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Essays on Inventory Control in Presence of Multiple Sourcing

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Essays on Inventory control in presence of multiple sourcing

Ph.D. Dissertation

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Introduction

Introduction

Supply chain management is an integrative approach for planning and control of materials and information flows with suppliers and customers as well as between different functions within a company. One particularly important element in supply chain management is the management of inventories and decreasing risks in this is very important so that the companies with high-performing supply chains (SC) keep essential competitive advantages and be able to offer a more punctual and a better service to the customer at lower costs than their competition.

A McKinsey Quarterly survey finds that supply chain risk is rising sharply. Nearly 80 percent of global executives surveyed by The McKinsey Quarterly say that the amount of supply chain risk faced by their companies has increased in the past five years. Refer to Mishra and Tadikamalla (2006). The central aspect of this work is the effect of lead-time, demand and supply variability on inventory management. We try to pool the risks caused by the stochastic environment in the inventory system by dual/multiple sourcing instead of single sourcing. The policy implemented is sometimes splitting orders simultaneously among two or more suppliers and sometimes using the second supplier as a backup or emergency source. In our investigations, we focus on the most common key performance indicator in SCM, the total costs. This aligns well with an important objective of SCM in days of a critical economic situation, namely cost reduction.

Tang (2006) gives a good overview of the different areas in a supply chain. He distinguishes four main SCM areas: supply management, demand management, product management, and information management. Based on his division our work is located in the area of supply management; it contributes to the topic of replenishment policies for a single-item warehouse with dual or multiple sourcing, which means two or in general more supply modes are available. The literature on multiple suppliers is ample. Minner (2003) gives a good review of that topic.

Although strategies like Just-In-Time and Total-Quality-Management often suggest that a manufacturer use a single supplier in order to build a long-term supplier relationship to improve the service quality, using multiple suppliers is still very popular in practice. According to a report by McMillan, Toyota and Honda had one supplier for 28 and 38%, respectively; another 39 and 44% had two suppliers, and the rest had three or more suppliers. US government defense agencies are mandated to maintain more than one source for all but very small procurements. (see Mishra and Tadikamalla (2006)).

Table 1 in Mishra and Tadikamalla (2006) summarizes the managerial issues discussed in the literature in favor of single sourcing and multiple sourcing. As we mentioned above our objective in dual sourcing is cost reduction through risk pooling between suppliers.

In dual/multiple sourcing one strategy that has received considerable attention in the last few years is the policy of simultaneously splitting replenishment orders among several suppliers. The main premise for using this policy is that the lead-times of the suppliers are stochastic, hence it make sense to split any replenishment order into several smaller orders, in order to hedge against the risk of possible long lead-times. In these models when deciding for replenishment, then the replenishment order is split into suborders and the suborders are issued simultaneously to the suppliers. Using this procedure the firm is able to reduce the safety stock needed to meet service targets or alternatively, the expected number of backorders for a prescribed level of safety stock. In addition, successive deliveries of smaller "split" orders will reduce cycle stock, too. Thus the total costs will be reduced. In most of the papers in this dissertation except one we have considered this policy for different problems (sometimes as an option for the buying company). In the following we briefly review the relevant literature in this area.

Sculli and Wu (1981) were among the first to show that splitting an order between two suppliers with independently normal distributed lead times reduces the reorder level and the buffer stock when compared with replenishment with only one supplier. Since then, many extensions and specializations have followed.

In order to assess the benefits of order splitting in an economic context it is necessary to consider models where the total cost for ordering, purchase prices, inventory holding, and stock out penalties are minimized. Ramasesh et al. (1991) seem to be the first to present such a cost minimization approach for two potential suppliers. They make a comparison between sole sourcing and dual sourcing with order splitting in the case of a regular (s, Q) replenishment policy. Both suppliers have identical lead-time distributions, being either uniform or exponential. Demand is assumed to be constant; shortages are backordered, and a penalty cost per item per time unit is incurred. They find that order splitting provides savings in holding and backordering costs.

In Ramasesh et al. (1993), these findings are extended to the case of suppliers with non-identical lead time distributions, purchasing prices, and order splitting portions. In their model, both suppliers are assumed to have exponential lead time distributions. Because of complexity of the models, researchers often use only dual sourcing instead of multiple sourcing in their models. One important model of order splitting which considers n suppliers for ordering belongs to Sedarage et al. (1999). Lead times of suppliers and demand arrival are probabilistic and the unit purchasing prices from different suppliers may be different, thus, the order quantities for different suppliers may vary as well. They developed a mathematical model to determine the reorder level and the order quantity for each supplier so that the expected total cost per time unit is minimized.

Tyworth and Ruiz-Torros (2000) considered transportation cost in their model and found that it can strongly influence the single versus dual sourcing decision under a broad set of realistic conditions.

In general, some authors such as Sculli and Wu (1981), Hayya et al. (1987), Kelle and Silver (1990a, 1990b), Sculli and Shum (1990), Fong et al. (2000), and Kelle and Miller (2001) have focused on statistical theories and methods for estimating the effect of splitting on the distribution function, mean, and variance of effective lead times. On the other hand, many authors have concentrated on economic analysis. They have developed long run average cost models to assess the performance of split models in relation to non-split models under common conditions. Most of these models include inventory holding cost, ordering cost, and shortage cost.

Thomas and tyworth (2006). have made a review of the literature in order splitting and the gaps in the literature.

§1.1 Structure of this Dissertation

Our work is organized in 6 chapters. In Chapter 1, this chapter, we give an introduction to the whole dissertation. Chapter 2 deals with determining optimal suppliers under stochastic lead times. Chapter 3 and 4 deal with investigating if it is optimal to order simultaneously to the suppliers or not. In Chapter 5 we model a real problem concerning making decision about using an emergency source. Finally in Chapter 6, we deal with the problem of supply risk in ordering policy. A summary of different papers (chapters) in this dissertation is as follows.

Paper 1: Modeling and analysis for determining optimal suppliers under stochastic lead times

When we look at the literature regarding simultaneously order splitting, we realize that in general these models are very complex. As a result, the researchers mostly prefer to consider only dual sourcing (compare to multiple sourcing). They mostly claim that dual sourcing is a representative of multiple sourcing. This is true to some extent. The problem is that sometimes firms are considering supplying from n suppliers, which can be more than two, because of managerial policies, cost reduction etc. Then those dual source models are not helpful enough to inform inventory managers about the optimal number of suppliers that should be used in supplying a certain product, and furthermore the optimal policy for ordering to these suppliers.

This was our motive for this paper. In this paper we study order splitting for any number of suppliers, namely n. We develop an analytical model for determining optimal order splitting and reorder level for n-supplier inventory systems. The item acquisition lead times of suppliers are random variables. As result, lead-time variability is the main premise for considering several suppliers in order to pool lead-time risk among the suppliers. Backorders are allowed in our model and shortage cost is charged per unit in shortage per time unit. In the main model we develop the demand rate is constant. Furthermore we develop a model for probabilistic demand as well, but our numerical studies are based on constant demand rate.

We use continuous review $(s, Q_1, \ldots, Q_n)$ policy for ordering; when the inventory level depletes to a reorder level, the total order is split among n suppliers. Since the suppliers have different characteristics, the quantity ordered to different suppliers may be different.

The problem is to determine the reorder level and quantity ordered to each supplier so that the expected total cost per time unit, including ordering cost, procurement cost, inventory holding cost, and shortage cost, is minimized. Because the model is general to be used with any given number of potential suppliers, we can find the optimal number of suppliers as well. As we mentioned earlier, our focus in optimality is total cost.

The only comparable model in the literature is that of Sedarage et al. (1999), which is an approximate model. We use the same assumptions as their model and try to develop a more exact model. Our extensive numerical results show the advantage of our model over the relevant models in the literature.

Paper 2: A semi-Markov decision model for a dual source inventory system with random lead-times: Structure of policy and impact of supply information

After more than 20 years of extensive study, the policy of pooling lead-time risk by simultaneously splitting replenishment orders among several suppliers continues to attract the attention of researchers. As we mentioned above, in these policies the risk of stochastic and long lead-times is pooled between two (or more) available suppliers. Using this policy the safety stock for a prescribed service level, as well as cycle stock, decreases. Recognizing that some, but not all, of the components of the ordering cost would be common to the vendors, we expect that the total ordering cost when orders are issued simultaneously be lower than that when they are not issued at the same time.

In this study we let this assumption about simultaneousness be relaxed. Thus we allow the decision maker to choose whether he will simultaneously issue two orders to both suppliers or he will first issue a single order to one supplier and then await further information concerning his inventory status before making replenishment to the other supplier. In the latter one, one supplier is kept as the main source and the other one is used occasionally.

We study a dual source system with non-identical suppliers. We are still using most of the assumptions used in the standard model but we model the problem as a semi-Markov decision problem. In our model lead-times are Erlangean distributed; demands follow Poisson process; any shortage will be lost; inventory is reviewed continuously. The problem is to determine the optimal ordering policy to each supplier so that the expected total cost per time unit, including ordering cost, inventory holding cost, and shortage cost, is minimized. One advantage of our model over the models in the literature is that now in one integrated model the decision maker can decide to use dual sourcing with order splitting, dual sourcing using emergency orders, or single sourcing alone, which to some respect makes our model more general in comparison with other relevant models in the literature. In addition, for the case of emergency ordering, usually in the literature it is assumed that regular orders are slower and less expensive than emergency orders. This is true in most cases. However, one should question. This assumption in practice imposes a restriction on the ordering process: the slower order has to be triggered first and then one can possibly place an emergency order if the lower reorder point is reached. We do not make this restriction because we believe there might exist cases in which the interplay of costs, lead times and demand could lead to a scenario where it is cheaper to not use the regular orders at all, for example.

Using significant numerical studies we will show that simultaneously order splitting is not always optimal policy. We numerically show that which policy (dual sourcing with order splitting, dual sourcing using emergency orders, or single sourcing) is optimal under different parameter inputs.

Another perspective that we provide in this problem is investigating the effect of supply information on the ordering policy and the value of visibility on the supply process. For this, we interpret the Erlangean lead-times as a supply process with some phases each of which follows an exponential distribution. In the primary model we consider, these phases (stages) are observable by the retailer. This is very common using new technologies like RFID or GPS. We call this type of visibility "order progress information" (OPI). Furthermore, we change our model to the case where there is no information on the order progress. One of the goals in the numerical studies is to compare the two cases.

Paper 3: A semi-Markov decision model for a dual source inventory system with random lead-times: An alternative compact formulation

In this paper we address the same problem as in Paper 2. In the model of Paper 2, there are many states in the semi-Markov decision problem to be considered for computing costs and in the introduced algorithm for obtaining the optimal costs the convergence speed is quite slow because of computational time.

In this paper using the same assumptions as was made in Paper 2 we make an alternative model with compact state space. Then using a new algorithm for obtaining the optimal solution we save a great amount of computational time. After comparing the results from both algorithms in Paper 2 and this paper using a significant number of tests, we conclude that the developed algorithm in this model not only works well regarding the accuracy of the results, but also it is much faster than the original algorithm in Paper 2.

Paper 4: Exploring the economic consequences of paying a supplier to keep reserve

This paper is developed on the basis of a contact to a leading Danish company. In the paper, we call it the focal company. The focal company has many of its suppliers located overseas with long replenishment lead-times. The products of the focal company have a very short life cycle, typically 1–2 years. In addition the demand is non-stationary with some initial growth in demand, followed by a peak-demand period, and the demand is gradually declining until the end of life cycle. Due to stochastic demand and high shortage costs the focal company needs to keep a rather large amount of safety stock. Therefore the focal company considers paying the supplier to have reserve storage, somewhere in Europe, and with lower lead-time but more procurement cost. Using this way the focal company can decrease its costly safety stock. The problem for the focal company is that because he has to pay to the supplier for having such reserve storage, the company wants to know how essential having this reserve or emergency storage is for them, regarding decreasing the costs. The idea is that the focal company more frequently has the possibility to order from the reserve storage and that, due to its closer position, the lead-time is also lower. So the focal company has both opportunities: ordering on a monthly basis from the production facility of supplier at overseas location or on a weekly basis to order from the reserve storage located somewhere in Europe. Of course, there must be some contractual agreement about how the supplier is obliged to refill the reserve storage.

We develop a dynamic programming model for this problem. We use periodic review inventory control with finite time horizon. There are two supply modes: normal and emergency. While there is no limitation to the normal order size, the reserve storage is capacitated and the company can either order all the reserve inventory or nothing. If the emergency storage is empty it takes some fixed time to be filled up again. Providing that the reserve storage is full, it can be used at any time period while normal orders can be issued at specific time periods. As a result, we use two different inventory review policies for these two supply modes. Both normal and emergency lead-times are fixed and known, but normal lead-times are larger than emergency lead-times. We develop an exact and an approximate model for this problem.

Our numerical studies show that using the aforementioned reserve storage is quite attractive. In addition, the approximate model works very well and is reliable.

Paper 5: A renewal-reward formulation of a dual source inventory problem where there is supply uncertainty with regards to the delivered quantities

Most studies on order splitting assume the only source of uncertainty concerns lead-time uncertainty. In these studies order splitting is often an attractive policy since the lead-time risk is pooled between different suppliers. Thus, if one supplier is unusually late in his delivery this is hopefully partly offset by another supplier being unusually fast in his delivery. In these studies there is no uncertainty concerning the ordered quantities. In this study, we are concerned with the situation where the risk does not come from lead-time uncertainty, but from yield or supply uncertainty. In fact in this paper we assume that the lead times are fixed and known, and we are interested to know if in this condition order splitting still is an optimal policy with regard to minimizing costs. Supply uncertainty can have many reasons. These include insufficient supply of raw materials, production of poor quality products, machine breakdown, workers strike, and so on. As a result the amount that the buying company receives is not the same as the quantity he has ordered. The issue for the companies is that sometimes they do not know about the exact amount that they are going to receive until they physically receive the order.

In our model demands follow poisson process. All shortages are backlogged, and the total costs include Ordering, purchasing, holding, and shortage costs. Using numerical tests it can be observed that dual sourcing with order splitting even when lead-times are fixed, but the uncertainty is in quantity delivered, is still an attractive policy in many cases.

2

Modeling and Analysis for Determining Optimal Suppliers Under Stochastic Lead times

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Modeling and Analysis for Determining Optimal Suppliers Under Stochastic Lead times

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Abstract

The policy of simultaneously splitting replenishment orders among several suppliers has received considerable attention in the last few years and continues to attract the attention of researchers. In this paper, we develop a mathematical model which considers multiple-supplier single-item inventory systems. The item acquisition lead times of suppliers are random variables. Backorder is allowed and shortage cost is charged based on not only per unit in shortage but also per time unit. Continuous review (s, Q) policy has been assumed. When the inventory level depletes to a reorder level, the total order is split among n suppliers. Since the suppliers have different characteristics, the quantity ordered to different suppliers may be different. The problem is to determine the reorder level and quantity ordered to each supplier so that the expected total cost per time unit, including ordering cost, procurement cost, inventory holding cost, and shortage cost, is minimized. We also conduct extensive numerical experiments to show the advantages of our model compared with the models in the literature. According to our extensive experiments, the model developed in this paper is the best model in the literature which considers order splitting for n-supplier inventory systems since it is the nearest model to the real inventory system.

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2.1. Introduction

Most of the inventory control models developed in the last few decades assume that an inventory item is replenished from a single supplier. However, there are often situations in which more than one supplier is necessary to sustain a desirable service standard or to reduce the total system cost when acquisition lead times are uncertain. The policy of simultaneously splitting replenishment orders among several suppliers has received considerable attention in the last few years and continues to attract the attention of researchers. This is because multiple-supplier sources can facilitate splitting an order to counter the variability of item arrivals.

Many of the authors who have worked in this field have concentrated on economic analysis or, more specifically, on developing long run average cost models to assess the performance of split models in relation to non-split models under common conditions. A major weakness in these studies is that almost all of them only consider order splitting when there are only two suppliers, i.e. dual sourcing. It means that they neglect the effect of order splitting on the system when there are more than two suppliers, i.e. the case of multiple suppliers. Most of the authors have concluded that dual sourcing in many cases is better than one sourcing, but are these results true when orders are split between more than two suppliers, or in general, n suppliers? The only study in the field of economic analysis which considers order splitting for n-supplier inventory systems is Sedarage et *al.* (1999). The analytical model developed in their paper only approximates the real inventory system and this approximation sometimes has a considerable error.

In this paper, we develop an analytical model for determining optimal order splitting and reorder level for n-supplier inventory systems. As we will show later, this model is the nearest model to the real inventory system; therefore, it can be used instead of that of Sedarage et *al.* (1999). We have also performed extensive numerical analyses and found some new results about the effect of the system parameters on the system performance measures when we use order splitting. Finally, the inventory system considered in this paper, like that of Sedarage et *al.* (1999), has the most general setting in the literature. Thus, the decision of whether to employ a single, dual, or multiple sourcing is also discussed in this paper.

The remainder of this paper is organized as follows. We will first present an overview of the literature of order split problem. Next, problem definition and the related formulation will be reviewed. Finally, computational results, and related analyses are introduced.

2.2. Overview of the literature

Sculli and Wu (1981) appear to be the first to present the concept of order splitting. They analyzed the two supplier situation with normally distributed lead times and used numerical integration to derive tables for the mean and standard deviation of the lead time demand. Their study, as well as that of others, indicated that a firm can achieve higher service levels for any level of safety stock by effectively distributing lead-time over several suppliers rather than only one individual supplier (See Thomas and Tyworth (2006)). Sculli and Shum (1990) extended the model for more than two suppliers with non-identical allocation portions of supply and gave expressions for the mean and variance of the effective lead times.

In order to assess the benefits of order splitting in an economic context it is necessary to consider models where the total cost for ordering, purchase prices, inventory holding, and stock out penalties are minimized. Ramasesh et *al.* (1991) seem to be the first to present such a cost minimization approach for two potential suppliers. In their model, suppliers are assumed to be of the same reliability, so the order quantity is evenly split between the suppliers. Both suppliers have identical lead time distributions, being either uniform or exponential. Demand is assumed to be constant, shortages are backordered, and a penalty cost per item per time unit is incurred. From the non-linear total cost functions, the optimal values for the reorder point s and the order quantity Q are delivered by numerical search. In Ramasesh et *al.* (1993), these findings are extended to the case of suppliers with non-identical lead time distributions, purchasing prices, and order splitting portions. In their model, both suppliers are assumed to have exponential lead time distributions. Chiang and Benton (1994) investigated a model with normally distributed demand and a two-supplier case where each supplier has an identically shifted exponential lead time distribution. In their model, both suppliers are assumed to be identical with respect to their lead time characteristics and purchase prices. Shortage cost is assumed to incur as a shortage penalty that is independent from the stock out duration. They showed that dual sourcing is better than single sourcing except for the situation where the order cost is high and lead time variability is low. Lau and Lau (1994) investigated a model with two suppliers in which the demand was assumed to be constant and the lead times followed beta distribution. They compared single sourcing and dual sourcing in such situation. The same analysis was conducted for normally distributed demands by Ganeshan et *al.* (1999). Chiang and Chiang (1996) used the concept of order splitting when there is one supplier from which inventory item is replenished. In their model, lead time is assumed to be constant. One of the most important models of order splitting in an economic context is the model of Sedarage et *al.* (1999). Their model is not restricted to only two suppliers. Lead times of suppliers and demand arrival are probabilistic and the unit purchasing prices from different suppliers may be different, thus, the order quantities for different suppliers may vary as well. They developed a mathematical model to determine the reorder level and the order quantity for each supplier so that the expected total cost per time unit, consisting of the fixed ordering cost, procurement cost, inventory holding cost, and shortage cost, is minimized. Tyworth and Ruiz-Torros (2000) considered transportation cost in their model and found that it can strongly influence the single versus dual sourcing decision under a broad set of realistic conditions. Their assumptions are similar to Ramasesh et *al.* (1993). Dullaert et *al.* (2005) used genetic algorithm to determine the best combination of different methods of transportation for a two supplier system. Recently Thomas and Tyworth (2006) performed a review on order splitting problems.

In general, some authors such as Sculli and Wu (1981), Hayya et *al.* (1987), Kelle and Silver (1990a, 1990b), Sculli and Shum (1990), Fong et *al.* (2000), and Kelle and Miller (2001) have focused on statistical theories and methods for estimating the effect of splitting on the distribution function, mean, and variance of effective lead times. On the other hand, many authors have concentrated on economic analysis. They have developed long run average cost models to assess the performance of split models in relation to non-split models under common conditions. Most of these models include inventory holding cost, ordering cost, and shortage cost. Our study is in this area. In order to better understand the characteristics and differences of each of these models, we have organized the relevant studies in Table 1.

As seen in Table 1, only Sedarage et *al.* (1999) have studied economic analysis of order splitting over n suppliers. In addition, their model is more general than the other models. However, the analytical model developed in their paper only approximates the inventory system and this approximation sometimes has a considerable error. In the next section, we try to introduce a new analytical model which is free from their restrictions, and is the nearest model to the real inventory system in the literature.

Table 1. Literature Review of Order Split Problems and Classifications.

Features / References	Problem Definition & Assumptions							Objective Function and its Component						Constraints	Outputs			
	# of Suppliers	Demand		Lead time		Purchasing Price	Type of Order Split	Objective Function Type	Objective Function Component						Order Quantity	Reorder Level	Split Ration	Others
		Deterministic	Probabilistic	Deterministic	Probabilistic				Holding Cost	Shortage Cost	Transportation Cost	Ordering Cost	Others					
Sedarage et *al.* (1999)	*n*		*		*	*d*	*d*	Cost	*	*		$K=K_1+n.K_2$	Purchasing		*	*	*	*n*
Tyworth and Ruiz-Torres (2000)	2	*			Exponential (*d*)	*d*	*d*	Cost	*	*	*	$\acute{A}=\alpha A$	Purchasing		*	*	*	
Mohebbi and Posner (1998)	2		Poisson		Exponential (*s*)	*s*	*s*	Cost	*	Lost Sale		$\acute{A}=\alpha A$		*SL*	*	*		
Chiang and Chiang (1996)	1		Normal	*			*s*	Cost	*			$\acute{A}=(1+r)A$		*SL*	*	*		L_1, L_2
Ganeshan et *al.* (1999)	2		Normal		*	*d*	*d*	Cost	*		*	$\acute{A}>A$	Purchasing, Pipeline Inv., Safety Stock	*SL*	*	*	*	Sales Price
Lau and Lau (1994)	2	*			Beta (*d*)	*d*	*d*	Cost	*			$\acute{A}=\alpha A$	Purchasing	*SL*	*	*	*	
Chiang and Benton (1994)	2		Normal		Shifted Exponential (*s*)	*s*	*s*	Cost	*	Shortag e Penalty					*	*		
Ramasesh et *al.* (1993)	2	*			Exponential (*d*)	*d*	*d*	Cost	*	*		$\acute{A}=\alpha A$			*	*	*	
Ryu and Lee (2003)	2	2			Exponential (*d*)	*s*	*d*	Cost	*	*		$\acute{A}=A(1+\rho)$	Lead-time Increasing/ Decreasing Cost		*	*	*	t_1, t_2
Ramasesh et *al.* (1991)	2	*			Exponential / Uniform (*s*)	*s*	*s*	Cost	*	*		$\acute{A}=\alpha A$			*	*		
Our model	*n*	*	*		*	*d*	*d*	Cost	*	*		$K=K_1+n.K_2$	Purchasing		*	*	*	*n*

d: different among suppliers; *s*: same among suppliers; t_1, t_2: lead-time reduction factors; *SL*: service level;

2.3. Problem Definition

2.3.1 Notations

The notations used in this paper are as follows:

- c_i, i=1, ..., n: the unit procurement cost, offered by supplier i.
- n: number of suppliers.
- h: unit inventory holding cost per time unit.
- π: unit shortage cost per time unit.
- $K_1 + n.K_2$: total ordering cost per cycle. K_1 is the constant portion of the ordering cost which is independent from the number of suppliers, and $n.K_2$ is portion of the ordering cost which depends on the number of suppliers.
- M: demand rate per time unit.
- Y_i, i=1, ..., n: acquisition lead time from supplier i which is a random variable with a density (or cumulative distribution) function of $g_i(t)$ (or $G_i(t)$).
- L_i, i=1, ..., n: lead time of a supplier for the ith delivery, or the time duration from the moment when the order is placed until the moment when the ith delivery from a supplier is made.
- X_i, i=1, ..., n: quantity demanded during lead time L_i which is a random variable.
- s: replenishment reorder level.
- Q_i, i=1, ..., n: ordering quantity for supplier i; thus, the total order quantity is $Q=\sum_{i=1}^{n} Q_i$..
- $IA_{i-1} \left| j_1, j_2, \ldots, j_{i-1} \right.$: inventory level right after the (i-1)th delivery, which can be positive or negative, assuming that the first i-1 deliveries have been made from suppliers $j_1, j_2, \ldots, j_{i-1}$, without taking into account any specific ordering of delivery arrival moments from suppliers $j_1, j_2, \ldots, j_{i-1}$.
- $IB_{i-1} \left| j_1, j_2, \ldots, j_{i-1} \right.$: inventory level right before the ith delivery, which can be positive or negative, assuming that the first i-1 deliveries have been made from suppliers $j_1, j_2, \ldots, j_{i-1}$ without taking into account any specific ordering of delivery arrival moments from suppliers $j_1, j_2, \ldots, j_{i-1}$.

2.3.2 Assumptions

Assumptions of the developed model are as follows:

- The number of suppliers can be any natural number $n \geq 1$.
- Continuous review (s, Q) policy has been assumed. It means that the main order Q is placed whenever the inventory position reaches the reorder level s.
- The main order is placed when the reorder level is reached and can be simultaneously split (not necessarily equally) among the n suppliers.
- Shortage cost is charged for per shortage per product unit per time unit.
- The supplier lead times are random variables Y_1, Y_2… Y_n of any distribution which are probabilistically independent and are not necessarily identically distributed.
- The unit time demand is constant.
- There is no crossover among main orders. Please note that by main order we mean the order which is placed once in each cycle. This main order with size Q in each cycle is split simultaneously among n suppliers. (Q_i to supplier i). Although we assume that main orders for different cycles do not cross over time and arrive sequentially, there is not any special sequence for splitted orders (Q_is) due to the nature of random lead times.

In reality, when lead times are stochastic, orders may not be received in the same sequence they are placed. Although the likelihood of order crossover would seem to be low, order crossover makes the analysis intractable. Therefore, it is usually assumed that the *main orders* do not cross in time.

The decisions to be taken are the order-split quantity for each supplier and the reorder level of stocked items, which minimize the total expected cost per time unit including the ordering cost, variable procurement cost, inventory holding cost, and shortage cost.

2.4. Problem Formulation (Model One)

In order to acquire the holding cost and shortage cost in a cycle, the interval between two consequent orders, we divide a cycle into a number of segments and compute the holding and shortage cost in each segment. Then, we can obtain the holding and shortage cost in a cycle by adding up those costs in each segment. If there are n suppliers, we divide the span of a cycle into n segments. Each segment is the span of two successive arrivals of orders. Therefore, the segment [i-1,i] is the distance between i-1th and i^{th} arrival.

For every segment [i-1,i] there are three possible states for inventory levels (or net inventories):

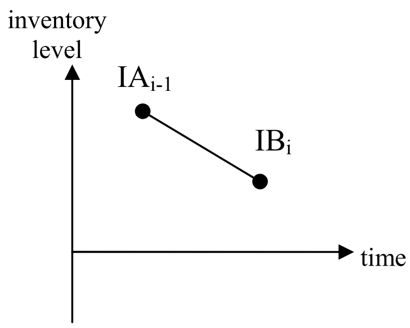

Figure 1. State A

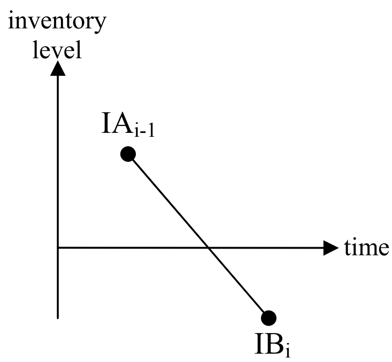

Figure 2. State B

- **State A**: according to Figure 1, in this state there is only holding cost which we show by $HC_{a_{i-1,i}}$. State A happens when $IB_i \geq 0$.
- **State B:** according to Figure 2, in this state there are both holding cost and shortage cost which are denoted by $HC_{b_{i-1,i}}$ and $SC_{b_{i-1,i}}$, respectively. State B happens when $IA_{i-1} \geq 0$ and $IB_i < 0$.
- **State C:** according to Figure 3, in this state there is only shortage cost which is denoted by $SC_{c_{i-1,i}}$. State C happens when $IA_{i-1} < 0$.

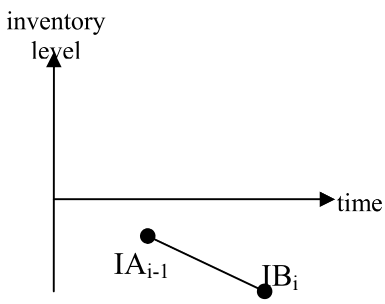

Figure 3. State C

Obviously, there is no other possible state for segment [*i*-1,*i*]. On the other hand, for segment [n,1], the distance between the n^{th} delivery of a given cycle and the first delivery of its next cycle, there are four possible states. This is due to the fact that although we assume that main orders (which are divided to n suppliers) are received in the same sequence they are placed, the inventory level after n^{th} delivery may be positive or negative. When the inventory level after the n^{th} delivery is below s or is negative, a portion of costs in segment $[n,1]$ is not computed in the current cycle and instead, since it is considered that net inventory level at the beginning of every cycle is equal to reorder level, s, (this assumption is in order to create a renewable cycle.) this portion of costs not considered in the current cycle will be accounted for in the next cycle. In long run, all costs will be calculated. (For more information see Ramasesh et *al.* (1991))

The four possible states for segment $[n,1]$ are as follows:

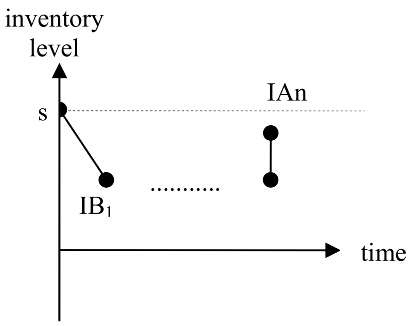

Figure 4. State A of Segment $[n,1]$

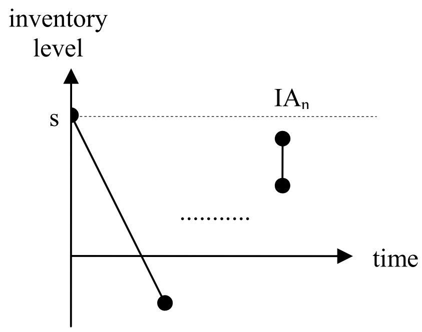

Figure 5. State B of Segment $[n,1]$

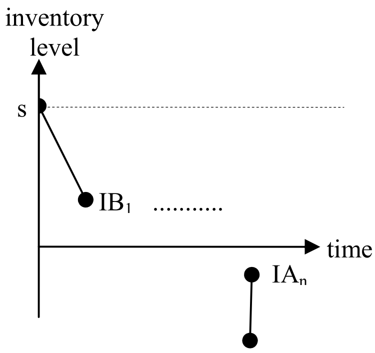

Figure 6. State C of Segment $[n,1]$

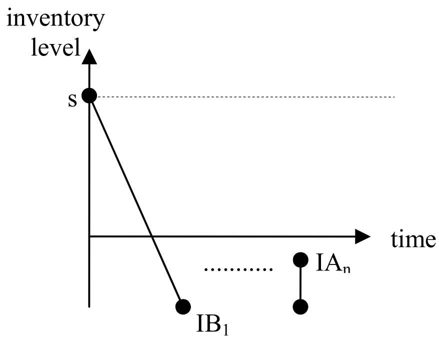

Figure 7. State D of Segment $[n,1]$

- **State A:** according to Figure 4, in this state there is only holding cost which is denoted by $HC_{a_{n,1}}$. This state happens when $IB_1 \geq 0$ and $IA_n \geq 0$.
- **State B:** according to Figure 5, in this state there are both holding cost and shortage cost which are denoted by $HC_{b_{n,1}}$ and $SC_{b_{n,1}}$, respectively. This state happens when $IB_1 < 0$ and $IA_n \geq 0$.

- **State C:** according to Figure 6, in this state there are both holding cost and shortage cost which are denoted by $HC_{c_{n,1}}$ and $SC_{c_{n,1}}$, respectively. Unlike state B, this state happens when $IB_1 \geq 0$ and $IA_n < 0$.
- **State D:** according to Figure 7, in this state $IB_1 < 0$ and $IA_n < 0$, so there is only shortage cost which is denoted by $SC_{d_{n,1}}$.

Clearly, there is no other possible state other than these four states for segment [$n,1$].

Now, we are ready to obtain the holding cost and the shortage cost in each segment. With regards to their definitions, we have:

$$IA_{i-1} \left| j_1,\ldots,j_{i-1} = s + \sum_{k=1}^{i-1} Q_{j_k} - x_{i-1}. \right. \tag{1}$$

$$IB_i \left| j_1,\ldots,j_{i-1} = s + \sum_{k=1}^{i-1} Q_{j_k} - x_i. \right. \tag{2}$$

And also:

$$IB_1 = s - x_1. \tag{3}$$

$$IA_n = s + Q - x_n; \tag{4}$$

where $x_i = ML_i$.

Furthermore, the joint density function L_{i-1} and L_i, assuming that the first i-1 deliveries have been made from suppliers j_1, j_2, …, j_{i-1}, without taking into account any specific ordering of delivery arrival moments from suppliers j_1, j_2, …, j_{i-1}, is given as follows:

$$f_{L_{i-1},L_i|j_1,\ldots,j_{i-1}}\left(l_{i-1},l_i\right) = \frac{\sum_{k=1}^{i-1}\left[g_{j_k}\left(l_{i-1}\right)\cdot\prod_{\substack{p=1\\p\neq k}}^{i-1}G_{j_p}\left(l_{i-1}\right)\times\left\{\sum_{q=i}^{n}\left[g_{i_q}\left(l_i\right)\cdot\prod_{\substack{t=i\\t\neq q}}^{n}\left(1-G_{j_t}\left(l_i\right)\right)\right]\right\}\right]}{P\left(j_1,\ldots,j_{i-1}\right)} \tag{5}$$

Now, we can calculate the holding and shortage cost for each of the possible states of segment [i-1,i]:

- **State A:** As stated before, in state A there is only holding cost. The following relation is apparent:

$$HC_{a_{i-1,i}} = \sum_{\{j_1,\ldots,j_{i-1}\}\subseteq\{1,2,\ldots,n\}} HC_{a_{i-1,i}} \left| j_1,\ldots,j_{i-1} \times P\left(j_1,\ldots,j_{i-1}\right). \right. \tag{6}$$

A similar relationship is valid for state B and state C. According to Relation (6), in order to have $HC_{a_{i-1,i}}$, we first have to compute $HC_{a_{i-1,i}} \left| j_1,\ldots,j_{i-1} \right.$. In state A, the time taken to drop the inventory level from $IA_{i-1} \left| j_1,\ldots,j_{i-1} \right.$ to $IB_i \left| j_1,\ldots,j_{i-1} \right.$ equals $\left(IA_{i-1}\left|j_1,\ldots,j_{i-1} - IB_i\right|j_1,\ldots,j_{i-1}\right)/\left(M\right)$; Therefore, the inventory holding cost in this interval is given as:

$$\frac{h}{2M}\left(IA_{i-1}\left|j_1,\ldots,j_{i-1} + IB_i\right|j_1,\ldots,j_{i-1}\right)\left(IA_{i-1}\left|j_1,\ldots,j_{i-1} - IB_i\right|j_1,\ldots,j_{i-1}\right); \tag{7}$$

But the probability that the holding cost in state A becomes as Relation (7) is that L_{i-1} and L_i have the values l_{i-1} and l_i , respectively. In state A, we have:

$$IB_i\left|j_1,\ldots,j_{i-1} \geq 0 \rightarrow s + \sum_{k=1}^{i-1} Q_{j_k} - x_i \geq 0 \rightarrow s + \sum_{k=1}^{i-1} Q_{j_k} - M \cdot L_i \geq 0 \rightarrow L_i \leq \frac{s + \sum_{k=1}^{i-1} Q_{j_k}}{M}\right.$$
$$and \quad L_{i-1} \leq L_i \Rightarrow L_{i-1} \in \left(0, L_i\right) \quad L_i \in \left(0, \frac{s + \sum_{k=1}^{i-1} Q_{j_k}}{M}\right) \tag{8}$$

With regards to these descriptions, we obtain $HC_{a_{i-1,i}} \left| j_1,\ldots,j_{i-1} \right.$ as:

$$HC_{a_{i-1,i}} \left| j_1,\ldots,j_{i-1} = \int_0^{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}} \int_0^{l_i} \frac{h}{2M}\left(IA_{i-1}\left|j_1,\ldots,j_{i-1} + IB_i\right|j_1,\ldots,j_{i-1}\right)\right. \tag{9}$$
$$\times\left(IA_{i-1}\left|j_1,\ldots,j_{i-1} - IB_i\right|j_1,\ldots,j_{i-1}\right) f_{L_{i-1},L_i|j_1,\ldots,j_{i-1}}\left(l_{i-1},l_i\right) d\,l_{i-1} d\,l_i\,.$$

Therefore, $HC_{a_{i-1,i}}$ is formulated as follows:

$$HC_{a_{i-1,i}} = \sum_{\{j_1,\ldots,j_{i-1}\}\subseteq\{1,2,\ldots,n\}} P\left(j_1,\ldots,j_{i-1}\right) \int_0^{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}} \int_0^{l_i} \frac{h}{2M}\left(IA_{i-1}\left|j_1,\ldots,j_{i-1} + IB_i\right|j_1,\ldots,j_{i-1}\right) \tag{10}$$
$$\times\left(IA_{i-1}\left|j_1,\ldots,j_{i-1} - IB_i\right|j_1,\ldots,j_{i-1}\right) f_{L_{i-1},L_i|j_1,\ldots,j_{i-1}}\left(l_{i-1},l_i\right) d\,l_{i-1} d\,l_i$$

- **State B:** in this state, there are both the holding cost and shortage cost. Similar to the demonstration given for state A, we obtain the following equations:

$$HC_{b_{i-1,i}} = \sum_{\{j_1,\dots,j_{i-1}\}\subseteq\{1,2,\dots,n\}} P\left(j_1,\dots,j_{i-1}\right) \int_{\left(s+\sum_{k=1}^{i-1}Q_{j_k}\right)/M}^{\infty} \int_{0}^{\left(s+\sum_{k=1}^{i-1}Q_{j_k}\right)/M} \frac{h}{2M}\left(IA_{i-1}\middle|j_1,\dots,j_{i-1}\right)^2 \times f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right)dl_{i-1}dl_i \tag{11}$$

$$SC_{b_{i-1,i}} = \sum_{\{j_1,\dots,j_{i-1}\}\subseteq\{1,2,\dots,n\}} P\left(j_1,\dots,j_{i-1}\right) \int_{\frac{s+\sum_{k=1}^{i-1}Q_{j_k}}{M}}^{\infty} \int_{0}^{\frac{s+\sum_{k=1}^{i-1}Q_{j_k}}{M}} \frac{\pi}{2M}\left(IB_i\middle|j_1,\dots,j_{i-1}\right)^2 \times f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right)dl_{i-1}dl_i. \tag{12}$$

- **State C:** in this state, there is only the shortage cost. Similar to the demonstration given for state A, we obtain the following equation for $SC_{c_{i-1,i}}$:

$$SC_{c_{i-1,i}} = \sum_{\{j_1,\dots,j_{i-1}\}\subseteq\{1,2,\dots,n\}} P\left(j_1,\dots,j_{i-1}\right) \int_{\frac{s+\sum_{k=1}^{i-1}Q_{j_k}}{M}}^{\infty} \int_{\frac{s+\sum_{k=1}^{i-1}Q_{j_k}}{M}}^{l_i} \frac{\pi}{2M}\left(IA_{i-1}\middle|j_1,\dots,j_{i-1} + IB_i\middle|j_1,\dots,j_{i-1}\right)\times\left(IB_i\middle|j_1,\dots,j_{i-1} - IA_{i-1}\middle|j_1,\dots,j_{i-1}\right) f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right)dl_{i-1}dl_i. \tag{13}$$

Using these equations, we can calculate the costs in segments [1,2], [2,3],…,[*n*-1,*n*]. Now, we formulate the costs in segment [*n*,1]:

- **State A:** As described before, in this state there is only the holding cost. Similar to the description given before, $HC_{a_{n,1}}$ is given by:

$$HC_{a_{n,1}} = \int_{l_1=0}^{\frac{s}{M}} \int_{l_n=l_1}^{\frac{s+Q}{M}} \frac{h}{2M}\left(IA_n + IB_1\right)\left(IA_n - IB_1\right) f_{L_1,L_n}\left(l_1,l_n\right)dl_n dl_1. \tag{14}$$

where $f_{L_1,L_n}\left(l_1,l_n\right)$ equals:

$$f_{L_1,L_n}(l_1,l_n)=\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[g_i(l_1)g_j(l_n)\prod_{\substack{k=1\\ k\neq i,j}}^{n}\left(G_k(l_n)-G_k(l_1)\right)\right]. \tag{15}$$

- **State B:** as stated before, in this state there are both holding cost and shortage cost, which can be obtained by:

$$HC_{b_{n,1}}=\int_{l_1=\frac{s}{M}}^{\frac{s+Q}{M}}\int_{l_n=l_1}^{\frac{s+Q}{M}}\frac{h}{2M}(IA_n)^2 f_{L_1,L_n}(l_1,l_n)\,dl_n\,dl_1. \tag{16}$$

$$SC_{b_{n,1}}=\int_{l_1=\frac{s}{M}}^{\frac{s+Q}{M}}\int_{l_n=l_1}^{\frac{s+Q}{M}}\frac{\pi}{2M}(IB_1)^2 f_{L_1,L_n}(l_1,l_n)\,dl_n\,dl_1. \tag{17}$$

- **State C:** as stated before, in this state there are both holding cost and shortage cost, which can be obtained by:

$$HC_{c_{n,1}}=-\int_{l_1=0}^{\frac{s}{M}}\int_{l_n=\frac{s+Q}{M}}^{\infty}\frac{h}{2M}(IB_1)^2 f_{L_1,L_n}(l_1,l_n)\,dl_n\,dl_1. \tag{18}$$

$$SC_{c_{n,1}}=-\int_{l_1=0}^{\frac{s}{M}}\int_{l_n=\frac{s+Q}{M}}^{\infty}\frac{\pi}{2M}(IA_n)^2 f_{L_1,L_n}(l_1,l_n)\,dl_n\,dl_1. \tag{19}$$

- **State D:** in this state, there is only shortage cost, which can be obtained by:

$$SC_{d_{n,1}}=\int_{l_n=\frac{s+Q}{M}}^{\infty}\int_{l_1=\frac{s}{M}}^{l_n}\frac{\pi}{2M}(IA_n+IB_1)(IB_1-IA_n)f_{L_1,L_n}(l_1,l_n)\,dl_1\,dl_n. \tag{20}$$

Eventually, the total inventory holding and shortage cost per cycle, denoted by *HC* and *SC*, is given as:

$$HC=\sum_{i=2}^{n}\left(HC_{a_{i-1,i}}+HC_{b_{i-1,i}}\right)+\left[HC_{a_{n,1}}+HC_{b_{n,1}}+HC_{c_{n,1}}\right]. \tag{21}$$

$$SC=\sum_{i=2}^{n}\left(SC_{b_{i-1,i}}+SC_{c_{i-1,i}}\right)+\left[SC_{b_{n,1}}+SC_{c_{n,1}}+SC_{d_{n,1}}\right]. \tag{22}$$

The total procurement cost per cycle equals $\sum_{i=1}^{n} c_i Q_i$. Finally, Since the cycle time is equal to $\sum_{i=1}^{n} Q_i \Big/ M$, the optimization model for minimizing the expected total cost per time unit, denoted by *TC*, can be given as follows:

$$\begin{aligned}\text{minimize } TC = M \Big/ \sum_{i=1}^{n} Q_i \Big\{ & K_1 + nK_2 + \sum_{i=1}^{n} c_i Q_i + \sum_{i=2}^{n} \left(HC_{a_{i-1,i}} + HC_{b_{i-1,i}} \right) + \left[HC_{a_{n,1}} + HC_{b_{n,1}} + HC_{c_{n,1}} \right] \\ & + \sum_{i=2}^{n} \left(SC_{b_{i-1,i}} + SC_{c_{i-1,i}} \right) + \left[SC_{b_{n,1}} + SC_{c_{n,1}} + SC_{d_{n,1}} \right] \Big\}\end{aligned} \quad (23)$$

The Complete version of this model is given in Appendix 1.

This is a non linear model which can be used with a range of distributions for lead times. If there are *n* potential suppliers, there will be n+1 decision variables in the model, which are *s* and Q_i. i=1,2,…,*n*.

It can be shown that $P\left(j_1,...,j_{i-1}\right)$ is expressed in terms of $g_i(t)$ and $G_i(t)$, (see Appendix A in Sedarage et al. (1999)).

2.5. Probabilistic Demand (Model Two)

In the main model, we assumed that the demand rate is constant. In this section, we consider the case that the demand per time unit is a random variable whose distribution function is closed under convolution (e.g., exponential, Gamma). We have:

- $f_p(m)$: demand density function in the time unit.
- $f_{p(L)}(m)$: demand density function in time period *L*. For example, for a exponential distribution $f_\rho(m) = \lambda \exp(-\lambda x)$ and $f_{\rho(L)}(m) = \lambda/L \exp(-x\, \lambda/L)$.
- *M*: expected value of demand rate per time unit. We have: $M = \int_0^\infty m \cdot f_\rho(m) d_m$

Now, we can calculate the holding cost and shortage cost for each state of segment [i-1,i]:

- **State A:** As mentioned before, in state A there is only holding cost. Assume that the first i-1 deliveries have been made from suppliers j_1, j_2, …, j_{i-1}, without taking into account any specific ordering of delivery arrival moments from suppliers j_1, j_2, …, j_{i-1}. As stated before, the inventory holding cost between $IA_{i-1}\left|j_1,...,j_{i-1}\right.$ and $IB_i\left|j_1,...,j_{i-1}\right.$ equals:

$$\frac{h}{2M}\left(IA_{i-1}\left|j_1,...,j_{i-1}\right.+IB_i\left|j_1,...,j_{i-1}\right.\right)\left(IA_{i-1}\left|j_1,...,j_{i-1}\right.-IB_i\left|j_1,...,j_{i-1}\right.\right); \tag{24}$$

If we assume that L_{i-1} and L_i have the values l_{i-1} and l_i ,respectively, the probability that the holding cost in state A becomes as Relation (24) is that demand quantities in the distances l_{i-1} and l_i, are equal to x_{i-1} and x_i , respectively. (Consult the definitions for $IA_{i-1}\left|j_1,...,j_{i-1}\right.$ and $IB_i\left|j_1,...,j_{i-1}\right.$). This probability is given as $f_{X_{i-1},X_i}\left(x_{i-1},x_i\right)$.We have:

$$f_{X_{i-1},X_i}\left(x_{i-1},x_i\right)=f_{X_{i-1}}\left(x_{i-1}\right)\cdot f_{X_i|X_{i-1}}\left(x_i\left|x_{i-1}\right.\right)=f_{\rho(l_{i-1})}\left(x_{i-1}\right)\cdot f_{\rho(l_i-l_{i-1})}\left(x_i-x_{i-1}\right). \tag{25}$$

Furthermore, as stated before, assuming that the first i-1 deliveries have been made from suppliers j_1, j_2, …, j_{i-1}, without taking into account any specific ordering of delivery arrival moments from suppliers j_1, j_2, …, j_{i-1}, the probability that L_{i-1} and L_i are equal to l_{i-1} and l_i , respectively, is $f_{L_{i-1},L_i|j_1,...,j_{i-1}}\left(l_{i-1},l_i\right)$.

Finally, in a similar way as the constant demand rate case, in the case of probabilistic demand we formulate the Relation (26) for $HC_{a_{i-1,i}}$:

$$\begin{aligned}HC_{a_{i-1,i}}=&\sum_{\{j_1,...,j_{i-1}\}\subseteq\{1,2,...,n\}}P\left(j_1,...,j_{i-1}\right)\int_0^{\infty}\int_0^{l_i}\int_0^{s+\sum_{k=1}^{i-1}Q_{j_k}}\int_0^{x_i}\frac{h}{2M}\left(IA_{i-1}\left|j_1,...,j_{i-1}\right.+IB_i\left|j_1,...,j_{i-1}\right.\right)\\&\times\left(IA_{i-1}\left|j_1,...,j_{i-1}\right.-IB_i\left|j_1,...,j_{i-1}\right.\right)f_{\rho(l_{i-1})}\left(x_{i-1}\right)dx_{i-1}f_{\rho(l_i-l_{i-1})}\left(x_i-x_{i-1}\right)dx_i\\&\times f_{L_{i-1},L_i|j_1,...,j_{i-1}}\left(l_{i-1},l_i\right)dl_{i-1}dl_i.\end{aligned} \tag{26}$$

- **State B:** in this state, there are both the holding cost and shortage cost. Similar to the demonstration given for state a, we obtain these relations in the case of probabilistic demand:

$$\begin{aligned}HC_{b_{i-1,i}}=&\sum_{\{j_1,...,j_{i-1}\}\subseteq\{1,2,...,n\}}P\left(j_1,...,j_{i-1}\right)\int_0^{\infty}\int_0^{l_i}\int_{s+\sum_{k=1}^{i-1}Q_{j_k}}^{\infty}\int_0^{s+\sum_{k=1}^{i-1}Q_{j_k}}\frac{h}{2M}\left(IA_{i-1}\left|j_1,...,j_{i-1}\right.\right)^2\\&\times f_{\rho(l_{i-1})}\left(x_{i-1}\right)dx_{i-1}f_{\rho(l_i-l_{i-1})}\left(x_i-x_{i-1}\right)dx_if_{L_{i-1},L_i|j_1,...,j_{i-1}}\left(l_{i-1},l_i\right)dl_{i-1}dl_i.\end{aligned} \tag{27}$$

$$SC_{b_{i-1,i}} = \sum_{\{j_1,\dots,j_{i-1}\}\subseteq\{1,2,\dots,n\}} P\left(j_1,\dots,j_{i-1}\right)\int_0^\infty\int_0^{l_i}\int_{s+\sum_{k=1}^{i-1}Q_{j_k}}^{\infty}\int_0^{s+\sum_{k=1}^{i-1}Q_{j_k}} \frac{\pi}{2M}\left(IB_i\left|j_1,\dots,j_{i-1}\right.\right)^2$$
$$\times f_{\rho(l_{i-1})}\left(x_{i-1}\right)dx_{i-1}f_{\rho(l_i-l_{i-1})}\left(x_i-x_{i-1}\right)dx_i f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right)dl_{i-1}dl_i. \tag{28}$$

- **State C:** in this state, there is only the shortage cost. Similar to the demonstration given for state A, we obtain the following equation for $SC_{c_{i-1,i}}$:

$$SC_{c_{i-1,i}} = \sum_{\{j_1,\dots,j_{i-1}\}\subseteq\{1,2,\dots,n\}} P\left(j_1,\dots,j_{i-1}\right)\int_0^\infty\int_0^{l_i}\int_{s+\sum_{k=1}^{i-1}Q_{j_k}}^{\infty}\int_{s+\sum_{k=1}^{i-1}Q_{j_k}}^{x_i} \frac{\pi}{2M}\left(IA_{i-1}\left|j_1,\dots,j_{i-1}\right.+IB_i\left|j_1,\dots,j_{i-1}\right.\right)$$
$$\times\left(IB_i\left|j_1,\dots,j_{i-1}\right.-IA_{i-1}\left|j_1,\dots,j_{i-1}\right.\right)f_{\rho(l_{i-1})}\left(x_{i-1}\right)dx_{i-1}f_{\rho(l_i-l_{i-1})}\left(x_i-x_{i-1}\right)dx_i$$
$$\times f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right)dl_{i-1}dl_i. \tag{29}$$

Using these equations, we can calculate the costs in segments [1,2], [2,3],…,[n-1,n]. Now, we obtain the costs in segment [n,1]:

- **State A:** As described before, in this state there is only the holding cost. Similar to the description given before, $HC_{a_{n,1}}$ is given by:

$$HC_{a_{n,1}} = \int_0^\infty\int_0^{l_n}\int_0^{s}\int_{x_1}^{s+Q} \frac{h}{2M}\left(IA_n+IB_1\right)\left(IA_n-IB_1\right)f_{\rho(l_n-l_1)}\left(x_n-x_1\right)dx_n f_{\rho(l_1)}\left(x_1\right)dx_1$$
$$\times f_{L_1,L_n}\left(l_1,l_n\right)dl_1dl_n. \tag{30}$$

- **State B:** as stated before, in state B there are holding and shortage cost, which can be obtained by:

$$HC_{b_{n,1}} = \int_0^\infty\int_0^{l_n}\int_s^{s+Q}\int_{x_1}^{s+Q} \frac{h}{2M}\left(IA_n\right)^2 f_{\rho(l_n-l_1)}\left(x_n-x_1\right)dx_n f_{\rho(l_1)}\left(x_1\right)dx_1 f_{L_1,L_n}\left(l_1,l_n\right)dl_1dl_n. \tag{31}$$

$$SC_{b_{n,1}} = \int_0^\infty\int_0^{l_n}\int_s^{s+Q}\int_{x_1}^{s+Q} \frac{\pi}{2M}\left(IB_1\right)^2 f_{\rho(l_n-l_1)}\left(x_n-x_1\right)dx_n f_{\rho(l_1)}\left(x_1\right)dx_1 f_{L_1,L_n}\left(l_1,l_n\right)dl_1dl_n. \tag{32}$$

- **State C:** as stated before, in this state there are both holding cost and shortage cost, which can be obtained by:

$$HC_{c_{n,1}} = -\int_0^{\infty}\int_0^{l_n}\int_0^{s}\int_{s+Q}^{\infty} \frac{h}{2M}\left(IB_1\right)^2 f_{\rho(l_n-l_1)}\left(x_n - x_1\right)dx_n f_{\rho(l_1)}\left(x_1\right)dx_1 f_{L_1,L_n}\left(l_1,l_n\right)dl_1 dl_n. \tag{33}$$

$$SC_{c_{n,1}} = -\int_0^{\infty}\int_0^{l_n}\int_0^{s}\int_{s+Q}^{\infty} \frac{\pi}{2M}\left(IA_n\right)^2 f_{\rho(l_n-l_1)}\left(x_n - x_1\right)dx_n f_{\rho(l_1)}\left(x_1\right)dx_1 f_{L_1,L_n}\left(l_1,l_n\right)dl_1 dl_n. \tag{34}$$

- **State D:** in this state there is only shortage cost, which can be obtained by:

$$SC_{d_{n,1}} = \int_0^{\infty}\int_0^{l_n}\int_{s+Q}^{\infty}\int_{s}^{x_n} \frac{h}{2M}\left(IA_n + IB_1\right)\left(IA_n - IB_1\right) f_{\rho(l_1)}\left(x_1\right)dx_1 f_{\rho(l_n-l_1)}\left(x_n - x_1\right)dx_n \times f_{L_1,L_n}\left(l_1,l_n\right)dl_1 dl_n. \tag{35}$$

Eventually, the total inventory holding cost and shortage cost per cycle for the case of probabilistic demand, denoted by *HC* and *SC*, can be obtained by adding up those costs in each segment.

Therefore, if we put these new values for inventory holding costs and shortage costs in a cycle in Model (23) the optimization model for minimizing the expected total cost per time unit, denoted by *TC*, when demand rate is a random variable, is obtained. The expanded version of this model is given in Appendix 2.

Model 2 is a nonlinear model in which lead times and demand rate are probabilistic.

2.6. Computational Results

The developed model in this paper is a general model that can be used with different lead time distributions. The only comparable model in the literature is that of Sedarage et *al.* (1999), which is an approximate model. To find the advantages of our model with regard to their model, in section 7.1, we compare both models with that of Ramasesh et *al.* (1993), which is an exact model assuming no crossover among orders. In section 7.2, we concentrate on order splitting among n suppliers, and seek to find the effect of the number of suppliers, *n*, on the system performance measures, such as reorder level *s*, total order quantity *Q*, and the expected total cost per time unit TC^*.

The Hessian matrix with the dual or multiple sourcing model turns out to be very complex even for the simplest case where there are only two potential identical suppliers whose unit procurement prices and lead time distributions are the same and the order quantity is split equally between them. (See Ramasesh et *al*. (1991)) Since then we resort to an extensive grid search to ensure global optimality of the total cost function. Moreover due to the fact that no closed form solution exists for the values of s and Q_i that minimizes the expected total cost function, we use a numerical procedure to find the optimal values of our NLP problem. Therefore we use a numerical search routine based on the sequential quadratic programming (SQP) algorithm and we used MAPLE 11 for finding the solution.

2.6.1. Comparison of the Developed Model against Similar Models

In this section, we compare our analytical model one with those of Ramasesh et *al.* (1993) and Sedarage et *al.* (1999) to evaluate the performance of our model compared to other models in the literature. To be able to compare the three models, we assume that the order is split between two suppliers, lead times are exponentially distributed with parameters λ_1 and λ_2, and the unit demand rate is constant. Since the number of suppliers is constant, we use A as the total ordering cost per cycle, instead of $K_1 + n.K_2$, similar to Ramasesh et *al.* (1993).

First, we investigate the effect of the lead time's mean and standard deviation on the system performance measures, such as reorder level s, order quantity Q_i for supplier i (i=1, 2), and the expected total cost per time unit TC^*. We set the following parameter values: c_1= 5, c_2= 4.95, A= 100, π= 10, M= 9600, λ_1= 24, h= 1. The results are given in Table 2. As shown in Table 2, in all experiments, the results of our developed model 1 and Ramasesh et *al.* (1993) are the same. There are two other columns in Table 2. As stated before, the total cost in Sedarage et *al.* (1999) only approximates the real total cost. To determine the real total cost when their model is used, we put the decision variable results obtained from their model in Ramasesh et *al.'s* (1993) model. Hence, the column "Real TC" is obtained. Thus, we can compare the performance of their model in each experiment with that of Ramasesh et *al.* (1993), or rather, with the real system provided that there is no order crossover. The percentages of these performance differences are shown in the last column. As shown in Table 2, Sedarage et *al.* (1999) model always underestimate the total cost compared with Ramasesh et *al.'s* (1993) model and our model. Additionally, when lead time means and variances increase, the differences between Sedarage et *al.'s* (1999) model and the other two models increase, i.e. its approximation of the inventory system deteriorates. As seen in Table 2, when lead time means and deviations increase, TC^* increases in all models.

Table 2. Investigation of the effect of the lead time mean and deviation, λ

	Model 1 & Ramasesh et *al.* (1993)				Sedarage et *al.* (1999)					
λ_2	s^*	Q_1^*	Q_2^*	TC^*	s^*	Q_1^*	Q_2^*	TC^*	Real *TC*	Difference (%)
24	104	660	1323	49133	84	186	1506	48780	49298	0.34
20	125	831	1272	49206	99	313	1452	48804	49398	0.39
16	150	1085	1181	49317	116	495	1387	48840	49546	0.46
12	175	1497	988	49498	134	786	1314	48903	49777	0.56
9	217	1952	589	49715	144	1160	1265	49002	50055	0.68

Then, we investigate the effect of *h* on the system performance measures. As seen in Table 3, when *h* increases, Q_i (*i*=1, 2) decreases while TC^* increases. According to Table 3, in all experiments our developed model coincides with Ramasesh et *al's.* (1993) model, and Sedarage et *al.'s* (1999) model underestimates the total cost compared with Ramasesh et *al.'s* (1993) model. Additionally, for smaller *h* values the differences between Sedarage et *al.* (1999) model and the other two models increase, i.e. its approximation of the inventory system deteriorates.

Table 3. Investigation of the effect of unit holding cost, *h*.

	Model 1 & Ramasesh et *al.* (1993)				Sedarage et *al.* (1999)					
h	s^*	Q_1^*	Q_2^*	TC^*	s^*	Q_1^*	Q_2^*	TC^*	Real *TC*	Difference (%)
1	150	1085	1181	49317	115	495	1387	48840	49547	0.47
2	65	938	843	49960	73	486	1000	49213	50170	0.42
3	21	840	708	50408	53	468	851	49451	50589	0.36
4	0	766	626	50754	40	451	772	49626	50915	0.32
5	0	700	561	51037	30	435	722	49762	51186	0.29

In Table 4, we investigate the effect of π on the system performance measures. As seen, when π increases, the difference between Sedarage et *al.*'s (1999) model, which is an approximate model, and the other two models increases, which means that the approximation of Sedarage et *al.'s* (1999) model of the inventory system deteriorates. Furthermore, in all of the experiments the results of our model coincide with those of Ramasesh et *al.* (1993). According to Table 4, when π increases, TC^* in all model increases.

Table 4. Investigation of the effect of unit shortage cost per time unit, π.

	Model 1 & Ramasesh et *al.* (1993)				Sedarage et *al.* (1999)					
π	s^*	Q_1^*	Q_2^*	TC^*	s^*	Q_1^*	Q_2^*	TC^*	Real *TC*	Difference (%)
10	150	1085	1181	49317	116	495	1387	48840	49546	0.46
20	302	1276	1130	49555	185	652	1287	48969	49954	0.81
30	395	1385	1112	49700	228	737	1242	49045	50246	1.10
40	463	1462	1105	49805	259	794	1215	49100	50485	1.37
50	515	1522	1102	49887	284	838	1197	49143	50687	1.60

Then, we investigate the effect of *A* on the system performance measures. As seen in Table 5, in all experiments our developed model coincides with that of Ramasesh et *al.* (1993).

Table 5. Investigation of the effect of ordering cost per cycle, *A*.

	Model 1 & Ramasesh et *al.* (1993)				Sedarage et *al.* (1999)					
A	s^*	Q_1^*	Q_2^*	TC^*	s^*	Q_1^*	Q_2^*	TC^*	Real TC	Difference (%)
50	216	980	983	49090	167	451	1132	48563	49329	0.49
100	150	1085	1181	49317	116	495	1387	48840	49546	0.46
200	73	1214	1509	49701	65	523	1838	49292	49915	0.43
300	26	1290	1800	50031	35	529	2229	49668	50234	0.41
400	0	1334	2073	50326	15	529	2577	49995	50519	0.38

Next, we investigate the effect of *M* on the system performance measures. As seen in Table 6, in all models when *M* increases, Q_i (i=1, 2) and TC^* increase as well. As shown in Table 6, in all experiments, the model developed in this paper has the same results as Ramasesh et *al.* (1993). Besides, Sedarage et *al.'s* (1999) model always underestimates the total cost compared with real inventory system and when *M* increases, the difference between their model and the other two models increases.

Table 6. Investigation of the effect of unit demand, *M*.

	Model 1 & Ramasesh et *al*. (1993)				Sedarage et *al*. (1999)					
M	s^*	Q_1^*	Q_2^*	TC^*	s^*	Q_1^*	Q_2^*	TC^*	Real TC	Difference (%)
5000	40	628	773	25871	35	272	939	25656	25983	0.43
6000	61	733	865	30972	50	323	1041	30702	31110	0.45
7000	83	833	995	36070	67	372	1139	35744	36234	0.45
8000	107	931	1043	41166	85	420	1236	40783	41355	0.46
9000	133	1028	1130	46261	104	467	1331	45819	46475	0.46

Clearly, in all experiments in this section our developed model coincides with Ramasesh et *al.'s* (1993) model. On the other hand, since Sedarage et *al.'s* (1999) model is only an approximation of real inventory system, it always underestimates the total cost. According to our extensive experiments, when the mean demand rate increases, or the ordering cost per cycle decreases ,or π increases, or lead time mean and variance increase, the difference between Sedarage et *al.'s* (1999) model and the other two model increases, i.e. its approximation of the inventory system deteriorates. On the other hand, our developed analytical model, like that of Sedarage et *al*. (1999), is a general model which can be used when there are more than two suppliers, or the suppliers' lead times are of any distribution, whereas as stated before, in Ramasesh et *al*.'s (1993) model, orders can be split only between two suppliers and both suppliers are assumed to have exponential lead time distributions. Thus, the model developed in this article has many advantages over the models in the literature. Observing Tables 2 to 6, one may conclude that there is not significant differences between the results of two models since the values in the last column are not high. The answer is that these tables are designed to show the trends in differences between the results of the two models when systems variables change, and as was discussed in prior lines when for instance π is high and/or mean demand rate is high etc. this difference will be much more significant. As an example, consider in an experiment we have the following parameter values: c_1= 1, c_2= 1.25, A= 50, π= 200, M= 10000, λ_1= 5, λ_2= 8, h= 1. In this problem the decision variables s, Q_1, Q_2 from our model (and Ramasesh et *al.'s* (1993) model would be 3077, 3012, and 5649 respectively, and TC^* in our model will be 19026. On the other hand, the decision variables for model Sedarage et *al*. (1999) are 1761, 3110, and 2999 respectively, and TC^* in their model will be 15214. If we put the decision variable results from their model into our model, the real total cost will be 26949. This shows the difference between the two models is more than 41% and that is a very significant difference.

2.6.2. Numerical Analysis for Order Splitting among *n* Suppliers

In this subsection, we concentrate on order splitting among n suppliers to find out:

- effect of number of suppliers, n, on the system performance measures, such as reorder level s, total order quantity Q, and the expected total cost per time unit TC^*.
- effect of system parameters on the optimal number of suppliers denoted by n^*, and on the system costs.

In order to reveal the general trend more clearly and to avoid possible over-complexity due to other factors, it is further assumed that suppliers have similar characteristics. In other words, lead times are identically and exponentially distributed, and unit purchasing costs, c_i, are all equal (we consider equal 0). Note that in this section, as stated in Section 3, the total ordering cost per cycle is given by $\$(K_1+ K_2)$.

(1) First, we investigate the effect of M. We set the following parameter values: K_1= 100, K_2= 50, π= 70, h= 1, and lead times, as mentioned before, are identically and exponentially distributed with λ_i= 16. Also, the unit time demand is constant. The results are given in Table 7. As seen in the table, for every n, when M increases, TC^* and order quantity to supplier i, Q_i, increases. We derived this result for two suppliers in the past section, and now, we see the same result for every n. Besides, when M increases, the optimal number of suppliers denoted by n^* increases as well. In other words, when demand rate increases, we must use more suppliers to decrease the costs.

(2) We then investigate the effect of ordering cost parameters K_1 and K_2. We set the following parameter values: M= 9000, π= 200, h= 1, and lead time, as mentioned, are identically and exponentially distributed with λ_i= 5. The results are given in Table 8. As shown in the table, for every n, with an increase in the ordering cost parameters, $TC^*$and order quantity Q_i increase for all suppliers. In addition, increasing the ordering cost parameters results in a decrease in the optimal number of suppliers, denoted by n^*. It means that when the ordering cost per cycle increases, we must use fewer suppliers to decrease the

annual system costs.

Table 7. Investigation of the effect of unit demand, *M,* on the system performance measures.

M	*n*	s^*	Q_i^*	TC^*
500	1	52	419	**440**
	2	14	232	445
	3	4	170	483
	4	0	138	524
	5	0	119	565
1000	1	123	613	675
	2	38	338	**641**
	3	14	244	680
	4	5	197	733
	5	0.5	169	787
2000	1	284	908	1069
	2	99	517	**948**
	3	44	363	969
	4	21	288	1026
	5	9	244	1094
3000	1	458	1151	1425
	2	169	683	1216
	3	80	476	**1208**
	4	42	370	1259
	5	22	309	1331
5000	1	826	1563	2088
	2	325	998	1716
	3	162	696	**1641**
	4	91	535	1665
	5	53	439	1728
15000	1	2841	3092	5115
	2	1224	2442	4048
	3	649	1760	3625
	4	395	1348	3470
	5	258	1089	**3436**

(3) Then we investigate the effect of *h*. We set the following parameter values: M= 1000, π= 200, K_1= 250, K_2= 500, λ_i= 5. As shown in Table 9, for every *n*, when *h* increases, TC* increases whereas the total order quantity *Q* decreases. Furthermore, when *h* increases, the optimal number of suppliers increases as well.

Table 8. the effect of ordering cost parameters on the system performance measures.

K_1	K_2	n	s^*	Q_i^*	TC^*
50	10	1	8521	2287	9901
		2	3798	4543	8151
		3	2093	3513	6929
		4	1374	2720	**6267**
	50	1	8353	2722	10044
		2	3747	4607	8230
		3	2067	3541	7031
		4	1354	2740	**6399**
	3000	1	6568	9360	14181
		2	2595	6880	**12772**
		3	1348	5008	13238
		4	811	4027	14334
100	10	1	8319	2812	10077
		2	3766	4583	8200
		3	2082	3525	6972
		4	1368	2726	**6309**
	100	1	8087	3453	10335
		2	3663	4717	8375
		3	2026	3587	7199
		4	1324	2771	**6603**
	500	1	7567	5083	11171
		2	3345	5179	9101
		3	1837	3827	8170
		4	1179	2953	**7861**
500	25	1	7637	4845	11035
		2	3544	4879	8609
		3	1993	3625	7345
		4	1313	2783	**6684**
	500	1	7282	6118	11812
		2	3233	5365	9442
		3	1789	3900	8480
		4	1151	2997	**8163**
	2000	1	6704	8656	13631
		2	2744	6430	11724
		3	1456	4623	**11649**
		4	894	3635	12247
1500	500	1	6852	7939	13089
		2	3021	5768	10250
		3	1688	4071	9233
		4	1089	3104	**8901**
	2000	1	6471	9889	14601
		2	2644	6725	12408
		3	1410	4774	**12288**
		4	867	3744	12857
	3000	1	6292	10948	15465
		2	2486	7273	**13694**
		3	1300	5226	14089
		4	784	4189	15129

Table 9. Investigation of the effect of unit holding cost, *h,* on the system performance measures.

h	*n*	s^*	Q_i^*	TC^*
1	1	665	1439	1906
	2	253	912	**1763**
	3	130	668	1877
	4	76	549	2051
5	1	471	760	5242
	2	176	550	4454
	3	80	395	**4342**
	4	39	307	4488
10	1	382	589	8039
	2	139	450	6725
	3	58	327	**6373**
	4	23	253	6418
20	1	291	460	12067
	2	101	367	10040
	3	36	270	9333
	4	8	210	**9208**

(4) We then investigate the effect of π. We set: M= 5000, h= 1, K_1= 250, K_2= 500, λ_i= 5. As it is evident from Table 10, for every n, when π increases, TC* and total order quantity Q increase as well. Furthermore, when π increases, n* increases too.

Table 10. Investigation of the effect of unit shortage cost per time unit, π, on the system performance measures.

π	*n*	s^*	Q_i^*	TC^*
20	1	1686	3799	4573
	2	537	2608	**4004**
	3	177	1873	4017
	4	28	1456	4238
50	1	2574	3799	5461
	2	996	2800	4604
	3	475	2012	**4453**
	4	250	1563	4574
100	1	3257	3799	6144
	2	1349	2967	5085
	3	703	2136	**4811**
	4	419	1653	4857
200	1	3945	3799	6832
	2	1704	3152	5585
	3	930	2275	5190
	4	589	1756	**5164**

5) Finally, we investigate the effect of the lead time mean and deviation on the system

performance measures. In order to reveal the effect of lead time on order splitting more clearly, we use Erlang distribution for lead times. Erlang distribution is one of the primary distributions used for building some other distributions. It has two parameters: the scale parameter b>0 and the shape parameter c>0, which is an integer. Its probability density function is $\frac{(x/b)^{c-1}\exp(-x/b)}{b(c-1)!}$. The mean and the variance of Erlang distribution are bc and b^2c, respectively.

We set the following parameter values: M= 5000, h= 1, π = 200, K_1= 250, K_2= 500. Furthermore, we assume that lead times are identically distributed. As it is evident from Table 11, we have solved the order splitting problem for different values of b and c and for different number of suppliers, n. Evidently, the experiments contain 2 groups. In the first group, c= 1 and with an increase in b, the mean and the variance of suppliers lead times increase. In the second group, c= 2 and the expected values of lead times are equal to the first group, but the lead time variances of this group are less than the first group. As a result, we can investigate the effect of changes in the mean and the variance of lead times separately.

Table 11. Investigation of the effect of the lead time on the system performance measures.

c	b	n	s^*	Q_i^*	TC^*
1	0.02	1	195	2840	**2936**
		2	51	1793	3537
		3	15	1405	4132
		4	1	1192	4669
	0.15	1	2804	3549	5636
		2	1167	2615	4737
		3	626	1868	**4607**
		4	388	1456	4761
	0.2	1	3945	3799	6832
		2	1704	3152	5585
		3	930	2275	5190
		4	589	1756	**5164**
2	0.01	1	142	2799	**2842**
		2	58	1785	3529
		3	31	1403	4142
		4	17	1191	4685
	0.075	1	1921	3197	4370
		2	972	2036	**4052**
		3	644	1500	4271
		4	481	1234	4624
	0.1	1	2699	3355	5062
		2	1402	2295	**4433**
		3	935	1660	4452
		4	702	1323	4686

As shown in Table 11, for every *n*, when the means and/or the variances of lead times increase, the total order quantity, *Q,* also increases. Furthermore, the optimal number of suppliers, denoted by n^*, increases as well. The effect of the lead time's mean and variance on *TC** is a little complex. According to the literature, when the lead time's mean and variance increase, the expected total cost per time unit increases. However, according to Table 11, this is not always true. Consider the case where c= 1 and b= 0.02. In this case, the lead time's mean and variance are 0.02 and 0.0004, respectively. Furthermore, in the case of c= 2 and b= 0.01, the lead time's mean and variance are 0.02 and 0.0002 respectively. As seen for n=4, in spite of the fact that in the second case the lead time variance is less, TC^* is higher than in the first case. In other words, for n= 4, the expected total cost per time unit increases from TC^*= 4669 to TC^*= 4685, although the lead time's variance decreases from 0.0004 to 0.0002. As another example, for n= 4 and c= 2, while for b=0.01, the expected total cost per time unit equals 4685, it is smaller for b= 0.07, where the lead time's mean and variance are greater, and equals to 4624. According to the experiments, for larger numbers of n, this occurrence is more likely to happen, and as we know when the number of suppliers increases, rises in the lead time's mean and variance has less effect on the lead time risk. Consequently, there is presumably another factor which compensates the effect of greater lead time's mean and variance on the total cost.

Now, we investigate the effect of the number of suppliers on the system performance measures. According to all of the tables in this section, when the number of suppliers, *n*, increases, and other parameters are constant, the total order quantity, *Q*, always increases. Furthermore, the total cost, *TC**, first decreases and then increases as the number of suppliers increases.

2.7. Conclusion

In this paper, we investigated general n- supplier inventory systems, in which supplier lead times are random variables. We developed an optimization model to determine both the reorder level and the order-split quantities simultaneously by minimizing the expected total cost, including procurement cost, ordering cost, inventory holding cost and shortage cost. We also compared our developed model with the models in the literature to highlight the relative advantages of our model.

The model developed in this paper is the only exact model in the literature that considers all possible number of suppliers for order splitting, assuming that there is no order crossover. In case of dual sourcing, our model obtained the same results as that of Ramasesh et *al.* (1993). Our extensive numerical experiments, described in section 7, to compare the model developed in this paper and that of Sedarage et *al.* (1999), resulted the following findings. Firstly their model always underestimates the total cost compared to ours. Secondly, when the mean demand rate increases, the ordering cost per cycle decreases, π increases, or the lead time's mean and variance increases, the difference between our analytical model and Sedarage et *al.*'s (1999) model increases; in other words, their model's approximation of the inventory system deteriorates.

There are several promising areas for further research. First, in this research, we did not consider transportation costs between suppliers and the buyer as an individual cost in our system. In fact, we considered it as part of the ordering cost; however, this is not always true. If transportation cost is added to total cost, we can decide about the means of transportation and the distance between suppliers and the buyer in our model. Another area for further research is considering the case when the suppliers have constraints in their capacity and deliveries. This constraint surely affects the selection of suppliers and splitting orders among them. Third, in our model suppliers' lead times are assumed to be independent of each other and of order quantities. This is not always true in reality. One important extension is considering n-supplier inventory systems when lead times are dependent.

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Appendix 1

minimize TC

$$
\begin{aligned}
&= \frac{M}{\sum_{i=1}^{n} Q_i} \Bigg\{ K_1 + nK_2 + \sum_{i=1}^{n} c_i Q_i \\
&+ \sum_{i=2}^{n} \Bigg\{ \sum_{\{j_1,\dots,j_{i-1}\}\subseteq\{1,\dots,n\}} P\left(j_1,\dots,j_{i-1}\right) \Bigg(\int_{0}^{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}} \int_{0}^{l_i} \frac{h}{2M}\left(2s + 2\sum_{k=1}^{i-1} Q_{j_k} - Ml_{i-1} - Ml_i\right) \\
&\times \left(M l_i - M l_{i-1}\right) f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right) d l_{i-1} d l_i + \int_{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}}^{\infty} \int_{0}^{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}} \frac{h}{2M} \\
&\times \left(s + \sum_{k=1}^{i-1} Q_{j_k} - M l_{i-1}\right)^2 f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right) d l_{i-1} d l_i \Bigg) \Bigg\} + \Bigg[\int_{l_1=0}^{\frac{s}{M}} \int_{l_n=l_1}^{\frac{s+Q}{M}} \frac{h}{2M} \\
&\times \left(2s + Q - Ml_n - Ml_1\right)\left(Q - Ml_n + Ml_1\right) f_{L_1,L_n}\left(l_1,l_n\right) d l_n d l_1 + \int_{l_1=\frac{s}{M}}^{\frac{s+Q}{M}} \int_{l_n=l_1}^{\frac{s+Q}{M}} \frac{h}{2M} \\
&\times \left(s + Q - Ml_n\right)^2 f_{L_1,L_n}\left(l_1,l_n\right) d l_n d l_1 - \int_{l_1=0}^{\frac{s}{M}} \int_{l_n=\frac{s+Q}{M}}^{\infty} \frac{h}{2M}\left(s - Ml_1\right)^2 f_{L_1,L_n}\left(l_1,l_n\right) d l_n d l_1 \Bigg] \\
&+ \sum_{i=2}^{n} \Bigg\{ \sum_{\{j_1,\dots,j_{i-1}\}\subseteq\{1,\dots,n\}} P\left(j_1,\dots,j_{i-1}\right) \Bigg(\int_{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}}^{\infty} \int_{0}^{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}} \frac{\pi}{2M}\left(s + \sum_{k=1}^{i-1} Q_{j_k} - M l_i\right)^2 \\
&\times f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right) d l_{i-1} d l_i + \int_{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}}^{\infty} \int_{\frac{s+\sum_{k=1}^{i-1} Q_{j_k}}{M}}^{l_i} \frac{\pi}{2M}\left(2s + 2\sum_{k=1}^{i-1} Q_{j_k} - M l_{i-1} - M l_i\right) \\
&\times \left(M l_{i-1} - M l_i\right) f_{L_{i-1},L_i|j_1,\dots,j_{i-1}}\left(l_{i-1},l_i\right) d l_{i-1} d l_i \Bigg) \Bigg\} + \Bigg[\int_{l_1=\frac{s}{M}}^{\frac{s+Q}{M}} \int_{l_n=l_1}^{\frac{s+Q}{M}} \frac{\pi}{2M}\left(s - M l_1\right)^2 \\
&\times f_{L_1,L_n}\left(l_1,l_n\right) d l_n d l_1 - \int_{l_1=0}^{\frac{s}{M}} \int_{l_n=\frac{s+Q}{M}}^{\infty} \frac{\pi}{2M}\left(s + Q - M l_n\right)^2 f_{L_1,L_n}\left(l_1,l_n\right) d l_n d l_1 \\
&+ \int_{l_n=\frac{s+Q}{M}}^{\infty} \int_{l_1=\frac{s}{M}}^{l_n} \frac{\pi}{2M} \times \left(2s + Q - Ml_n - Ml_1\right)\left(M l_n - M l_1 - Q\right) f_{L_1,L_n}\left(l_1,l_n\right) d l_n d l_1 \Bigg] \Bigg\}.
\end{aligned}
$$

s.t. $Q_i > 0$ for $i = 1,2,\dots,n,\ s \ge 0$.

Appendix 2

$$\min TC$$

$$= \frac{M}{\sum_{i=1}^{n} Q_i} \left\{ K_1 + nK_2 + \sum_{i=1}^{n} c_i Q_i \right.$$

$$+ \sum_{i=2}^{n} \left\{ \sum_{\{j_1,\ldots,j_{i-1}\} \subseteq \{1,2,\ldots,n\}} P(j_1,\ldots,j_{i-1}) \left\{ \int_0^{\infty} \int_0^{l_i} \int_0^{s+\sum_{k=1}^{i-1} Q_{j_k}} \int_0^{x_i} \frac{h}{2M} \left(2s + 2\sum_{k=1}^{i-1} Q_{j_k} - x_{i-1} - x_i \right) \right. \right.$$

$$\times (x_i - x_{i-1}) f_{\rho(l_{i-1})}(x_{i-1}) dx_{i-1} f_{\rho(l_i - l_{i-1})}(x_i - x_{i-1}) dx_i f_{L_{i-1},L_i | j_1,\ldots,j_{i-1}}(l_{i-1}, l_i) dl_{i-1} dl_i$$

$$+ \int_0^{\infty} \int_0^{l_i} \int_{s+\sum_{k=1}^{i-1} Q_{j_k}}^{\infty} \int_0^{s+\sum_{k=1}^{i-1} Q_{j_k}} \frac{h}{2M} \left(s + \sum_{k=1}^{i-1} Q_{j_k} - x_{i-1} \right)^2 f_{\rho(l_{i-1})}(x_{i-1}) dx_{i-1} f_{\rho(l_i - l_{i-1})}(x_i - x_{i-1}) dx_i$$

$$\left. \left. \times f_{L_{i-1},L_i | j_1,\ldots,j_{i-1}}(l_{i-1}, l_i) dl_{i-1} dl_i \right\} \right\} + \left[\int_0^{\infty} \int_0^{l_n} \int_0^{s} \int_{x_1}^{s+Q} \frac{h}{2M} (2s + Q - x_1 - x_n)(Q + x_1 - x_n) \right.$$

$$\times f_{\rho(l_n - l_1)}(x_n - x_1) dx_n f_{\rho(l_1)}(x_1) dx_1 f_{L_1,L_n}(l_1, l_n) dl_1 dl_n$$

$$+ \int_0^{\infty} \int_0^{l_n} \int_s^{s+Q} \int_{x_1}^{s+Q} \frac{h}{2M} (s + Q - x_n)^2 f_{\rho(l_n - l_1)}(x_n - x_1) dx_n f_{\rho(l_1)}(x_1) dx_1 f_{L_1,L_n}(l_1, l_n) dl_1 dl_n$$

$$\left. - \int_0^{\infty} \int_0^{l_n} \int_0^{s} \int_{s+Q}^{\infty} \frac{h}{2M} (s - x_1)^2 f_{\rho(l_n - l_1)}(x_n - x_1) dx_n f_{\rho(l_1)}(x_1) dx_1 f_{L_1,L_n}(l_1, l_n) dl_1 dl_n \right]$$

$$+ \sum_{i=2}^{n} \left\{ \sum_{\{j_1,\ldots,j_{i-1}\} \subseteq \{1,2,\ldots,n\}} P(j_1,\ldots,j_{i-1}) \left\{ \int_0^{\infty} \int_0^{l_i} \int_{s+\sum_{k=1}^{i-1} Q_{j_k}}^{\infty} \int_0^{s+\sum_{k=1}^{i-1} Q_{j_k}} \frac{\pi}{2M} \left(s + \sum_{k=1}^{i-1} Q_{j_k} - x_n \right)^2 \right. \right.$$

$$\times f_{\rho(l_{i-1})}(x_{i-1}) dx_{i-1} f_{\rho(l_i - l_{i-1})}(x_i - x_{i-1}) dx_i f_{L_{i-1},L_i | j_1,\ldots,j_{i-1}}(l_{i-1}, l_i) dl_{i-1} dl_i$$

$$+ \int_0^{\infty} \int_0^{l_i} \int_{s+\sum_{k=1}^{i-1} Q_{j_k}}^{\infty} \int_{s+\sum_{k=1}^{i-1} Q_{j_k}}^{x_i} \frac{\pi}{2M} \left(2s + 2\sum_{k=1}^{i-1} Q_{j_k} - x_{i-1} - x_i \right) (x_{i-1} - x_i)$$

$$\left. \left. \times f_{\rho(l_{i-1})}(x_{i-1}) dx_{i-1} f_{\rho(l_i - l_{i-1})}(x_i - x_{i-1}) dx_i f_{L_{i-1},L_i | j_1,\ldots,j_{i-1}}(l_{i-1}, l_i) dl_{i-1} dl_i \right\} \right\}$$

$$+ \left[\int_0^{\infty} \int_0^{l_n} \int_s^{s+Q} \int_{x_1}^{s+Q} \frac{\pi}{2M} (s - x_1)^2 f_{\rho(l_n - l_1)}(x_n - x_1) dx_n f_{\rho(l_1)}(x_1) dx_1 f_{L_1,L_n}(l_1, l_n) dl_1 dl_n \right.$$

$$- \int_0^{\infty} \int_0^{l_n} \int_0^{s} \int_{s+Q}^{\infty} \frac{\pi}{2M} (s + Q - x_n)^2 f_{\rho(l_n - l_1)}(x_n - x_1) dx_n f_{\rho(l_1)}(x_1) dx_1 f_{L_1,L_n}(l_1, l_n) dl_1 dl_n$$

$$+ \int_0^{\infty} \int_0^{l_n} \int_{s+Q}^{\infty} \int_s^{x_n} \frac{h}{2M} (2s + Q - x_n - x_1)(Q + x_1 - x_n) f_{\rho(l_1)}(x_1) dx_1 f_{\rho(l_n - l_1)}(x_n - x_1) dx_n$$

$$\left. \left. \times f_{L_1,L_n}(l_1, l_n) dl_1 dl_n \right] \right\}$$

subject to $Q_i > 0$, for $i = 1,2,\ldots,n$, $s \geq 0$.

3

A semi-Markov decision model for a dual source inventory system with random lead-times: Structure of policy and impact of supply information

A semi-Markov decision model for a dual source inventory system with random lead-times: Structure of policy and impact of supply information

Soheil Abginehchi* Christian Larsen*

Abstract

We consider an inventory system with two potential supply sources. At each supplier there can be no more than one outstanding order, and the replenishment lead-times are independent and Erlangean distributed. We use semi-Markov decision theory to model the system. In this paper we view the problem from two perspectives. The first deals with the structure of the optimal policy when there is order progress information in the system. Our developed model allows the decision-maker the possibility to issue orders to both suppliers simultaneously or to use one supplier as main source and the other as emergency supplier or even to use solely one supplier. Through numerical analyses we explore when each of these three options are optimal. The second perspective concerns the case where the actual progress of a replenishment order cannot be monitored. After having computed the optimal policy under the false assumption of full information, we show that it can be modified to a policy that fits the real assumption. We use numerical tests to explore the performance of this modified policy compared to the ideal (but maybe unrealistic) policy.

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§3.1 Introduction

After more than 20 years of extensive study, the policy of pooling lead-time risk by simultaneously splitting replenishment orders among several suppliers continues to attract the attention of researchers.

The main premise for studying inventory control models with several suppliers is that the lead-times of the suppliers are stochastic, hence it make sense to split any replenishment order into several smaller orders, in order to hedge against the risk of possible long lead-times. In those dual/multiple source models, it is assumed that when deciding for replenishment, the replenishment order is split into suborders and the suborders are issued simultaneously to the suppliers. In this study we let this assumption about simultaneousness be relaxed. We study a dual source system with non-identical suppliers. We are still using most of the assumptions used in the standard model but we model the problem as a semi-Markov decision problem.

Our research is concerned with providing two perspectives on the problem. The first perspective is related to the structure of optimal policy and whether it is of an advantage to make replenishments simultaneously. Our model deals with the environment that allows the decision-maker to choose whether he will simultaneously issue orders to both suppliers or whether he will first issue a single order to one supplier and then await further information concerning his inventory status before making a replenishment order to the other supplier (the latter having some resemblance to an emergency order system). In our cost structure we have a joint replenishment cost, thus making it of some advantage to do joint ordering. So when considering this perspective, we make some numerical analyses where we explore how large this cost figure must be in order to make it in-optimal to deviate from making replenishments simultaneously.

The second perspective concerns whether the supply process is observable and the value of information regarding the visibility of supply process. In the Markov decision model we assume that there are some observable phases which an outstanding order has to go through until it arrives at the stock of the retailer, and that the retailer has information about the current phase of the outstanding order. We call this type of visibility "order progress information" (OPI). One solution to enable this kind of tracking is the sensor technology known as radiofrequency identification (RFID), which utilizes small transponders that can be affixed to a variety of surfaces on spare parts, entire products, containers, or any kind of equipment. Other high-tech means of capturing OPI include global positioning system (GPS) modules or GSM/UMTS (universal mobile telecommunication system) network nodes as used in cell-phones.

On the other hand, this assumption may be unrealistic or the retailer may not have the information about these phases for an outstanding order. Therefore we also propose more simple policies where information about the current phase of any outstanding order is ignored, and compare the performance of such a policy with the best possible policy in order to evaluate the impact of supply information.

The remainder of the paper is organized as follows. In section 3.2 we review the relevant literature. Section 3.3 discusses the assumptions of our model. In section 3.4 the semi-Markov decision model for our problem is developed. The structure of the optimal policy is discussed in section 3.5. In section 3.6 we develop the modified model without order progress information. The algorithms for solving these models are introduced in section 3.7. The numerical studies are performed in section 3.8. Finally section 3.9 contains our conclusions from this study, and suggestions for further research.

§3.2 Literature review

There are two streams of research related to this paper. One focuses on inventory models when there are more than one main source for the same product, for example when dual sourcing is implemented. The purpose is to achieve improved delivery performance by splitting orders between independent suppliers. Normally the lead-times of suppliers in these models are random so by using different suppliers the risk is pooled between suppliers. Sculli and Wu (1981) appear to be the first to present the concept of order splitting. In such models the main order is split into suborders and these suborders are placed simultaneously among different suppliers when the reorder level is reached.

Ramasesh et al. (1991) seems to be the first to consider the order splitting approach in a cost minimization, or economic context. In their model and in models by other researchers who have assessed the benefits of order splitting in an economic context, the total cost for ordering, purchasing prices, inventory holding, and stock out penalties are minimized. Their model deals with the case where there are two potential suppliers, both of which have identical lead-time distributions, being either uniform or exponential, and demand is assumed to be constant. Using numerical search they claim that dual sourcing is often better than single sourcing.

A similar approach has been used subsequently by other researchers, see Mohebbi and Posner (1998), Chiang and Chiang (1996), Tyworth and Ruiz-Torres (2000), Ryu and Lee (2003). Recently Abginehchi and Farahani (2010) use order splitting in a stochastic inventory system with n potential suppliers and through numerical search they find the optimal number of suppliers for minimizing the total cost comprising fixed and variable ordering cost and inventory holding and backorder costs. In their model lead-times are random variables that follow any of known probability functions. This model is an improvement on the work by Sedarage et al. (1999) with the same assumptions.

The general research results of the above are that first, simultaneous order splitting is a way to reduce the safety stock needed to meet service targets or alternatively to reduce the expected number of backorders for a prescribed level of safety stock. Second, because two or more smaller orders are issued instead of one big order size the cycle stock will be reduced. Third, the incremental ordering cost of the second and subsequent orders may be relatively small in a variety of settings. (Thomas and Tyworth (2006)). For the case of dual sourcing, the cost of placing two split orders simultaneously in articles is usually of the form αK, in which K is the cost of placing one single order and $1 \leq \alpha \leq 2$. Recognizing that some, but not all, of the components of the ordering cost would be common to the vendors, we expect that the total ordering cost should be between K and $2K$, or αK. In our model we use $K + K_1 + K_2$ for the cost of placing two split orders simultaneously, and $K + K_j$, $j = 1, 2$ for the cost for placing an order to supplier j. This is the approach used by Sedarage et al. (1999) and Abginehchi and Farahani (2010).

The order splitting literature has some common features. In these researches the only way of placing orders is through splitting orders simultaneously between suppliers. In our paper, however, we do not restrict ourselves to this assumption and let the model decide if splitting orders simultaneously is optimal or whether it is better to place one order and then we await further information about inventory level before placing an order to the second supplier. Consequently we consider the effect of information when deciding about ordering policy contrary to the pure order splitting literature in which such information does not play any role. Another point is that in most previous models the inventory system has been modeled using non-linear programming. In our paper, however, we use a semi-Markov decision model for modeling the inventory system. We still use most of the common assumptions in the literature.

The other stream of research related to this paper is emergency ordering. In many large-scale inventory systems, options exist for expediting orders through the use of alternative modes of resupply. In emergency situations, for instance, orders that are normally shipped by sea or land and thus have relatively long lead-times could be shipped by air by which lead-times are lower but cost is higher as well. There are various versions of this problem. Some researchers have studied the single planning problem with an option of emergency ordering after having more information about the stochastic demand in the planning period. On the other hand, there is a stream of research that considers periodic review inventory system with emergency orders (Chiang (2001), Chiang and Gutierrez (1998), Feng et al. (2006)). Chiang (2003) considers periodic review inventory with two supply modes for ordinary and emergency orders. In the paper, unlike most earlier periodic review models, supply lead-time does not need to be a multiple of the review period and they let the review period be possibly larger than lead-time.

Some papers consider problems in which there is an option for expediting the outstanding order. As a result the fixed lead-time l will be converted into the faster but costlier emergency one with lead-time $l_e < l$. This option is usually chosen in cases of low inventory. The works by Allen and D'Esopo (1968), Dohi et al. (06), Dohi et al. (1999), Duran et al. (2004), Chiang (2002) are all in this group. The literature on expediting orders with two random lead-times proves that stochastic lead-times seem to significantly influence the cost-minimal policy. Our work is different from theirs in that outstanding orders are not expedited here.

Most related to our work are problems on continuous review inventory systems with two supply modes. Moinzadeh and Nahmias (1988) is one of the primary papers which has addressed the emergency ordering. They consider an inventory control with two suppliers with constant lead-times where the emergency lead-time is lower than the normal one. The policy parameters are (Q_1, Q_2, R_1, R_2), which is a natural extension of the continuous review (R, Q) policy. It is assumed that there is never more than a single order outstanding of each type. In our model we use the same assumption. Their model further assumes that an emergency order is issued only if it arrives before the normal outstanding order. This is a sensible assumption when lead-times are constant like in their model but in our model we do not have a restriction about the timing of placing orders to each supplier since the lead-times are assumed to be random. In their model demands that are not met are backordered. In our model we use the lost sale assumption. Moinzadeh and Nahmias (1988) use some approximations to find the expected inventory and shortages in a cycle. According to their numerical studies using the model with two supply modes is the most dominating over simple supply mode model when stock out costs are very large and setup costs are small.

Mohebbi and Posner (1999) use the (Q_1, Q_2, R_1, R_2) policy introduced by Moinzadeh and Nahmias (1988) for a continuous review inventory system with lost sales. In this system demands follow a compound Poisson distribution and lead-times for both regular and emergency orders are assumed to be exponentially distributed. Only one order of each type can be outstanding at any point in time. They use the level-crossing method to find the stationary distribution function of inventory level and thereby obtaining long-run average of total cost. Due to the complex form of total expected costs in their model they resort to numerical search to find the optimal values of the control policy parameters, although they admit the existence of local minima that may raise doubts about the optimality of their obtained results.

Johansen and Thorstenson (1998) consider an option for using two supply modes in which normal orders have constant lead-times (L) and are controlled by continuous review (R, Q) policy. Emergency orders, on the other hand, have constant but much shorter lead-times ($1/L$). Normal orders can be issued when no order is outstanding. Unmet demands are backorders. When there is an outstanding normal order, we have the option for issuing emergency order when the remaining time of the phase is j/L time units, where $j = L, L-1, \ldots, 1$. Then the state in this phase is shown by (i, j) where i is net inventory. The policy for emergency orders specifies the size $q(i, j)$ of the emergency order to be issued for each state (i, j). Although demand follows Poisson process during the normal-order lead-time, demand is approximated by a Bernoulli process. In our model, however, orders can be placed to each of the two suppliers at any time.

Axsater (2007) considers two supply modes in a continuous review inventory system. For normal orders the continuous review (R, Q) policy is used. At any time the warehouse can also use emergency orders with any size with additional cost. Demands follow compound Poisson process. Both lead-times are constant and the emergency mode supply lead-time is lower than that of the normal mode. Unmet demands are backordered. Given the reorder point policy for normal replenishments, the decision problem is to determine the timing and size of the emergency replenishments. He uses a heuristic for this problem. The heuristic is only able to find the suitable emergency replenishment policy.

Ng et al. (2001) consider a special case of emergency ordering in a two level supply chain with two warehouses and two retailers. Each retailer uses its own warehouse as the main source of supply but in the case that the warehouse is out of stock there is an opportunity to order from the other supplier with higher transportation cost or the retailer's order can be backordered in its own warehouse until the warehouse receives its order from an outside supplier. Transportation times from each warehouse to each retailer are the same and constant. Also the lead-times of warehouses are constant and equal. Any stock-out at both levels is completely backordered. There is a transportation cost for each facility per order. Ordering replenishment policy both at the retailers and at the warehouses are continuous review (R, Q) policy. This problem can be classified as an expediting orders problem, since by having the opportunity to transfer the retailer's order to the supplier which has stock on hand, the retailer is expediting its order by paying higher transportation cost. Because of the complexity of the system they use the results of simulation study to estimate the performance of their models.

Gaukler et al. (2008) consider a continuous review (R, Q) system when there is visibility in the supply process. The lead-time of the ordinary order is comprised of some stages and an outstanding order remains in each stage with a known distribution function. The retailer can track the current stage of the outstanding order. Using the outstanding order's progress information the retailer can decide when to place an emergency order with a known constant lead-time. For this problem they develop an approximate model, but because they cannot find an analytical approach for constructing an average cost expression even for the simplest case of exponential sojourn times they use simulation. Through a numerical study they compare this policy to the traditional (R, Q) without order progress information and conclude that their policy is most beneficial when lead-times are long and very uncertain.

Recently Giri and Dohi (2009) argue that managers need not only pay attention to the economical aspects; they have to take the reliability of the inventory system into account as well. They define reliability as the probability that the stock is not depleted until a pre-specified time. In the model they introduce, if the stock is depleted up to some time t_0 the emergency order with a shorter fixed lead-time is placed, otherwise at time t_0 a regular order with a longer fixed lead-time is placed. They try to find the optimal inventory policy (t_0^*, Q^*) that maximizes the cost effectiveness and they define cost effectiveness as the mean time interval that the stock is not depleted in one cycle divided by total expected costs in a cycle.

In all these models the supply lead-time for the emergency mode is lower than the lead-time of the normal mode. In addition both lead-times for normal and emergency supply modes are considered to be constant and predetermined. In our model, however, lead-times for each supplier are random variables, which is closer to reality. Furthermore, unlike the models in the literature, the main and emergency suppliers are not predetermined and the model decides which supplier is the main source and which one is a backup or emergency source. In other words, it is possible that a supplier with higher lead-time is chosen as the main source because of the conditions of the problem as we will see in numerical tests. It is also possible that the model decides the two suppliers must be the main source as discussed before. None of these situations have been considered in the related literature.

§3.3 assumptions

The demand is a Poisson process with intensity λ. We assume that there is a fixed order size $q_j, j = 1, 2$ when making a replenishment order to supplier j. Furthermore, we assume that only one order can be outstanding at each supplier. As long as this constraint is met, the retailer can issue an order to each of the two suppliers or it can make two orders simultaneously at any time. If only a replenishment order to supplier j ($j = 1, 2$) is incurred then the replenishment cost is $K + K_j$. Here K is the constant portion of the ordering cost which is independent of the number of suppliers, and K_j, ($j = 1, 2$) is the portion of the ordering cost which depends on the supplier we are ordering to. On the other hand, if the replenishment orders are issued simultaneously a replenishment cost of $K + K_1 + K_2$ is incurred. This is a reasonable assumption which some authors like Sedarage et al. (1999) and Abginehchi and Farahani (2010) have used, because by splitting orders simultaneously some, but not all, of the components of the ordering cost would be common to the vendors. Thus, the cost parameter K is a sort of joint order cost and if it is large compared to the two supplier-specific order costs K_1 and K_2 then there is an incentive to make replenishments simultaneously. The inventory system incurs inventory costs at a rate h per unit per unit time. Any demand that can not be met immediately is lost and incurs a cost π per unit.

We assume that the lead-time of supplier j is R_j-phased Erlangean distributed with mean R_j/μ_j. In our primary model we assume that these phases or stages are observable by the retailer, which means that we interpret the supply process to be in R_j phases with identical phase durations and independent distributions with mean $1/\mu_j$. The supply process starts in phase R_j, then proceeds to phase $R_j - 1$ and so forth. When phase 1 is terminated the outstanding order arrives to the inventory.

§3.4 Mathematical model

In order to model the above-mentioned inventory system we use the semi-Markov decision model. The state space is defined as triple (i, r_1, r_2) where i is the on-hand inventory and $r_j (j = 1, 2)$ attain the values $0, 1, \ldots, R_j$. When $r_j \geq 1$ it means that an order is under way from supplier j and it is in phase r_j. If $r_j = 0$ it means that no orders are currently at supplier j. In order to have a finite state space and facilitate the computations we assume that $i \leq i_{\max}$. If $i_{\max}$ is chosen sufficiently large it possesses no limitations in the optimal decisions.

When observing state (i, r_1, r_2) two decisions are made, $a_1(i, r_1, r_2)$ and $a_2(i, r_1, r_2)$. The decision variable $a_j(i, r_1, r_2)$ is binary. If 1 it means that a replenishment order of size q_j is issued to supplier j and if 0 it means no replenishment decision is made. In order to fulfil the requirement that only one order from each supplier can be outstanding the restriction on $a_j(i, r_1, r_2)$ is that it can (but does not have to) only be 1 if $r_j = 0$. This limitation is only for modeling purposes and in reality when expected lead-times are small enough compared to order sizes there is no need for the limitation of having a maximum of one outstanding order and our model very well can describe the reality.

When observing state (i, r_1, r_2) and making decision (a_1, a_2) then the expected time until the next decision epoch are given by

$$\tau_{(i,r_1,r_2)}(a_1,a_2) = \frac{1}{\lambda + \sum_{j=1}^{2} \mu_j \mathrm{I}_{\{r_j(1-a_j)+R_j a_j \geq 1\}}} \tag{3.1}$$

where I_A is an indicator function which is 1 if condition A is true and 0 otherwise.

Furthermore, when observing state (i, r_1, r_2) and making decision (a_1, a_2) then the total expected costs until the next decision epoch is given by

$$c_{(i,r_1,r_2)}(a_1,a_2) = \frac{hi + \lambda\pi \mathrm{I}_{\{i=0\}}}{\lambda + \sum_{j=1}^{2} \mu_j \mathrm{I}_{\{r_j(1-a_j)+R_j a_j \geq 1\}}} + K\max(a_1,a_2) + \sum_{j=1}^{2} K_j a_j. \tag{3.2}$$

Finally, the probability that at the next decision epoch the system will be in the state (i', r_1', r_2') if decision (a_1, a_2) is chosen in the present state (i, r_1, r_2) is given by the following equations. If next state is so that

$$i' = \max(i-1,0);\, r_j' = r_j(1-a_j) + R_j a_j;\, j = 1,2$$

then

$$\Pr_{(i,r_1,r_2)(i',r_1',r_2')}(a_1,a_2) = \frac{\lambda}{\lambda + \sum_{j=1}^{2} \mu_j \mathrm{I}_{\{r_j(1-a_j)+R_j a_j \geq 1\}}}. \tag{3.3}$$

If next state is so that

$$\begin{aligned} i' &= \min(i+q_1, i_{\max})\mathrm{I}_{\{r_1(1-a_1)+R_1 a_1 = 1\}} + i\mathrm{I}_{\{r_1(1-a_1)+R_1 a_1 \neq 1\}}; \\ r_1' &= r_1(1-a_1) + R_1 a_1 - 1;\, r_2' = r_2(1-a2) + R_2 a_2, \end{aligned}$$

then

$$\Pr_{(i,r_1,r_2)(i',r_1',r_2')}(a_1,a_2) = \frac{\mu_1 \mathrm{I}_{\{r_1(1-a_1)+R_1 a_1 \geq 1\}}}{\lambda + \sum_{j=1}^{2} \mu_j \mathrm{I}_{\{r_j(1-a_j)+R_j a_j \geq 1\}}} \tag{3.4}$$

If in the next state

$$\begin{aligned} i' =& \min(i + q_2, i_{\max})\mathrm{I}_{\{r_2(1-a_2)+R_2a_2=1\}} + i\mathrm{I}_{\{r_2(1-a_2)+R_2a_2\neq1\}}; \\ r_1' =& r_1(1 - a_1) + R_1a_1; r_2' = r_2(1 - a2) + R_2a_2 - 1 \end{aligned}$$

then

$$\Pr_{(i,r_1,r_2)(i',r_1',r_2')}(a_1, a_2) = \frac{\mu_2 \mathrm{I}_{\{r_2(1-a_2)+R_2a_2\geq1\}}}{\lambda + \sum_{j=1}^{2} \mu_j \mathrm{I}_{\{r_j(1-a_j)+R_ja_j\geq1\}}}. \tag{3.5}$$

And otherwise the probability is 0.

Note that the state space limitation implies that if the incoming order makes the inventory level go above $i_{\max}$ all items in excess of $i_{\max}$ are discarded.

A stationary policy R is a rule which adds to each state (i, r_1, r_2) a single action $R_{(i,r_1,r_2)} \in A((i, r_1, r_2))$ and prescribes to always take this action whenever the system is observed in state (i, r_1, r_2) at a decision epoch. $A((i, r_1, r_2))$ is the set of possible actions for each observed state (i, r_1, r_2). For example, for state $(i, 0, 0)$ the set of possible actions is $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Now let

$$X_n = \text{the state of the system at the } n\text{th decision epoch.}$$

Define the random variable $Z(t)$ by

$$Z(t) = \text{the total costs incurred up to time } t, t \geq 0.$$

Denote by $\mathrm{E}_{(i,r_1,r_2),R}$ the expectation operator when the initial state $X_0 = (i, r_1, r_2)$ and policy R is used. Then the limit $g_{(i,r_1,r_2)}(R) = \lim_{t\to\infty} \frac{1}{t}\mathrm{E}_{(i,r_1,r_2),R}[Z(t)]$ in our problem equals the long-run actual average cost per time unit and does not depend on the initial state $X_0 = (i, r_1, r_2)$ (Tijms (2003a)). A stationary policy R^* is said to be optimal decision policy if $g_{(i,r_1,r_2)}(R^*) \leq g_{(i,r_1,r_2)}(R)$ for all initial states and stationary policies R.

For a given pair of order sizes (q_1, q_2) the optimal decision policy as well as the resulting minimum costs $C(q_1, q_2)$ can be found using different algorithms found in the literature. For a general review of various algorithms see Tijms (2003a). We use the value-iteration algorithm which is claimed to be an efficient algorithm by Tijms (2003). Furthermore we use an altered version of this algorithm called the *modified iteration-policy algorithm* (Puterman (2005)). Our aim is to establish which is the more efficient algorithm.

Having found the optimal policy for a given (q_1, q_2), the optimal pair (q_1, q_2) can be found as well by varying q_1 and q_2 and repeating this procedure. These algorithms are discussed in section 3.7.

§3.4.1 Model for exponentially distributed lead-times

In the special case when lead-times follow exponential distribution the main model introduced in this section can be simplified as follows. When $R_j = 1$ the lead-time of supplier j is exponentially distributed with mean $1/\mu_j$, $j = 1, 2$. Other assumptions remain unchanged. The state space is again defined as three dimensional and is given by (i, r_1, r_2) where i is the on-hand inventory and r_j, $j = 1, 2$ is a binary where $r_j = 1$ means that an order is under way from supplier j and if $r_j = 0$ it means that no orders are currently at supplier j. When observing state (i, r_1, r_2) two decisions are made, $a_1(i, r_1, r_2)$ and $a_2(i, r_1, r_2)$, respectively. The decision variable $a_j(i, r_1, r_2)$, $j = 1, 2$ is binary and has the same definition as before.

$$\tau_{(i,r_1,r_2)}(a_1, a_2) = \frac{1}{\lambda + \sum_{j=1}^{2} \mu_j \max(a_j, r_j)}. \tag{3.6}$$

Moreover, when observing state (i, r_1, r_2) and making decision (a_1, a_2) then the total expected costs until the next decision epoch are given by

$$c_{(i,r_1,r_2)}(a_1, a_2) = \frac{hi + \lambda\pi \mathrm{I}_{\{i=0\}}}{\lambda + \sum_{j=1}^{2} \mu_j \max(a_j, r_j)} + K \max(a_1, a_2) + \sum_{j=1}^{2} K_j a_j. \tag{3.7}$$

Finally, the probability that at the next decision epoch the system will be in the state (i', r'_1, r'_2) if decision (a_1, a_2) is chosen in the present state (i, r_1, r_2) is given by

$$\Pr_{(i,r_1,r_2)(i',r'_1,r'_2)}(a_1, a_2) =$$
$$\begin{cases} \frac{\lambda}{\lambda+\sum_{j=1}^{2}\mu_j \max(a_j,r_j)} & i' = \max(i-1, 0); r'_j = \max(a_j, r_j), j = 1, 2 \\ \frac{\mu_1 \max(a_j,r_j)}{\lambda+\sum_{j=1}^{2}\mu_j \max(a_j,r_j)} & i' = \min(i+q_1, i_{\max}); r'_1 = 0; r'_2 = \max(a_2, r_2) \\ \frac{\mu_2 \max(a_j,r_j)}{\lambda+\sum_{j=1}^{2}\mu_j \max(a_j,r_j)} & i' = \min(i+q_2, i_{\max}); r'_1 = \max(a_1, r_1); r'_2 = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.8)$$

§3.5 Structure of the optimal policy

In this section we discuss the structure of the optimal policy. First we consider more simple case of exponential lead-times.

We conjecture that in the optimal policy the following must hold:

A. For any pair (r_1, r_2) it holds that $a_j(i, r_1, r_2)$ is non-increasing in i, $j = 1, 2$.

B. For any i it holds that $a_1(i, 0, 0) \geq a_1(i, 0, 1)$ and $a_2(i, 0, 0) \geq a_2(i, 1, 0)$.

Based on these two premises we argue that the optimal policy can be characterized by the quadruple (u, s, c_1, c_2) where u is an integer that can attain the values from 0 to 2 and the three other parameters are integers that can attain values from -1 to $i_{\max}$. The first two parameters prescribe what to do if there are currently no outstanding orders. It states that when the inventory level drops down to level s then a replenishment order of size q_u should be issued to supplier u (in case $u > 0$) and in case $u = 0$ there should be issued simultaneous replenishment orders of sizes q_1 and q_2 to both suppliers. It also means that if $s = -1$ then no replenishment orders will be issued at all. The last two parameters in the quadruple prescribe what to do if one replenishment order has already been issued. Parameter c_1 prescribes that if there is already a replenishment order at supplier 2 then a replenishment order should be issued to supplier 1 if the inventory level drops down to level c_1 (in case $c_1 = -1$ this option is never exploited). Similarly, parameter c_2 prescribes that if there is already a replenishment order at supplier 1 then a replenishment order should be issued to supplier 2 if the inventory level drops down to level c_2 (in case $c_2 = -1$ this option is never exploited). Notice that $c_1 = c_2$ does not have any meaning concerning splitting orders simultaneously, due to the fact that issuing orders simultaneously is impossible when there is already an outstanding order. A mere order splitting happens when $u = 0, s \geq 0, c_1 = c_2 = -1$ then all replenishments are issued simultaneously to both suppliers. The other extreme case that may happen is single sourcing when $u = 1$ or 2, $s \geq 0$, and $c_{3-u} = -1$. Then orders are only issued to supplier u.

Proposition 3.1. *The following holds in optimal policy:* $c_1 \leq s$ *and* $c_2 \leq s$ *if* $u = 0$ *and if* $u > 0$*, then* $c_u \leq s$ *and* $c_{3-u} < s$*.*

Proof. Our proof is the following. Define $s = \max\{i : i_{\max} \geq i \geq 0, (a_1(i,0,0) = 1 \vee a_2(i,0,0) = 1)\}$ where $s = -1$ in case the set is empty. If $s = -1$ define $u = 0$. Otherwise, define $u = 1$ if $a_1(s,0,0) > a_2(s,0,0)$, $u = 2$ if $a_1(s,0,0) < a_2(s,0,0)$ and $u = 0$ if $a_1(s,0,0) = a_2(s,0,0)$. Define $c_1 = \max\{i : a_1(i,0,1) = 1\}$ and $c_2 = \max\{i : a_2(i,1,0) = 1\}$. If $s = -1$ then $c_1 = c_2 = -1$ as well (based on B). So assume $s \geq 0$. If $u = 0$ then it follows from B that $c_1 \leq s$ and $c_2 \leq s$. If $u > 0$, assume without loss of generality that $u = 1$. It holds from B that $c_1 \leq s$. Because $a_2(s,0,0) = 0$ it holds from B that $a_2(s,1,0) = 0$ and from A that $c_2 < s$. □

§3.5.1 Erlangean distributed lead-times

Now it is easy to discuss the structure of optimal policies for Erlangean distributed lead-times. In the optimal policy the following conjectures must hold.

I. For any pair (r_1, r_2) it holds that $a_j(i, r_1, r_2)$ is non-increasing in i, $j = 1, 2$.

II. For any i it holds that $a_1(i,0,1) \leq \cdots \leq a_1(i,0,R_2) \leq a_1(i,0,0)$ and $a_2(i,1,0) \leq \cdots \leq a_2(i,R_1,0) \leq a_2(i,0,0)$.

(II) states that in optimal policy if we need not make an order at some inventory level i when there is no outstanding order, then there is no need to make an order when there is already an outstanding order. Likewise, assume that the inventory on-hand is at some i and in the optimal policy we do not make any order to supplier $j, j = 1, 2$ when there is already an outstanding order from supplier $3 - j$ at phase r_{3-j}. Then, based on our conjecture in (II) in the optimal policy, the retailer will not make any order to supplier $3 - j$ when there is already an outstanding order from supplier $3 - j$ at any phase $r'_{3-j}, r'_{3-j} \leq r_{3-j}$.

Again based on these two premises we argue that the optimal policy can be characterized by $(u, s, c_1^1, \ldots, c_1^{R_2}, c_2^1, \ldots, c_2^{R_1})$ where u is an integer that can attain the values from 0 to 2 and other $R_1 + R_2 + 1$ parameters are integers that can attain values from -1 to $i_{\max}$. The first two parameters prescribe what to do if there currently are no outstanding orders and the definitions are the same as above. $c_1^{r_2}$ prescribes that if there is already a replenishment order at supplier 2 and it is in phase r_2, then a replenishment order should be issued to supplier 1 if the inventory level drops down to level $c_1^{r_2}$. Similarly parameter $c_2^{r_1}$ prescribes that if there is already a replenishment order at supplier 1 and it is in phase r_1, then a replenishment order should be issued to supplier 2 if the inventory level drops down to level $c_2^{r_1}$.

Proposition 3.2. *The following holds in optimal policy:* $c_1^{r_2} \leq s, r_2 = 1, \ldots, R_2$ *and* $c_2^{r_1} \leq s, r_1 = 1, \ldots, R_1$ *if* $u = 0$ *and if* $u > 0$*, then* $c_u^{r_u} \leq s$ *and* $c_{3-u}^{r_{3-u}} < s$.

Proof. Please see the proof for Proposition 4.1. □

§3.6 The model without OPI

There are many cases where there is no visibility in information about order progress. Furthermore the lead-time may not really consist of different phases, but only one whole stage which follows Erlang distribution. Thus the assumption of observable stages in a supply process may not always hold. We therefore now focus our research on how to modify the optimal policy, assuming observable phases, to a policy that does not assume that the phases of the supply processes can be observed. In particular, the focus is to investigate by how much the costs of the policy increase by this. This idea is very much inspired by the work by Ha (2000).

In the model without OPI, state space can be defined by (i, r_1, r_2), where $r_1, r_2 \in \{0, 1\}$ and $0 \leq i \leq i_{\max}$. i is defined as in our main model. When $r_j = 0, j = 1, 2$ it means that there is no outstanding order from supplier j. On the other hand, when $r_j = 1, j = 1, 2$ it means there is an outstanding order from supplier j. This order can be in any phase but because we lack OPI the retailer does not know its current phase (or maybe there are not distinct phases); we just know there is an order outstanding.

In order to convert the optimal policy with OPI to the modified policy without OPI we proceed as follows. The idea is that after finding the optimal decision policy assuming observable lead-time phases, if the majority of decisions $a_1(i, 0, r_2), r_2 = 1, \ldots, R_2$ for a specific i is 1 in the optimal policy then for unobservable phases we choose $a_1(i, 0, 1) = 1$. Otherwise we use $a_1(i, 0, 1) = 0$. The reason is that when the retailer does not know the current phase of an outstanding order (or maybe there are not phases), the retailer makes the same decision for all values of $r_2 = 1, 2, \ldots, R_2$. We implement the same approach in order to obtain $a_2(i, 1, 0)$. When there is no outstanding order the actions taken for the main optimal policy and the new policy are the same. When there is already an outstanding order from one of the suppliers, $(i, 0, r_2)$ or $(i, r_1, 0)$, the algorithm decides about taking action in these states based on the information from optimal policy.

Table 3.1 compares the two policies. In this table, three different areas can be distinguished. In two of these areas in optimal policy, the action taken for $a_1(i, 0, 1)$ is the same as $a_1(i, 0, r_2)$ in optimal policy with OPI, for all positive values of r_2 (1 in the first area and 0 for the third area). As a result decision values for the modified policy are the same as optimal policy with OPI. As seen in Table 3.1, the changes in decision between the two policies take place in the middle area where only some of decisions $a_2(i, r_1, 0)$, $r_1 = 1, 2, \ldots, R_1$ are the same. In Section 3.8 we investigate the increase in costs by using this modified policy instead of the main one. When the order progresses through different stages but with no OPI to allow the retailer

Table 3.1: comparing decision policies with OPI and without OPI

i	r_2					$a_1(i,0,1)$
0	1	1	1	1	1	1
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	0	1	1	1	1	1
4	0	0	1	1	1	1
5	0	0	0	1	1	0
6	0	0	0	0	1	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0

to track the order, the cost difference is equivalent to the value of the OPI for the retailer

§3.7 Specifications of algorithms

In this section we introduce algorithms for solving the models developed in this paper: The model with OPI and the model without OPI.

First we consider the main model, with OPI. We use two developed algorithms for solving the model. These algorithms are value-iteration algorithm and modified-policy iteration algorithm. We then compare the two algorithms with respect to *time*.

A representation of the value-iteration algorithm can be seen in Tijms (2003a). It uses the recursive solution approach from dynamic programming. It computes recursively a sequence of value functions approximating the minimal average cost per time unit. The value functions provide lower and upper bounds on the minimal average cost which converge to the minimal average cost, see Tijms (2003a). The algorithm for a specific q_1 and q_2 is indicated in Algorithm 1.

The bounds m_n and M_n are non-decreasing and non-increasing, respectively, in the iteration counter n. When the algorithm terminates the average costs of the final policy are less than $100\epsilon\%$ above the theoretical minimum. Having found the average costs and the final policy for a particular q_1 and q_2, by changing the values of q_1 and q_2 we are able to find the optimal q_1^* and q_2^*.

Algorithm 1 Value-iteration algorithm

Step 0. Define tolerance parameter ϵ. Choose function $V_0(i,r_1,r_2) := 0$ for all $(i,r_1,r_2) \in I$. Choose $\tau := 1/(\lambda+\mu_1+\mu_2)$. Let $n := 1$.

Step 1. (value-iteration step) For each $(i,r_1,r_2) \in I$ proceed as follows

Step1–1. For each $a_1, a_2 \in A(i,r_1,r_2)$ compute

$$\vartheta(i,r_1,r_2,a_1,a_2) = \frac{c_{(i,r_1,r_2)}(a_1,a_2)}{\tau_{(i,r_1,r_2)}(a_1,a_2)} + \frac{\tau}{\tau_{(i,r_1,r_2)}(a_1,a_2)} \sum_{(i',r_1',r_2') \in I} \Pr_{(i,r_1,r_2)(i',r_1',r_2')}(a_1,a_2) \times V_{n-1}(i',r_1',r_2') + \left(1 - \frac{\tau}{\tau_{(i,r_1',r_2')}(a_1,a_2)}\right) V_{n-1}(i,r_1',r_2'). \quad (3.9)$$

Step1–2. Let

$$V_n(i,r_1,r_2) = \min_{a_1,a_2}\{\vartheta(i,r_1,r_2,a_1,a_2)\}$$

Let $R(n)$ be any stationary policy such that the action $(a_1,a_2) = R_{(i,r_1,r_2)}(n)$ minimizes the right-hand side of the equation for $V_n(i,r_1,r_2)$ for each state (i,r_1,r_2).

Step 2. (bounds on the minimal cost) Compute the bounds

$$m_n := \min_{(i,r_1,r_2)\in I}\{V_n(i,r_1,r_2) - V_{n-1}(i,r_1,r_2)\},$$
$$M_n := \max_{(i,r_1,r_2)\in I}\{V_n(i,r_1,r_2) - V_{n-1}(i,r_1,r_2)\}.$$

Step 3. (stopping test) **If** $0 \le M_n - m_n \le \epsilon m_n$ **then** stop with policy $R(n)$

Step 4. (continuation) $n := n+1$ and repeat *step 1.*

The modified policy-iteration algorithm, on the other hand, combines both features of the standard policy-iteration algorithm and the value-iteration algorithm (See Puterman (2005a)). The policy-iteration algorithm requires that in each iteration a system of linear equations of the same size as the state space be solved. Whether it is of type value-iteration or policy-iteration depends on the number of partial policy evaluations $J(n)$. This algorithm proceeds as is indicated in Algorithm 2.

In this algorithm in each iteration n, using value-iteration algorithm, value functions $V_n(i, r_1, r_2)$ and a stationary policy $R(n)$ are acquired. Then these results are used in a partial policy-evaluation to enhance estimations for the value functions $V_n(i, r_1, r_2)$. This evaluation is carried out iteratively, which is repeated $J(n)$ times at iteration n. In the next iteration, these new value functions are used in the value-iteration algorithm again to give new estimations for value functions and stationary policies, and so on. Like policy-iteration, the algorithm contains an improvement step, and an evaluation step; however, the evaluation is not done exactly. Instead it is performed iteratively (Puterman (2005a)).

If $J(n)$ is kept equal to *zero* in all iterations, then the modified policy-iteration algorithm is just the value-iteration algorithm. On the other hand, letting $J(n)$ approach infinity shows that it also includes policy-iteration as a special case. Here we are concerned with how to set the numbers $J(n)$. In Puterman (2005a) three approaches are briefly introduced. In our algorithm we follow the approach that requires two input parameters n_{crit} and J. We set $J(n) = 0$ for $n \leq n_{\text{crit}}$ and $J(n) = J$ for $n > n_{\text{crit}}$.

After many numerical tests it turns out that in general, the modified policy-iteration algorithm presented in Algorithm 2 is much faster than the value-iteration algorithm presented in Algorithm 1. In Table 3.2 we have made a comparison between the different strategies based on the problem with the following parameter values: $h = 10$, $\lambda = 10$, $\pi = 350$, $K = 20$, $K_1 = 50$, $K_2 = 60$, $\mu_1 = 10$, $\mu_2 = 8$, $q_1 = 40$ and $q_2 = 20$. In problem 1 we set $R_1 = 15$, $R_2 = 12$, and in problem 2 they are $R_1 = 7$, $R_2 = 3$. The tests are performed with a computer with CPU T7300 and

Algorithm 2 Modified policy-iteration algorithm

Require: $\epsilon, n_{\text{crit}}, J$
  $n \leftarrow 1$
  $\tau \leftarrow 1/(\lambda + \mu_1 + \mu_2)$
  $V_0(i, r_1, r_2, q_1, q_2) \leftarrow 0, \forall (i, r_1, r_2) \in I$
  for all feasible (q_1, q_2) **do**
    repeat {This loop runs for incremental iterations of a given (q_1, q_2)}
      for all $(i, r_1, r_2) \in I$ **do**
        for all $a_1, a_2 \in A(i, r_1, r_2)$ **do**
          compute $\vartheta(i, r_1, r_2, a_1, a_2)$ from Equation 3.9
        end for
        $V_n(i, r_1, r_2) = \min_{a_1,a_2}\{\vartheta(i, r_1, r_2, a_1, a_2)\}$
        Let $R(n)$ be any stationary policy such that the action $(a_1, a_2) = R_{(i,r_1,r_2)}(n)$ minimizes the right-hand side of the equation for $V_n(i, r_1, r_2)$ for each state (i, r_1, r_2). Call these actions $a_1^n(i, r_1, r_2)$ and $a_2^n(i, r_1, r_2)$
      end for
      $m_n \leftarrow \min_{(i,r_1,r_2)\in I}\{V_n(i, r_1, r_2) - V_{n-1}(i, r_1, r_2)\}$,
      $M_n \leftarrow \max_{(i,r_1,r_2)\in I}\{V_n(i, r_1, r_2) - V_{n-1}(i, r_1, r_2)\}$
      $n' \leftarrow 1$, define $W_0(i, r_1, r_2) \leftarrow V_n(i, r_1, r_2), \forall (i, r_1, r_2) \in I$
      while $n' \leq J(n)$ **do**
        for all $(i, r_1, r_2) \in I$ **do**
          $W_{n'}(i, r_1, r_2) \leftarrow \vartheta(i, r_1, r_2, a_1^n(i, r_1, r_2), a_2^n(i, r_1, r_2))$
        end for
        $n' \leftarrow n' + 1$
      end while
      $V_{n+1}(i, r_1, r_2) \leftarrow W_{J(n)}(i, r_1, r_2)$
      $n \leftarrow n + 2$
    until $M_n - m_n \leq \epsilon m_n$
    $\text{TotalCost}(q_1, q_2) := (m_n + M_n)/2$
    $\text{OptimalPolicy}(\text{q}_1, \text{q}_2) = R(n)$
  end for
  $\text{TotalCost}^* = \min\{\text{TotalCost}(q_1, q_2)\}$,
  $\Rightarrow q_1^*$ and q_2^* and the related optimal policy are obtained

4 GB memory in Turbo mode.

Table 3.2: Different strategies in modified iteration-policy

Problem	n_{crit}	Time $J = 10$	Time $J = 50$
1	0	$19' : 42''$	$21' : 16''$
	10	$19' : 46''$	$21' : 53''$
	20	$20' : 40''$	$21' : 23''$
2	0	$2' : 42''$	$3' : 9''$
	10	$2' : 49''$	$3' : 4''$
	20	$2' : 12''$	$3' : 0''$

First we executed the pure value-iteration algorithm (Algorithm 1). Using a tolerance parameter $\epsilon = 0.001$, the algorithm takes $23'$:$50''$ for problem 1 and $3'$:$40''$ for problem 2 to obtain the result. Then we applied the modified policy-iteration algorithm with different choices of n_{crit} and J (Table 3.2). Comparing algorithms 1 and 2 we conclude that although the value-iteration algorithm is known to be an efficient algorithm (Tijms (2003a)), it is inefficient compared to the modified policy-iteration algorithm which combines features of both value-iteration and policy-iteration. In fact the rate of convergence between lower and upper bounds, m_n and M_n in Algorithm 2 is much higher. A minor improvement is achieved by changing n_{crit} or J. Finally, the optimum values of J and n_{crit} change from problem to problem, but our observations from different problem tests show that normally n_{crit} between 7 and 20 and J between 10 and 30 give the optimal speed to the algorithm.

Algorithm 3 is concerned with the modified policy without OPI. In this algorithm, first we assume that we have full information over stages of lead-times. With this assumption we are able to find the optimal policy using either value-iteration, Algorithm 1, or modified policy-iteration, Algorithm 2, which means that for each state (i, r_1, r_2) we obtain the optimal actions taken (a_1^*, a_2^*). Then in the next step we change this policy to the policy that assumes no OPI in the system. As we explained before, in the new policy state space is defined by (i, r_1, r_2), where $r_1, r_2 \in \{0, 1\}$. When there are no outstanding orders the actions taken in the algorithm for main optimal policy and the new policy are the same as mentioned in Section 3.6. When there is already an outstanding order from one of the suppliers, $(i, 0, 1)$ or $(i, 1, 0)$, the algorithm decides about taking action in these states for the modified policy based on the information from optimal policy. When $\sum_{r_2=1}^{R_2} a_1^*(i, 0, r_2) > R_2/2$ it means when there is an outstanding order from supplier 2 in one of its lead-time phases, in most cases the optimal policy is to make an order to supplier 1. In the modified policy without OPI the algorithm therefore chooses to make an order to supplier 1 when there is an outstanding order from supplier 2 (which can be in any phase of its lead-time). The same approach is used for $a_1(i, 1, 0)$.

Finally, after finding the new policy for the model without OPI, we can find the average total costs of this policy.

Algorithm 3 The algorithm for the model without OPI

Require: ϵ

Find the optimal actions $a_1^*(i, r_1, r_2)$ and $a_2^*(i, r_1, r_2)$ for the model with OPI using either Algorithm 1 or Algorithm 2.
for all $(i, r_1, r_2) \in I$ **do**
  $a_{1\text{new}}(i, 1, 1) \leftarrow 0$
  $a_{2\text{new}}(i, 1, 1) \leftarrow 0$
  $a_{1\text{new}}(i, 0, 0) \leftarrow a_1{}^*(i, 0, 0)$
  $a_{2\text{new}}(i, 0, 0) \leftarrow a_2{}^*(i, 0, 0)$
  if $\sum_{r_2=1}^{R_2} a_1^*(i, 0, r_2) > R_2/2$ **then**
    $a_{1\text{new}}(i, 0, 1) \leftarrow 1$
  else
    $a_{1\text{new}}(i, 0, 1) \leftarrow 0$
  end if
  if $\sum_{r_1=1}^{R_1} a_2^*(i, r_1, 0) > R_1/2$ **then**
    $a_{2\text{new}}(i, 1, 0) \leftarrow 1$
  else
    $a_{2\text{new}}(i, 1, 0) \leftarrow 0$
  end if
end for

§3.8 Numerical studies

In this section we focus on finding the behavior of the model in different parameter settings concerning selecting single sourcing or dual sourcing, and in the case of dual sourcing splitting orders simultaneously or first issuing a single order to one supplier and then awaiting further information concerning the inventory status before making a replenishment order to the other supplier. Furthermore, we compare the performance of the modified policy without OPI and the optimal policy in presence of OPI in a number of test problem sets.

The logic applied when selecting test problems is as follows. To limit our numerical experiments we have to keep some parameters fixed. So we could keep λ and h fixed. In order to make the stock-out scenario unattractive we could set π sufficiently high. We could introduce: $K_{Av} = K+(K_1+K_2)/2$, an average replenishment cost and $L_{Av} = \frac{1}{2}(\frac{R_1}{\mu_1} + \frac{R_2}{\mu_2})$, an average lead-time. If we temporarily assumed everything deterministic and a single supplier, having K_{Av} as the fixed replenishment cost, having $\lambda = h$ and fixed lead-time L_{Av} would give an $EOQ = \sqrt{(2K + K_1 + K_2)}$. As EOQ is also a guideline for how large the state space can be (EOQ is lower bound on $i_{\max}$) we try to keep EOQ moderate, say 20–50.

First, we investigate the effect of ordering cost parameters, K, K_1 and K_2, on the system performance measures. Let the lead-times be exponentially distributed, that is, $R_1 = R_2 = 1$. We set the following parameter values: $h = 10$, $\lambda = 10$, $\pi = 200$, $\mu_1 = 0.4$, $\mu_2 = 0.2$. The results are given in Tables 3.3 and 3.4. In Table 3.3, the values for K, K_1, K_2, are chosen such that $K_{Av} = 800$. On the other hand, in Table 3.4 their values are chosen such that $K_{Av} = 400$. Both tables are separated into three sub-tables: when $K_1 = K_2$, or $K_1 = 2K_2$ or $2K_1 = K_2$. In each of them we determine threshold values of K, K_1, K_2 that show when order splitting is no longer optimal. As shown in Table 3.3 in the case of $K_1 = 2K_2$, only when K is quite low compared to K_1 and K_2 dual sourcing is not optimal anymore. The reason is that although the second suppliers lead-time is twice that of the first supplier, its reduced ordering cost compensates for that and makes order splitting an attractive option in many cases. This is true about Table 3.4 as well. In both tables as long as simultaneously ordering is optimal, s remains almost constant, but the values of s in Table 3.4 are generally higher than in Table 3.3. The reason is that when orders cost is low, we prefer to order more often and in smaller order quantities (q_1 and q_2).

Table 3.3: effect of ordering cost parameters when K_{Av} = 800

K	K_1	K_2	u	s	c_1	c_2	q_1^*	q_2^*	Total cost
700	100	100	0	21	16	6	45	34	637.9
500	300	300	0	20	16	8	45	35	655.4
400	400	400	0	20	17	9	44	35	663.2
200	600	600	0	19	17	10	45	35	680.0
150	650	650	0	20	17	11	44	35	683.4
145	655	655	0	20	17	11	44	35	683.9
50	750	750	1	27	16	9	48	38	684.1
650	100	200	0	21	16	6	44	35	640.6
500	200	400	0	21	17	7	44	34	651.4
275	350	700	0	20	18	8	44	37	667.2
200	400	800	1	30	17	7	46	38	666.9
50	500	1000	1	30	17	6	46	39	662.0
650	200	100	0	21	16	8	43	36	644.1
500	400	200	0	20	16	10	43	37	658.6
200	800	400	0	19	15	13	47	35	686.1
125	900	450	0	19	15	13	48	34	692.4
50	1000	500	2	35	14	13	48	39	692.5

Table 3.4: effect of ordering cost parameters when K_{Av} = 400

K	K_1	K_2	u	s	c_1	c_2	q_1^*	q_2^*	Total cost
300	100	100	0	23	20	13	39	31	575.1
200	200	200	0	23	20	14	39	31	583.8
100	300	300	0	23	21	16	38	31	591.9
65	335	335	0	23	21	16	39	29	594.4
50	350	350	1	29	20	15	39	32	594.5
325	50	100	0	24	21	13	38	29	571.9
250	100	200	0	24	21	14	36	32	576.9
175	150	300	0	24	22	15	35	31	582.1
115	190	380	0	23	21	14	39	30	586.3
100	200	400	1	31	20	13	38	34	585.7
25	250	500	1	31	21	13	38	33	583.2
325	100	50	0	24	20	14	38	31	574.1
250	200	100	0	23	20	15	39	31	581.5
175	300	150	0	23	19	16	41	30	589.1
100	400	200	0	23	20	17	40	30	596.6
64	448	224	0	23	20	18	39	30	600.1
25	500	250	2	35	19	18	42	28	600.3

Another interesting parameter is the lead-time of each supplier. Here by changing the parameters μ_1 and μ_2 we investigate the effect of lead-times on the inventory system. We set the following parameter values: $h = 10$, $\lambda = 10$, $\pi = 350$, $K = 200$, $K_1 = 400$, and $K_2 = 800$. The results are given in Table 3.5 for two values for L_{Av}, 3.75 and 6. Simultaneous order splitting is again preferred as the effect of higher ordering cost for the second supplier (1000) is offset by the high value of μ_2 (lower lead-time), thereby making order splitting the better option. With a rise in μ_1 and a fall in μ_2 while order splitting is still more attractive, this stability becomes more precarious which causes an increase in s as a reply to rise in the risk. On the other hand when dual sourcing without simultaneously splitting orders is optimal then the higher μ_1 is, the lower s will be, which is a a consequence of reduced risk in the system because of lower lead-time. Note that in this case (i.e. dual sourcing without simultaneous order splitting), the main source for the retailer is the supplier with the shorter lead-time and the other supplier is only used occasionally as an emergency source. That is why we claim that by increasing μ_1 the risk in the inventory system decreases, despite the fact that the second supplier's lead-time is increasing.

Finally when $\mu_1 = 1.2$ and $\mu_2 = 0.15$ dual sourcing is not preferable anymore and orders are only given to the supplier with lower lead-times. In general total expected costs are higher when $L_{Av} = 6$ in comparison with $L_{Av} = 3.75$. Concerning changes in q_1 and q_2, our conjecture is that they are related to u and s meaning that when s increases, or risk increases, in each order the preference is to issue a larger quantity, which is sensible.

When we model the inventory system using Erlangean distributed lead-times, two important parameters to consider are R_1 and R_2. Using Erlang distribution one has more control over the mean and variance of supply lead-times. In Table 3.6 we change the values of μ_1 and R_1 in each of the numerical tests so that the average lead-time for each supplier remains constant. There are two subsets in Table 3.6, in one of which the expected lead-time of supplier 1 is kept at 1.25 and the other one's lead-time is 1. In the other subset the expected lead-time of supplier 1 is kept at 0.25 and the other one's lead-time is 0.2. We set the following parameter values:

Table 3.5: effect of lead-time parameters

μ_1	μ_2	u	s	c_1	c_2	q_1^*	q_2^*	Total Cost
0.15	1.2	0	14	1	13	24	46	621.8
0.2	0.4	0	29	22	24	37	48	783
0.3	0.24	1	43	26	21	47	47	804.7
0.4	0.2	1	39	25	15	49	40	762.5
0.55	0.176	1	33	22	8	47	37	699.1
0.8	0.16	1	25	18	0	46	29	622.5
1.2	0.15	1	18	11	-1	43	-	541.6
0.09	1.125	0	16	0	15	24	48	631.8
0.1	0.5	0	28	15	26	33	57	778.6
0.125	0.25	0	40	31	35	47	62	938.3
0.15	0.1875	0	43	37	36	52	61	977.5
0.25	0.125	1	52	37	26	58	53	922.5
0.5	0.1	1	36	28	8	51	37	740.4

$h = 10$, $\lambda = 10$, $\pi = 350$, $K = 150$, $K_1 = 650$, and $K_2 = 650$. In Table 3.6, c_j^r is the reorder level to supplier j, $j = 1, 2$ when there is already an outstanding order from supplier $3 - j$ and it is in phase r of its supply process.

In the first dataset with higher L_{Av}, when $R_1 = 1$ dual sourcing with the second supplier as the main source is optimal. When increasing R_1 to 2, the main source will change to the first supplier and now supplier 2 is used as an emergency source in the optimal solution. By increasing R_1 we can observe a trend in which supplier 1 becomes more and more important as the main source and supplier 2 is used less and less as the second source. For example when $\mu_1 = 8$, $\mu_2 = 1$, $R_1 = 10$, and $R_2 = 1$, the second supplier is not even used when there is an outstanding order in the first five phases of the supply process and it will only be used if the inventory level is too low and the outstanding order from the other supplier has just been released (phase 10) up to phase 6. As is seen there is a tendency that for higher values of R_1, in the optimal policy finally sole sourcing is chosen (the first supplier). We can see that in the dataset with lower L_{Av} this conjecture comes true. As is seen in the case of exponential lead-times ($\mu_1 = 4$, $\mu_2 = 5$, $R_1 = 1$, and $R_2 = 1$) single sourcing is chosen in the optimal solution, and that is sensible because supplier 2 has the lower lead-time. Just by changing R_2 to 2, supplier 2 is not used anymore as our source and single sourcing with the first supplier is preferred and , and the result remains unchanged as R_2 increases. The implication from Table 3.6 is that if R_1 and R_2 are large then the lead-times are close to be deterministic. As random lead-times are a main motivator for introducing dual sourcing we expect that when R_1 and R_2 become larger it is optimal to use only one supplier. In all problem tests in Table 3.6, the average lead-time of supplier 1 is always longer than the other one, but as R_1 becomes larger, the variability in lead-time gets less and that is why the larger lead-time is offset and the supplier becomes the more attractive.. Finally the expected total cost per time unit decreases when R_1 or R_2 increases.

Table 3.6: Effect of number of lead-time phases

μ_1	μ_2	R_1	R_2	u	s	c_1^1	c_2^1	c_2^2	c_2^3	c_2^4	c_2^5	c_2^6	c_2^7	c_2^8	c_2^9	c_2^{10}	q_1^*	q_2^*	Total cost
0.8	1	1	1	2	18	3	6										28	43	587.6
1.6	1	2	1	1	20	8	0	6									43	24	571.7
2.4	1	3	1	1	20	8	0	2	6								44	21	550.6
3.2	1	4	1	1	20	9	0	0	3	7							44	19	537.3
4	1	5	1	1	19	9	-1	-1	1	4	7						44	18	527.2
4.8	1	6	1	1	19	10	-1	-1	-1	2	5	7					44	17	519.7
8	1	10	1	1	19	10	-1	-1	-1	-1	-1	1	3	4	6	7	43	15	502.6
4	5	1	1	2	5	-1	0										-	42	457.2
8	5	2	1	1	5	0	-1	-1									42	12	453.5
12	5	3	1	1	5	-1	-1	-1	-1								42	32	447.7
16	5	4	1	1	5	1	-1	-1	-1	-1							41	8	444.7
20	5	5	1	1	5	-1	-1	-1	-1	-1	-1						41	18	442.8
24	5	6	1	1	5	-1	-1	-1	-1	-1	-1	-1					41	49	441.5
32	5	8	1	1	5	-1	-1	-1	-1	-1	-1	-1	-1	-1			41	20	439

Finally, in the last set of problem tests we are interested in evaluating the effect of considering non-observable stages in total costs. In other words, we would like to know the importance of order tracking. If there is no visibility in information about order progress or if the lead-time does not consist of different phases, but only one whole stage, then the expected total cost for the optimal policy is the lower bound on the real system costs, and any policy will cause higher expected costs. We have already discussed how from the general policy we can construct a modified policy (it is of type (u, s, c_1, c_2)) that is applicable under the assumption of non-observable supply processes. In case each supply process only has one stage, the issue of observability is irrelevant. It is therefore reasonable to assume that the difference in cost will increase when the number of stages increases (but keeping L_{Av} constant). In the problem tests in Table 3.7 we have kept the average lead-time of each supplier constant and made a comparison between the optimal expected total cost assuming observable stages and total expected costs assuming non-observable stages using the modified policy introduced above. As seen in the last column of Table 3.7, order tracking is most important if there are many stages in the supply process and the difference in expected costs increases when R_1 (the number of phases in the supply process) increases. On the other hand, when the number of stages in the supply process is moderate this modified policy is quite safe and its results are very close to optimal.

§3.9 Conclusion

In this paper we have considered the problem of optimal policy for a single-item inventory system in which there are two supply options available, and the lead-times from the suppliers are random variables with Erlang distribution. A long-run average cost model has been formulated using a semi-Markov decision process. When there is fixed order sizes to each supplier, the optimal policy is of the form (u, s, c_1, c_2) and it determines whether to implement order splitting simultaneously or dual sourcing with emergency ordering or single sourcing in optimal policy. In this respect our problem is more general than other similar problems in the literature that only deal with one of these areas. Furthermore, we extended our problem to include the case where order progress stages are not observable and we have introduced a modified policy for the situation where the entire lead-time is Erlangean distributed. In the numerical section we made a comparison between the two models. Using this comparison we can assess the value of order progress information. In addition we have numerically examined the effect of different system parameters on the structure of the optimal policy.

Erlang distribution can be fit to many data sets, which makes our model more useful in practice. We can even use it when lead-time(s) is(are) deterministic. In this

Table 3.7: Effect of considering non-observable stages

μ_1	μ_2	R_1	R_2	u	s	c_1^1	c_2^1	c_2^2	c_2^3	c_2^4	c_2^5	c_2^6	c_2^7	c_2^8	c_2^9	c_2^{10}	c_2^{11}	c_2^{12}	c_2^{13}	c_2^{14}	c_2^{15}	q_1^*	q_2^*	Total Cost OP	Total Cost MP	% of change
0.3	1.5	1	1	2	13	0	9															22	49	591.8	591.8	0.0
0.6	1.5	2	1	0	12	3	6	11														29	38	588.4	593.2	0.8
0.9	1.5	3	1	0	13	5	4	9	12													32	36	582.7	588.0	0.9
1.2	1.5	4	1	0	13	6	2	7	11	13												33	36	577.8	590.3	2.2
1.5	1.5	5	1	1	39	13	0	6	9	12	14											38	29	570.7	596.1	4.5
1.8	1.5	6	1	1	40	15	0	4	8	11	13	15										39	27	563.4	605.3	7.4
2.1	1.5	7	1	1	41	16	0	3	7	10	12	14	15									40	26	555.3	597.7	7.6
2.4	1.5	8	1	1	41	17	0	2	5	8	11	13	14	16								40	25	548.7	603.3	10.0
2.7	1.5	9	1	1	41	18	0	1	5	7	10	12	13	15	16							40	24	543.1	595.4	9.6
4.5	1.5	15	1	1	42	21	-1	-1	0	3	5	7	8	10	11	13	14	15	16	17	17	40	21	520.0	590.0	13.5
9	1.5	30	1	…									…								…	40	17	494.0	602.7	22

situation, we can presume that the deterministic lead-times are virtually Erlangean distributed with many phases, and the outstanding order stays in each phase for an exponentially distributed time which has a mean close to zero. Thereby we can use our model for formulating the system with deterministic lead-time(s).

There are some areas for improvement. First of all, the model is quite slow in finding the optimal solution, specially when R_1 and R_2, parameters in the Erlang function, are large causing a huge state space or when we have to search more combinations of order sizes to find the optimal q_1^* and q_2^*. This situation is even worse when we want high precision in our final solution. In our numerical examples this time could easily be in excess of eight hours. So speeding up the model and algorithm could be of major interest. We expect to cover this in our next paper.

When we consider OPI and observable stages (phases), our model only takes into account the very special case of identically and exponentially distributed times for each stage. This assumption obviously is limiting. Investigating the optimal policy structure for more general situations is left for future research.

In the numerical studies section we performed a number of tests, but more tests can be done regarding the structure of the optimal policy in different conditions and comparing the two models (with and without OPI). However, it should be noticed that these tests are generally very time-consuming. This is why we hope to publish new and more efficient algorithms and models in our next paper.

Finally, refinements or extensions of our present model should also be considered. An interesting generalization is to consider demand processes other than the pure Poisson process. Considering backorders instead of lost-sale could be another interesting area for future research.

4

A semi-Markov decision model for a dual source inventory system with random lead-times: An alternative compact formulation

A semi-Markov decision model for a dual source inventory system with random lead-times: An alternative compact formulation

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Abstract

In this paper we develop a new model for the problem introduced by Abginehchi and Larsen (2012). The model addresses the problem of dual sourcing when lead-times follow Erlang distributions. The objective is to minimize total expected costs comprising holding, shortage and ordering costs. We use the same assumptions as in the paper by Abginehchi and Larsen (2012) and develop a semi-Markov decision model for this problem. We use the results from their model as our benchmark, and we conclude that our model gives the same results as the model by Abginehchi and Larsen (2012). We also develop a heuristic algorithm for obtaining the optimal solution. Our numerical tests show that this heuristic algorithm is quite precise and the results from that have infinitesimal deviation from the optimal policy.

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§4.1 Introduction

Abginehchi and Larsen (2012) address the problem of dual sourcing using semi-Markov decision process when lead-times are Erlangean distributed and they use the modified value iteration algorithm to find the optimal solution for their model. Although the modified value iteration algorithm helps find the optimal solution faster, long computing time is still needed to find the optimal solution. The reason is that the algorithm requires a lot of calculations to be done for all states. The state space in their model is defined as triple (i, r_1, r_2), where $0 \leq i \leq i_{\max}$, and $r_j = 0, 1, \ldots R_j$ for $j = 1, 2$. They define i as inventory on-hand. When $r_j \geq 1$ it means that an order is under way from supplier j and it is in phase r_j. If $r_j = 0$ it means that no orders are currently at supplier j. If we define $i_{\max}$ to limit i values so that the state space is finite for each value of q_1 and q_2, there are $(R_1 + 1)(R_2 + 1)(i_{\max} + 1)$ states to be checked when lead-times of the two suppliers follow Erlang distribution, which causes finding the final solution be a time consuming process.

In this model we use the same assumptions as used in their problem but try to change their model by lowering the state space. We still use semi-Markov decision model to model our problem. Abginehchi and Larsen (2012) have already done many test problems to find out ordering policies in different situations. These policies can be divided into three types of single sourcing, dual sourcing where one supplier is the main source and the other as emergency source, and finally dual sourcing using order splitting simultaneously. Since Abginehchi and Larsen (2012) have addressed these situations in their numerical studies, this paper does not deal with policy structure but focuses on finding ways to improve the convergence speed through an alternative model formulation.

Thus, our aim here is to make a new model to the problem they have defined in order to speed up the algorithm for finding the optimal solution. (We can compare between the two models with respect to the time spent for reaching the optimal solution.) In order to reach this goal we try to lower the state space and as a result decrease the number of calculations for each given q_1 and q_2.

The remainder of this paper is organized as follows. In section (4.2) we describe the assumptions and develop a mathematical model when lead-times follow exponential distribution, then in section (4.3) we generalize the mathematical model for Erlangean distributed lead-times. In section (4.4), a heuristic algorithm for our developed semi-Markov model is developed and its efficiency is evaluated in comparison with the algorithm used by Abginehchi and Larsen (2012). Finally, we conclude with some directions for future work in section (5.6).

§4.2 Model

In order to understand the new model better, we first consider the case of dual sourcing when lead-times follow exponential distribution with the same assumptions as in the main model by Abginehchi and Larsen (2012). Although this case is a special case of the general Erlangean distributed lead-times case we prefer to start with this simpler case so that the idea for reformulation of the problem can be understood better.

In the case of exponentially distributed lead-times, in the Abginehchi and Larsen (2012) model each decision epoch is the time point of either an arriving customer demand or an arriving outstanding replenishment order from a supplier, and then we have a three-dimensional state space (i, r_1, r_2) where i is the on-hand inventory and $r_j (j = 1, 2)$ is a binary where $r_j = 1$ means that an order is under way from supplier j and if $r_j = 0$ it means that no orders are currently at supplier j. We reduce the decision epochs and consequently the state space by noticing that if we have an outstanding order from a supplier even if a demand happens we do not have any option to decide about ordering to that supplier, due to the fact that according to our assumptions only one order can be outstanding from each supplier at any time $(r_j = 0, 1; j = 1, 2)$.

However, in our developed model, the next decision epoch can be one of these time moments:

- If there is not any outstanding order $(r_j = 0; j = 0, 1)$ the next decision epoch is the time of a demand arrival.

- If there is only one outstanding order from either supplier 1 or 2 the next decision epoch is the time of a demand or outstanding order arrival.

- If our current decision causes both r_1 and r_2 to turn to 1 then the next decision epoch is the time of an order arrival.

The state space in our new model is therefore two-dimensional and is defined by (i, r) where i is the same as before, but r is defined as below.

$$r = \begin{cases} 0 & \text{if there is not no outstanding order,} \\ j, j = 1, 2 & \text{if an order is under way from supplier } j. \end{cases} \tag{4.1}$$

Before going any further we define the following notations. These notations are used both for the model for exponentially distributed lead-times, this section, and the extended model for Erlangean distributed lead-times, next section.

$L_j, j = 1, 2$ random variable of the lead-time of supplier j which has the probability density function $g_{L_j}(x_j)$ and cumulative distribution function $G_{L_j}(x_j)$. According to our assumptions in general $g_{L_j}(x_j)$ has Erlangean distribution.

$\tau_{(i,r)}(a_1, a_2)$ the expected time until the next decision epoch when the current state is (i, r) and decision (a_1, a_2) is made.

$c_{(i,r)}(a_1, a_2)$ the total expected costs until the next decision epoch when the current state is (i, r) and decision (a_1, a_2) is made.

$ASC(i|r_1, r_2)$ the average shortage cost until the next decision epoch when after making decision in the current decision epoch we have outstanding orders from both suppliers in phases r_1 and r_2, and as a result the next decision epoch is the time of one of the orders arriving. For the exponentially distributed lead-times, $r_1 = r_2 = 1$.

$AHC(i|r_1, r_2)$ the average holding cost until the next decision epoch when after making decision in the current decision epoch we have outstanding orders from both suppliers in phases r_1 and r_2, and as a result the next decision epoch is the time of one the orders arriving. For the exponentially distributed lead-times, $r_1 = r_2 = 1$.

$\Pr_{(i,r)(i',r')}(a_1, a_2)$ the probability of going from state (i, r) to state (i', r') when decision (a_1, a_2) is made.

D_x random variable of demand during lead-time x.

$f^{(1)}_{L_{r_1,r_2}}$ the probability density function of the first effective remaining lead-time when the outstanding orders from suppliers 1 and 2 are in phase r_1 and phase r_2 respectively. In other words, it is the probability density function of the minimum of the two Erlangean distributions.

$E\left(L_{r_1,r_2}\right)$ The expected value of L_{r_1,r_2} which is a random variable with probability $f^{(1)}_{L_{r_1,r_2}}$.

Now we proceed with model formulation for exponentially distributed lead-times. When observing state (i, r) decisions $a_1(i, r)$ and $a_2(i, r)$ are made. Decision variables a_j, $j = 1, 2$ are binary. If 1 it means that a replenishment order of size q_j is issued to the supplier j and if 0 it means no replenishment decision is made. For the exponential case we have $0 \leq a_1 \leq \mathrm{I}_{\{r \neq j\}}, j = 1, 2$, where $\mathrm{I}_{\{A\}}$ is an indicator function which is 1 if condition A is true and 0 otherwise.

Then the expected time until the next decision epoch is given by

$$\tau_{(i,0)}(0, 0) = 1/\lambda \tag{4.2}$$

$$\tau_{(i,0)}(1, 0) = 1/(\lambda + \mu_1) \tag{4.3}$$

$$\tau_{(i,0)}(0, 1) = 1/(\lambda + \mu_2) \tag{4.4}$$

$$\tau_{(i,0)}(1, 1) = 1/(\mu_1 + \mu_2) \tag{4.5}$$

$$\tau_{(i,1)}(0,0) = 1/(\lambda+\mu_1) \tag{4.6}$$

$$\tau_{(i,1)}(0,1) = 1/(\mu_1+\mu_2) \tag{4.7}$$

$$\tau_{(i,2)}(0,0) = 1/(\lambda+\mu_2) \tag{4.8}$$

$$\tau_{(i,2)}(1,0) = 1/(\mu_1+\mu_2) \tag{4.9}$$

To understand how these expected times are obtained consider Equation (4.2). After making the decision $(0,0)$ when we are in state $(i,0)$ the next decision epoch is the time a demand arrives and so the expected time until the next decision epoch in this circumstance is $\frac{1}{\lambda}$. As another example consider Equation (4.7). In state $(i,1)$ after making the decision there will be two outstanding orders from suppliers. According to our definition the next decision epoch is consequently the time when one of the orders arrives, so the expected time is $\frac{1}{\mu_1+\mu_2}$. Other equations above are derived in the same manner.

When observing state (i,r) and making decision (a_1,a_2) the total expected costs until the next decision epoch are given by

$$c_{(i,0)}(0,0) = \frac{hi}{\lambda} + \pi I_{\{i=0\}} \tag{4.10}$$

$$c_{(i,0)}(1,0) = K + K_1 + \frac{hi}{\lambda+\mu_1} + \pi I_{\{i=0\}}\frac{\lambda}{\lambda+\mu_1} \tag{4.11}$$

$$c_{(i,0)}(0,1) = K + K_2 + \frac{hi}{\lambda+\mu_2} + \pi I_{\{i=0\}}\frac{\lambda}{\lambda+\mu_2} \tag{4.12}$$

$$c_{(i,0)}(1,1) = K + K_1 + K_2 + ASC(i|1,1) + AHC(i|1,1) \tag{4.13}$$

$$c_{(i,1)}(0,0) = \frac{hi}{\lambda+\mu_1} + \pi I_{\{i=0\}}\frac{\lambda}{\lambda+\mu_1} \tag{4.14}$$

$$c_{(i,1)}(0,1) = K + K_2 + ASC(i|1,1) + AHC(i|1,1) \tag{4.15}$$

$$c_{(i,2)}(0,0) = \frac{hi}{\lambda+\mu_2} + \pi I_{\{i=0\}}\frac{\lambda}{\lambda+\mu_2} \tag{4.16}$$

$$c_{(i,2)}(1,0) = K + K_1 + ASC(i|1,1) + AHC(i|1,1) \tag{4.17}$$

In order to understand how these expected costs are obtained consider (4.10). Since there is no outstanding order and the decision is to not issue any order, the next decision epoch is the time when a demand arrives. (Refer to (4.2)) Consequently the expected costs until the next decision epoch comprise holding cost of i units in stock and in the case of running out of inventory ($i = 0$) shortage cost of one unit of inventory. In Equations (4.11) and (4.12) because we make an order of size q_1 or q_2, in addition to holding cost or shortage cost for one unit of inventory we have ordering cost to supplier 1 ($K + K1$) or to supplier 2 ($K + K2$), and the next decision epoch is the time of either a demand or an order arrival. Now consider any of Equations (4.13), (4.15), or (4.17). Note that in these equations since exponential distribution is a special case of Erlang distribution with only one phase, for r_1 and r_2 in $ASC(i|r_1, r_2)$ and $AHC(i|r_1, r_2)$ definition we use 1.In these equations the next decision epoch is the time of an order arrival. Using the idea of $ASC(i|, r_2)$ and $AHC(i|r_1, r_2)$ we omit many states in the model by Abginehchi and Larsen (2012) that have no effect on our final optimal solution. In the following we explain how to obtain $ASC(i|1, 1)$ and $AHC(i|1, 1)$. This is very much inspired by the work by Rosling (2002). In the inventory system considered in this paper the customers arrive according to a Poisson process with intensity λ where each customer demands one unit. The inventory system incurs inventory costs at a rate h per unit per unit time. Any demand that can not be met immediately is lost and incurs a cost π per unit. Now assume the inventory system at time 0 has an on-hand inventory level i and there are two outstanding orders under way. If no replenishment orders arrive in-between, then the expected on-hand inventory level at time point ω is

$$OH(i, \omega) = \sum_{d=0}^{i} (i - d) \Pr(D_\omega = d) \tag{4.18}$$

The first time one of the outstanding orders arrives is x with probability function $f^{(1)}_{L_{1,1}}(x)$, which is an exponential variable with rate $\mu_1 + \mu_2$. Thus, the average holding cost until the random time point x is obtained by (4.19).

$$AHC(i|1, 1) = h \int_0^\infty \left[\int_0^x OH(i, \omega)\, \mathrm{d}\omega \right] f^{(1)}_{L_{1,1}}(x)\, \mathrm{d}x \tag{4.19}$$

On the other hand, if no replenishment orders arrive in between, then the expected number of shortages happened at time point ω is

$$SH(i,\omega) = \sum_{d=i}^{\infty} (d-i)\Pr(D_\omega = d) \tag{4.20}$$

Thus, since the shortage is in the form of lost sale, the average shortage cost until the above-mentioned point x is obtained by (4.21).

$$ASC(i|1,1) = \pi \int_0^\infty SH(i,x) f^{(1)}_{L_{1,1}}(x)\,\mathrm{d}x \tag{4.21}$$

When observing state (i,r) and making decision (a_1,a_2) the probability that at the next decision epoch, the system is in state (i',r') is given by the following relations.

$$\Pr_{(i,0)(\max(i-1,0),0)}(0,0) = 1 \tag{4.22}$$

(4.22) holds because when after making the decision in the current decision epoch there are no outstanding orders, the next decision epoch happens when a demand arrives which always causes next state to be $(\max(i-1,0),0)$ if the current state after making the decision is $(i,0)$.

$$\Pr_{(i,0)(\max(i-1,0),1)}(1,0) = \frac{\lambda}{\lambda+\mu_1} \tag{4.23}$$

$$\Pr_{(i,1)(\max(i-1,0),1)}(0,0) = \frac{\lambda}{\lambda+\mu_1} \tag{4.24}$$

$$\Pr_{(i,0)(\max(i-1,0),2)}(0,1) = \frac{\lambda}{\lambda+\mu_2} \tag{4.25}$$

$$\Pr_{(i,2)(\max(i-1,0),2)}(0,0) = \frac{\lambda}{\lambda+\mu_2} \tag{4.26}$$

In (4.23) and (4.24), either there is an outstanding order from supplier 1 after making the decision in the current decision epoch or there is already an outstanding order from this supplier before making the decision and the probability of a demand arriving before the order arrives is calculated. Equations (4.25) and (4.26) have the same explanation but this time for supplier 2.

$$\mathrm{Pr}_{(i,0)(\min(i+q_1,i_{\max}),0)}\,(1,0) = \frac{\mu_1}{\lambda+\mu_1} \tag{4.27}$$

$$\mathrm{Pr}_{(i,1)(\min(i+q_1,i_{\max}),0)}\,(0,0) = \frac{\mu_1}{\lambda+\mu_1} \tag{4.28}$$

$$\mathrm{Pr}_{(i,0)(\min(i+q_2,i_{\max}),0)}\,(0,1) = \frac{\mu_2}{\lambda+\mu_2} \tag{4.29}$$

$$\mathrm{Pr}_{(i,2)(\min(i+q_2,i_{\max}),0)}(0,0) = \frac{\mu_2}{\lambda+\mu_2} \tag{4.30}$$

When in the current decision epoch there is an outstanding order from supplier 1 either before making a decision or after, the probability of the outstanding order arriving before a demand happens is shown in (4.27) and (4.28). When this outstanding order belongs to supplier 2 the probabilities are obtained by Equations (4.29) and (4.30).

(4.31), (4.33) and (4.34) calculate the probability that after making decision in the current decision epoch there are two outstanding orders, which are the maximum number of outstanding orders allowable in the model, and in the next decision epoch the order from supplier 1 arrives first and demand during this time is d_1. This equation holds when $d_1 < i$. In order to obtain (4.32) from the right hand side of (4.31) we can compute (4.31) using integration by parts. If we define $U = x^{d_1}$ and $\mathrm{d}V = \mathrm{e}^{-(\mu_1+\mu_2+\lambda)x}$, using the formula $\int U\,\mathrm{d}V = UV - \int V\,\mathrm{d}U$ iteratively we finally obtain (4.32).

$$\begin{aligned}
\mathrm{Pr}_{(i,0)(\min(i-d_1+q_1,i_{\max}),2)}\,(1,1) &= \int_0^\infty g_{L_1}(x)\,(1-G_{L_2}(x))\,\mathrm{Pr}\,(D_x = d_1)\,\mathrm{d}x \\
&= \int_0^\infty \mu_1 \mathrm{e}^{-(\mu_1+\mu_2+\lambda)x} \frac{(\lambda x)^{d_1}}{d_1!}\,\mathrm{d}x \qquad (4.31)\\
&= \frac{\mu_1 \lambda^{d_1}}{(\mu_1+\mu_2+\lambda)^{d_1+1}}, \quad d_1 < i \qquad (4.32)
\end{aligned}$$

$$\mathrm{Pr}_{(i,1)(\min(i-d_1+q_1,i_{\max}),2)}\,(0,1) = \frac{\mu_1 \lambda^{d_1}}{(\mu_1+\mu_2+\lambda)^{d_1+1}}, \quad d_1 < i \tag{4.33}$$

$$\mathrm{Pr}_{(i,2)(\min(i-d_1+q_1,i_{\max}),2)}\,(1,0) = \frac{\mu_1 \lambda^{d_1}}{(\mu_1+\mu_2+\lambda)^{d_1+1}}, \quad d_1 < i \tag{4.34}$$

On the other hand, when the outstanding order from supplier 2 arrives first we have

Equations (4.35)–(4.37) for such a probability.

$$\Pr_{(i,0)(\min(i-d_2+q_2,i_{\max}),1)}(1,1) = \int_0^\infty g_{L_2}(x)\left(1 - G_{L_1}(x)\right)\Pr\left(D_x = d_2\right)\,\mathrm{d}x$$
$$= \int_0^\infty \mu_2 \mathrm{e}^{-(\mu_1+\mu_2+\lambda)x}\frac{(\lambda x)^{d_2}}{d_2!}\,\mathrm{d}x$$
$$= \frac{\mu_2\lambda^{d_2}}{(\mu_1+\mu_2+\lambda)^{d_2+1}}, \quad d_2 < i \tag{4.35}$$
$$\Pr_{(i,1)(\min(i-d_2+q_2,i_{\max}),1)}(0,1) = \frac{\mu_2\lambda^{d_2}}{(\mu_1+\mu_2+\lambda)^{d_2+1}}, \quad d_2 < i \tag{4.36}$$
$$\Pr_{(i,2)(\min(i-d_2+q_2,i_{\max}),1)}(1,0) = \frac{\mu_2\lambda^{d_2}}{(\mu_1+\mu_2+\lambda)^{d_2+1}}, \quad d_2 < i \tag{4.37}$$

When between the two outstanding orders the outstanding order from supplier 1 arrives first and $d_1 \geq i$, since no backorders are allowed, i turns to q_1 after order arrival. Its probability is shown in Equations (4.38)–(4.40). Analogously (4.41)–(4.43) show the probabilities of when the outstanding order from supplier 2 arrives first and $d_2 \geq i$.

$$\Pr_{(i,0)(q_1,2)}(1,1) = \sum_{d_1=i}^{\infty}\left[\int_0^\infty \mu_1 e^{-(\mu_1+\mu_2+\lambda)x}\frac{(\lambda x)^{d_1}}{d_1!}\,\mathrm{d}x\right]$$
$$= \sum_{d_1=i}^{\infty}\left[\frac{\mu_1\lambda^{d_1}}{(\mu_1+\mu_2+\lambda)^{d_1+1}}\right]$$
$$= \frac{\mu_1}{\mu_1+\mu_2+\lambda}\sum_{d_1=i}^{\infty}\left(\frac{\lambda}{\mu_1+\mu_2+\lambda}\right)^{d_1}$$
$$= \left(\frac{\mu_1}{\mu_1+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_1+\mu_2}\right)^i \tag{4.38}$$
$$\Pr_{(i,1)(q_1,2)}(0,1) = \left(\frac{\mu_1}{\mu_1+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_1+\mu_2}\right)^i \tag{4.39}$$

$$\mathrm{Pr}_{(i,2)(q_1,2)}(1,0) = \left(\frac{\mu_1}{\mu_1+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_1+\mu_2}\right)^i \tag{4.40}$$

$$\begin{aligned}\mathrm{Pr}_{(i,0)(q_2,1)}(1,1) &= \sum_{d_2=i}^{\infty}\left[\int_0^{\infty} \mu_2 e^{-(\mu_1+\mu_2+\lambda)x}\frac{(\lambda x)^{d_2}}{d_2!}\,\mathrm{d}x\right] \\ &= \sum_{d_2=i}^{\infty}\left[\frac{\mu_2\lambda^{d_2}}{(\mu_1+\mu_2+\lambda)^{d_2+1}}\right] \\ &= \left(\frac{\mu_2}{\mu_1+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_1+\mu_2}\right)^i \end{aligned} \tag{4.41}$$

$$\mathrm{Pr}_{(i,1)(q_2,1)}(0,1) = \left(\frac{\mu_2}{\mu_1+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_1+\mu_2}\right)^i \tag{4.42}$$

$$\mathrm{Pr}_{(i,2)(q_2,1)}(1,0) = \left(\frac{\mu_2}{\mu_1+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_1+\mu_2}\right)^i \tag{4.43}$$

Now we have all the information for our semi-Markov decision model and we can use a known algorithm to solve it. In next section we expand our model to the case of Erlangean distributed lead-times.

§4.3 Model for Erlangean distributed lead-times

Abginehchi and Larsen (2012) expand their main model for the case of Erlangean distributed lead-times. Lowering the time for reaching the optimal solution is even more important here because it is much more time consuming than the special case of exponential lead-times. To achieve this goal there are two options: improving the model by decreasing the state space; and improving the algorithm to find the solution faster. First we try to improve the model.

In this section we try to make a new model for dual sourcing when supplier lead-times follow Erlang distribution using a semi-Markov decision model. We assume that the lead-time of supplier j has an R_j-phased Erlang distribution with mean $\frac{R_j}{\mu_j}$. We interpret the supply process to be in R_j phases and the duration of each of the phases is identical and independent distribution with mean $1/\mu_j$. The supply process starts in phase R_j, then proceeds to phase $R_j - 1$ and so forth. When phase 1 is terminated the outstanding order arrives to the inventory. Like their model, we assume that the phases of the supply processes are observable.

On the other hand, in the model by Abginehchi and Larsen (2012), each decision epoch is the time of either an arriving customer demand or terminating one phase for an outstanding replenishment order from a supplier and going to the next phase. Then we have a three-dimensional state space (i, r_1, r_2) where i is the on-hand inventory and $r_j (j = 1, 2)$ attain the values $0, 1, \ldots, R_j$. $r_j \geq 1$ means that an order is under way from supplier j and it is in phase r_j.

Based on our understanding from the previous section on the case of exponential lead-times we reduce the decision epochs and consequently the state space in our new model. We use the fact that if there are two outstanding orders, one from each supplier, then there is no option for deciding about ordering. This is independent of the present phase of the orders. Analogously to what we did in the exponential case, the next decision epoch can be one of these time moments:

1. If there is no outstanding replenishment order, then the next decision epoch is the time of a demand arrival.

2. If there is only one outstanding order from supplier j and it is in phase $r_j (j = 1, 2)$, then the next decision epoch is the time of a demand arrival or departure of the outstanding order from phase r_j to phase $r_j - 1$. If $r_j = 0$ it means arrival of the outstanding replenishment order.

3. If there are outstanding orders from both suppliers, then the next decision epoch is the time of the first outstanding order arriving.

So we define our deduced state space as two-dimensional (i, r) in which i has the same definition as before, but now r is defined as follows.

$$r = \begin{cases} 0 & \text{if there is no outstanding order,} \\ r_1, r_1 = 1, \ldots, R_1 & \text{if an order is under way from supplier 1 and it} \\ & \text{is in phase } r_1 \\ r_2, r_2 = -1, \ldots, -R_2 & \text{if an order is under way from supplier 2 and it} \\ & \text{is in phase } -r_2 \end{cases}$$

Using the new state space we save a lot of unnecessary states that exist in the straightforward model by Abginehchi and Larsen (2012) from considering in the developed model here. $(R_1+1)(R_2+1)(i_{\max}+1)$ states in the original model are now reduced to $(R_1+R_2+1)(i_{\max}+1)$ states in our new model.

When observing state (i,r) and making decision (a_1,a_2) the expected time until the next decision epoch is given by the following relations. Recall that L_{r_1,r_2} is the random variable of the first effective remaining lead-time when the outstanding orders from supplier 1 and 2 are in phase r_1 and r_2 respectively.

$$\tau_{(i,0)}(0,0) = \frac{1}{\lambda} \tag{4.44}$$

$$\tau_{(i,0)}(1,0) = \frac{1}{(\lambda+\mu_1)} \tag{4.45}$$

$$\tau_{(i,+r_1)}(0,0) = \frac{1}{(\lambda+\mu_1)} \tag{4.46}$$

$$\tau_{(i,0)}(0,1) = \frac{1}{(\lambda+\mu_2)} \tag{4.47}$$

$$\tau_{(i,-r_2)}(0,0) = \frac{1}{(\lambda+\mu_2)} \tag{4.48}$$

$$\tau_{(i,0)}(1,1) = \mathrm{E}\left(L_{R_1,R_2}\right) \tag{4.49}$$

$$\tau_{(i,+r_1)}(0,1) = \mathrm{E}\left(L_{r_1,R_2}\right) \tag{4.50}$$

$$\tau_{(i,-r_2)}(1,0) = \mathrm{E}\left(L_{R_1,r_2}\right) \tag{4.51}$$

(4.44) shows the expected time until the next decision epoch when there is no order under way. As for (4.45) and (4.46), the next decision epoch is the time when a demand arrives or an outstanding order from supplier 1 departs from the current phase to the preceding phase or the order arrives in case the current phase is 1. The difference is that in (4.45) the current decision epoch is the time when the order has just been given to supplier 1, but in (4.46) the outstanding order is already in phase $+r_1$ in the current decision epoch. When this outstanding order belongs to supplier 2 we have (4.47) and (4.48).

Equations (4.49)–(4.51) occur when in the current decision epoch after making a decision there are two outstanding orders from both suppliers. In this case the next decision epoch is the time of first outstanding order arriving. Consider $\tau_{(i,0)}(1,1)$. For finding $\tau_{(i,0)}(1,1)$ we must note that the buyer just decided to order to both suppliers, which means that these outstanding orders from supplier 1 and supplier 2 are in phases R_1 and R_2 respectively, and $\tau_{(i,0)}(1,1)$ is the expected time for the first effective lead-time.

In order to find $E\left(L_{r_1,r_2}\right)$ we first define $g_{r_j}^{(j)}(x)$ as the probability density function of the remaining lead-time of the outstanding order from supplier j, $j = 1,2$ when the order is in phase r_j. It has Erlang density function with parameters μ_j and r_j $G_{r_j}^{(j)}(x)$ is its cumulative distribution function. Equations (4.52) and (4.53) hold.

$$g_{r_j}^{(j)}(x) = \frac{(\mu_j x)^{r_j-1}\mu_j \mathrm{e}^{-\mu_j x}}{(r_j-1)!} \tag{4.52}$$

$$\begin{aligned} G_{r_j}^{(j)}(x) &= 1 - \mathrm{e}^{-\mu_j x}\left(\sum_{i=0}^{r_j-1}(\mu_j x)^i/i!\right) \\ &= 1 - \frac{\Gamma(r_j,\mu_j x)}{(r_j-1)!} \end{aligned} \tag{4.53}$$

In (4.53), $\Gamma(a,z)$ is defined as an incomplete Gamma function and it is obtained by

$$\Gamma(a,z) = (a-1)!\exp(-z)\left(1 + \frac{z}{1!} + \cdots + \frac{z^{a-1}}{(a-1)!}\right) \tag{4.54}$$

As seen the remaining lead-time of the outstanding order from a supplier has Erlang distribution. Here we used memoryless property of exponential distribution in each phase. Now we are able to find the relation for $f_{L_{r_1,r_2}}^{(1)}$. According to its definition (4.55) holds for $f_{L_{r_1,r_2}}^{(1)}$ (refer to Ross (1997)).

$$f_{L_{r_1,r_2}}^{(1)}(x) = g_{r_1}^{(1)}(x)\left(1 - G_{r_2}^{(2)}(x)\right) + g_{r_2}^{(2)}(x)\left(1 - G_{r_1}^{(1)}(x)\right) \tag{4.55}$$

If we substitute their values, finally we will have the following relation for $f_{L_{r_1,r_2}}^{(1)}$:

$$f_{L_{r_1,r_2}}^{(1)}(x) = \frac{x^{(r_1-1)}\mu_1^{(r_1)}\mathrm{e}^{(-x\mu_1)}\Gamma(r_2,x\mu_2) + x^{(r_2-1)}\mu_2^{(r_2)}\mathrm{e}^{(-x\mu_2)}\Gamma(r_1,x\mu_1)}{\Gamma(r_2)\Gamma(r_1)} \tag{4.56}$$

Having the formula for $f^{(1)}_{L_{r_1,r_2}}$ we can easily find the expected value $E\left(L_{r_1,r_2}\right)$. After simplification we obtain

$$\mathrm{E}\left(L_{r_1,r_2}\right)=\frac{r_1}{\mu_1}+\frac{\left(\frac{r_2}{\mu_2}-\frac{r_1}{\mu_1}\right)\left(\sum_{i=0}^{r_1-1}\binom{r_1+r_2}{i}\mu_1^i\mu_2^{(r_1+r_2-i)}\right)-r_1\binom{r_1+r_2}{r_2}\mu_1^{r_1-1}\mu_2^{r_2}}{(\mu_1+\mu_2)^{r_1+r_2}} \tag{4.57}$$

When observing state (i,r) and making decision (a_1,a_2) then the total expected costs until the next decision epoch are given by

$$c_{(i,0)}(0,0) = \frac{hi}{\lambda}+\pi\mathrm{I}_{\{i=0\}} \tag{4.58}$$

$$c_{(i,0)}(1,0) = K+K_1+\frac{hi}{\lambda+\mu_1}+\pi\mathrm{I}_{\{i=0\}}\frac{\lambda}{\lambda+\mu_1} \tag{4.59}$$

$$c_{(i,+r_1)}(0,0) = \frac{hi}{\lambda+\mu_1}+\pi\mathrm{I}_{\{i=0\}}\frac{\lambda}{\lambda+\mu_1} \tag{4.60}$$

$$c_{(i,0)}(0,1) = K+K_2+\frac{hi}{\lambda+\mu_2}+\pi\mathrm{I}_{\{i=0\}}\frac{\lambda}{\lambda+\mu_2} \tag{4.61}$$

$$c_{(i,-r_2)}(0,0) = \frac{hi}{\lambda+\mu_2}+\pi\mathrm{I}_{\{i=0\}}\frac{\lambda}{\lambda+\mu_2} \tag{4.62}$$

$$c_{(i,0)}(1,1) = K+K_1+K_2+ASC(i|R_1,R_2)+AHC(i|R_1,R_2) \tag{4.63}$$

$$c_{(i,+r_1)}(0,1) = K+K_2+ASC(i|r_1,R_2)+AHC(i|r_1,R_2) \tag{4.64}$$

$$c_{(i,-r_2)}(1,0) = K+K_1+ASC(i|R_1,r_2)+AHC(i|R_1,r_2) \tag{4.65}$$

(4.58) shows the expected costs until the next decision epoch when there is no outstanding order after making a decision. In this case the expected time between the two decision epochs is obtained by (4.58). In (4.59) and (4.60) after making decision there is one outstanding order from supplier 1. The difference is that in (4.59) we just issue an order to supplier 1 but in (4.60) there is already an order underway which is in phase r_1. In both (4.59) and (4.60) the next decision epoch is the time when a demand arrives or when the outstanding order departs from the current phase, R_1 in (4.59) and r_1 in (4.60), to the preceding phase. These explanations are valid for (4.61) and (4.62) with the difference that the outstanding order now belongs to supplier 2. Equations (4.63)–(4.65) show the expected costs when there are two outstanding orders after making a decision. As a result the next decision epoch is when one of them arrives. $AHC(i|r_1, r_2)$ and $ASC(i|r_1, r_2)$ are calculated in Appendix 2 and 3.

When observing state (i, r) and making decision (a_1, a_2) then the probability that at the next decision epoch, the system is in state (i', r') is given by the following relations.

$$\Pr_{(i,0)(\max(i-1,0),0)}(0,0) = 1 \tag{4.66}$$

$$\Pr_{(i,0)(\max(i-1,0),1)}(1,0) = \frac{\lambda}{\lambda+\mu_1} \tag{4.67}$$

$$\Pr_{(i,+r_1)(\max(i-1,0),+r_1)}(0,0) = \frac{\lambda}{\lambda+\mu_1} \tag{4.68}$$

$$\Pr_{(i,0)(\max(i-1,0),-R_2)}(0,1) = \frac{\lambda}{\lambda+\mu_2} \tag{4.69}$$

$$\Pr_{(i,-r_2)(\max(i-1,0),-r_2)}(0,0) = \frac{\lambda}{\lambda+\mu_2} \tag{4.70}$$

$$\Pr_{(i,0)(\min\{(i+q_1)\mathrm{I}_{\{R_1=1\}}+i\mathrm{I}_{\{R_1\neq 1\}},i_{\max}\},R_1-1)}(1,0) = \frac{\mu_1}{\lambda+\mu_1} \tag{4.71}$$

$$\Pr_{(i,r_1)(\min\{(i+q_1)\mathrm{I}_{\{r_1=1\}}+i\mathrm{I}_{\{r_1\neq 1\}},i_{\max}\},r_1-1)}(0,0) = \frac{\mu_1}{\lambda+\mu_1} \tag{4.72}$$

$$\Pr_{(i,0)(\min\{(i+q_2)\mathrm{I}_{\{R_2=1\}}+i\mathrm{I}_{\{R_2\neq 1\}},i_{\max}\},-R_2+1)}(0,1) = \frac{\mu_2}{\lambda+\mu_2} \tag{4.73}$$

$$\Pr_{(i,-r_2)(\min((i+q_2)\mathrm{I}_{\{r_2=1\}}+i\mathrm{I}_{\{r_2\neq 1\}},i_{\max}),-r_2+1)}(0,0) = \frac{\mu_2}{\lambda+\mu_2} \tag{4.74}$$

The explanation for (4.66)–(4.70) is given above. In (4.71) just after making a decision there is an outstanding order from supplier 1, which apparently is in phase R_1, and this equation shows the probability of the order arriving before a demand arrival. When the inventory on-hand in the current decision epoch is i the inventory on-hand in next decision epoch will be either $i + q_1$ in the special case of $R_1 = 1$ or i otherwise. When $R_1 = 1$ this equation is the same as (4.27). In (4.72) there is already an underway order from supplier 1 in its r_1 phase ($r_1 = 1, \ldots, R_1$), and the equation shows the probability that the order arrives before a demand. Again based on the phase r_1 in the next decision epoch the inventory on hand will be either $i + q_1$ or i. When the outstanding order belongs to supplier 2 we use (4.73) and (4.74) instead.

To obtain $\Pr_{(i,0)(i-d_1+q_1,-r2)}(1,1)$, notice that this is the probability that after being in state $(i,0)$ and making the decision $(1,1)$, the outstanding order from supplier 1 arrives first (or in other words, departs from phase R_1 to phase 0) while the outstanding order from supplier 2 departs from phase R_2 to phase r_2. In the meantime the demand is d_1 when $d_1 = 0, 1, 2, \ldots, q_1 - 1$. Therefore using probability theory we can obtain

$$\Pr_{(i,0)(\min(i-d_1+q_1,i_{\max}),-r_2)}(1,1) = \int_0^\infty g^{(1)}_{R_1}(x) \int_0^x g^{(2)}_{R_2-r_2}(t_1)\left(1 - G^{(2)}_1(x-t_1)\right) \mathrm{d}t_1 \Pr(D_x = d_1)\,\mathrm{d}x, \quad d_1 \le i - 1. \tag{4.75}$$

in which $\Pr(D_x = d_1)$ is obtained by

$$\Pr(D_x = d_1) = \frac{\mathrm{e}^{-\lambda x}(\lambda x)^{d_1}}{d_1!} \tag{4.76}$$

To understand why this is the probability we are looking for assume that it takes x time units for the order from supplier 1 to depart from phase R_1 to phase 0, or arrival, with probability $g^{(1)}_{R_1}(x)$. At time x the order from supplier 2 is already in phase r_2 which means that it has traveled $R_2 - r_2$ phases. $g^{(2)}_{R_2-r_2}(t_1)\left(1 - G^{(2)}_1(x-t_1)\right)$ indicates the probability that it takes t_1 time units for the order from supplier 2 to travel from phase R_2 to phase r_2 ($t_1 \le x$) and the time needed to travel one more phase is greater than $x - t_1$. Consequently at time x this order will be in phase r_2. Furthermore for this whole time x we want the demand to be d_1 which has the probability $\Pr(D_x = d_1)$. For the sake of simplicity we use the notation $\Pr(N^{\mu_j}_x = r_j - r'_j)$ instead of $\int_0^x g^{(j)}_{r_j-r'_j}(t_1)\left(1 - G^{(j)}_1(x-t_1)\right)\mathrm{d}t_1$ from now on. We can interpret $N^{\mu_j}_x$ as follows:

$N_x^{\mu_j}$ The number of arrivals in a Poisson process with intensity μ_j seen over an interval of length x

Thus, (4.75) can be rewritten as (4.77)

$$\Pr_{(i,0)(\min(i-d_1+q_1,i_{\max}),-r_2)}(1,1) = \int_0^\infty g_{R_1}^{(1)}(x)\Pr(N_x^{\mu_2} = R_2 - r_2)\Pr(D_x = d_1)\,\mathrm{d}x,\ d_1 \le i-1. \tag{4.77}$$

(4.78) has the same explanation but now applies when the order from supplier 2 arrives first.

$$\Pr_{(i,0)(\min(i-d_2+q_2,i_{\max}),+r_1)}(1,1) = \int_0^\infty g_{R_2}^{(2)}(x)\Pr(N_x^{\mu_1} = R_1 - r_1)\Pr(D_x = d_2)\,\mathrm{d}x,$$
$$d_2 = 0,\ldots,i-1 \tag{4.78}$$

(4.79) shows the situation where there is already an outstanding order from supplier 1 in its r_1 phase, $r_1 = 1,\ldots,R_1$, and after making a decision there is an outstanding order from supplier 2 in phase R_2 too and it shows the probability that the order from supplier 1 arrives first, while the other outstanding order departs to phase r_2. On the other hand, in (4.80) the outstanding order from supplier 2 arrives first and the other order departs from phase r_1 to phase r_1'. We can explain (4.81) and (4.82) in a similar manner.

$$\Pr_{(i,r_1)(\min(i-d_1+q_1,i_{\max}),-r_2)}(0,1) = \int_0^\infty g_{r_1}^{(1)}(x)\Pr(N_x^{\mu_2} = R_2 - r_2)\Pr(D_x = d_1)\,\mathrm{d}x,$$
$$d_1 = 0,1,\ldots,i-1 \tag{4.79}$$

$$\Pr_{(i,r_1)(\min(i-d_2+q_2,i_{\max}),+r_1')}(0,1) = \int_0^\infty g_{R_2}^{(2)}(x)\Pr(N_x^{\mu_1} = r_1 - r_1')\Pr(D_x = d_2)\,\mathrm{d}x,$$
$$d_2 = 0,1,\ldots,i-1 \tag{4.80}$$

$$\text{Pr}_{(i,-r_2)(\min(i-d_1+q_1,i_{\max}),-r_2')}(1,0) = \int_0^\infty g_{R_1}^{(1)}(x)\text{Pr}(N_x^{\mu_2} = r_2 - r_2')\,\text{Pr}(D_x = d_1)\,\mathrm{d}x,$$
$$d_1 = 0, 1, \ldots, i-1 \quad (4.81)$$

$$\text{Pr}_{(i,-r_2)(\min(i-d_2+q_2,i_{\max}),+r_1)}(1,0) = \int_0^\infty g_{r_2}^{(2)}(x)\text{Pr}(N_x^{\mu_1} = R_1 - r_1)\,\text{Pr}(D_x = d_2)\,\mathrm{d}x,$$
$$d_2 = 0, 1, \ldots, i-1. \quad (4.82)$$

When the demand in the current decision epoch is greater than or equal to the current inventory on-hand we use (4.83) to (4.88).

$$\text{Pr}_{(i,0)(q_1,-r_2)}(1,1) = \sum_{d_1=i}^{\infty}\left[\int_0^\infty g_{R_1}^{(1)}(x)\text{Pr}(N_x^{\mu_2} = R_2 - r_2)\text{Pr}(D_x = d_1)\,\mathrm{d}x\right] \quad (4.83)$$

$$\text{Pr}_{(i,0)(q_2,+r_1)}(1,1) = \sum_{d_2=i}^{\infty}\left[\int_0^\infty g_{R_2}^{(2)}(x)\text{Pr}(N_x^{\mu_1} = R_1 - r_1)\text{Pr}(D_x = d_2)\,\mathrm{d}x\right] \quad (4.84)$$

$$\text{Pr}_{(i,r_1)(q_1,-r_2)}(0,1) = \sum_{d_1=i}^{\infty}\left[\int_0^\infty g_{r_1}^{(1)}(x)\text{Pr}(N_x^{\mu_2} = R_2 - r_2)\text{Pr}(D_x = d_1)\,\mathrm{d}x\right] \quad (4.85)$$

$$\text{Pr}_{(i,r_1)(q_2,+r_1')}(0,1) = \sum_{d_2=i}^{\infty}\left[\int_0^\infty g_{R_2}^{(2)}(x)\text{Pr}(N_x^{\mu_1} = r_1 - r_1')\text{Pr}(D_x = d_2)\,\mathrm{d}x\right] \quad (4.86)$$

$$\text{Pr}_{(i,-r_2)(q_1,-r_2')}(1,0) = \sum_{d_1=i}^{\infty}\left[\int_0^\infty g_{R_1}^{(1)}(x)\text{Pr}(N_x^{\mu_2} = r_2 - r_2')\text{Pr}(D_x = d_1)\,\mathrm{d}x\right] \quad (4.87)$$

$$\Pr_{(i,-r_2)(q_2,r_1)}(1,0) = \sum_{d_2=i}^{\infty}\left[\int_0^{\infty} g_{r_2}^{(2)}(x)\Pr(N_x^{\mu_1} = R_1 - r_1)\Pr(D_x = d_2)\,\mathrm{d}x\right] \quad (4.88)$$

In Appendix 1, the transition probabilities are further specified.

Now we have all the information for our semi-Markov decision model and we can use a known algorithm like value iteration to solve the model.

§4.4 Improving the algorithm for finding the optimal solution

In this section we introduce a heuristic algorithm to find the solution for the model developed in this paper. Abginehchi and Larsen (2012) use the modified value iteration algorithm to find the optimal solution for the semi-Markov decision model. Although this algorithm is faster than other similar algorithms, it still is very time consuming. In the modeling section of the present paper we developed an altered model which has lower state space than the original one. In this section we now introduce an approximate algorithm which is faster than the modified value iteration algorithm for our problem. The basis of our algorithm here is still the modified value iteration algorithm but we have changed it and constructed a more efficient algorithm. A representation of the algorithm is illustrated in Algorithm 1.

Algorithm 1 A heuristic algorithm based on modified value iteration algorithm

Require: $\epsilon, n_{\text{crit}}, q_{1\min}, q_{2\min}, q_{1\max}, q_{2\max}, J$

$n \leftarrow 1$

$\tau \leftarrow 1/(\lambda + \mu_1 + \mu_2)$

$q_1 \leftarrow q_{1\min}; q_2 \leftarrow q_{2\min}$

repeat {This loop runs for different values of (q_1, q_2)}

if $q_1 = q_{1\min} \wedge q_2 = q_{2\min}$ **then**

for all $(i, r) \in I$ **do**

$V_0(i, r, q_1, q_2) \leftarrow 0$

end for

else

$V_0(i, r, q_1, q_2) \leftarrow V_{n_{\max}}(i, r, q_1 - 1, q_2)$ if the previous examined order sizes were $(q_1 - 1, q_2)$

or $V_0(i, r, q_1, q_2) \leftarrow V_{n_{\max}}(i, r, q_1, q_2 - 1)$ if the previous examined order sizes were $(q_1, q_2 - 1)$

end if

repeat {This loop runs for incremental iterations of a given (q_1, q_2)}

for all $(i, r) \in I$ **do**

for all $a_1, a_2 \in A(i, r)$ **do**

if not $\{(r = 0, a_1 = 1, a_2 = 1)$**or**$(r > 0, a_1 = 0, a_2 = 1)$**or**$(r < 0, a_1 = 1, a_2 = 0)\}$ **then**

compute $\vartheta(i, r, a_1, a_2, q_1, q_2)$ from (4.89)

else

compute *approximate* value of $\vartheta(i, r, a_1, a_2, q_1, q_2)$ using Equations (4.97), (4.101)–(4.103), (4.108)–(4.111), and (4.116)–(4.119)

end if

end for

$V_n(i, r, q_1, q_2) = \min_{a_1, a_2}\{\vartheta(i, r, a_1, a_2, q_1, q_2)\}$

Let $R(n)$ be any stationary policy such that the action $(a_1, a_2) = R_{(i,r)}(n)$ minimizes the right-hand side of the equation for $V_n(i, r, q_1, q_2)$ for each state (i, r). Call these actions $a_1^n(i, r, q_1, q_2)$ and $a_2^n(i, r, q_1, q_2)$

end for

$m_n \leftarrow \min_{(i,r)\in I}\{V_n(i, r) - V_{n-1}(i, r)\}, \quad M_n \leftarrow \max_{(i,r)\in I}\{V_n(i, r) - V_{n-1}(i, r)\}$

if $M_n - m_n > \epsilon m_n$ **then**

$n' \leftarrow 1$, define $W_0(i, r_1, r_2) \leftarrow V_n(i, r_1, r_2), \forall (i, r_1, r_2) \in I$

while $n' \leq J(n)$ **do**

for all $(i, r_1, r_2) \in I$ **do**

$W_{n'}(i, r, q_1, q_2) \leftarrow \vartheta(i, r, a_1^n(i, r, q_1, q_2), a_2^n(i, r, q_1, q_2), q_1, q_2)$

end for

$n' \leftarrow n' + 1$

end while

$V_{n+1}(i, r, q_1, q_2) \leftarrow W_{J(n)}(i, r, q_1, q_2)$

$n \leftarrow n + 2$

end if

until $M_n - m_n \leq \epsilon m_n$

$\text{TotalCost}(q_1, q_2) := (m_n + M_n)/2$

$\text{OptimalPolicy}(\mathrm{q}_1, \mathrm{q}_2) = R(n)$

$n_{\max} \leftarrow n$

Change the value of q_1 and/or q_2 by 1

until $q_1 = q_{1\max}$ **and** $q_2 = q_{2\max}$

$\text{TotalCost}^* = \min\{\text{TotalCost}(q_1, q_2)\}$, for all feasible q_1 and q_2

$\Rightarrow q_1^*$ and q_2^* and the respective optimal policy are obtained

---

In our algorithm, $\vartheta(i, r, a_1, a_2)$ is obtained by

$$\vartheta(i, r, a_1, a_2) = \frac{c_{(i,r)}(a_1, a_2)}{\tau_{(i,r)}(a_1, a_2)} + \frac{\tau}{\tau_{(i,r)}(a_1, a_2)} \sum_{(i',r')\in I} \Pr_{(i,r)(i',r')}(a_1, a_2) \times V_{n-1}(i', r') + \left(1 - \frac{\tau}{\tau_{(i,r)}(a_1, a_2)}\right) V_{n-1}(i, r). \quad (4.89)$$

A problem about finding ϑ for all combinations of (i, r, a_1, a_2) is that when $(r = 0, a_1 = 1, a_2 = 1)$ or $(r > 0, a_1 = 0, a_2 = 1)$ or $(r < 0, a_1 = 1, a_2 = 0)$, computing $\vartheta(i, r, a_1, a_2)$ is very time-consuming and as a result the speed of the algorithm declines. To overcome this issue for such cases we acquire the approximate value of $\vartheta(i, r, a_1, a_2)$ using Equations (4.97), (4.101)–(4.103), (4.108)–(4.111), and (4.116)–(4.119), which saves a great deal of time in the algorithm. Our idea of making an approximation to the value function of a dynamic program or semi-Markov decision process is to some extent in the similar spirit as seen in Adelman (2003).

First consider the case when $r < 0$ and $a_1 = 1$ and $a_2 = 0$. In this case the part $\sum_{(i',r')\in I} \Pr_{(i,r)(i',r')}(a_1, a_2) \cdot V_{n-1}(i', r')$ in $\vartheta(i, r, a_1, a_2)$ can be simplified as follows. First, let

$$\xi_1(i, r, 1, 0) = \sum_{r_2=1}^{-r} \sum_{d=0}^{i-1} \Pr_{(i,r)(i-d+q_1,-r_2)}(1, 0) \cdot V_{n-1}(i - d + q_1, -r_2) \tag{4.90}$$

$$\xi_2(i, r, 1, 0) = \sum_{r_2=1}^{-r} \Pr_{(i,r)(q_1,-r_2)}(1, 0) \cdot V_{n-1}(q_1, -r_2) \tag{4.91}$$

$$\xi_3(i, r, 1, 0) = \sum_{r_1=1}^{R_1} \sum_{d=0}^{i-1} \Pr_{(i,r)(i-d+q_2,r_1)}(1, 0) \cdot V_{n-1}(i - d + q_2, r_1) \tag{4.92}$$

$$\xi_4(i, r, 1, 0) = \sum_{r_1=1}^{R_1} \Pr_{(i,r)(q_2,r_1)}(1, 0) \cdot V_{n-1}(q_2, r_1) \tag{4.93}$$

We have

$$\sum_{(i',r')\in I} \Pr_{(i,r)(i',r')}(a_1, a_2) \times V_{n-1}(i', r') =$$

$$\xi_1(i, r, a_1, a_2) + \xi_2(i, r, a_1, a_2) + \xi_3(i, r, a_1, a_2) + \xi_4(i, r, a_1, a_2) \tag{4.94}$$

In the following our aim is to acquire some recursive relationships for $\xi_j(i, r, a_1, a_2)$, $j = 1, \ldots, 4$ based on the already computed values of $\xi_j(i', r', a_1, a_2)$.

In proposition 4.1 we show how we can obtain a recursive relationship for $\xi_1(i, r, 1, 0)$.

Proposition 4.1. *Assume a state where* $(r < 0, a_1 = 1, a_2 = 0)$. *In addition, consider the following approximations:*

$$V_{n-1}(i - d + q_1, -r_2) \simeq V_{n-1}(i - 1 - d + q_1, -r_2), \quad 0 \leq d \leq i - 2 \tag{4.95}$$

$$V_{n-1}(1 + q_1, -r_2) \simeq V_{n-1}(1 + q_1, -r_2 - 1), \quad 1 \leq r_2 \leq -r \tag{4.96}$$

Using the above assumptions the following relationship holds.

$$\xi_1(i,r,1,0) = \xi_1(i-1,r,1,0) + \xi_1(i,r-1,1,0) - \xi_1(i-1,r-1,1,0) - \mathrm{Pr}_{(i,r-1)(1+q_1,-1)}(1,0)\cdot V_{n-1}(1+q_1,-1) \quad (4.97)$$

Proof. By expanding their expressions it can easily be shown that

$$\mathrm{Pr}_{(i,r)(i-d+q_1,-r_2)}(1,0) = \mathrm{Pr}_{(i-1,r)(i-1-d+q_1,-r_2)}(1,0) \quad (4.98)$$

(4.98) and (4.95) lead us to

$$\xi_1(i,r,1,0) - \xi_1(i-1,r,1,0) = \sum_{r_2=1}^{-r} \mathrm{Pr}_{(i,r)(1+q_1,-r_2)}(1,0)\cdot V_{n-1}(1+q_1,-r_2). \quad (4.99)$$

Analogously,

$$\xi_1(i,r-1,1,0) - \xi_1(i-1,r-1,1,0) = \sum_{r_2=1}^{-r+1} \mathrm{Pr}_{(i,r-1)(1+q_1,-r_2)}(1,0)\cdot V_{n-1}(1+q_1,-r_2). \quad (4.100)$$

Now if we subtract (4.100) from (4.99) and use the approximation (4.96), Equation (4.97) will be obtained. □

Using the same approach for $\xi_2(i,r,1,0)$ and $\xi_3(i,r,1,0)$ and $\xi_4(i,r,1,0)$ we can obtain

$$\xi_2(i,r,1,0) = \xi_2(i-1,r,1,0) + \xi_2(i,r-1,1,0) - \xi_2(i-1,r-1,1,0) + \mathrm{Pr}_{(i,r-1)(1+q_1,-1)}(1,0)\cdot V_{n-1}(q_1,-1) \quad (4.101)$$

$$\xi_3(i,r,1,0) = \xi_3(i-1,r,1,0) + \frac{\lambda(-r)}{\mu_2(i-1)}\Big(\xi_3(i-1,r-1) - \xi_3(i-2,r-1)\Big) \quad (4.102)$$

$$\xi_4(i,r,1,0) = \xi_4(i-1,r,1,0) + \frac{\lambda(-r)}{\mu_2(i-1)}\Big(\xi_4(i-1,r-1) - \xi_4(i-2,r-1)\Big). \quad (4.103)$$

Then we consider the case when $r > 0$, $a_1 = 0$ and $a_2 = 1$. We define

$$\xi_1(i, r, 0, 1) = \sum_{r_1=1}^{r} \sum_{d=0}^{i-1} \Pr_{(i,r)(i-d+q_2,r_1)}(0, 1) \cdot V_{n-1}(i - d_1 + q_2, r_1) \tag{4.104}$$

$$\xi_2(i, r, 0, 1) = \sum_{r_1=1}^{r} \Pr_{(i,r)(q_2,r_1)}(0, 1) \cdot V_{n-1}(q_2, r_1) \tag{4.105}$$

$$\xi_3(i, r, 0, 1) = \sum_{r_2=1}^{R_2} \sum_{d=0}^{i-1} \Pr_{(i,r)(i-d+q_1,-r_2)}(0, 1) \cdot V_{n-1}(i - d + q_1, -r_2) \tag{4.106}$$

$$\xi_4(i, r, 0, 1) = \sum_{r_2=1}^{R_2} \Pr_{(i,r)(q_1,-r_2)}(0, 1) \cdot V_{n-1}(q_1, -r_2) \tag{4.107}$$

Again in this case (4.94) holds.

Like what we did for the preceding case, we can acquire

$$\xi_1(i, r, 0, 1) = \xi_1(i - 1, r, 0, 1) + \xi_1(i, r - 1, 0, 1) - \xi_1(i - 1, r - 1, 0, 1) + \Pr_{(i,r)(1+q_2,1)}(0, 1)V_{n-1}(1 + q_2, 1) \tag{4.108}$$

$$\xi_2(i, r, 0, 1) = \xi_2(i - 1, r, 0, 1) + \xi_2(i, r - 1, 0, 1) - \xi_2(i - 1, r - 1, 0, 1) - \Pr_{(i,r)(1+q_2,1)}(0, 1)V_{n-1}(q_2, 1) \tag{4.109}$$

$$\xi_3(i, r, 0, 1) = \xi_3(i - 1, r, 0, 1) + \frac{\lambda r}{\mu_1(i - 1)}\Big(\xi_3(i - 1, r + 1, 0, 1) - \xi_3(i - 2, r + 1, 0, 1)\Big) \tag{4.110}$$

$$\xi_4(i, r, 0, 1) = \xi_4(i - 1, r, 0, 1) + \frac{\lambda r}{\mu_1(i - 1)}\Big(\xi_4(i - 1, r + 1, 0, 1) - \xi_4(i - 2, r + 1, 0, 1)\Big) \tag{4.111}$$

To obtain the above equations two groups of approximations

$$V_{n-1}(i, r) \simeq V_{n-1}(i - 1, r) \quad \text{and} \quad V_{n-1}(i, r) \simeq V_{n-1}(i, r - 1), \quad i \geq 1, r \geq 2$$

have been used.

Finally we consider the case when $r = 0, a_1 = 1, a_2 = 1$. We define

$$\xi_1(i,0,1,1) = \sum_{r_2=1}^{R_2}\sum_{d=0}^{i-1} \Pr_{(i,0)(i-d+q_1,-r_2)}(1,1)\cdot V_{n-1}(i-d+q_1,-r_2) \quad (4.112)$$

$$\xi_2(i,0,1,1) = \sum_{r_2=1}^{R_2} \Pr_{(i,0)(q_1,-r_2)}(0,1)\cdot V_{n-1}(q_1,-r_2) \quad (4.113)$$

$$\xi_3(i,0,1,1) = \sum_{r_1=1}^{R_1}\sum_{d=0}^{i-1} \Pr_{(i,0)(i-d+q_2,r_1)}(1,1)\cdot V_{n-1}(i-d+q_2,r_1) \quad (4.114)$$

$$\xi_4(i,0,1,1) = \sum_{r_1=1}^{R_1} \Pr_{(i,0)(q_2,r_1)}(1,1)\cdot V_{n-1}(q_2,r_1). \quad (4.115)$$

Again in this case (4.94) holds.

We can obtain the following relations for $\xi_2(i,r,1,1)$, $\xi_2(i,r,1,1)$, $\xi_3(i,r,1,1)$ and $\xi_4(i,r,1,1)$.

$$\xi_1(i,0,1,1) = \xi_1(i-1,0,1,1) + \xi_1(i,-R_2,1,0) - \xi_1(i-1,-R_2,1,0) \quad (4.116)$$

$$\xi_2(i,0,1,1) = \xi_2(i-1,0,1,1) + \xi_2(i,-R_2,1,0) - \xi_2(i-1,-R_2,1,0) \quad (4.117)$$

$$\xi_3(i,0,1,1) = \xi_3(i-1,0,1,1) + (\xi_3(i,-R_2,1,0) - \xi_3(i-1,-R_2,1,0)) \quad (4.118)$$

$$\xi_4(i,0,1,1) = \xi_4(i-1,0,1,1) + \xi_4(i,-R_2,1,0) - \xi_4(i-1,-R_2,1,0) \quad (4.119)$$

Now we have all the information needed for the algorithm. As seen we have obtained some recursive relationships in order to decrease the computational time. The following pseudocode shows how to acquire $\xi_j(i,r,a_1,a_2)$ recursively.

for $i = 0 \to i_{\max}$ **do**
  for $r = -R_2 \to R_1$ **do**
    for all $a_1, a_2 \in A(i,r)$ **do**
      if $i = 0, 1$ **or** $r = -R_2$ **then**
        find $\xi_j(i,r,a_1,a_2)$ directly
      else
        find $\xi_j(i,r,a_1,a_2)$ using Equations (4.97), (4.101)–(4.103), (4.108)–(4.111), and (4.116)–(4.119)

end if

end for

end for

end for

Algorithm 1 combines features of both the standard policy iteration algorithm and the value iteration algorithm. Whether it is of type value iteration or policy-iteration depends on the number of partial policy evaluations $J(n)$. For more information regarding that refer to Puterman (2005b).

Value-iteration algorithms begin with choosing value functions $V_0(i,r)$ such that $0 \leq V_0(i,r) \leq \min_{a_1,a_2}\{c_{(i,r)}(a_1,a_2)/\tau_{(i,r)}(a_1,a_2)\}$ for all (i,r). Then in each iteration these value functions improve and lead to better upper and lower bounds for the optimal solution. Since we have to repeat this process for all combinations of q_1 and q_2, where $q_{j_{\min}} \leq q_j \leq q_{j_{\max}}$, $j = 1,2$, the value iteration algorithm and the modified policy iteration algorithm which are used in the paper by Abginehchi and Larsen (2012) take many unnecessary steps for improving V_n functions. What we do in Algorithm 1 is that for each given q_1 and q_2 we use the final value functions V_n when order sizes are (q_1-1, q_2) or (q_1, q_2-1) and use them as initial values for $V_0(i,r)$ for the problem with order sizes (q_1, q_2). It turns out that by using this method the algorithm converges to the optimal solution much faster. Based on many problem tests we can conclude that although m_n and M_n are not monotonic when we use Algorithm 1, they always converge to the optimal expected costs.

In addition, since computing $V_n(i,r)$ in some cases is time consuming, we have used some approximations in these situations, which have been discussed in detail above.

In order to compare the two models, the current model and the model proposed by Abginehchi and Larsen (2012), we made some problem tests and solved them with both models. It must be noted that the model developed in this paper uses the same assumptions as the model by Abginehchi and Larsen (2012) and if the same exact algorithm like value-iteration is used both models must give the same optimal solutions. That is true and we examined it for several numerical tests to validate our model.

Furthermore, we compare the two models with regard to *time*. For the model by Abginehchi and Larsen (2012) we used the modified value iteration algorithm, which is shown by them as an efficient algorithm, with $J = 10$ and $n_{\text{crit}} = 10$. For our model we used Algorithm 1 with the same values for J and n_{crit}. Since in the algorithms we want to find optimal order sizes, q_1^* and q_2^*, by searching among all possible values between $q_{j_{\min}}$ and $q_{j_{\max}}$, $j = 1, 2$, we are interested in seeing the effect of the number of examined order sizes on the time of the algorithm. For example when q_1 varies from 40 to 49 and q_2 varies from 20 to 29 then the number of examined pair order sizes is 100. By increasing the lower and upper bound of $q_j, j = 1, 2$ by very small amounts this number grows very fast and causes the algorithm to run very slowly. Figure (5.1) compares the two models with regard to time. The input parameters are as follows: $h = 1$, $\lambda = 1$, $\pi = 50$, $K = 20$, $K_1 = 50$, $k_2 = 60$, $\mu_1 = 0.03$, $\mu_2 = 0.1$, $R_1 = 6$, $R_2 = 6$, $i_{\max} = 160$, and $\epsilon = 0.001$. As seen, both algorithms have linear speed rate. When the two models are used for one specific order size for each of the suppliers then the model by Abginehchi and Larsen (2012) is a little faster; the time spent for our model is 42′′ and for the model by Abginehchi and Larsen is 25′′. As can be observed in Figure 1 this slowness is compensated very fast when the number of examined q_1 and q_2 rises. Due to the stochastic environment for real problems it may be necessary to examine thousands of paired order sizes to find the optimal solution. In such cases speed is very important and using a faster algorithm and better model can save so much time. For example, if in Figure 1 the number of pairs (q_1, q_2) is 900 (for instance q_1 and q_2 each varies among 30 values), and the algorithms keep their linear trends for response time, then their algorithm takes more than 7 hours while ours takes only 1.7 hours which is a big difference.

The final question to answer is how different the results of our developed algorithm are from the optimal solution. In the 200 tests we performed the results from our developed algorithm were very close to the optimal policy. As seen from Figure 2 the results from our algorithm are very close to optimal policy and in most cases

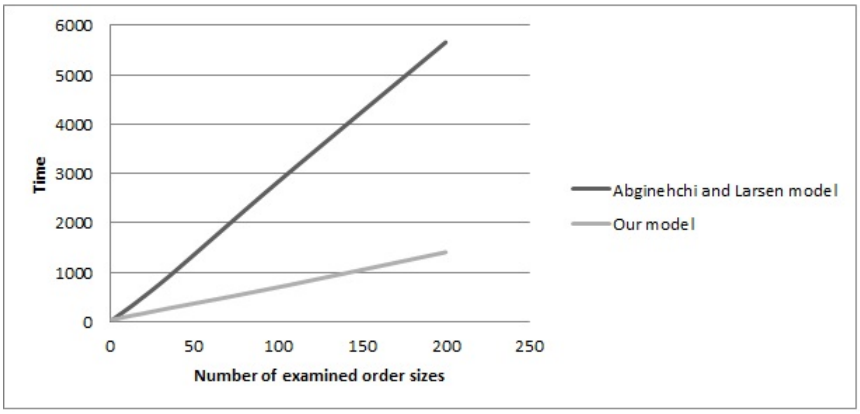

Figure 4.1: Comparing the speed of the two algorithms

the difference is less than 2.8%. This shows how precise our algorithm is.

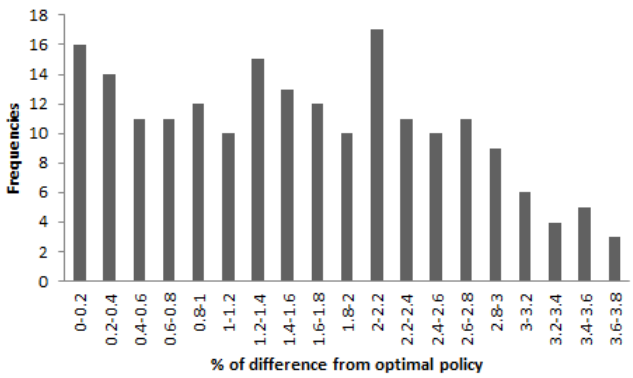

Figure 4.2: The difference of our algorithm compared to the optimal policy

One might suspect that for a specific (q_1, q_2) due to using approximations for $\xi_j(i, r, a_1, a_2)$, $j = 1, \ldots, 4$ the computed value functions $V_n(i, r, q_1, q_2)$ in the algorithm become more inaccurate in each iteration which has a direct effect on the accuracy of the final result. That does not happen in the algorithm. Notice that in each iteration $V_n(i, r, q_1, q_2) = \min_{a_1,a_2}\{\vartheta(i, r, a_1, a_2, q_1, q_2)\}$. Based on our algorithm some of these $\vartheta(i, r, a_1, a_2, q_1, q_2), a_1, a_2 = 0, 1$ are computed approximately while the rest does not use the introduced approximations. As a result after some iterations the value of $V_n(i, r, q_1, q_2)$ might be less or more accurate. In order to indicate that, we solved a problem with the same input parameters as mentioned earlier except that we set the values of the order sizes to $q_1 = 40$ and $q_2 = 20$.

We solved the problem both with our algorithm and with the pure value iteration algorithm introduced in Tijms (2003b). Figure (4.3) shows the difference between the value functions $V_n(30, -3, q_1, q_2)$ obtained from the two algorithms (exact and approximate) in different iterations. The lower the difference is, the more accurate $V_n(30, -3, q_1, q_2)$ has been computed from our algorithm.

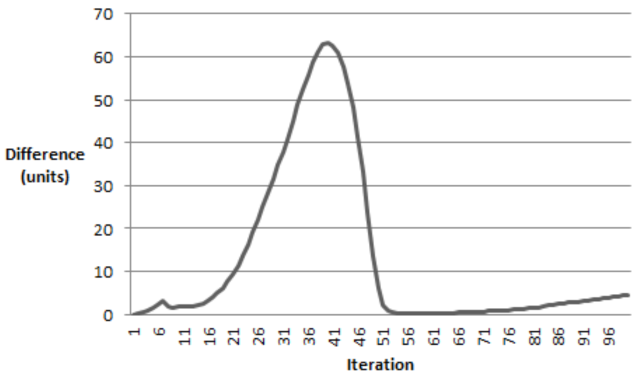

Figure 4.3: Comparing the accuracy of obtained value functions in Algorithm 1

As can be seen from Figure (4.3) after some iteration ($n = 38$) the value function $V_n(i, r, q_1, q_2)$ gets more and more accurate which confirms our argument. As explained above our algorithm was not only generally faster, but quite accurate in all tests performed.

§4.5 Conclusion

This paper addresses the same problem with similar assumptions as done in Abginehchi and Larsen (2012). We have considered the problem of optimal policy for a single-item inventory system in which there are two supply options available, and in which the lead-times from the suppliers are random variables with Erlang distribution. We use a semi-Markov decision process to formulate a long-run average cost model. The model in this paper is optimal policy and leads to the same results as the model by Abginehchi and Larsen (2012), but its state space is much lower. In addition we have introduced an approximate algorithm for finding the optimal solution by approximating some of the value functions. The point in this algorithm is that it uses previous information for finding the optimal solution to the current (q_1, q_2). The algorithm is quite precise and in all numerical tests performed the results are at most 3.5 % different from the optimal policy.

What we did in this article was to reduce the time for obtaining the optimal policy through a new model with lower state space and a customized algorithm for solving that. The results were quite impressive and specially for large problems a great amount of time is saved. The point is that we did not change any of the assumptions of the main model by Abginehchi and Larsen (2012) and yet we found a more efficient model.

There may be some heuristic ways for acquiring the results even faster. One main way to do that is not to search all the possible order sizes for finding the optimal $q_j^*, j = 1, 2$. For example one method is to keep $q_j, j = 1$ or 2 fixed and find the optimal q_{3-j} and then do the opposite and so on until they converge to some q_1^* and q_2^* and we call them the optimal order sizes. In many cases the results from this heuristical method are really the optimal values, but anyway this is a heuristical method and in some cases does not lead to optimal policy. Because of the objective of this paper we did not intervene in these approximate policies, but it remains an open problem for different heuristic methods to be considered later.

One interesting improvement area is to include $q_j, j = 1, 2$ in the model as decision variables. Neither our model nor Abginehchi and Larsen (2012)'s model has considered this, and they both find optimal policy for given (q_1, q_2).

Finally, extensions of our present model should also be considered. It would be especially interesting to extend the model for use in situations when lead-times follow a phase distribution other than Erlang distribution in order to make it more general and useful for environments with order progress information (OPI). Furthermore our compact formulation method used for dual sourcing could be applied to other inventory control systems too.

Appendix 1

Here we calculate the probabilities in Equations (4.75) to (4.88).

First consider $\Pr_{(i,0)(\min(i-d_1+q_1,i_{\max}),-r_2)}(1,1)$ in (4.75). We have

$$1 - G_1^{(2)}(x - t_1) = \mathrm{e}^{-\mu_2(x-t_1)}, \tag{4.120}$$

and

$$g_{R_2-r_2}^{(2)}(t_1) = \frac{(t_1\mu_2)^{(R_2-r_2-1)} \cdot \mathrm{e}^{(-t_1\mu_2)} \cdot \mu_2}{(R_2 - r_2 - 1)!}, \tag{4.121}$$

so the inner integral would be:

$$\int_0^x g_{R_2-r_2}^{(2)}(t_1)\left(1 - G_1^{(2)}(x - t_1)\right) \mathrm{d}t_1 = \frac{{\mu_2}^{R_2-r_2}\mathrm{e}^{-\mu_2 x}}{(R_2 - r_2 - 1)!} \int_0^x t_1^{(R_2-r_2-1)} \,\mathrm{d}t_1 = \frac{\mu_2^{(R_2-r_2)} x^{(R_2-r_2)}\mathrm{e}^{-\mu_2 x}}{(R_2 - r_2)!}. \tag{4.122}$$

Hence the probability is obtained by

$$\begin{aligned}
&\Pr_{(i,0)(\min(i-d_1+q_1,i_{\max}),-r_2)}(1,1) = \\
&\quad \int_0^\infty \frac{(x\mu_1)^{R_1-1}\mathrm{e}^{-x\mu_1}\mu_1}{(R_1 - 1)!} \times \frac{(x\mu_2)^{R_2-r_2}\mathrm{e}^{-x\mu_2}}{(R_2 - r_2)!} \times \mathrm{e}^{-\lambda x}\frac{(\lambda x)^{d_1}}{d_1!}\,\mathrm{d}x \\
&= \frac{\mu_1^{R_1}\mu_2^{R_2-r_2}\lambda^{d_1}}{(R_1 - 1)!(R_2 - r_2)!d_1!} \int_0^\infty x^{(R_1+R_2-r_2-1+d_1)} \exp\left(-x(\mu_1 + \mu_2 + \lambda)\right) \mathrm{d}x \\
&= \frac{\mu_1^{R_1}\mu_2^{R_2-r_2}\lambda^{d_1}}{(R_1 - 1)!(R_2 - r_2)!d_1!} \times \frac{(R_1 + R_2 - r_2 - 1 + d_1)!}{(\mu_1 + \mu_2 + \lambda)^{(R_1+R_2-r_2+d_1)}}
\end{aligned} \tag{4.123}$$

Using the same procedure for other equations we have :

$$\Pr_{(i,0)(i-d_2+q_2,+r_1)}(1,1) = \frac{{\mu_1}^{R_1-r_1}{\mu_2}^{R_2}\lambda^{d_2}}{(R_1 - r_1)!\,(R_2 - 1)!d_2!} \times \frac{(R_1 + R_2 - r_1 - 1 + d_2)!}{(\mu_1 + \mu_2 + \lambda)^{(R_1+R_2-r_1+d_2)}} \tag{4.124}$$

$$\Pr_{(i,r_1)(i-d_1+q_1,-r_2)}(0,1) = \frac{{\mu_1}^{r_1}{\mu_2}^{R_2-r_2}\lambda^{d_1}}{(r_1-1)!\,(R_2-r_2)!d_1!}\cdot\frac{(r_1+R_2-r_2-1+d_1)!}{(\mu_1+\mu_2+\lambda)^{(r_1+R_2-r_2+d_1)}} \tag{4.125}$$

$$\Pr_{(i,r_1)\left(i-d_2+q_2,+r_1'\right)}(0,1) = \frac{{\mu_1}^{r_1-r_1'}{\mu_2}^{R_2}\lambda^{d_2}}{(r_1-r')!\,(R_2-1)!d_2!}\times\frac{\left(r_1+R_2-r_1'-1+d_2\right)!}{(\mu_1+\mu_2+\lambda)^{\left(r_1+R_2-r_1'+d_2\right)}} \tag{4.126}$$

$$\Pr_{(i,-r_2)\left(i-d_1+q_1,-r_2'\right)}(1,0) = \frac{{\mu_1}^{R_1}{\mu_2}^{r_2-r_2'}\lambda^{d_1}}{(R_1-1)!\left(r_2-r_2'\right)!d_1!}\times\frac{\left(R_1+r_2-r_2'-1+d_1\right)!}{(\mu_1+\mu_2+\lambda)^{\left(R_1+r_2-r_2'+d_1\right)}} \tag{4.127}$$

$$\Pr_{(i,-r_2)(i-d_2+q_2,r_1)}(1,0) = \frac{{\mu_1}^{R_1-r_1}{\mu_2}^{r_2}\lambda^{d_2}}{(R_1-r_1)!\,(r_2-1)!d_2!}\times\frac{(R_1+r_2-r_1-1+d_2)!}{(\mu_1+\mu_2+\lambda)^{(R_1+r_2-r_1+d_2)}} \tag{4.128}$$

As for $\Pr_{(i,0)(q_1,-r_2)}(1,1)$ we have:

$$\begin{aligned}\Pr_{(i,0)(q_1,-r_2)}(1,1) &= \sum_{d_1=i}^{\infty}\left[\frac{{\mu_1}^{R_1}{\mu_2}^{R_2-r_2}\lambda^{d_1}}{(R_1-1)!\,(R_2-r_2)!d_1!}\times\frac{(R_1+R_2-r_2-1+d_1)!}{(\mu_1+\mu_2+\lambda)^{(R_1+R_2-r_2+d_1)}}\right]\\ &= \frac{{\mu_1}^{R_1}\mu_2^{R_2-r_2}}{(R_1-1)!\,(R_2-r_2)!}\sum_{d=i}^{\infty}\frac{\lambda^d\,(R_1+R_2-r_2-1+d)!}{(\mu_1+\mu_2+\lambda)^{R_1+R_2-r_2+d}}\end{aligned} \tag{4.129}$$

If $i=0$ then we have:

$$\Pr_{(0,0)(q_1,-r_2)}(1,1) = \frac{{\mu_1}^{R_1}{\mu_2}^{R_2-r_2}\,(R_1+R_2-r_2-1)!}{(R_1-1)!\,(R_2-r_2)!(\mu_1+\mu_2)^{R_1+R_2-r}} \tag{4.130}$$

And for $i > 0$ we have the following relation:

$$\Pr_{(i,0)(q_1,-r_2)}(1,1) = \Pr_{(i-1,0)(q_1,-r_2)}(1,1) - \frac{{\mu_1}^{R_1}\mu_2^{R_2-r_2}}{(R_1-1)!\,(R_2-r_2)!} \times \left(\frac{\lambda^{i-1}\,(R_1+R_2-r_2+i-2)!}{(i-1)!(\mu_1+\mu_2+\lambda)^{R_1+R_2-r_2+i-1}}\right) \quad (4.131)$$

Using the same method we have the following equations.

$$\Pr_{(i,0)(q_2,+r_1)}(1,1) = \Pr_{(i-1,0)(q_2,+r_1)}(1,1) - \frac{{\mu_1}^{R_1-r_1}\mu_2^{R_2}}{(R_1-r_1)!\,(R_2-1)!} \times \left(\frac{\lambda^{i-1}\,(R_1+R_2-r_1+i-2)!}{(i-1)!(\mu_1+\mu_2+\lambda)^{R_1+R_2-r_1+i-1}}\right) \quad (4.132)$$

where

$$\Pr_{(0,0)(q_2,+r_1)}(1,1) = \frac{{\mu_1}^{R_1-r_1}\mu_2^{R_2}}{(R_1-r_1)!\,(R_2-1)!} \times \frac{(R_1+R_2-r_1-1)!}{(\mu_1+\mu_2)^{R_1+R_2-r_1}}. \quad (4.133)$$

$$\Pr_{(i,r_1)(q_1,-r_2)}(0,1) = \Pr_{(i-1,r_1)(q_1,-r_2)}(0,1) - \frac{{\mu_1}^{r_1}{\mu_2}^{R_2-r_2}}{(r_1-1)!\,(R_2-r_2)!} \times \frac{\lambda^{i-1}\,(r_1+R_2-r_2+i-2)!}{(i-1)!(\mu_1+\mu_2+\lambda)^{(r_1+R_2-r_2+i-1)}} \quad (4.134)$$

where

$$\Pr_{(0,r_1)(q_1,-r_2)}(0,1) = \frac{{\mu_1}^{r_1}{\mu_2}^{R_2-r_2}}{(r_1-1)!\,(R_2-r_2)!}\frac{(r_1+R_2-r_2-1)!}{(\mu_1+\mu_2)^{r_1+R_2-r_2}}. \quad (4.135)$$

$$\Pr_{(i,r_1)\left(q_2,+r_1'\right)}(0,1) = \Pr_{(i-1,r_1)\left(q_2,+r_1'\right)}(0,1) - \frac{{\mu_1}^{r_1-r_1'}{\mu_2}^{R_2}}{\left(r_1-r_1'\right)!\,(R_2-1)!} \times \frac{\lambda^{i-1}\left(r_1+R_2-r_1'+i-2\right)!}{(i-1)!(\mu_1+\mu_2+\lambda)^{\left(r_1+R_2-r_1'+i-1\right)}} \quad (4.136)$$

where

$$\Pr_{(0,r_1)(q_2,+r_1')}(0,1) = \frac{{\mu_1}^{r_1-r_1'}{\mu_2}^{R_2}\left(r_1+R_2-r_1'-1\right)!}{\left(r_1-r_1'\right)!\,(R_2-1)!(\mu_1+\mu_2)^{r_1+R_2-r_1'}}. \tag{4.137}$$

$$\Pr_{(i,-r_2)(q_1,-r_2')}(1,0) = \Pr_{(i-1,-r_2)(q_1,-r_2')}(1,0) - \frac{{\mu_1}^{R_1}{\mu_2}^{r_2-r_2'}}{(R_1-1)!\left(r_2-r_2'\right)!} \times \frac{\lambda^{i-1}\left(R_1+r_2-r_2'+i-2\right)!}{(i-1)!(\mu_1+\mu_2+\lambda)^{\left(R_1+r_2-r_2'+i-1\right)}} \tag{4.138}$$

where

$$\Pr_{(0,-r_2)(q_1,-r_2')}(1,0) = \frac{{\mu_1}^{R_1}{\mu_2}^{r_2-r_2'}}{(R_1-1)!\left(r_2-r_2'\right)!} \times \frac{\left(R_1+r_2-r_2'-1\right)!}{(\mu_1+\mu_2)^{\left(R_1+r_2-r_2'\right)}}. \tag{4.139}$$

$$\Pr_{(i,-r_2)(q_2,r_1)}(1,0) = \Pr_{(i-1,-r_2)(q_2,r_1)}(1,0) - \frac{{\mu_1}^{R_1-r_1}{\mu_2}^{r_2}}{(R_1-r_1)!\,(r_2-1)!} \times \frac{\lambda^{i-1}\,(R_1+r_2-r_1+i-2)!}{(i-1)!(\mu_1+\mu_2+\lambda)^{(R_1+r_2-r_1+i-1)}} \tag{4.140}$$

where

$$\Pr_{(0,-r_2)(q_2,r_1)}(1,0) = \frac{{\mu_1}^{R_1-r_1}{\mu_2}^{r_2}}{(R_1-r_1)!\,(r_2-1)!} \times \frac{(R_1+r_2-r_1-1)!}{(\mu_1+\mu_2)^{(R_1+r_2-r_1)}}. \tag{4.141}$$

Appendix 2

In Appendix 2 we compute $AHC(i|r_1,r_2)$:

$$AHC(i|r_1,r_2) = h\int_{x=0}^{\infty}\int_{\omega=0}^{x}\sum_{d_1=0}^{i} \mathrm{e}^{-\lambda\omega}\frac{(\lambda\omega)^{d_1}}{d_1!}(i-d_1)\,\mathrm{d}\omega f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \tag{4.142}$$

If we define $\Psi(i) = \int_0^x \sum_{d_1=0}^{i} \mathrm{e}^{-\lambda\omega}\frac{(\lambda\omega)^{d_1}}{d_1!}(i-d_1)\,\mathrm{d}\omega$, we know from the exponential case section that

$$\Psi(i) = \Psi(i-1) + \frac{i}{\lambda} - \frac{1}{\lambda}\sum_{p=1}^{i} \frac{(i-p+1)}{(p-1)!} e^{-\lambda x}(\lambda x)^{p-1} \tag{4.143}$$

Now consider the two terms of $f^{(1)}_{L_{r_1,r_2}}$. We call the first term $f^{(1)}_1(x)$ and the second term $f^{(1)}_2(x)$. We define

$$f^{(1)}_1 = \frac{x^{(r_1-1)}\mu_1^{(r_1)} e^{(-x\mu_1)}\Gamma(r_2, x\mu_2)}{(r_1-1)!(r_2-1)!}$$
$$f^{(1)}_2 = \frac{x^{(r_2-1)}\mu_2^{(r_2)} e^{(-x\mu_2)}\Gamma(r_1, x\mu_1)}{(r_1-1)!(r_2-1)!}$$

$$f^{(1)}_{L_{r_1,r_2}}(x) = \frac{x^{r_1-1}\mu_1^{r_1} e^{-x\mu_1}\Gamma(r_2, x\mu_2)}{(r_1-1)!(r_2-1)!} + \frac{x^{r_2-1}\mu_2^{r_2} e^{-x\mu_2}\Gamma(r_1, x\mu_1)}{(r_1-1)!(r_2-1)!}. \tag{4.144}$$

We define

$$AHC_1(i\,|r_1, r_2) = h\int_0^\infty \left(\frac{i}{\lambda} - \frac{1}{\lambda}\sum_{p=1}^{i} \frac{(i-p+1)}{(p-1)!} e^{-\lambda x}(\lambda x)^{p-1}\right) \frac{x^{r_1-1}{\mu_1}^{r_1} e^{-x\mu_1}\Gamma(r_2, x\mu_2)}{(r_1-1)!\,(r_2-1)!}\, dx, \tag{4.145}$$

$$AHC_2(i\,|r_1, r_2) = h\int_0^\infty \left(\frac{i}{\lambda} - \frac{1}{\lambda}\sum_{p=1}^{i} \frac{(i-p+1)}{(p-1)!} e^{-\lambda x}(\lambda x)^{p-1}\right) \frac{x^{r_2-1}{\mu_2}^{r_2} e^{-x\mu_2}\Gamma(r_1, x\mu_1)}{(r_1-1)!\,(r_2-1)!}\, dx\cdot \tag{4.146}$$

Finally we have the following relation for $AHC(i|r_1, r_2)$.

$$\begin{aligned} AHC(i|r_1, r_2) &= h\int_0^\infty \Psi(i) f^{(1)}_{L_{r_1,r_2}}(x)\, dx \\ &= AHC_1(i|r_1, r_2) + AHC_2(i|r_1, r_2) + h\int_0^\infty \Psi(i-1) f^{(1)}_{L_{r_1,r_2}}(x)\, dx \\ &= AHC_1(i|r_1, r_2) + AHC_2(i|r_1, r_2) + AHC(i-1|r_1, r_2) \end{aligned} \tag{4.147}$$

Now we try to find simple relations for $AHC_1(i|r_1, r_2)$ and $AHC_2(i|r_1, r_2)$ defined in (4.145) and (4.146).

$$
\begin{aligned}
&AHC_1(i|r_1, r_2) \\
&\quad = \frac{i}{\lambda}\int_0^\infty \frac{x^{r_1-1}{\mu_1}^{r_1}\mathrm{e}^{-x\mu_1}\Gamma(r_2, x\mu_2)}{(r_1-1)!\,(r_2-1)!}\,\mathrm{d}x \\
&\qquad - \int_0^\infty \frac{1}{\lambda}\sum_{p=1}^{i}\left[\frac{(i-p+1)}{(p-1)!}\mathrm{e}^{-\lambda x}(\lambda x)^{p-1}\right]\frac{x^{r_1-1}{\mu_1}^{r_1}\mathrm{e}^{-x\mu_1}\Gamma(r_2, x\mu_2)}{(r_1-1)!\,(r_2-1)!}\,\mathrm{d}x
\end{aligned}
\tag{4.148}
$$

The first term of (4.148) is obtained by

$$
\frac{i}{\lambda}\int_0^\infty \frac{x^{r_1-1}{\mu_1}^{r_1}\mathrm{e}^{-x\mu_1}\Gamma(r_2, x\mu_2)}{(r_1-1)!\,(r_2-1)!}\,\mathrm{d}x =
\frac{{\mu_1}^{r_1} i}{\lambda(r_1-1)!}\left(\frac{(r_1-1)!}{(\mu_1+\mu_2)^{r_1}} + \frac{\mu_2\,(r_1)!}{1!(\mu_1+\mu_2)^{r_1+1}} + \cdots + \frac{{\mu_2}^{r_2-1}(r_1+r_2-2)!}{(r_2-1)!(\mu_1+\mu_2)^{r_1+r_2-1}}\right)
$$

For finding the second term of (4.148) we need to compute the following expression.

$$
\begin{aligned}
&\frac{1}{\lambda}\int_0^\infty \frac{(i-p+1)}{(p-1)!}\mathrm{e}^{-\lambda x}(\lambda x)^{p-1} \times \frac{x^{r_1-1}{\mu_1}^{r_1}\mathrm{e}^{-x\mu_1}\Gamma(r_2, x\mu_2)}{(r_1-1)!\,(r_2-1)!}\,\mathrm{d}x \\
&= \frac{(i-p+1)\,\lambda^{p-1}{\mu_1}^{r_1}}{\lambda(p-1)!\,(r_1-1)!}\int_0^\infty \mathrm{e}^{-x(\lambda+\mu_1+\mu_2)}x^{(r_1+p-2)}\left(1 + \frac{(x\mu_2)^1}{1!} + \cdots + \frac{(x\mu_2)^{r_2-1}}{(r_2-1)!}\right)\mathrm{d}x \\
&= \frac{(i-p+1)\,\lambda^{p-1}{\mu_1}^{r_1}}{\lambda(p-1)!\,(r_1-1)!}\left(\frac{\mu_2^0\,(r_1+p-2)!}{0!(\lambda+\mu_1+\mu_2)^{r_1+p-1}} + \cdots \right.\\
&\qquad\qquad \left. + \frac{\mu_2^{r_2-1}(r_1+r_2+p-3)!}{(r_2-1)!(\lambda+\mu_1+\mu_2)^{r_1+r_2+p-2}}\right)
\end{aligned}
$$

As a result, $AHC_1(i|r_1, r_2)$ will be

$$
\begin{aligned}
AHC_1(i|r_1, r_2) = {}& \frac{{\mu_1}^{r_1} i}{\lambda(r_1-1)!}\sum_{k=0}^{r_2-1}\left[\frac{{\mu_2}^k(r_1+k-1)!}{k!(\mu_1+\mu_2)^{r_1+k}}\right] \\
&- \sum_{p=1}^{i}\left[\frac{(i-p+1)\,\lambda^{p-1}{\mu_1}^{r_1}}{\lambda(p-1)!\,(r_1-1)!} \times \sum_{j=0}^{r_2-1}\left\{\frac{{\mu_2}^j(r_1+p+j-2)!}{j!(\lambda+\mu_1+\mu_2)^{r_1+p+j-1}}\right\}\right]
\end{aligned}
$$

Using the same method $AHC_2(i|r_1, r_2)$ is obtained by

$$AHC_2\,(i\,|r_1, r_2\,) = \frac{{\mu_2}^{r_2} i}{\lambda\,(r_2 - 1)!} \sum_{k=0}^{r_1-1} \left[\frac{{\mu_1}^k\,(r_2 + k - 1)!}{k!(\mu_1 + \mu_2)^{r_2+k}} \right] - \sum_{p=1}^{i} \left[\frac{(i - p + 1)\,\lambda^{p-1} {\mu_2}^{r_2}}{\lambda\,(p - 1)!\,(r_2 - 1)!} \times \sum_{j=0}^{r_1-1} \left\{ \frac{{\mu_1}^j\,(r_2 + p + j - 2)!}{j!(\lambda + \mu_1 + \mu_2)^{r_2+p+j-1}} \right\} \right]$$

$AHC_1(i|r_1, r_2)$ and $AHC_2(i|r_1, r_2)$ with the current form are time consuming to find. We can improve the form as follows. First consider $AHC_1(i|r_1, r_2)$. It is a function of i. We have $AHC_1(0|r_1, r_2) = 0$. If $i = 1$ then

$$AHC_1(1|r_1, r_2) = \left(\frac{\mu_1^{r_1}}{\lambda(r_1 - 1)!} \times \sum_{j=0}^{r_2-1} \frac{\mu_2^j (r_1 + j - 1)!}{j!(\mu_1 + \mu_2)^{r_1+j}} \right) - \Theta(1)$$

where

$$\Theta(i) = \sum_{j=0}^{r_2-1} \frac{\lambda^{i-2} \mu_1^{r_1} \mu_2^j (r_1 + i + j - 2)!}{(i - 1)!(r_1 - 1)!j!(\lambda + \mu_1 + \mu_2)^{r_1+i+j-1}}.$$

Using induction it can be shown that the following relation for $AHC_1(i|r_1, r_2)$ holds.

$$AHC_1(i|r_1, r_2) = 2AHC_1(i - 1|r_1, r_2) - AHC_1(i - 2|r_1, r_2) - \Theta(i);\; i \geq 2. \quad (4.149)$$

If we use the same procedure for $AHC_2(i|r_1, r_2)$ we obtain

$$AHC_2(i|r_1, r_2) = 2AHC_2(i - 1|r_1, r_2) - AHC_2(i - 2|r_1, r_2) - \Theta(i);\; i \geq 2. \quad (4.150)$$

Appendix 3

In this Appendix we find a tractable formula for $ASC(i|r_1, r_2)$. Regarding computation time, it is much easier to compute $ASC(i|r_1, r_2)$ using the already computed

$ASC(i-1|r_1,r_2)$. From the main text we have:

$$ASC(i|r_1,r_2) = \int_0^\infty \left\{ \sum_{d_1=i}^{\infty} \pi\,(d_1 - i)\,\mathrm{e}^{-\lambda x} \frac{(\lambda x)^{d_1}}{d_1!} \right\} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \tag{4.151}$$

If inventory on-hand is $i-1$ then the above equation changes to

$$\begin{aligned} ASC(i-1|r_1,r_2) &= \int_0^\infty \left\{ \sum_{d_1=i-1}^{\infty} \pi\,(d_1 - i + 1)\,\mathrm{e}^{-\lambda x} \frac{(\lambda x)^{d_1}}{d_1!} \right\} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \\ &= 0 + \int_0^\infty \left\{ \sum_{d_1=i}^{\infty} \pi\,(d_1 - i)\,\mathrm{e}^{-\lambda x} \frac{(\lambda x)^{d_1}}{d_1!} \right\} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \\ &\quad + \int_0^\infty \left\{ \sum_{d_1=i}^{\infty} \pi \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{d_1}}{d_1!} \right\} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \\ &= ASC(i|r_1,r_2) + \int_0^\infty \left\{ \sum_{d_1=i}^{\infty} \pi \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{d_1}}{d_1!} \right\} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \end{aligned}$$

Likewise for $ASC(i-2|r_1,r_2)$ we have

$$\begin{aligned} ASC(i-2|r_1,r_2) &= ASC(i-1|r_1,r_2) + \int_0^\infty \left\{ \sum_{d_1=i-1}^{\infty} \pi \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{d_1}}{d_1!} \right\} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \\ &= ASC(i-1|r_1,r_2) + \int_0^\infty \left\{ \pi \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{i-1}}{(i-1)!} + \sum_{d_1=i}^{\infty} \pi \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{d_1}}{d_1!} \right\} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \\ &= ASC(i-1|r_1,r_2) + (ASC(i-1|r_1,r_2) - ASC(i|r_1,r_2)) \\ &\quad + \int_0^\infty \frac{\pi \mathrm{e}^{-\lambda x}(\lambda x)^{i-1}}{(i-1)!} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x \end{aligned}$$

In order to calculate $\int_0^\infty \frac{\pi \mathrm{e}^{-\lambda x}(\lambda x)^{i-1}}{(i-1)!} f^{(1)}_{L_{r_1,r_2}}(x)\,\mathrm{d}x$ we use $f^{(1)}_1(x)$ and $f^{(1)}_2(x)$ defined in Appendix 2. We have

$$\begin{aligned} &\int_0^\infty \frac{\pi \mathrm{e}^{-\lambda x}(\lambda x)^{i-1}}{(i-1)!} f^{(1)}_1(x)\,\mathrm{d}x \\ &= \int_0^\infty \frac{\pi \mathrm{e}^{-\lambda x}(\lambda x)^{i-1} x^{r_1-1}\mu_1^{r_1}\mathrm{e}^{-x\mu_1}\Gamma(r_2, x\mu_2)}{(i-1)!(r_1-1)!(r_2-1)!}\,\mathrm{d}x \\ &= \frac{\pi\lambda^{i-1}\mu_1^{r_1}}{(i-1)!(r_1-1)!} \int_0^\infty \mathrm{e}^{-x(\lambda+\mu_1+\mu_2)} x^{i+r_1-2} \left(1 + \frac{x\mu_2}{1!} + \cdots + \frac{x{\mu_2}^{r_2-1}}{(r_2-1)!}\right)\mathrm{d}x \end{aligned}$$

If we use the formula $\int_0^\infty \mathrm{e}^{-\lambda\omega}(\lambda\omega)^p\,\mathrm{d}\omega = p!/\lambda$ to solve the above equation then finally we obtain

$$\int_0^\infty \frac{\pi e^{-\lambda x}(\lambda x)^{i-1}}{(i-1)!} f_1^{(1)}(x)\,dx = \frac{\pi\lambda^{i-1}\mu_1^{r_1}}{(i-1)!(r_1-1)!}\sum_{p=0}^{r_2-1}\frac{\mu_2^p(i+r_1-2+p)!}{p!(\lambda+\mu_1+\mu_2)^{i+r_1-1+p}} \quad (4.152)$$

Using the same procedure we are able to find the formula for $\int_0^\infty \frac{\pi e^{-\lambda x}(\lambda x)^{i-1}}{(i-1)!} f_2^{(1)}(x)\,dx$, which means

$$\int_0^\infty \frac{\pi e^{-\lambda x}(\lambda x)^{i-1}}{(i-1)!} f_2^{(1)}(x)\,dx = \frac{\pi\lambda^{i-1}\mu_2^{r_2}}{(i-1)!(r_2-1)!}\sum_{p=0}^{r_1-1}\frac{\mu_1^p(i+r_2-2+p)!}{p!(\lambda+\mu_1+\mu_2)^{i+r_2-1+p}} \quad (4.153)$$

Finally we gain the below relation for $ASC(i|r_1, r_2)$:

$$\begin{aligned} ASC(i|r_1, r_2) &= 2ASC(i-1|r_1, r_2) - ASC(i-2|r_1, r_2) \\ &+ \frac{\pi\lambda^{i-1}\mu_1^{r_1}}{(i-1)!(r_1-1)!}\sum_{p=0}^{r_2-1}\frac{\mu_2^p(i+r_1-2+p)!}{p!(\lambda+\mu_1+\mu_2)^{i+r_1-1+p}} \\ &+ \frac{\pi\lambda^{i-1}\mu_2^{r_2}}{(i-1)!(r_2-1)!}\sum_{p=0}^{r_1-1}\frac{\mu_1^p(i+r_2-2+p)!}{p!(\lambda+\mu_1+\mu_2)^{i+r_2-1+p}} \end{aligned} \quad (4.154)$$

5

Exploring the economic consequences of paying a supplier to keep reserve

Exploring the economic consequences of paying a supplier to keep reserve

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Abstract

We consider a single-item, periodic-review inventory control problem in which discrete stochastic demand must be satisfied. The time horizon of our problem is finite. There are two modes of supply: normal and reserve (emergency). While there is no limitation to the normal order size, the reserve storage is capacitated and the company can either order all the reserve inventory or nothing. If the emergency storage is empty it takes some fixed time to be filled up again. Providing that the reserve storage is full, it can be used at any time period while normal orders can be issued at specific time periods. Normal lead-times are larger than emergency lead-times but both are fixed and known. All shortages are back ordered. We use a stochastic dynamic programming to model the problem. Furthermore, we develop an approximate model for the same problem which has turned out to be faster than the original model. Then numerically we make sensitivity analyses with respect to key parameters like emergency order size and emergency unit cost. The paper is developed on the basis of a contact to a leading Danish provider of communications solutions

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†blah blah

§5.1 Introduction

Using dual or multiple sourcing is a common way for hedging against risks in an inventory system, as a consequence of the stochastic environment. Especially in recent years, quite a number of researchers have addressed the problem of having two (or more) supply modes for delivering the goods to the retailer. In this paper we deal with a periodic review inventory system in which there are two supply modes: regular and emergency.

The paper is developed on the basis of a contact to a leading Danish provider of communications solutions. In the following the company is denoted the *focal company*. The focal company has many of its suppliers located overseas with long replenishment lead-times and many of the products delivered are specific for the focal company. Therefore the supplier produces only upon receipt of an order from the focal company. The products of the focal company have a very short life cycle, typically no more than two years. In addition, the demand is non-stationary with some initial growth in demand, followed by a peak-demand period; then demand is gradually declining and finally it is phased out. Due to uncertainty about the demand in combination with the long lead-time, the focal company needs to keep a rather large amount of safety stock. Therefore the focal company is considering to pay the supplier to have reserve storage somewhere in Europe, so that the focal company can decrease its costly safety stock and occasionally pull from the reserve storage if needed. The crucial point is of course for the focal company to know how much it should pay the supplier for offering such a reserve storage opportunity and how to design the contract parameters. Therefore the authors were contacted and asked if they could give some feedback on this.

The idea was to develop a dynamic programming model for the new set-up and to show how to make sensitivity analyses with respect to some contract parameters in order to give the focal company a useful tool for any future negotiations with the supplier.

The paper is organized as follows. First there is a review of relevant literature in Section 5.2. In Section 5.3 we outline the dynamic programming model. In Section 5.4 we outline an alternative model. In Section 5.5, we review the data sets we have been using in our analysis. As the computation times are a bit long despite the approximations, we have limited ourselves to studying two data sets. Using these data sets we do sensitivity analyses with respect to key input parameters. Furthermore, in Section 5.5, we compare the optimal model and approximate models with respect to speed and precision. Finally, Section 5.6 contains concluding remarks in addition to some reflections concerning the applicability of the developed models for the focal company.

§5.2 Literature review

Studies in the literature addressing this problem are either policy-evaluation or policy-optimization studies. In policy-optimization studies the goal is to find the true optimal policy and solve specific instances of the problem under consideration (Chiang and Gutierrez (1998)). On the other hand policy-evaluation studies use broader assumptions and consider a specific policy form and devise methods for evaluating it. This paper contributes to the policy-evaluation literature.

Chiang and Gutierrez (1998) and Chiang (2001) deal with almost the same problem. They develop a dynamic programming model over a finite planning horizon for a periodic review inventory system in which both regular and emergency orders exist. Regular orders are placed at the beginning of an order cycle which consists of a number of periods, while emergency orders can be placed at the start of each period. The length of an order cycle is exogenously determined. In Chiang (2001), the two lead-times for regular and emergency orders differ by exactly one period, while in Chiang and Gutierrez (1998) the two lead-times differ by more than one period but less than the order-cycle length. All unsatisfied demands are backordered. they develop a stochastic dynamic programming model to obtain the optimal order-up-to levels. Their optimal policy is valid only when there are no fixed regular and emergency ordering costs. In our model, as in the Chiang (2001) regular orders can be placed at the beginning of an order cycle consisting of a number of predetermined periods. However, the policy for placing emergency orders is different between the two models. Here, although reviewing inventory for placing emergency orders is continuous, as in the Chiang (2001) model, because there is capacity restriction for emergency source, after each time the retailer places an emergency order he has to wait a number of periods until the source refills itself. As a result emergency orders cannot be placed in all periods .

In the work by Teunter and Vlachos (2001) they only allow emergency orders that arrive one, two, or up to some fixed number of time units before a regular order arrives. This is because the probability of a stock-out shortly before receiving a regular order is the highest. Their study is a generalization of that of Chiang and Gutierrez (1998). If the policy allows that emergency orders arrive at each of time units in a review period then the policy introduced by Chiang and Gutierrez (1998) is obtained.

In the periodic review inventory system introduced by Tagaras and Vlachos (2001) regular and emergency orders are placed based on a base stock policy with a deterministic lead-time, and only one emergency order is placed per cycle as late as possible so that it arrives before the end of the review period, when the likelihood of a stock out is highest. It is therefore a special case of the work by Teunter and Vlachos (2001) and they develop an approximate cost model for this system.

Sometimes allowing for shortage is completely unacceptable in the system. Huggins and Olsen (2010) consider such a situation in a periodic review inventory system with infinite horizon. At the beginning of each period a decision about regular order is made. After realization of demand in that period the manager must now decide how many expedited orders to produce overnight, given that backlog demand is not allowed at the start of the following day. They show that the optimal regular production policy is (s, S). They allow for multiple modes of transportation with each mode having a fixed cost component, a unit cost component, and a fixed cost for each load defined by a cargo capacity. They develop optimality properties and dynamic programming algorithms to compute an optimal solution. For cases with only two modes of transportation, their algorithms are polynomial.

§5.3 Model formulation

We consider a product with a limited lifetime, subdivided into time periods $0, 1, \ldots, T$. The time period can be considered as a week and the time horizon typically 1–2 years, meaning T is say 96 weeks. We have for each time period developed a probabilistic forecast specified by the random variables D_t, $t = 0, 1, \ldots, T$. For simplicity we assume these to be uncorrelated and we also assume no updating of those forecasts during the lifetime of the product. Note that some of the initial D's could be *zero* as we also want ramp-up effects to be part of the model. At the beginning of each time period it is possible to make replenishment decisions. We assume that there are two suppliers.

Supplier **N**, the normal source, will only accept order requests every R time periods, where $R > 1$. As an example R could be a month which is then 4 weeks. Specifically, Supplier **N** will only accept replenishments issued in time periods t which fulfills that $t \bmod R = 0$ (that is, in time period $0, R, 2R$ etc.). We assume that the lead-time is equal to the review period (could probably be generalized to a case where the lead-time is an integral number multiplied by R, for example $3R$) such that a replenishment issued at the start of time period t is received in the inventory at the start of time period $t + R$ (so this time index is also divisible by R). The replenishment unit cost is v_N.

Supplier **E**, the emergency source, keeps a reserve inventory of q units. If the focal company wants to use this reserve storage, the whole storage must be replenished. So either the focal company orders 0 or orders q units. When issuing a replenishment order from the emergency source, it takes L_E time periods before the order arrives at the inventory. We assume $L_E < R$. The unit replenishment cost is v_E. It takes S time periods (where S is a positive integer) before the emergency source can refill the reserve storage and it can be exploited again. Thus, when exploiting the reserve storage, it takes some time before it can be exploited again. In principle S can be below L_E or above L_E where the former implies that there can be several orders (each of size q) in transit between Supplier **E** and the inventory of the focal company.

The focal company is therefore charged an inventory cost rate c per time period per unit on the storage. They are also charged a unit refill cost f when refilling the reserve storage. Finally in each time period the end-of-period on-hand inventory at the focal company is charged an inventory unit cost h, while units on the back-order list at the end of the time period are charged a unit cost b.

This problem can be considered as a dual source inventory model. Two specific aspects of this problem are as follows. Firstly, because of limited life time the steady state is not an issue here. Secondly, the two suppliers are handled differently, where Supplier **N** is operated by a periodic review control policy and Supplier **E** by a continuous review control policy (for simplicity and of practicality though, our model uses discrete time). Using different policies for normal and emergency orders is not uncommon. An example of this is the refilling of ATM machines with bills where for normal orders a periodic review (s, S) policy is used and for emergency ones a continuous (r, Q) policy. Please look at Miranda and Muñoz (2005) for more information.

We can model this decision problem by *dynamic programming*. Let the triple (i, y, u) denote the state space. The first component, i, denotes the net-inventory at the focal company (that is on-hand inventory − backorder list) as well as the amount on-order from supplier **E**. The component y denote the amount on order from Supplier **N**, where if $t \bmod R = 0$ it means the shipment has just arrived to the inventory (but not yet counted into the inventory of the focal company). The component u denotes the availability date of the reserve inventory. Contrary to the first two components, its range depends on the *current* time period t, because it attains the values $t, t+1, \ldots, \min\{t+S, T-L_E+1\}$. If $u = t$ it means the storage is available for usage, that is, it is filled with q units. If $t < u < \min\{t+S, T-L_E+1\}$ it means that it is currently not available for usage but it is under way to be refilled. If $u = t+S$, it is not available for use and no effort is taken to refill it. As $t+S$

can exceed the time horizon (where $T - L_E$ is the last time period where a decision can be made), the state variable u is defined such that it can never be larger than $T - L_E + 1$. As a consequence of this (see later in the specification of decision variable a_3) no refill decision is taken at time periods t where $t + S > T - L_E$. Thus when $t = T - L_E$ then the state variable u can only attain two values: $T - L_E$ (there is reserve storage available for usage) or $T - L_E + 1$ (there is no reserve storage available for usage). For later ease of the notation define

$$u_{\max}(t) = \min\{t + S, T - L_E + 1\}. \tag{5.1}$$

We assume we can only order an integral number of batches, each of size ψ. Accordingly, y is therefore measured in batches instead of in units.

When observing the state (i, y, u) at the start of time period t, three decisions a_1, a_2 and a_3 must be taken. Decision a_1 concerns how much to order from Supplier **N** (in batches). Assume some (artificial) maximum amount M of batches can be ordered; the constraint here is

$$0 \leq a_1 \leq M \cdot \mathrm{I}_{\{\{t \bmod R = 0\}\}} \tag{5.2}$$

The notation $\mathrm{I}_{\{B\}}$ is a binary variable which attains the value 1 if condition B is fulfilled and 0 otherwise. Hence, the condition just states that only in the time periods with $t \bmod R = 0$ are we allowed to order from Supplier **N**.

Decision a_2 concerns ordering from Supplier **E**, the emergency source, and because we can only order the full amount or nothing, we can let this attain the values 0 or 1. Also, we can only order from Supplier **E** if the storage is available. Therefore the restriction on a_2 is

$$a_2 \in \{0, 1\} \quad \text{and} \quad a_2 \leq \mathrm{I}_{\{u=t\}} \tag{5.3}$$

Decision a_3 concerns whether to refill the reserve storage. It is only meaningful to do this if it is currently empty and no refill decision has been taken so far ($u =$

$t+S$) or if it has been decided to empty it ($a_2 = 1$). Furthermore it is only meaningful to refill the reserve storage if $t + S \leqslant T - L_E$. Therefore the restriction on a_3 is

$$a_3 \in \{0, 1\} \quad \text{and} \quad a_3 \leq \mathrm{I}_{\{u=t+S \vee a_2=1\}} \times \mathrm{I}_{\{t+S \leq T-L_E\}}. \tag{5.4}$$

Because the state variable i is a sort of inventory position, also including amounts on orders from supplier **E**, we can first assess the inventory cost impact (of at the beginning of time period t after having observed the state (i, y, u) and made the decisions a_1, a_2 and a_3) at the end of time period $t + L_E$. The net inventory at that time (ignoring the outstanding order from Supplier **N**) is

$$i + a_2 q - \sum_{j=t}^{t+L_E} D_j. \tag{5.5}$$

Furthermore we must also keep track of whether the outstanding shipment (of amount y) from Supplier **N** has arrived to the inventory before or in the time period $t + L_E$. This is the case if

$$\mathrm{I}_{\{\mathrm{Int}(\frac{t+L_E}{R})>\mathrm{Int}(\frac{t}{R})\}} \times \mathrm{I}_{\{t \bmod R \neq 0\}} + \mathrm{I}_{\{t \bmod R=0\}} = 1 \tag{5.6}$$

Here Int denotes the integer remainder. This is a very mathematic way of expressing a much more simple fact. We should include the current outstanding shipment from the Supplier **N** if either we are just about to receive the shipment from Supplier **N** (mathematically if $t \bmod R = 0$) or we shall receive the shipment during the next L_E time periods. This happens if there is an integer k such that $t < kR \leq t + L_E$ (mathematically if $\mathrm{Int}(\frac{t+L_E}{R}) > \mathrm{Int}\left(\frac{t}{R}\right)$).

Therefore, when $t + L_E \leq T$ the expected inventory and back order costs can be assessed as

$$h \cdot \mathrm{E}\left[i + a_2 q - \sum_{j=t}^{t+L_E} D_j + y\psi\left(I_{\{\mathrm{Int}(\frac{t+L_E}{R})>\mathrm{Int}(\frac{t}{R})\}} \times I_{\{t \bmod R \neq 0\}} + I_{\{t \bmod R=0\}}\right)\right]^+ +$$
$$b \cdot \mathrm{E}\left[i + a_2 q - \sum_{j=t}^{t+L_E} D_j + y\psi\left(I_{\{\mathrm{Int}\left(\frac{t+L_E}{R}\right)>\mathrm{Int}(\frac{t}{R})\}} \times I_{\{t \bmod R \neq 0\}} + I_{\{t \bmod R=0\}}\right)\right]^- \quad (5.7)$$

The notation x^+ and x^- means $\max(x, 0)$ and $\max(-x, 0)$, respectively.

The other costs (procurement, keeping the reserve storage and refilling it) are specified by

$$v_N \psi a_1 + v_E a_2 q + f q a_3 + c q \mathrm{I}_{\{u=t, a_2=0\}} \quad (5.8)$$

We must also specify the states we will be in at the start of the next time period $t+1$. The new state variable i_{new} is random as it depends on the demand observed in time period t. It can be specified as

$$i_{\text{new}} = i + a_2 q - D_t + y\psi \mathrm{I}_{\{t \bmod R=0\}} \quad (5.9)$$

In principle i_{new} can be infinite, which is not possible to cope with computationally. Therefore we need to truncate the state variable i by two pre-determined parameters $i_{\max}$ and $i_{\min}$ where the first is positive and the second is negative, such that we always have $i_{\min} \leq i \leq i_{\max}$. This should also hold for the new state variable i_{new} which is therefore specified as:

$$i_{\text{new}} = \max\left(i_{\min},\ \min(i_{\max}, i + a_2 q - D_t + y\psi \mathrm{I}_{\{t \bmod R=0\}})\right). \quad (5.10)$$

In order to secure this, we introduce two artificial cost components. The unit cost of fast delivery, π_F, and the unit cost of discarding, π_D. It means that if we (hypothetically) at the start of time period $t+1$ should have $i + a_2 q - D_t + y\psi \mathrm{I}_{\{t \bmod R=0\}} < i_{\min}$ then we instantaneously buy some units at a price π_F to bring i_{new} at level $i_{\min}$. Similarly if $i + a_2 q - D_t + y\psi \mathrm{I}_{\{t \bmod R=0\}} > i_{\max}$ then we instantaneously discard some units (either some at the focal company inventory or some units in transit) at a price π_D to bring i_{new} at level $i_{\max}$. The expected costs of fast-buying and discarding is:

$$\pi_F \mathrm{E}[i_{\min} - i - a_2 q + D_t - y\psi \mathrm{I}_{\{t \bmod R=0\}}]^+ + \pi_D \mathrm{E}[i + a_2 q - D_t + y\psi \mathrm{I}_{\{t \bmod R=0\}} - i_{\max}]^+ \tag{5.11}$$

The point is that if the interval $(i_{\min}, i_{\max})$ is chosen sufficiently wide and the artificial cost parameters π_F and π_D are chosen sufficiently large, then the optimization model will secure that fast-buying and discarding happens with a probability close to zero.

The new state variable y_{new} is either the old one (because the shipment is not yet received) or the new order decision. Thus it is

$$y_{\text{new}} = y\mathrm{I}_{\{t \bmod R \neq 0\}} + a_1 \mathrm{I}_{\{t \bmod R=0\}}. \tag{5.12}$$

The new state variable u_{new} is one of the following: the old one u (because already a replenishment order has been issued), $t + S$ (because a refill decision has been taken), $t + 1$ (because in the current time period no action was taken to use the available reserve storage) or $u_{\max}(t + 1)$ (because the storage has been emptied, or it was empty with no outstanding refill decision, but no refill decision has been made in the current time period). Furthermore we must also keep track of whether we are close to the time horizon. Thus it is

$$\begin{aligned} u_{\text{new}} &= u\mathrm{I}_{\{t<u<u_{\max}(t)\}} + (t+S)\mathrm{I}_{\{a_3=1\}} + (t+1)\mathrm{I}_{\{u=t \wedge a_2=0\}} \\ &\quad + u_{\max}(t+1)\mathrm{I}_{\{(u=u_{\max}(t) \wedge a_3=0) \vee (u=t \wedge a_2=1 \wedge a_3=0)\}}. \end{aligned} \tag{5.13}$$

We can now by $V_t(i, y, u)$ specify the expected cost incurred during the remaining lifetime of the product by acting optimal in all decision epochs. By combining all the information above this function can be formulated in a mathematical optimization model wherein the random future scenario is expressed by making expectations on the function V_{t+1}. Later at the end of this section we show an outline of this. We also need to specify a terminal function. As $T - L_E$ is the last time period where a decision is made, the terminal function is specified by the function V_{T-L_E+1}. If we assume a unit cost γ_1 of scrapping all inventories left over and a unit cost γ_2 if at the end of the time horizon we have a backlog, we can at the start of time period $T - L_E + 1$ assess the expected terminal costs by

$$\begin{aligned} V_{T-L_E+1}(i, y, T - L_E + 1) &= \gamma_1 \mathrm{E}\left[i + y\psi - \sum_{t=T-L_E+1}^{T} D_t\right]^+ \\ &\quad + \gamma_2 \mathrm{E}\left[i + y\psi - \sum_{t=T-L_E+1}^{T} D_t\right]^- . \end{aligned} \tag{5.14}$$

Ultimately we want to assess $V_0(0, 0, S)$ the expected cost of starting prior to the sales season with no inventories, no outstanding orders and no reserve storage.

§5.3.1 Outline of dynamic programming algorithm

Collect all cost expressions

$$\begin{aligned} & h\mathrm{E}\left[i + a_2 q - \sum_{j=t}^{t+L_E} D_j + y\left(\mathrm{I}_{\{\mathrm{Int}(\frac{t+L_E}{R})>\mathrm{Int}(\frac{t}{R})\}} \times \mathrm{I}_{\{t \bmod R \neq 0\}} + \mathrm{I}_{\{t \bmod R = 0\}}\right) \times \psi\right]^+ \\ & + b\mathrm{E}\left[i + a_2 q - \sum_{j=t}^{t+L_E} D_j + y\left(\mathrm{I}_{\{\mathrm{Int}(\frac{t+L_E}{R})>\mathrm{Int}(\frac{t}{R})\}} \times \mathrm{I}_{\{t \bmod R \neq 0\}} + \mathrm{I}_{\{t \bmod R = 0\}}\right) \times \psi\right]^- \\ & + \ \pi_F \mathrm{E}\left[i_{\min} - i - a_2 q + D_t - y\psi \mathrm{I}_{\{t \bmod R = 0\}}\right]^+ \\ & + \ \pi_D \mathrm{E}\left[i + a_2 q - D_t + y\psi \mathrm{I}_{\{t \bmod R = 0\}} - i_{\max}\right]^+ \\ & \qquad + \ v_N a_1 + v_E a_2 q + f q a_3 + c q \mathrm{I}_{\{u = t \wedge a_2 = 0\}} \end{aligned} \tag{5.15}$$

into a cost function $C_t(i, y, u, a_1, a_2, a_3)$.

For $t = T - L_E, T - L_E - 1, \ldots, 0$ do the optimization

$$V_t(i, y, u) = \min_{a_1, a_2, a_3} \{C_t(i, y, u, a_1, a_2, a_3) + \mathrm{E}[V_{t+1}(i_{\mathrm{new}}, y_{\mathrm{new}}, u_{\mathrm{new}})]\}, \tag{5.16}$$

where the minimization is done under the constraints

$$\begin{aligned}
&0 \leq a_1 \leq M \times \mathrm{I}_{\{t \bmod R=0\}} \\
&a_2 \in \{0,1\} \text{ and } a_2 \leq \mathrm{I}_{\{u \leq t\}} \\
&a_3 \in \{0,1\} \text{ and } a_3 \leq \mathrm{I}_{\{u=t+S \,\vee\, a_2=1\}} \times \mathrm{I}_{\{t+S \leq T-L_E\}}. \qquad (5.17)
\end{aligned}$$

The terminal function is obtained by

$$\begin{aligned}
V_{T-L_E+1}(i, T-L_E+1) &= \gamma_1 \mathrm{E}\left[i + y\psi - \sum_{t=T-L_E+1}^{T} D_t\right]^+ \\
&\quad + \gamma_2 \mathrm{E}\left[i + y\psi - \sum_{t=T-L_E+1}^{T} D_t\right]^- . \qquad (5.18)
\end{aligned}$$

The aim is to assess $V_0(0,0,S)$ and the optimal decision policy herein. Furthermore, to investigate how the contract parameters υ_E, c, f and q affect $V_0(0,0,S)$ and the optimal decision policy herein.

§5.4 An alternative model

In the main model introduced in the preceding section the state space can be very large. In the main model the state space is defined by triple (i, y, u) where i denotes the net inventory at the focal company as well as the amount on order from supplier **E**. The component y denotes the amount on order from Supplier **N**, and the component u denotes the availability date of the reserve inventory.

It turned out that the original dynamic programming model is a bit cumbersome in terms of computation time. In this section we introduce a new model in which our aim is to reduce the state space and consequently the computational time. Let (i, u) denote the state space. The second component is as previously defined. The first component is the inventory position of the focal company; that is the net-inventory at the focal company plus all outstanding orders from any of the two suppliers. Now in the new situation the decisions only occur at the beginning of the time period t, where either $t \bmod R = 0$ or $R - L_E \leq (t \bmod R) \leq R - 1$ (we are throughout this section assuming that $L_E \geq 1$). In the following subsections we discuss each of these cases and the actions taken in each time period.

§5.4.1 Consider the case $t \bmod R = 0$

When at the start of time period t observing state (i, u) the following actions are taken:

a_1: It concerns the number of batches to order from Supplier **N** at the start of time period t.

$a_2(r)$: This set of actions concerns whether to issue a replenishment order to Supplier **E** at the start of time period $t + r; r = 0, 1, \ldots, R - L_E - 1$.

$a_3(r)$: This set of actions concerns whether to refill the reserve storage at the beginning of time period $t + r; r = 0, 1, \ldots, R - L_E - 1$.

In order to acquire the restrictions on the actions taken in this case, first we need to define the vector $u(r), r = 0, \ldots, R - L_E - 1$ which specifies the availability date of reserve storage at the beginning of time period $t + r$. We have

$$u(0) = u \tag{5.19}$$

$$\begin{aligned} u(r+1) = {} & u(r)\mathrm{I}_{\{t<u(r)<u_{\max}(t)\}} + (t+r+S)\mathrm{I}_{\{a_3(r)=1\}} + (t+r+1)\mathrm{I}_{\{u(r)=t \wedge a_2(r)=0\}} \\ & + u_{\max}(t+r+1)\mathrm{I}_{\{\{u(r)=u_{\max}(t) \wedge a_3=0\} \vee \{u(r)=t \wedge a_2(r)=1 \wedge a_3(r)=0\}\}}. \end{aligned} \tag{5.20}$$

The restrictions on the actions are as follows.

$$a_1 \leq M \tag{5.21}$$

$$a_2(r) \in \{0, 1\} \text{ and } a_2(r) \leq \mathrm{I}_{\{u(r)=t+r\}}; r = 0, \ldots, R - L_E - 1 \tag{5.22}$$

$$a_3(r) \in v\{0, 1\} \text{ and } a_3(r) \leq \mathrm{I}_{\{u(r)=t+r+S \vee a_2(r)=1\}}\mathrm{I}_{\{t+r+S \leq T-L_E\}}, r = 0, \ldots, R - L_E - 1 \tag{5.23}$$

We could also find all the above relations based on u which is the information we have in the current time period. As an example, the restriction for $a_2(1)$ can be rewritten as $a_2(1) \leq \mathrm{I}_{\{u=t \wedge a_2(0)=0\}} + \mathrm{I}_{\{u=t+1\}}$.

As seen at time periods t such that $t \bmod R = 0$, actions are taken for $R - L_E$ consecutive time periods starting from t. having observed the state (i, u) and making actions $a_1, a_2(r)$, and $a_3(r), r = 0, \ldots, R - L_E - 1$, the expected inventory and backorder costs until the next decision epoch is $\sum_{r=0}^{R-L_E-1} K_{t+r}$ where

$$K_{t+r} = h\mathrm{E}\left[i + q\sum_{r'=0}^{R-L_E-1} a_2(r') + \sum_{j=t}^{t+r+L_E} D_j\right]^+ + b\,\mathrm{E}\left[i + q\sum_{r'=0}^{R-L_E-1} a_2(r') + \sum_{j=t}^{t+r+L_E} D_j\right]^- \tag{5.24}$$

Along the lines of what we did in Section 5.3, K_{t+r} is obtained based on the net inventory at time period $t + r + L_E$.

The expected costs of procurement, and of keeping the reserve storage and refilling it until the next decision epoch is specified by

$$v_N a_1 + v_E q \sum_{r=0}^{R-L_E-1} a_2(r) + fq \sum_{r=0}^{R-L_E-1} a_3(r) + cq \sum_{r=0}^{R-L_E-1} \mathrm{I}_{\{u(r)=t+r \wedge a_2(r)=0\}} \tag{5.25}$$

The next decision epoch occurs when $t \bmod R = R - L_E$ and the new state variables $(i_{\text{new}}, u_{\text{new}})$ are given as follows.

$$i_{\text{new}} = \max\left(i_{\min}, \min\left(i_{\max}, i + a_1\psi + q\sum_{r=0}^{R-L_E-1} a_2(r) - \sum_{j=t}^{t+R-L_E-1} D_j\right)\right) \tag{5.26}$$

and

$$u_{\text{new}} = u(R - L_E) \tag{5.27}$$

Again we could also obtain the relation for u_{new} directly from u.

As we did in the previous section, in order to truncate the state variable i by two pre-determined parameters $i_{\max}$ and $i_{\min}$ such that we always have $i_{\min} \leq i \leq i_{\max}$ we use fast-buying and discard. The expected discard and fast-buying costs until the decision epoch is

$$\pi_F \mathrm{E}\left[i_{\min} - i - a_1\psi - q\sum_{r=0}^{R-L_E-1} a_2(r) + \sum_{j=t}^{t+R-L_E-1} D_j\right]^+ \\ + \pi_D \mathrm{E}\left[i - i_{\max} + a_1\psi + q\sum_{r=0}^{R-L_E-1} a_2(r) - \sum_{j=t}^{t+R-L_E-1} D_j\right]^+ \quad (5.28)$$

§5.4.2 Consider the case $R - L_E \leq (t \bmod R) \leq R - 1$

In this case we only need to take actions for the current time period. Hence, at the start of time period t observing state (i, u) the following actions are taken:

a_2**:** It concerns whether to issue a replenishment order of size q to Supplier **E** at the start of time period t.

a_3**:** It concerns whether to refill the reserve storage at the beginning of time period t.

The restrictions on these two decisions are as in the original model.

In this case the outstanding order from Supplier **N** always arrives before time period $t + L_E$. The proof for that is as follows. $R - L_E \leq (t \bmod R) \leq R - 1$ states that there is an integer k such that $kR - L_E \leq t \leq kR - 1$. Consequently $kR \leq t$. Furthermore, $t < t + 1 < kR$. As a result the relation $t < kR \leq t + L_E$ always holds.

The expected inventory and back order costs can be assessed as

$$h\mathrm{E}\left[i + a_2 q - \sum_{j=t}^{t+L_E} D_j\right]^+ + b\mathrm{E}\left[i + a_2 q - \sum_{j=t}^{t+L_E} D_j\right]^- \quad (5.29)$$

The comparison between Equation 5.29 and Equation 5.7 confirms that we do not need to keep track of y anymore in our new model.

The expected costs of procurement, keeping the reserve storage and refilling is specified by

$$v_N a_1 + v_E a_2 q + f q a_3 + c q \mathrm{I}_{\{u=t \wedge a_2=0\}} \quad (5.30)$$

Like in the original model, the next decision epoch in this case is the next time period,$t+1$, and the new state variables are

$$i_{\text{new}} = \max(i_{\min}, \min(i_{\max}, i + a_2 q - D_t)) \tag{5.31}$$

$$\begin{aligned} u_{\text{new}} &= u\mathrm{I}_{\{t<u<u_{\max}(t)\}} + (t+S)\mathrm{I}_{\{a_3=1\}} + (t+1)\mathrm{I}_{\{u=t\wedge a_2=0\}} \\ &\quad + u_{\max}(t+1)\mathrm{I}_{\{(u=u_{\max}(t)\wedge a_3=0)\vee(u=t\wedge a_2=1\wedge a_3=0)\}}. \end{aligned} \tag{5.32}$$

And finally the expected discarding and fast-buying costs are

$$\pi_F \mathrm{E}[i_{\min} - i - a_2 q + D_t]^+ + \pi_D \mathrm{E}[i + a_2 q - D_t - i_{\max}]^+ \tag{5.33}$$

§5.4.3 Outline of dynamic programming algorithm

Collect all cost terms into a cost function $C_t(i, u, a_1, a_2(r)\ r = 0, \ldots, R - L_E - 1, a_3(r)\ r = 0, \ldots, R - L_E - 1)$ when $t \mod R = 0$ and cost function $C_t(i, u, a_2, a_3)$ when $R - L_E \leq (t \bmod R) \leq R - 1$. Then proceed backward in time using the recursion. Then for $t = T - L_E, T - L_E - 1, \ldots, 0$ proceed backward in time using the recursion

$$\begin{aligned} V_t(i, u) = \min\{&C_t(i, u, a_1, a_2(r) r = 0, \ldots, R - L_E - 1, a_3(r) r = 0, \ldots, R - L_E - 1) + \\ &\mathrm{E}[V_{t+R-L_E}(i_{\text{new}}, u_{\text{new}})]\}, \text{ when } t \bmod R = 0 \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} V_t(i, u) = \min\{&C_t(i, u, a_2, a_3) + \mathrm{E}[V_{t+1}(i_{\text{new}}, u_{\text{new}})]\}; \\ &\text{when } R - L_E \leq (t \bmod R) \leq R - 1, \end{aligned} \tag{5.35}$$

where the minimization is with respect to the decision variables $(a_1, a_2(r)\ r = 0, \ldots, R - L_E - 1, a_3(r)\ r = 0, \ldots, R - L_E - 1)$ and (a_2, a_3) respectively and is done under the constraints (5.21), (5.22), (5.23), (5.3), and (5.4). The terminal function is obtained as the original model. At the end of this section we point out that the alternative model introduced in this section is an approximate model, where at some decision epochs the model locks some future decisions, thereby decreasing the state space. One of our objectives in the next section is to evaluate how far the results of this model are from optimality.

§5.5 numerical results

In order to test the computational procedures two data sets have been developed. One is provided by the focal company and our propositions to the company was based on this data-set, and the second one is self-developed, partly inspired by information received by the company.

§5.5.1 The data set derived from the focal company

The data set from the focal company is as follows in Table 5.1. In addition, the forecasts for demand in different months are given in Table 5.2. In order to study

Table 5.1: Data on parameter values provided by the focal company

Variable	Notation	Value	Unit
Lifetime horizon	T	78	weeks
Forecast	D_t		
Order interval supplier N (Normal)	R	1	month (or 4 weeks)
Replenishment unit cost, normal supplier	υ_N	385	DKK
Replenishment unit cost, emergency supplier	υ_E	404.25	DKK
Reserve quantity at supplier E (Emergency)	q	6000	units
Replenishment lead-time supplier N	L_N	4	weeks
Replenishment lead-time supplier E	L_E	2	weeks
Replenishment lead-time to refill reserve storage	S	6	weeks
Inventory cost rate of reserve storage (per time period per unit)	c	0.925481	DKK/unit/week
Refill cost when refilling reserve storage, (per unit)	f	0	DKK/unit
Inventory cost rate at the focal company (per unit per end of time period)	h	1.850962	DKK/unit/week
Back-order cost (per unit per end of time period)	b	25	DKK/unit/week
Unit cost of scrapping (excess inventory at end of lifetime horizon)	γ_1	385	

the inventory system using this data set, we set some assumptions as follows. In our experiments we decided to scale the demand data by 100 (for example the expected demand in May $X + 1$ is now 195.97). We use one week as the time unit. In order to match with the previous information about a lifetime of 18 months (= 78 weeks) it must be assumed that November $X+1$ (with an expected demand of 130 (new) units) is the last month with any demand. We also assume that the months: Oct X, Jan $X + 1$, Mar $X+1$, May $X+1$, Aug $X+1$ and Oct $X+1$, each have 5 weeks. It was assumed that each week in a month has the same expected demand. In principle, when the number of weeks is uneven, we should strictly speaking let the re-order interval R vary between being 4 or 5 (R is 1 month), but that would be extremely difficult for computation. Therefore it is assumed that $R = 4$ weeks. Also in order to be able to study any ramp-up effects, we insert an additional 8 weeks with zero demand as the first 8 time periods. As the model requires a computation of the probability distribution of random variables like $\sum_{j=t}^{t+L_E} D_j$, which can be cumbersome to carry out, it is assumed that the underlying demand model is a non-stationary Poisson process. This means that all random variables D_j are Poisson distributed, which means that also the aforementioned $\sum_{j=t}^{t+L_E} D_j$ is Poisson distributed. In order to further scale down the costs, it was decided to put $\upsilon_N = 0$ and $\upsilon_E = 19.25$.

Table 5.2: Demand forecasts from the data set by the focal company

Month	Forecast	Month	Forecast
June X	16961	March X+1	19271
July X	16063	April X+1	14850
August X	16353	May X+1	19597
September X	16626	June X+1	15955
October X	20275	July X+1	13273
November X	15915	August X+1	12517
December X	20999	September X+1	13515
January X+1	17586	October X+1	12749
February X+1	16535	November X+1	13000

§5.5.2 The self-constructed data set

In order not to experiment solely with a single data set which from a methodological point of view could be problematic, we decided to construct another data set, much inspired by the data set of the focal company. Here we put $T = 96$, $S = 4$, $R = 4$, $L_E = 2$, $h = 1$, $b = 5$, $\upsilon_E = 5$, $\upsilon_N = 0$, $c = 0.5$, $\gamma_1 = 7$, $\gamma_2 = 10$ and $f = 0$. In addition, we assume D_t is 0 for periods $t = 0, \ldots, 7$, but for the periods $t = 8, \ldots, T$, D_t is Poisson distributed with mean $10 + 0.0025(T - t)(t - 8)$.

In the following subsection we perform a number of tests for both data sets using both optimal and approximate algorithms.

§5.5.3 Computational results

The dynamic programming algorithms for both algorithms are programmed as a macro (in Visual Basic) in Excel. Besides the basic inputs, the user also specifies an actual start state and whether one wants to see detailed outputs, in terms of policy structure and simulations (including a single sample path). As output the model then provides a computation of the expected cost from the current state until the end of the horizon and what is the eminent optimal action to initiate. As an assessment of whether the various artificial constraints specified by M, $i_{\min}$ and $i_{\max}$ are set properly, the model also provides information on: the expected number of times from the current state until the end of the horizon that an order of M batches from the Normal supplier will be made (and if high it indicates M is set too low), the expected number of items that will be discarded from the current state until the end of the horizon (and if high indicating that the parameter $i_{\max}$ is set too low) and the expected number of items that will be fast-bought from the current state until the end of the horizon (and if high indicating that the parameter $i_{\min}$ is set too high). As it is of great importance to get information about how much the reserve storage is used, the spreadsheet model also outputs the expected number of times, from the current state until termination, that an order to the reserve storage will be issued. Finally one can optionally see how the decisions a_1 for the current value of t (only relevant if $t \mod R = 0$) and u depend on i and y, how the decisions a_2 for the current value of t and u (only relevant if $u = t$) depend on i and y and how the decisions a_3 for the current value of t and u (only relevant if $u = t$ or $u = t + S$) depend on i and y. Using this option also gives some insight into the policy structure.

The tests are performed with a computer with CPU T7300 and 4 GB memory in Turbo mode.

The first group of tests that we were interested in was the optimal value of the emergency order size, q. It turned out that by using the current data-set from the focal company, the reserve storage is not attractive to use. The reason is the high value of $v_E - v_N$ which is 19.25 in the data-set provided by the focal company. Therefore, in order to be able to observe the effect of different parameters on the optimal policy, we have chosen $V_E - v_N = 3$ in all tests which use the data from the focal company, unless otherwise is stated.

Tables 5.3 and 5.4 show the effect of reserve order size q on the system performance measures. Each test is performed using both optimal and approximate models and the results are compared in the tables. In all of the performed tests in this section, simulation has been done as well. Simulation results confirm our models, but in order to be brief we have not included them in our tables.

As seen in Tables 5.3 and 5.4, when q is large enough and when $q = 0$ the reserve storage is not used any more. The optimal q is between these two extremes. In Table 5.3 the difference of the total expected costs when we use optimal order size compared to when we do not use the reserve storage at all is at least 3.53%.

Notice that using the optimal value of q does not necessarily mean that the expected number of reserve storage is the highest. For example, as seen in Table 5.3, the minimum expected cost is obtained when $q = 30$ while the maximum expected use of reserve storage is obtained when $q = 60$.

In all of these tests the difference between the original optimization model and the approximate model is very modest and in the worst case the difference between them is 4.13%. Furthermore, using the optimal model, the average time spent for obtaining the results is 38 minutes for the data set provided by the focal company and 12 minutes for the self-constructed data set. for the approximate model these times are 26 and 7 minutes, respectively. the times depend on the borders of the state space, meaning $i_{\min}$, $i_{\max}$, and M. These bounds are used in order to have finite state space and to cope with the problem computationally. Their absolute values must be chosen large enough so that they do not have any effect on the results of the problems. For our experiments we used $i_{\min} = -80$, $i_{\max} = 310$, $M = 24$, and $\psi = 10$ for the data set provided by the company and $i_{\min} = -80$, $i_{\max} = 300$, $M = 15$, and $\psi = 10$ for our self-constructed data set.

Table 5.3: $V_0(0, 0, S)$ as a function of q for the data provided by the focal company (after scaling)

	Total expected costs			
q	Optimal model	Approximate model	Difference (%)	Expected reserve use
0	17012.7	17012.6	0.05	0.00
10	16678.6	15988.9	4.13	5.25
20	16506.4	16021.7	2.94	6.57
30	16431.1	16301.2	0.79	7.79
40	16455.0	16437.5	0.11	8.27
50	16584.4	16537.4	0.28	8.72
60	16654.9	16606.5	0.29	8.97
70	16733.8	16718.0	0.09	8.00
100	16967.9	16968.0	0.00	1.00
120	17012.7	17021.6	0.05	0.00

Table 5.4: $V_0(0, 0, S)$ as a function of q for the self-constructed data set

	Total Expected Costs			
q	Optimal model	Approximate model	Difference (%)	Expected reserve use
0	2423.7	2423.8	0.00	0.00
1	2420.6	2423.2	0.11	3.38
3	2418.5	2422.8	0.18	1.97
5	2418.4	2423.3	0.20	1.25
10	2420.9	2423.8	0.12	0.37
20	2423.6	2423.8	0.01	0.02
30	2423.7	2423.8	0.00	0.00
40	2423.7	2423.8	0.00	0.00
50	2423.7	2423.8	0.00	0.00
60	2423.7	2423.8	0.00	0.00

In order to compare the speed of the two models more precisely, we did many tests using different input parameters. One important factor for the speed of the models is the number of time periods in the problem. Figure 5.1 compares the speed of the two models with respect to different time horizons. Other input parameter values are derived from the data set provided by the focal company. Figure 5.2

shows how choosing different values for M can change the running time of the algorithms for approximate and optimal models. Although border values like $i_{\max}$, $i_{\min}$, and M are used only for having a finite state space, they have a direct effect on the running time of the algorithms. As seen in both the above-mentioned figures, the approximate algorithm is always faster than the optimal one and the speed rate for both models is almost linear.

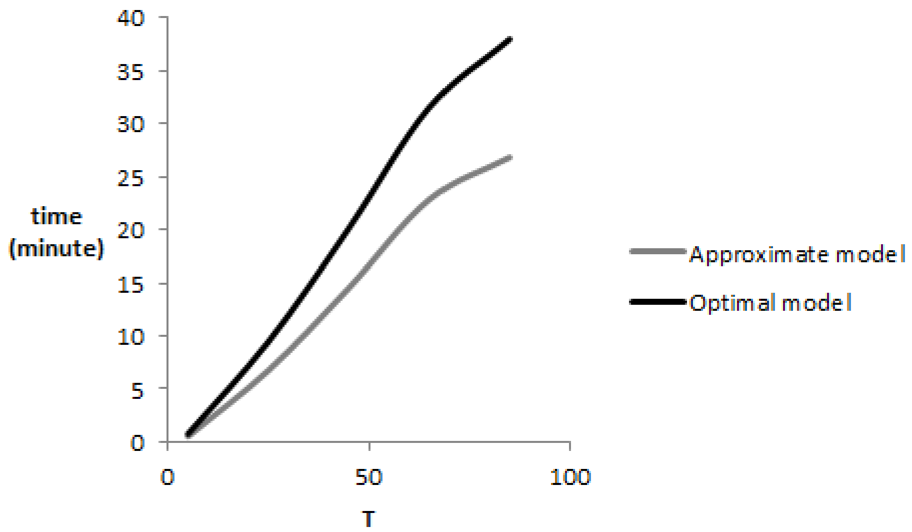

Figure 5.1: The speed of the models in different time horizons (the focal company data set)

Next, we are interested in finding out the amount the focal company is willing to pay for reserve items, or in other words, we are interested in the effect of $v_E - v_N$ on the system costs and the resource use in the optimal policy. As of now we use the optimal obtained order size, $q = 30$, for the evaluations concerning the focal company data set. Tables 5.5 and 5.6 show the results of our tests. Obviously when $v_E - v_N$ is reduced from the originally assessed values to a lower value, then reserve storage is of more advantage. For the case of Table 5.5, the reserve storage is attractive for all tested values of $v_E - v_N$. For the self constructed data-set, the reserve storage is almost unattractive when $v_E - v_N = 5$ or more.

In Tables 5.7 and 5.8 we investigate the effect of the parameter value S on the system performance measures. As seen in both tables, the lower value of S, the

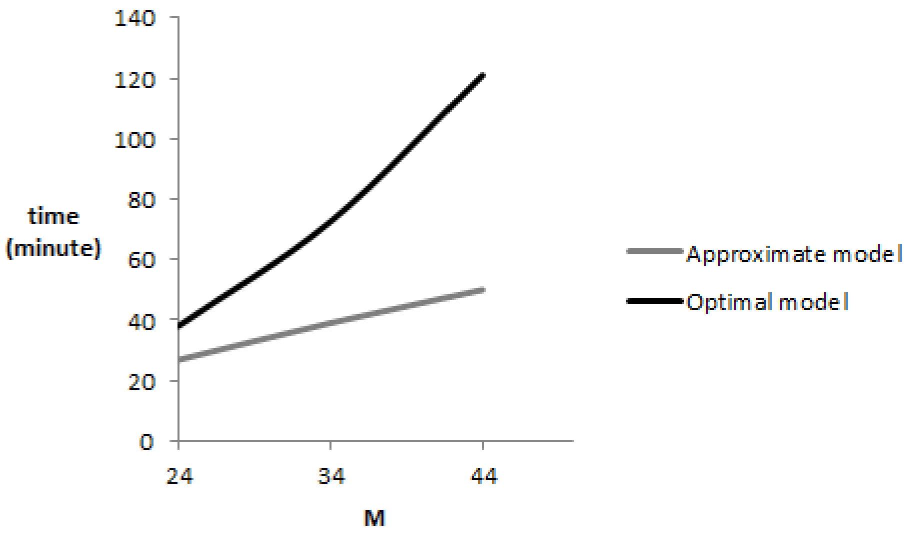

Figure 5.2: The speed of the two models when the size of state space changes (the focal company data set)

Table 5.5: $V_0(0, 0, S)$ as a function of $v_E - v_N$ for the data provided by the focal company

$v_E - v_N$	Total expected costs	Expected reserve use
0	15652.5	9.83
1	15938.3	8.56
2	16190.3	8.25
3	16431.1	7.79
4	16654.6	7.10
5	16853.7	6.07

Table 5.6: $V_0(0, 0, S)$ as a function of $v_E - v_N$ for the self-constructed data set

$v_E - v_N$	Total expected costs	Expected reserve use
0	1963.7	20.69
1	2157.6	17.22
2	2304.3	11.06
3	2381.9	4.65
4	2412.1	1.66
5	2420.9	0.37

more use of reserve storage and the lower expected costs.

Table 5.7: $V_0(0,0,S)$ as a function of S–the focal company data set

S	Total expected costs	Expected reserve use
6	16431.1	7.79
5	16428.3	8.48
4	16022.5	15.91
3	15873.3	15.58
2	15787.2	15.67

Table 5.8: $V_0(0,0,S)$ as a function of S–the self-constructed data set

S	Total expected costs	Expected reserve use
4	2423.7	0.00
3	2420.9	0.37
2	2416.7	0.74

§5.6 Conclusion

This paper is developed on the basis of a contact to a leading Danish company. The products of the focal company have a very short life cycle, and they are exposed to non-stationary stochastic demand. Most of the suppliers are located overseas with long replenishment lead-times. As a result, the focal company considers paying the supplier to have reserve storage, somewhere in Europe, so that the focal company can decrease its costly safety stock and occasionally pull from the reserve storage if needed. We made a dynamic programming model in order to analyze the company's situation and provide a tool for them to answer their questions about using this reserve option. It turned out that the model was a bit cumbersome in terms of computation time. Therefore an alternative model was developed too.

The results of the numerical analyses indicate that this emergency storage option that the supplier can provide for the focal company is quite attractive (on condition that $V_E - v_N$ changes) with respect to costs.

Our tests show that the approximate model performs reasonably well compared to the optimal dynamic programming model. Therefore it can be safely used in decision making. Its speed is an important factor in making it a good alternative model.

Although the developed model of this paper is specific to the communication industries, there are other industries which also almost fit our problem. One of them, which is mentioned by a number of researchers, is the case of the fashion industry, where many firms use manufacturing facilities located in Asian countries because of lower production costs there. However it takes weeks for receiving the order. Fashion goods usually have a limited life time. Many fashion industry firms still maintain some domestic factories in which their lead-times are lower, but at higher cost. As a result the firm can use them as an emergency source.

It is definitely the most problematic part of the model that the decision variable a_2 is binary variable, meaning either the whole reserve storage is ordered or nothing happens. Thus, in our models (as this deficiency of course also carries over to the approximate model) the parameter q is at the same time both an order size as well as the maximum reserve storage level. Therefore, our models are inflexible, forcing the order size to be equal to the maximum reserve storage level. In reality, often you will have more flexibility allowing the order size to be less than or equal to the maximum reserve storage level. In general, when more flexibility (relaxing the constraint set) is allowed into a cost minimization model, the lower will be the optimized costs. So therefore one could hypothesize that the cost figures stated in the tables in this paper are conservative, and the optimal costs found in a model where ordering from the reserve storage does not necessarily imply emptying would give better results. Based on this observation a recommendation to the focal company could be: given the data provided in Section 5.5.1 and Table 5.3 a solution could be dimensioning the reserve storage to size 40 (putting it to 30 would probably be too limiting taking into consideration that order sizes are smaller than the dimension of the reserve storage). An upper bound on the costs is then about 16455 and presumably the costs will be lower. An attractive future research would be making a more flexible model concerning the reserve storage ordering policy.

6

A renewal-reward formulation of a dual source inventory problem where there is supply uncertainty with regards to the delivered quantities

A renewal-reward formulation of a dual source inventory problem where there is supply uncertainty with regards to the delivered quantities

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Abstract

Supply uncertainty can both concern the timing and or the delivery of a quantity. Uncertainty concerning timing has received considerable attention. In this paper we study the supply uncertainty concerning the quantity delivered. There are two potential suppliers with fixed and known lead-times L_1 and L_2 . The inventory is facing demand occurring from a Poisson process with intensity λ. There is uncertainty about the real size the retailer receives from each supplier. Specially when the order size to supplier i is Q_i, the delivered quantity is random variables B_i which in our model follows binomial distribution. All shortages are backordered. Each time the inventory level hits the reorder point two orders are replenished to the suppliers, simultaneously. The objective is to minimize the total expected costs per time unit comprising ordering costs, purchasing cost, holding cost, and shortage cost. Using numerical tests, we compare between dual sourcing and single sourcing with regard to minimizing costs. The numerical studies show that dual sourcing in this case is quite attractive.

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§6.1 Introduction

Most studies on order splitting assume the only source of uncertainty concerns the timing of the receipts. Therefore as lead-times are uncertain it is of advantage to split the orders to two (or more) suppliers in order to pool the lead-time uncertainty. Thus, if one supplier is unusually late in his delivery this is hopefully partly offset by another supplier being unusually fast in his delivery. In these studies there is no uncertainty concerning the ordered quantities. That is, you receive exactly the same amount as you ordered. In this paper we seek to reverse the setting. Instead of having the source of uncertainty to be *timing* we assume it is *quantity*. Thus, the lead-times are known with certainty, however, due to quality problems, the amounts received will be random and less than the amounts ordered. In this "opposite" setting, and for simplicity only assuming only two suppliers, we investigate whether it is of advantage also here to apply order splitting.

Tang (2006) classifies the work in the area of allocating order quantity among selected suppliers according to different types of operational risks:uncertain demands; uncertain supply yields; uncertain supply lead-times; and uncertain supply costs. Our work fits in the second category. Supply or yield uncertainty specially recently has attracted a number of researchers. Several factors, such as unexpected machine breakdowns, the shortages in material availability, process adjustments, strikes, etc., make the treatment of supply uncertainty an important issue in the analysis of inventory problems. Gerchak et al. (1988) analyze a finite horizon problem with stationary demand distribution and show that order-up-to policies are not optimal. Henig and Gerchak (1990) developed a periodic review inventory model with random supply yield. They demonstrated the existence of a time-dependent critical inventory level such that an order is placed if and only if the periods starting inventory is below this level. Agrawal and Nahmias (1998) develop a model to determine optimal lot sizes and the optimal number of suppliers when there is supply uncertainty. In their model the suppliers are identical and demand rate is constant. They show that there is a trade-off between the fixed costs associated with each selected supplier and the yield loss costs. Yano and Lee (1995) do a through review over models for random yield.

In extreme case uncertainty in yield may mean supply disruption, but normally it has the form that the quantity received is a random proportion of the quantity requisitioned.

§6.2 The Model

An inventory is facing demand occurring from a Poisson process with intensity λ customer arrivals per time unit (that is, each customer demands exactly one unit). The inventory has two suppliers, in the following indexed 1 and 2. When deciding to replenish the inventory, there are simultaneously issued orders of size Q_1 and Q_2, respectively, to the two suppliers. We assume Q_1 and Q_2 are given and predetermined. Both suppliers suffer from quality problems thus the delivered quantities are random variables B_1 and B_2, respectively. We will assume that the random variable B_i is binomial distributed with number parameter Q_i and probability parameter $p_i, i = 1, 2$ where the probability parameters are known by the operator of the inventory system. The actual number delivered will first be detected upon receipt of the order. Both suppliers have constant lead-times respectively L_1 and L_2. Without loss of generality we assume $L_1 \leq L_2$. We will assume there can only be one pair of outstanding orders at the time and that replenishment orders are issued simultaneously when the inventory drops to level R. In case upon receipt of the replenishment order of Supplier 2 the inventory is still below level R then an emergency order (with zero lead-time) at a unit cost c_e is issued together with the normal replenishment orders Q_1 and Q_2. Thereby, the inventory level is always at level R when issuing the normal replenishment orders, and therefore time epochs upon where normal replenishments are issued are regeneration points. The time between two consecutive regeneration points is a cycle. If the cost parameters are set in accordance with the assumption that at there is at most one pair of outstanding replenishment orders then the probability of ever doing any emergency ordering is almost zero, and it just serves as a technical assumption to force through that the Markov chain, describing the evolution of the Markov chain, regenerates itself each time a new pair of replenishment orders are issued. The policy variable is R. The cost parameters concerning replenishing from the suppliers are: fixed replenishment costs A_1 and A_2; unit replenishment costs c_1 and c_2. In addition the inventory system is incurred an inventory cost at rate h per unit and a backorder cost at rate b per unit.

Consider a random time interval where there occur no replenishments and inventory starts at level y_1 and at the end of the time interval the inventory level is y_2. The expected accumulated inventory and back-order costs in this time interval, denoted $IB(y_1, y_2)$ is

$$IB(y_1, y_2) = \begin{cases} \frac{h}{\lambda}\left[y_2 + \frac{(y_1-y_2)(y_1-y_2+1)}{2}\right] & y_2 \geq 0 \\ \frac{1}{\lambda}\left[h\frac{y_1(y_1+1)}{2} + b\frac{y_2(y_2-1)}{2}\right] & y_2 < 0 \leq y_1 \\ \frac{b}{\lambda}\left[-y_1 + \frac{(y_1-y_2)(y_1-y_2+1)}{2}\right] & y_1 < 0 \end{cases} \tag{6.1}$$

Here is used that the considered time interval can be decomposed into a sum of $y_2 - y_1$ time interval each being exponentially distributed with mean $1/\lambda$. For a similar argument see Tijms (2003a) page 66.

Furthermore, Denote by $BI(I, P)$ the expected accumulated inventory and back-order costs collected over a time interval of length P starting with an inventory level I and where no replenishments are received in the time interval. It holds that

$$BI(I, P) = \begin{cases} \int_0^P \left[h\sum_{y=0}^{I}(I-y)e^{-\lambda t}\frac{(\lambda t)^y}{y!} + b\sum_{y=I}^{\infty}(y-I)e^{-\lambda t}\frac{(\lambda t)^y}{y!}\right] \mathrm{d}t & I > 0 \\ bP(-I + 1/2\lambda P) & I \leq 0 \end{cases} \tag{6.2}$$

The upper part in this expression can be rewritten to Equation 6.3.

$$bP(-I + 1/2\lambda P) + \frac{b+h}{\lambda}\sum_{y=0}^{I-1}(I-y)\left[1 - e^{-\lambda P}\sum_{j=0}^{y}\frac{(\lambda P)^j}{j!}\right] \tag{6.3}$$

Let by the random variable D_L denote the aggregate demand in a time interval of length L. Thus, this random variable is Poisson distributed with mean λL.

The expected cycle length is

$$EL = L_2 + \frac{1}{\lambda} E\left[\max\{B_1 + B_2 - D_{L_2}, 0\}\right] \tag{6.4}$$

Note that this term does not depend on the re-order level R.

The expected cost incurred during a cycle is then

$$EC(R) = A_1 + A_2 + c_1 E[B_1] + c_2 E[B_2] + \sum_{x=0}^{Q_1} \sum_{y=0}^{Q_2} \Pr(B_1 = x) \Pr(B_2 = y) g(x, y) \tag{6.5}$$

Where

$$g(x, y) = BI(R, L_1) + \sum_{u=0}^{\infty} \Pr(D_{L_1} = u) \Big\{ BI(R - u + x, L_2 - L_1) + \tag{6.6}$$

$$\sum_{v=0}^{\infty} \Big[\Pr(D_{L_2 - L_1} = v)(IB(R + \max\{x + y - u - v, 0\}, R) + c_e \max\{u + v - x - y, 0\}) \Big] \Big\} . \tag{6.7}$$

is the expected accumulated inventory and backorder costs in the cycle given known receipts of x units from Supplier 1 and y units from Supplier 2. This term can be rewritten to

$$g(x, y) = BI(R, L_1) + \sum_{u=0}^{\infty} \Pr(D_{L_1} = u) BI(R - u + x, L_2 - L_1) +$$

$$\sum_{v=0}^{\infty} \Pr(D_{L_2} = v) \left[IB(R + \max\{x + y - v, 0\}, R) + c_e \max v - x - y, 0 \right] \tag{6.8}$$

Please notice that here we have assumed that the retailer only pays for the amounts that actually are delivered. When delivered quantities have binomial distributions with parameters Q_i and $p_i, i = 1, 2$, the expected purchasing costs in a cycle is $c_1 p_1 Q_1 + c_2 p_2 Q_2$.

The best policy is then the one that minimizes $\gamma(R) = EC(R)/EL$, that is the expected cost per time unit (or simply the cost rate).

As we later will compare the order-split case, derived so far, to the case of single sourcing we now state the similar expressions for the single source case.

In the case of only one supplier, for instance Supplier 1, the expected cycle length is

$$EL = L_1 + \frac{1}{\lambda}E[\max\{B_1 - D_{L_1}, 0\}] \tag{6.9}$$

while the expected total costs incurred during a cycle is

$$EC(R) = A_1 + c_1E[B_1] + \sum_{x=0}^{Q_1} \Pr(B_1 = x)g(x), \tag{6.10}$$

where

$$g(x) = BI(R, L_1) + \sum_{v=0}^{\infty} \Pr(D_{L_1} = v)[IB(R + \max\{x - v, 0\}, R) + c_e \max\{v - x, 0\}] \tag{6.11}$$

The corresponding expressions, if only supplier 2 is used, is of course in analog with this.

§6.3 The numerical experiment

The numerical experiment has been designed as follows. We let $\lambda = h = 3$. We assume the two suppliers are equal in respect to costs so $A_1 = A_2 = 700$ and $c_1 = c_2 = 10$. The unit cost of expediting c_e is set to 100. We assume $L_1 = 8$ and $L_2 = 10$. As can be seen in the model outline we assume the order sizes are given, for instance governed by practical considerations (a full truck load) or simply requested by the suppliers and not subject for negotiations (or optimization). As all studies of EOQ type cost function reveal that the cost function is very insensitive to the order size. Therefore we do not feel that this assumption is so constraining for our experiment. We assume that $Q_1 = Q_2 = 30$. Note that this value is chosen with some emphasis on the EOQ formula. If single sourcing is applied the EOQ should be $\sqrt{2 \times 700} \simeq 37$ to each of the suppliers. We let the probability parameters p_1 and p_2 be varied and for different combinations of p_1 and p_2 find the optimal reorder level and expected total costs.

Obviously if $p_1 = p_2 = 1$ then order splitting is of no advantage as Supplier no 1 should be preferred as he has shorter lead-time. So a point of interest is to see how much the probability parameters must be raised in order to make order splitting advantageous. This is illustrated in Table 6.1. In this table two scenarios have been considered. The first one is the case that the retailer only pays for the amounts that actually are delivered. The purchasing cost in this case has the form that we mentioned in the model section. The other scenario is when the retailer pays for the whole order size. Thus, the purchasing costs for dual sourcing is $c_1 Q_1 + c_1 Q_2$.

Based on the results from Table 6.1, one can observe that the quality of the products, or in other word supply uncertainty, has a direct effect in deciding between dual and single sourcing. The results show that when p_1 is significantly greater that p_2 or when p_1 is very close to 1 then only using Supplier 1 as the main source is optimal; on the other hand, when p_1 is very low compared to p_2 then using only Supplier 2 as the main source is optimal, in spite of having larger lead-time. In all other combinations of quality probabilities dual sourcing is optimal. Table 6.1 shows that these results are common in both scenarios. Please note that even in the case that $p_1 = p_2 = 0.8$, although Supplier 1 has better specifications because of lower lead-time, still dual sourcing is optimal.

Regarding optimal value of R, one can observe that it is quite insensitive to the changes in quality probabilities. In fact, in the costs function, the optimal R does not depend on p_2. as a result, only when p_1 changes R might change slowly. Furthermore, the optimal value of R in the objective function does not depend on unit costs c_1 and c_2. That describes why the optimal R in both scenarios are the same for each combination of p_1 and p_2.

§6.4 Conclusion

In this paper we considered a dual sourcing problem with supply uncertainty. We use order splitting policy for ordering to the suppliers. Mostly this policy is used for inventory control model with stochastic lead-times. Hence it makes sense

Table 6.1: Comparing dual and single sourcing under two scenarios

Scenario 1: Purchasing cost is for all units ordered								Scenario 2: Purchasing cost is for units delivered					
Probability parameters		Order splitting		Only supplier 1		Only supplier 2		Order splitting		Only supplier 1		Only supplier 2	
p_1	p_2	R^*	$\gamma(R^*)$	R^*	$\gamma(R^*)$	R^*	$\gamma(R^*)$	R^*	$\gamma(R^*)$	R^*	$\gamma(R^*)$	R^*	$\gamma(R^*)$
0.8	0.95	19	153.88	20	176	25	168.96	19	149.59	20	169.1	25	167.53
0.8	0.9	19	156.47	20	176	25	181.19	19	151.18	20	169.1	25	178.29
0.8	0.8	19	162.59	20	176	25	208.13	19	155.09	20	169.1	25	202.21
0.8	0.7	19	170.14	20	176	25	237.02	19	160.14	20	169.1	25	228.07
0.8	0.6	19	179.51	20	176	25	266.72	19	166.68	20	169.1	25	254.74
0.8	0.5	19	191.21	20	176	25	296.68	19	175.17	20	169.1	25	281.68
0.95	0.8	19	155.4	20	143.6	25	208.13	19	151.11	20	142.05	25	202.21
0.9	0.8	19	157.52	20	152.7	25	208.13	19	152.22	20	149.48	25	202.21
0.8	0.8	19	162.59	20	176	25	208.13	19	155.09	20	169.1	25	202.21
0.7	0.8	19	169.03	20	205.7	25	208.13	19	159.04	20	194.91	25	202.21
0.6	0.8	19	177.31	20	240	25	208.13	19	164.48	20	225.24	25	202.21
0.5	0.8	20	188.19	20	276.6	25	208.13	20	172.14	20	257.97	25	202.21
0.4	0.8	20	202.47	20	314.2	25	208.13	20	182.82	20	291.48	25	202.21
0.3	0.8	21	221.25	20	351.5	25	208.13	21	197.66	20	325.21	25	202.21
0.2	0.8	22	245.13	20	389	25	208.13	22	217.45	20	358.97	25	202.21

to split any replenishment order into several smaller orders, in order to hedge against the risk of possible long lead-times. In those dual/multiple source models, it is assumed that when deciding for replenishment, the replenishment order is split into suborders and the suborders are issued simultaneously to the suppliers. In those problems, often there is no uncertainty concerning the ordered quantities. In this study however, we consider the inventory model that the risk is not in lead-time variability but in supply uncertainty. In order to model this uncertainty in the amount the retailer really receives we use binomial probability function which is a common way in the literature for modeling supply or yield uncertainty. The actual number delivered will first be detected upon receipt of the order.

Our numerical results show that when the uncertainty is only about quantities delivered, and not lead-times, dual sourcing with order splitting is still an attractive policy in many situations. Only when one supplier has very bad conditions regarding lead-time and supply quality or a supplier has much better conditions compared with the other supplier single sourcing is favorable. To sum up, dual/multiple sourcing is an attractive policy not only to pool lead-time risk between different suppliers but also to pool supply risk between them.

There are some areas for future study. First, in this study we assumed that order sizes are fixed and are not subject to negotiation. As a result we optimized our model with respect to reorder level, R. One may would like to optimize total costs with respect to order sizes Q_1 and Q_2 too and see if the results still hold.

Another point of interest is considering the cases where the retailer is informed about the real order quantity he will receive prior to receipt of them. As we mentioned earlier, in the model in this paper we assumed the real value of variables B_1 and B_2 just at the time of receipt of the orders. This is of course an extreme case. It would be interesting to consider the cases where this information is given to the retailer at a time between ordering and delivery.

Bibliography

Abginehchi, S. and R. Z. Farahani (2010). Modeling and analysis for determining optimal suppliers under stochastic lead times. *Applied Mathematical Modelling 34*(5), 1311–1328.

Abginehchi, S. and C. Larsen (2012). *Essays on Inventory control in presence of multiple sourcing*. Ph. D. thesis, Aarhus University, Business and Social Sciences, Fuglesangs Allé 4, 8210 Aarhus, Denmark.

Adelman, D. (2003). Price-directed replenishment of subsets: Methodology and its application to inventory routing. *Manufacturing & Service Operations Management 5*(4), 348 – 371.

Agrawal, N. and S. Nahmias (1998). Rationalization of the supplier base in the presence of yield uncertainty. *Production and Operations Management*, 291–308.

Allen, S. G. and D. A. D'Esopo (1968). An ordering policy for stock items when delivery can be expedited. *Operations Research 16*(4), 880.

Axsater, S. (2007). A heuristic for triggering emergency orders in an inventory system. *European Journal of Operational Research 176*(2), 880 – 891.

Chiang, C. (2001). A note on optimal policies for a periodic inventory system with emergency orders. *Computers & Operations Research 28*(2), 93 – 103.

Chiang, C. (2002). Ordering policies for inventory systems with expediting. *The Journal of the Operational Research Society 53*(11), pp. 1291–1295.

Chiang, C. (2003). Optimal replenishment for a periodic review inventory system with two supply modes. *European Journal of Operational Research 149*(1), 229 – 244.

Chiang, C. and W. Benton (1994). Sole sourcing versus dual sourcing under stochastic demands and lead times. *Naval Research Logistics*, 609–24.

Chiang, C. and W.-C. Chiang (1996). Reducing inventory costs by order splitting in the sole sourcing environment. *The Journal of the Operational Research Society 47*(3), pp. 446–456.

Chiang, C. and G. J. Gutierrez (1998). Optimal control policies for a periodic review inventory system with emergency orders. *Naval Research Logistics (NRL) 45*(2), 187–204.

Dohi, T., N. Kaio, and S. Osaki (1995–06). Continuous review cyclic inventory models with emergency order. *Journal of the Operations Research Society of Japan 38*(2), 212–229.

Dohi, T., H. Okamura, and S. Osaki (1999). Optimal order-limit policies for an (r, q) inventory system. *IMA Journal of Management Mathematics 10*(2), 127–145.

Duran, A., G. Gutierrez, and R. I. Zequeira (2004). A continuous review inventory model with order expediting. *International Journal of Production Economics 87*(2), 157–169.

Feng, Q., S. Sethi, H. Yan, and H. Zhang (2006). Optimality and nonoptimality of the base-stock policy in inventory problems with multiple delivery modes. *Journal of Industrial And Management Optimization 2*(1), 19–42.

Fong, D., V. Gempesaw, and J. Ord (2000, JAN 1). Analysis of a dual sourcing inventory model with normal unit demand and Erlang mixture lead times. *EUROPEAN JOURNAL OF OPERATIONAL RESEARCH 120*(1), 97–107.

Ganeshan, R., J. Tyworth, and Y. Guo (1999, MAR). Dual sourced supply chains: the discount supplier option. *Transportation Research Part E-Logistics And Transportation Review 35*(1), 11–23.

Gaukler, G. M., O. Özer, and W. H. Hausman (2008). Order progress information: Improved dynamic emergency ordering policies. *Production & Opera-*

tions Management 17(6), 599 – 613.

Gerchak, Y., R. Vickson, and M. Parlar (1988, JUN). Periodic Review Production Models With Variable Yield And Uncertain DEMAND. *IIE Transactions 20*(2), 144–150.

Giri, B. and T. Dohi (2009). Cost-effective ordering policies for inventory systems with emergency order. *Computers & Industrial Engineering 57*(4), 1336 – 1341.

Ha, A. (2000, JAN). Stock rationing in an M/E-k/1 make-to-stock queue. *Management Science 46*(1), 77–87.

Hayya, J., D. Christy, and A. Pan (1987). Reducing inventory uncertainty: a reorder point system with two vendors. *Production and Inventory Management*, 43–8.

Henig, M. and Y. Gerchak (1990, JUL-AUG). The Structure Of Periodic Review Policies In The Presence Of Random Yield. *Operations Research 38*(4), 634–643.

Huggins, E. L. and T. L. Olsen (September/October 2010). Inventory control with generalized expediting. *Operations Research 58*(5), 1414–1426.

Johansen, S. G. and A. Thorstenson (1998). An inventory model with poisson demands and emergency orders. *International Journal of Production Economics 56–57*(0), 275 – 289. ¡ce:title¿Production Economics: The Link Between Technology And Management¡/ce:title¿.

Kelle, P. and P. Miller (2001, MAY 6). Stockout risk and order splitting. *INTERNATIONAL JOURNAL OF PRODUCTION ECONOMICS 71*(1-3, SI), 407–415. 10th International Symposium on Inventories, BUDAPEST, HUNGARY, AUG, 1998.

Kelle, P. and E. Silver (1990, OCT). Safety Stock Reduction By Order Splitting. *Naval Research Logistics 37*(5), 725–743.

Lau, H.-S. and A. H.-L. Lau (1994). Coordinating two suppliers with offsetting lead time and price performance. *Journal of Operations Management*, 327–37.

Miranda, A. K. and D. F. Muñoz (2005). Dss to manage atm cash under periodic review with emergency orders. In *Proceedings of the 37th conference on Winter simulation*, WSC '05, pp. 2595–2599. Winter Simulation Conference.

Mohebbi, E. and M. J. Posner (1998). Sole versus dual sourcing in a continuous-review inventory system with lost sales. *Computers & Industrial Engineering 34*(2), 321–336.

Mohebbi, E. and M. J. Posner (1999). A lost-sales continuous-review inventory system with emergency ordering. *International Journal of Production Economics 58*(1), 93 – 112.

Moinzadeh, K. and S. Nahmias (1988). A continuous review model for an inventory system with two supply modes. *Management Science 34*(6), pp. 761–773.

Ng, C., L. Y. Li, and K. Chakhlevitch (2001). Coordinated replenishments with alternative supply sources in two-level supply chains. *International Journal of Production Economics 73*(3), 227 – 240.

Puterman, M. L. (2005a). *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc.

Puterman, M. L. (2005b). *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc.

Ramasesh, R., J. Ord, and J. Hayya (1993, Mar). Dual Sourcing With Nonidentical Suppliers. *Naval Research Logistics 40*(2), 279–288.

Ramasesh, R. V., J. K. Ord, J. C. Hayya, and A. Pan (1991). Sole versus dual sourcing in stochastic lead-time (s, q) inventory models. *Management Science 37*(4), pp. 428–443.

Rosling, K. (November/December 2002). Inventory cost rate functions with non-linear shortage costs. *Operations Research 50*(6), 1007–1017.

Ross, S. (1997). *A First Course in Probability* (fifth ed.). Prentice Hall.

Ryu, S. W. and K. K. Lee (2003). A stochastic inventory model of dual sourced supply chain with lead-time reduction. *International Journal of Production Economics 81–82*(0), 513–524. Proceedings of the Eleventh International Symposium on Inventories.

Sculli, D. and Y. Shum (1990, SEP). Analysis Of A Continuous Review Stock-Control Model With Multiple Suppliers. *Journal Of The Operational Research Society 41*(9), 873–877.

Sculli, D. and S. Y. Wu (1981). Stock control with two suppliers and normal lead times. *The Journal of the Operational Research Society 32*(11), pp. 1003–1009.

Sedarage, D., O. Fujiwara, and H. T. Luong (1999). Determining optimal order splitting and reorder level for n-supplier inventory systems. *European Journal of Operational Research 116*(2), 389–404.

Tagaras, G. and D. Vlachos (2001). A periodic review inventory system with emergency replenishments. *Management Science 47*(3), pp. 415–429.

Tang, C. S. (2006, OCT). Perspectives in supply chain risk management. *International Journal Of Production Economics 103*(2), 451–488.

Teunter, R. and D. Vlachos (2001). An inventory system with periodic regular review and flexible emergency review. *IIE Transactions 33*(8), 625–635.

Thomas, D. J. and J. E. Tyworth (2006). Pooling lead-time risk by order splitting: A critical review. *Transportation Research Part E: Logistics and Transportation Review 42*(4), 245 – 257.

Tijms, H. C. (2003a). *A First Course in Stochastic Models*. John Wiley & Sons, Ltd.

Tijms, H. C. (2003b). *A First Course in Stochastic Models*. John Wiley & Sons, Ltd.

Tyworth, J. and A. Ruiz-Torres (2000). Transportation's role in the sole- versus dual-sourcing decision. *International Journal of Physical Distribution & Logistics Management 30*(2), 128–144.

Yano, C. and H. Lee (1995, Mar-Apr). Lot-Sizing With Random Yields - A Review. *Operations Research 43*(2), 311–334.

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