for constants $c_1 \in \mathbb{C}^*$ and $c_2 \in \mathbb{C}$. These correspond to lifts of real straight lines on U' towards the singularity, which hit a point on the exceptional fibre corresponding to the complex tangent direction their velocity represents, and then continue along the same real direction. Since the exceptional fibre is reached in finite time, the metric on the complement of the exceptional fibre in $\widetilde{W}_2$ is not complete.

In fact, such straight ray geodesics are the only geodesics reaching the exceptional fibre $\mathbb{CP}^1$. To see this, note first that, as long as $\dot{p}$ is not constant, (8.11) implies that $\dot{q}$ cannot be zero. Dividing equation (8.12) by $\dot{q}$ (assumed to be nonzero) and extracting the real part of the resulting equation, we obtain a new differential equation,

$$\frac{\ddot{q}}{\dot{q}} + \frac{\ddot{\bar{q}}}{\dot{\bar{q}}} + \frac{2\dot{p}}{p} + \frac{2\dot{\bar{p}}}{\bar{p}} + \frac{2(2+|q|^2)(\bar{q}\dot{q} + q\dot{\bar{q}})}{1+4|q|^2+|q|^4} = 0, \tag{8.14}$$

which can be integrated to conclude that

$$(1 + 4|q|^2 + |q|^4)|p|^4|\dot{q}|^2 \tag{8.15}$$

is another integral of motion. Thus for p to reach zero, $\dot{q}$ would have to blow up, which cannot happen. Initial conditions that try to reach the exceptional fibre with initial velocities having nontrivial tangent component along the $\mathbb{CP}^1$ will be forced to flow rapidly around this 2-sphere as they approach it transversely.

In terms of vortex motion, the effect of the singularity is that motion along the special linear system is suppressed. So whenever two vortices reach points on the surface that are related by the hyperelliptic involution, they will be unable to move to neighbouring pairs of points that are also related by the involution. In particular, it will be impossible to make vortices collide head-on onto a Weierstraß point of the surface: these are precisely the branch points of the two-fold holomorphic branched cover $\Sigma \to \mathbb{CP}^1$, and geodesics through them are tangentially preserved by the hyperelliptic involution near the branch point.

A On the notion of Bergman metric

Here we give a brief account of the notion of Bergman metric, around which there is occasionally some confusion in the literature. The name is always used to refer to a metric on a complex manifold that is invariant under local holomorphic maps, but it may refer to distinct objects, as we shall now review.

For most authors, the Bergman metric arises in the theory of several complex variables and is defined on complex domains using the (intrinsic) Bergman kernel that we now describe [19]. Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$, i.e. a connected open subset, where global complex coordinates $z = (z_1, \ldots, z_n)$ are defined. The Hilbert space of square-integrable functions on $\mathcal{D}$ (with the L^2 inner product

$$\langle f_1, f_2 \rangle_{L^2} := \int_{\mathcal{D}} f_1(z)\overline{f_2(z)}\, \mathrm{d}\mu(z) \tag{A.1}$$

coming from the Lebesgue measure μ on $\mathbb{C}^n$) has a distinguished subspace, namely, the set of L^2-functions f that are holomorphic. This subspace is closed and thus inherits a Hilbert space structure from (A.1). For each point $z \in \mathbb{C}^n$, the evaluation map

$$\mathrm{ev}_z : f \mapsto f(z) \in \mathbb{C} \tag{A.2}$$

is continuous and linear, so the Riesz representation theorem implies that this map can be described by an integral operator:

$$\mathrm{ev}_z(f) = \int_{\mathcal{D}} f(z) K(z, z')\, \mathrm{d}\mu(z')\,. \tag{A.3}$$

The Bergman kernel is the function of $2n$ complex variables K. If we let u_j denote elements of an orthonormal basis of the space of holomorphic L^2-functions, indexed by the integers $j \in \mathbb{N}$, one can show [19] that

$$K(z, z') = \sum_{j=1}^{\infty} u_j(z)\overline{u_j(z')}\,. \tag{A.4}$$

So K is holomorphic in the first n and antiholomorphic in the last n variables, and to emphasise this we shall denote it as $K(z, \bar{z}')$ henceforth.

If we restrict the Bergman kernel to the diagonal and set

$$\mathrm{d}s^2_{(1)} = \sum_{k,l=1}^{n} \frac{\partial^2 \log K(z,\bar{z})}{\partial z_k \partial \overline{z_l}}\, \mathrm{d}z_k \mathrm{d}\overline{z_l}\,, \tag{A.5}$$

we obtain a Kähler metric on $\mathcal{D}$, which is easily seen to be invariant under holomorphic maps. However, if $n = 1$ we could also have set (writing z instead of z_1)

$$\mathrm{d}s^2_{(2)} = K(z, \bar{z})\, \mathrm{d}z\, \mathrm{d}\bar{z} \tag{A.6}$$

to obtain another Kähler metric invariant under holomorphic maps. Notice that there is no analogue of $\mathrm{d}s^2_{(2)}$ if $n > 1$. Standard textbooks [19] call $\mathrm{d}s^2_{(1)}$ the Bergman metric of $\mathcal{D}$, but this designation is also used for $\mathrm{d}s^2_{(2)}$, which in Bergman's most famous book [4] is simply called 'the invariant metric'. Notice that the Kähler form $\omega_{(1)}$ of $\mathrm{d}s^2_{(1)}$ turns out to be $\omega_{(1)} = -\rho_{(2)}$, that is, minus the Ricci form of the Kähler structure associated to $\mathrm{d}s^2_{(2)}$, cf. [18].

Now suppose that instead of a domain $\mathcal{D} \subset \mathbb{C}$ we have a closed Riemann surface Σ. The space of square-integrable functions on Σ is very large, but the only ones that are holomorphic are the constants. So in this case the above notions of Bergman metric (depending on the Bergman kernel as defined) are vacuous. The way out is to work with holomorphic 1-forms instead of holomorphic functions. If Σ has genus $g > 1$, the space $H^0(\Sigma, K_\Sigma)$ of global holomorphic 1-forms on Σ is nontrivial and has complex dimension g. Choosing a basis $\zeta_1, \ldots, \zeta_g$, we can set

$$\mathrm{d}s^2_{(3)} = \sum_{j=1}^{g} \zeta_j \overline{\zeta_j} \tag{A.7}$$

and obtain a metric on Σ. A metric of this type is also sometimes referred to as a Bergman metric [17, 18], and this is the notion that we use in this paper. (These metrics can be extended to the study of smooth curves in positive characteristic, and they are the starting point for Arakelov geometry [10].) Notice that the basis of $H^0(\Sigma, K_\Sigma)$ can be changed by a linear transformation $S \in \mathrm{GL}_g(\mathbb{C})$, and then a new metric will be obtained whose coefficients with respect to the original basis are related by the hermitian matrix SS^*. So the space of Bergman metrics of type (A.7) has real dimension g^2, even though there is twice as much freedom to choose a basis of holomorphic 1-forms on a compact Riemann surface of genus g. Each such metric is associated to a basis of $H^0(\Sigma, K_\Sigma) \cong \mathbb{C}^g$, or alternatively to a hermitian inner product on this complex vector space. One can show that all these metrics have nonpositive curvature, and the curvature vanishes precisely at the Weierstraß points of Σ in the case where Σ is hyperelliptic.

If z is a local coordinate, one has $\zeta_j = \zeta_j(z)\mathrm{d}z$ for some local holomorphic function $z \mapsto \zeta_j(z)$ for each j, and then

$$\mathrm{d}s^2_{(3)} = \sum_{j=1}^{g} \zeta_j(z)\overline{\zeta_j(z)}\,\mathrm{d}z\,\mathrm{d}\bar{z}\,. \tag{A.8}$$

Comparing (A.8) and (A.6), one sees that the Bergman kernel (A.4), which does not exist on Σ, is being replaced by the quantity $\sum_{j=1}^{g}\zeta_j(z)\overline{\zeta_j(z)}$. Notice that we could use this ersatz Bergman kernel to define another metric on Σ mimicking $\mathrm{d}s^2_{(1)}$ above in most cases, but this would not be a genuine metric everywhere if Σ were hyperelliptic, by our observation above about vanishing curvature.

The arbitrariness in the Bergman metrics (A.7) can be avoided if we orthonormalise the holomorphic 1-forms ζ_j with respect to the inner product (4.15). This amounts to restricting to one particular metric of type (A.7), namely, the one associated to the polarisation of the Jacobian, which defines a hermitian inner product on $H^0(\Sigma, K_\Sigma)$ by duality. This is precisely the Bergman metric (5.29) that we have shown to describe the motion of one vortex on Σ in the dissolving limit.

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