

Heterotic Non-Kähler Geometries via Polystable Bundles on Calabi-Yau Threefolds

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Abstract

In arXiv:1008.1018 it is shown that a given stable vector bundle V on a Calabi-Yau threefold X which satisfies $c_2(X) = c_2(V)$ can be deformed to a solution of the Strominger system and the equations of motion of heterotic string theory. In this note we extend this result to the polystable case and construct explicit examples of polystable bundles on elliptically fibered Calabi-Yau threefolds where it applies. The polystable bundle is given by a spectral cover bundle, for the visible sector, and a suitably chosen bundle, for the hidden sector. This provides a new class of heterotic flux compactifications via non-Kähler deformation of Calabi-Yau geometries with polystable bundles. As an application, we obtain examples of non-Kähler deformations of some three generation GUT models.

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1 Introduction

The geometry of the internal six-dimensional manifold of a supersymmetric compactification of the heterotic string with non-zero torsion is characterized by a hermitian $(1, 1)$ -form ω , a holomorphic $(3, 0)$ -form Ω and a non-abelian gauge field strength F . Supersymmetry and anomaly cancellation constrain possible triples (ω, Ω, F) which must satisfy

$$F^{2,0} = F^{0,2} = 0, \quad F \wedge \omega^2 = 0, \quad (1.1)$$

$$d(\|\Omega\|_{\omega} \omega^2) = 0, \quad (1.2)$$

$$i\partial\bar{\partial}\omega - \alpha'(\text{tr}(R \wedge R) - \text{tr}(F \wedge F)) = 0. \quad (1.3)$$

This system of equations (usually called *the Strominger system*) provides the necessary and sufficient conditions for $N = 1$ space-time supersymmetry in four dimensions [26]. Moreover, in [26] it is shown that given a solution of this system, the physical fields can be expressed in terms of (ω, Ω, F) .

First examples of solutions of the Strominger system have been obtained in [21, 19, 20, 24]. One difficulty in obtaining smooth solutions of the Strominger system lies in the fact that many theorems of Kähler geometry and thus methods of algebraic geometry do not apply. Therefore one approach to obtain solutions is to simultaneously perturb Kähler and Hermitian-Yang-Mills (HYM) metrics and so avoid the direct construction of non-Kähler manifolds with stable bundles. This approach has been used in [24] where it is shown that a deformation of the holomorphic structure on the direct sum of the tangent bundle and the trivial bundle and of the Kähler form of a given Calabi-Yau threefold leads to a smooth solution of the Strominger system whereas the original Calabi-Yau space is perturbed to a non-Kähler space. For other constructions we refer to [13, 22, 8, 9].

The aim of this note is to obtain solutions of the Strominger system using a perturbative method which has been inspired by the method developed in [24] and recently used to construct solutions in [1]. Our starting point will be here a solution of the Strominger system in the large radius limit, given by any polystable vector bundle $V_1 \oplus V_2$ on a Calabi-Yau threefold X with $SU(3)$ holonomy which satisfies the second Chern class constraint

$$c_2(X) = c_2(V_1) + c_2(V_2), \quad (1.4)$$

where V_2 corresponds to a gauge bundle in the *hidden sector* of the $E_8 \times E_8$ heterotic string theory. This solution is then perturbed to a solution of the system using the implicit function theorem similarly as in [1]. Moreover, the solutions obtained satisfy

also the equations of motion derived from the effective action of the heterotic string. For this we use a result of [16] and extend the original Strominger system as in [1] by an instanton condition

$$R^{2,0} = R^{0,2} = 0, \quad R \wedge \omega^2 = 0. \quad (1.5)$$

This condition is obtained in [16] considering the equations of motion of the heterotic string up to two-loops in sigma model perturbation theory as exact. Now having obtained a supersymmetric solution to the low energy field theory approximation of heterotic string theory, one can argue that such solutions can be completed to solutions to all orders of the fully α' -corrected equations (cf. [27, 28, 29]) using non-renormalization theorems for the effective four-dimensional superpotential.

In this note we construct examples where the deformation result applies, given by a class of polystable bundles on an elliptically fibered Calabi-Yau threefold with V_1 given by a particular spectral cover bundle and V_2 given by a suitably chosen pullback bundle. Note that due to the appropriate choice of the hidden sector bundle V_2 we avoid the otherwise necessary introduction of background five-branes which would lead to a further modification of the anomaly constraint (for a discussion on this see [1]). For another class of hidden sector bundles we obtain examples of non-Kähler deformations of some three generation GUT models.

2 Solutions via Polystable Bundles

In this section we prove an existence result for the Strominger system and the equations of motion in the polystable case, adapting a previous result in [1]. As in [16], we consider the equations of motion of the heterotic string up to two-loops in sigma model perturbation theory as exact. We use the standard terminology in the mathematical literature and, for simplicity, we restrict to the case in which the Calabi-Yau threefold has strict $SU(3)$ holonomy.

Let X be a compact complex threefold with nowhere vanishing $(3, 0)$ -holomorphic form Ω . Let V be a holomorphic vector bundle over X satisfying $c_2(V) = c_2(X)$ with rank r and $c_1(V) = 0$. Consider the coupled system given by (1.1), (1.2), (1.3) and (1.5) where F is the curvature of a (unitary) connection A on V , ω is a hermitian form on X and R is the curvature of a ω -unitary connection ∇ on TX .

Suppose that X admits a Calabi-Yau metric ω_0 with strict $SU(3)$ holonomy and irre-

ducible Chern connection ∇_0 and that V is polystable with respect to $[\omega_0]$. Then,

$$V = \bigoplus_{j=1}^k V_j \tag{2.1}$$

is a direct sum of stable bundles with respect to $[\omega_0]$. We claim that there exists a family of solutions $(A_\lambda, \omega_\lambda, \nabla_\lambda)$ of (1.1), (1.2), (1.3) and (1.5) parameterized by $\lambda \in]\lambda_0, +\infty[$ such that $(A_\lambda, \omega_\lambda/\lambda, \nabla_\lambda)$ converges to a HYM connection on V , the Calabi-Yau metric ω_0 and its Chern connection ∇_0 , respectively. In order to prove this result, following [1] we rescale the hermitian form ω by a positive constant $\lambda > 0$ and consider a new system of equations, the ϵ -system, defined by the change of variable $\alpha' \longrightarrow \epsilon = \alpha'/\lambda$ in (1.3). Therefore, any solution of the ϵ -system with $\epsilon > 0$ is related to the original system after rescaling. In the limit $\lambda \longrightarrow \infty$ a solution of the ϵ -system is given by a degree zero polystable holomorphic vector bundle on a Calabi-Yau threefold. This way the pair (X, V) is regarded as a solution of the initial system in the large radius limit.

We prove now that (X, V) , the given solution for $\epsilon = 0$, can be perturbed to a solution with small $\epsilon > 0$, i.e., with large λ . More precisely, the Kähler form of the given Calabi-Yau threefold is perturbed to a conformally balanced Hermitian form on the fixed complex manifold while also perturbing its Chern connection and the unique HYM connection on the bundle V , whereas we preserve the HYM condition. The argument is analogue to that in [1], so we give a brief sketch here and refer to [1] for more details. We consider a 1-parameter family of maps between (suitable) fixed Hilbert spaces

$$\mathbf{L}^\epsilon: \mathcal{V}_1 \longrightarrow \mathcal{V}_2,$$

indexed by $\epsilon \in \mathbb{R}$ which acts on triples $(A, \omega, \nabla) \in \mathcal{V}_1$ and whose zero locus corresponds to solutions of the ϵ -system. Roughly speaking, \mathbf{L}^ϵ is defined by sending a triple (A, ω, ∇) to the 4-tuple given by the LHS of (1.1), (1.2), (1.3) (with $\alpha' \longrightarrow \epsilon = \alpha'/\lambda$) and (1.5). This way, the map \mathbf{L}^ϵ splits into a direct sum

$$\mathbf{L}^\epsilon = \mathbf{L}_1 \oplus \mathbf{L}_2^\epsilon \oplus \mathbf{L}_3 \oplus \mathbf{L}_4$$

and the equation $\mathbf{L}^\epsilon = 0$ with $\epsilon \neq 0$ corresponding to the ϵ -system is related to a simpler one at $\epsilon = 0$ which has a known solution $(A_0, \omega_0, \nabla_0)$. As the initial bundle V is polystable we start with a reducible connection $A_0 = \bigoplus^j A_j$, with irreducible HYM pieces A_j corresponding to the direct sum decomposition (2.1) into stable subbundles. Then, for the perturbative argument we consider connections A which are direct sum of connections on the subbundles E_j (for the definition of the Hilbert spaces $\mathcal{V}_1, \mathcal{V}_2$ above) and split the map $\mathbf{L}_1 = \bigoplus^j \mathbf{L}_{1,j}$ in the obvious way. Then, the differential

$$\delta_0 \mathbf{L}^0: \mathcal{V}_1 \longrightarrow \mathcal{V}_2,$$

at the initial solution is an isomorphism (after composition with a suitable orthogonal projection) and an implicit function theorem argument shows that there is a solution of $\mathbf{L}^\epsilon = 0$ nearby $(A_0, \omega_0, \nabla_0) \in \mathcal{V}_1$ in an open neighborhood of $\epsilon = 0 \in \mathbb{R}$. Therefore, we obtain the following result.

Theorem 2.1. *Let V be a degree zero holomorphic vector bundle over a compact Calabi-Yau threefold (X, ω_0) with holonomy equals to $SU(3)$. If $c_2(V) = c_2(X)$ and V is polystable with respect to $[\omega_0]$, then there exists $\lambda_0 \gg 0$ and a C^1 curve*

$$] \lambda_0, +\infty[\ni \lambda \longrightarrow (A_\lambda, \omega_\lambda, \nabla_\lambda)$$

of solutions of the Strominger system and the equations of motion derived from the effective heterotic string action such that $\bar{\partial}_\lambda$ is isomorphic to $\bar{\partial}_0$ for all λ and $(A_\lambda, \frac{\omega_\lambda}{\lambda}, \nabla_\lambda)$ converges uniformly to the solution $(A_0, \omega_0, \nabla_0)$ when $\lambda \longrightarrow \infty$.

The perturbation provided by Theorem 2.1 leaves the holomorphic structure of V unchanged while the one on TX is shifted by a complex gauge transformation and so remains isomorphic to the initial one. Since the initial bundle is polystable the solutions of the hermitian-Yang-Mills equations obtained in the perturbation may be reducible.

3 Application to Elliptic Calabi-Yau Threefolds

One motivation for this note was to apply the method developed in [1] to the widely explored class of spectral cover bundles which has been originally constructed in [17, 18]. As shown in [25], these bundles are not capable to solve $c_2(X) = c_2(V)$ but do solve the generalized anomaly constraint $c_2(X) - c_2(V) = [W]$ with $[W]$ an effective curve class [17]. Here $[W] \neq 0$ corresponds to the presence of background five-branes in the heterotic vacuum. One possibility to avoid background five-branes (as these cause a delta function contribution to the anomaly equation, cf. discussion in [1]) is to search for stable bundles whose second Chern class is equal to $[W]$ corresponding to turning on a *hidden sector* bundle which can then be used to break part of the hidden E_8 gauge symmetry. (The general question to give sufficient conditions for the existence of stable bundles with prescribed Chern classes has been studied in [15] and [2]).

Following this approach, in this section we will construct a class of polystable bundles $V \oplus \pi^*E$ where V is given by a spectral cover bundle and π^*E a pullback bundle. For this we first briefly recall the spectral cover construction of [17] and then prove that π^*E can be chosen to be stable with respect to the same Kähler class ω as V , that is,

$\omega = t\omega_0 + \pi^*\omega_B$ (cf. below). We then provide conditions such that $V \oplus \pi^*E$ satisfies the second Chern class constraint (1.4) imposed by anomaly cancellation.

On elliptically fibered Calabi-Yau threefolds $\pi: X \rightarrow B$ with section σ a general construction of stable vector bundles was given in [17]. We will consider here the case of stable $SU(n)$ bundles V . Such bundles can be explicitly constructed from a divisor C in X , called the spectral cover, and a line bundle L on C . The spectral cover C is a surface in X that is an n -fold cover of the base B . More precisely, C is an element of the linear system $|n\sigma + \pi^*\eta|$ of $\mathcal{O}_X(n\sigma + \pi^*\eta)$ whereas η is a curve class in B . Let us briefly recall the conditions on the spectral data (C, L) such that V is a stable $SU(n)$ bundle. For more details see [17].

For $SU(n)$ bundles the first Chern class of L is given by [17]

$$c_1(L) = n\left(\frac{1}{2} + \mu\right)\sigma + \left(\frac{1}{2} - \mu\right)\pi^*\eta + \left(\frac{1}{2} + n\mu\right)\pi^*c_1(B) \quad (3.1)$$

since $c_1(L)$ must be an integer class, it follows that: if n is odd, then μ is strictly half integral and for n even an integral μ requires $\eta \equiv c_1 \pmod{2}$ while a strictly half-integral μ requires $c_1(B)$ even.

Now given the spectral data (C, L) , there is a unique $SU(n)$ bundle V which can be constructed canonically in two ways. As in [17], V can be directly constructed as

$$V = \pi_{1*}(\pi_2^*L \otimes \mathcal{P})$$

where \mathcal{P} is the associated Poincaré bundle and π_i are the two projections of the fiber product $X \times_B C$ onto the two factors X and C . Equivalently, one can construct V directly from the spectral data using a relative Fourier-Mukai transform (for a survey see [4]).

In Theorem 7.1 in [18] it is shown that for irreducible C the bundle V is stable with respect to the Kähler class $\omega = t\omega_0 + \pi^*\omega_B$ with ω_0 some fixed Kähler class in X and ω_B some Kähler class in B and t a sufficiently small positive real number. To assure that the linear system $|n\sigma + \pi^*\eta|$ contains an irreducible surface C one has to demand that the linear system $|\eta|$ be base-point free in B and that $\eta - nc_1(B)$ is effective in B [17].

The second Chern classes of X and V are given by [17]

$$\begin{aligned} c_2(X) &= 12\sigma \cdot \pi^*c_1(B) + \pi^*(c_2(B) + 11c_1(B)^2) \\ c_2(V) &= \pi^*\eta \cdot \sigma - \frac{n^3 - n}{24}\pi^*c_1(B)^2 + \frac{1}{2}\left(\mu^2 - \frac{1}{4}\right)n\pi^*\eta \cdot (\pi^*\eta - n\pi^*c_1(B)). \end{aligned}$$

As we are interested in solving the topological constraint with spectral cover bundles and $[W] = 0$ (cf. above) we need to find another class of ω -stable bundles. For this let

us consider pullback bundles which have been also studied in [3]. However, in this note we will not assume that the base of the elliptically fibered Calabi-Yau threefold has an ample anti-canonical line bundle as it was assumed in the proof of Lemma 5.1 in [3]. Note that in [3] the loss of generality in B allowed to freely specify t in $\omega = t\omega_0 + \omega_B$ which was required to construct stable extension bundles in a second step.

Let X be an elliptic Calabi-Yau threefold $\pi: X \rightarrow B$ with section σ and let E be a given ω_B -stable rank r vector bundle on B (a rational surface) with $c_1(E) = 0$ and $c_2(E) = k$. We get the following result:

Theorem 3.1. π^*E is stable on X with respect to $\omega = t\omega_0 + \pi^*\omega_B$ for t sufficiently small.

Proof. Let F be a subsheaf of π^*E with $0 < \text{rank } F < r$. We need to show that $\mu(F) < 0$. From Lemma 4.1 in [3] we get for a subsheaf F of π^*E that $c_1(F) \cdot \pi^*\omega_B^2 \leq 0$ and $c_1(F) \cdot \omega_0 \cdot \pi^*\omega_B < 0$. Now using Lemma 7.2 in [18] we have for all nonzero subsheaves F of π^*E that $\frac{c_1(F) \cdot \pi^*\omega_B^i \cdot \omega_0^{2-i}}{\text{rank } F} \leq A$ with $0 \leq i \leq 2$ and A only depending on π^*E , ω_0 and $\pi^*\omega_B$. We can thus conclude that for t chosen sufficiently small $\mu(F) < 0$. \square

So far we have constructed a class of polystable vector bundles $V \oplus \pi^*E$ with V a given spectral cover bundle. Now if we impose the following conditions (set $c_i(B) = c_i$)

$$\eta = 12c_1 \tag{3.2}$$

$$k = c_2 + 11c_1^2 + \frac{n^3 - n}{24}c_1^2 - \left(\mu^2 - \frac{1}{4}\right)\frac{n}{2}\eta(\eta - nc_1) \tag{3.3}$$

then $c_2(X) = c_2(V) + c_2(\pi^*E)$ is satisfied and thus gives examples of polystable bundles which satisfy the hypothesis of Theorem 2.1.

It remains to prove that a ω_B -stable rank r bundle E on B with $c_1(E) = 0$ and $c_2(E) = k$ and k given by (3.3) actually exists on B . In [7] it is shown that the moduli space of vector bundles of rank r , $c_1(E) = 0$ and $c_2(E) = k$ is not empty if $k > \max(1, p_g)(r + 1)$ (where p_g denotes the geometric genus of B). Thus in our case we have to assure that (using $\eta = 12c_1$ and $c_1^2 + c_2 = 12$ for B rational)

$$12 + \left(10 + \frac{n^3 - n}{24} - \left(\mu^2 - \frac{1}{4}\right)(72 - 6n)n\right)c_1^2 > r + 1. \tag{3.4}$$

An example is given by spectral bundles with n odd and $\mu = \pm\frac{1}{2}$ or with n even, $\mu = 0$ and c_1 even solve this condition ($r \leq 8$ is assumed, for breaking the hidden E_8 gauge symmetry).

4 Towards GUT Model Building

The heterotic string provides a natural setting for building realistic models of particle physics. In recent years much progress has been made in the construction of physically interesting Calabi-Yau compactifications with stable vector bundles which lead to GUT models and models with a particle content close to the Standard Model (for recent progress see, [6], [10], [11]). However, it is an open problem whether these phenomenologically interesting models can be generalized to torsional heterotic backgrounds. If this were possible, then it would allow to address the question of solving the anomaly equation and not only its integrability condition (which is necessary but not sufficient) as it is done in most of the present constructions, and it would also allow to address the question of moduli stabilization.

In this section we will study this problem for some three generation GUT models. We will construct a polystable bundle which leads to three generation models of unbroken gauge group $SU(5)$ or E_6 in the visible sector and which then can be perturbed using the result of Theorem 2.1 to a solution of the Strominger system and the equation of motion (in lowest order of α' , cf. above). Therefore the GUT models qualify as torsional heterotic backgrounds.

The net number of generations N_{gen} is determined by the index of the Dirac operator coupled to the visible sector gauge bundle V_{vis} and is given by the third Chern class of V_{vis}

$$N_{gen} = \frac{1}{2}c_3(V_{vis}).$$

For spectral cover bundles one finds (cf. [5, 12, 14]) that N_{gen} is given by

$$N_{gen} = \mu\pi^*\eta(\pi^*\eta - n\pi^*c_1(B))\sigma.$$

Now obviously the above examples with $\eta = 12c_1$ (and say $\mu = \pm\frac{1}{2}$) lead to rather large generation numbers. To get more flexibility for GUT model building we will consider now another possibility. As above, we will construct a polystable bundle

$$V = V_{vis} \oplus V_{hid},$$

with V_{vis} given by a spectral cover $SU(n_1)$ bundle. For the hidden bundle we take the polystable bundle

$$V_{hid} = W \oplus \pi^*E,$$

with W an $SU(n_2)$ spectral cover bundle and π^*E an $SU(r)$ ω_B -stable pullback bundle. Alternatively, one could consider a stable deformation of $W \oplus \pi^*E$ which would then lead

to an irreducible structure group of the hidden bundle and thus give different unbroken hidden gauge groups. Necessary and sufficient conditions for the existence of such a deformation have been obtained for polystable bundles in [23].

The corresponding Chern classes are given by $c_2(\pi^*E) = k$ and

$$\begin{aligned} c_2(V_{vis}) &= \pi^*\eta_1 \cdot \sigma - \frac{n_1^3 - n_1}{24} \pi^*c_1(B)^2 + \frac{1}{2}(\mu_1^2 - \frac{1}{4})n_1\pi^*\eta_1 \cdot (\pi^*\eta_1 - n_1\pi^*c_1(B)) \\ c_2(W) &= \pi^*\eta_2 \cdot \sigma - \frac{n_2^3 - n_2}{24} \pi^*c_1(B)^2 + \frac{1}{2}(\mu_2^2 - \frac{1}{4})n_2\pi^*\eta_2 \cdot (\pi^*\eta_2 - n_2\pi^*c_1(B)). \end{aligned}$$

Now similar as above, if we impose the conditions

$$\eta_1 + \eta_2 = 12c_1 \quad (4.1)$$

$$k = c_2 + 11c_1^2 + \sum_{i=1}^2 \left[\frac{n_i^3 - n_i}{24} c_1^2 - (\mu_i^2 - \frac{1}{4}) \frac{n_i}{2} \eta_i (\eta_i - n_i c_1) \right] \quad (4.2)$$

then $c_2(X) = c_2(V_{vis}) + c_2(V_{hid})$ is satisfied and thus gives examples of polystable bundles which satisfy the hypothesis of Theorem 2.1. Moreover, as above in (3.4) we have here also to assure that $k > r + 1$ which guarantees the existence of stable rank r bundles E on B with the Chern classes $c_1(E) = 0$ and $c_2(E) = k$.

For concreteness, we will now show that the construction leads to three generation models of unbroken gauge groups E_6 and $SU(5)$ in the visible sector, that is, we specify $SU(n_1)$ bundles V_{vis} with $n_1 = 3, 5$. For simplicity, we will fix the base B to be given by the Hirzebruch surface \mathbb{F}_1 with Chern classes $c_1(\mathbb{F}_1) = 2b + 3f$, $c_2(\mathbb{F}_1) = 4$ (and with the intersection relations $b^2 = -1$, $bf = 1$ and $f^2 = 0$). Moreover, we will assume that $r \leq 8$.

Example 4.1. We obtain an unbroken E_6 gauge group for $n_1 = 3$ and $\eta_1 = 11b + 10f$ and $\mu_1 = \pm \frac{1}{2}$ giving three generations. Therefore by (4.1) we get $\eta_2 = 13b + 26f$ and the instanton number of E is then given by

$$k = 100 + \frac{n_2^3 - n_2}{24} c_1^2 - (\mu_2^2 - \frac{1}{4}) \frac{n_2}{2} \eta_2 (\eta_2 - n_2 c_1) \quad (4.3)$$

thus if we take $n_2 = 3$ and $\mu_2 = \pm \frac{1}{2}$ giving $k = 108$ and all conditions are satisfied.

Example 4.2. An unbroken $SU(5)$ gauge group in the visible sector is obtained for $n_1 = 5$ and $\eta_1 = 12b + 15f$ and $\mu_1 = \pm \frac{1}{2}$ giving three generations. By (4.1) we get $\eta_2 = 12b + 21f$ and the instanton number of E is then given by

$$k = 132 + \frac{n_2^3 - n_2}{24} c_1^2 - (\mu_2^2 - \frac{1}{4}) \frac{n_2}{2} \eta_2 (\eta_2 - n_2 c_1) \quad (4.4)$$

as $\eta_2 \equiv c_1 \pmod{2}$ we can take here $n_2 = 2$ and $\mu_2 = 0$ giving $k = 197$ which guarantees the existence of the required rank two bundle E on B .

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