ABSENCE OF EMBEDDED EIGENVALUES FOR RIEMANNIAN LAPLACIANS

by K. Ito and E. Skibsted

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ABSTRACT. In this paper we study absence of embedded eigenvalues for Schrödinger operators on non-compact connected Riemannian manifolds. A principal example is given by a manifold with an end (possibly more than one) in which geodesic coordinates are naturally defined. In this case one of our geometric conditions is a positive lower bound of the second fundamental form of angular submanifolds at infinity inside the end. Another condition may be viewed (at least in a special case) as being a bound of the trace of this quantity, while similarly, a third one as being a bound of the derivative of this trace. In addition to geometric bounds we need conditions on the potential, a regularity property of the domain of the Schrödinger operator and the unique continuation property. Examples include ends endowed with asymptotic Euclidean or hyperbolic metrics studied previously in the literature.

CONTENTS

1. INTRODUCTION AND RESULTS

Let (M, g) be a non-compact connected Riemannian manifold of dimension $d \geq 1$ (possibly incomplete), and H the Schrödinger operator on the Hilbert space $\mathcal{H} = L^2(M)$:

$$H = H_0 + V; \quad H_0 = -\tfrac{1}{2}\triangle = \tfrac{1}{2}p_i^* g^{ij} p_j, \quad p_i = -\mathrm{i}\partial_i.$$

We introduce four conditions under which we prove that a self-adjoint realization of H does not have eigenvalues greater than some computable constant. For the Euclidean case the theory boils down to absence of positive eigenvalues which is a well studied subject, see e.g. [RS, FHH2O, JK]. Our conditions appear rather weak and allow for application to manifolds with boundary (possibly caused by metric or potential singularities). In particular, to our knowledge, they are weaker than

conditions used so far in the literature on the subject, cf. e.g. [Me, MZ, Do, Ku1, Ku2]. The present work is applied in a companion paper [IS] in which scattering theory is studied for a general class of metrics. Our conditions are also weaker than the conditions of [IS].

The first condition we impose guarantees intuitively that (M, g) has at least one "expanding end".

Condition 1.1. There exists an unbounded real-valued function $r \in C^\infty(M)$, $r(x) \geq 1$, such that uniformly in $x \in M$ (i.e. all limits below are meant to be uniform in $x \in M$):

(1) The following inequality holds,

$$\limsup_{r \to \infty} |\mathrm{d}r| < \infty. \tag{1.1}$$

(2) There exist constants $c > 0$, $\tilde{c} \in [c/2, c)$ and $r_0 \geq 1$ such that

$$\nabla^2 r^2 \geq cg \text{ for } r \geq r_0, \tag{1.2}$$

and

$$\liminf_{r \to \infty} (r\partial^r |\mathrm{d}r|^2 + \tilde{c}|\mathrm{d}r|^2) > 0, \quad \lim_{r \to \infty} \partial^r |\mathrm{d}r|^2 = 0, \tag{1.3}$$

where $\partial^r = \mathrm{i}p^r = \nabla r = \operatorname{grad} r$ denotes the gradient vector field for r, i.e.

$$\partial^r f = (\partial_i r)g^{ij}(\partial_j f), \quad f \in C^\infty(M).$$

(3) There exists a decomposition $\triangle r^2 = \rho_1 + \rho_2$ such that

$$\lim_{r \to \infty} \rho_1 = 0, \quad \limsup_{r \to \infty} r^{-1}|\rho_2| < \infty, \quad \limsup_{r \to \infty} |\mathrm{d}\rho_2| < \infty. \tag{1.4}$$

Note that the subsets $\{x \in M | \ r(x) \leq \tilde{r}\}$, $\tilde{r} \geq 1$, may not be compact (this is similar to [Ku1, Ku2], see Subsection 2.2). In particular the function r could model a distance function within a fixed single *end* of M extended to be bounded outside, in particular bounded in other ends of M. Also note that for an exact distance function (1.1) and (1.3) are trivially fulfilled, and in that case the above operator ∂^r is identified as the geodesic radial derivative ∂_r, see Subsection 2.2.

Condition 1.2. There exists a decomposition $V = V_1 + V_2$, $V_1 \in L^2_{\mathrm{loc}}(M)$, $V_2 \in C^1(M)$ and V_1, V_2 real-valued, such that uniformly in $x \in M$:

$$\lim_{r \to \infty} rV_1 = 0, \quad \limsup_{r \to \infty} |V_2| < \infty, \quad \limsup_{r \to \infty} r\partial^r V_2 \leq 0. \tag{1.5}$$

Note that under Condition 1.2 the subspace $C_\mathrm{c}^\infty(M) \subseteq \mathcal{D}(V)$ and whence that H is defined at least on $C_\mathrm{c}^\infty(M)$. However under Conditions 1.1 and 1.2 this operator is not necessarily essentially self-adjoint. Note that (M, g) is allowed to be incomplete and that V is allowed to be unbounded. For instance (M, g) could be the interior of a Riemannian manifold with boundary and for essentially self-adjointness we would then need a symmetric boundary condition. Lack of essential self-adjointness could also originate from unboundedness of V in some end. To fix a self-adjoint extension we first choose a non-negative $\chi \in C^\infty(\mathbb{R})$ with

$$\chi(r) = \begin{cases} 0 & \text{for } r \leq 1, \\ 1 & \text{for } r \geq 2, \end{cases}$$

and then set

$$\chi_\nu(r) = \chi(r/\nu), \quad \nu \geq 1. \tag{1.6}$$

We shall henceforth consider the function χ_ν as being composed with the function r from Condition 1.1. In this sense particularly $\chi_\nu \in C^\infty(M)$.

Condition 1.3. The operator H defined on $C_c^\infty(M)$ (by Condition 1.2) has a self-adjoint extension, denoted by H again, such that for any $\psi \in \mathcal{D}(H)$ there exists a sequence $\psi_n \in C_c^\infty(M)$ such that for all large $\nu \geq 1$

$$\|\chi_\nu(\psi - \psi_n)\| + \|\chi_\nu(H\psi - H\psi_n)\| \to 0 \quad \text{as } n \to \infty.$$

Note that Condition 1.3 is fulfilled if (M, g) is complete and V is bounded. In that case indeed H is essentially self-adjoint on $C_c^\infty(M)$, see Proposition 2.1 for a more general result.

As a global condition we impose for this self-adjoint extension *the unique continuation property.*

Condition 1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$, $E \in \mathbb{R}$, and $\phi(x) = 0$ in some open subset, then $\phi(x) = 0$ in M.

In Section 2 we shall discuss various models satisfying Conditions 1.1–1.4. We define a "critical" energy,

$$E_0 = \limsup_{r \to \infty} \left(V + \frac{|\mathrm{d}\rho_2|^2}{32(c-\tilde{c})\tilde{c}} \right). \tag{1.7}$$

Note that the smallest possible value of E_0 under variation of $\tilde{c}$ in (1.3) is attained at $\tilde{c} = c/2$. For examples in Subsection 2.2 (for which for simplicity $V = 0$) we can use this $\tilde{c}$ and verify that the essential spectrum $\sigma_{\mathrm{ess}}(H_0) = [E_0, \infty)$, see Remark 2.3 1). Whence for these examples indeed E_0 is critical regarding absence of eigenvalues as stated more generally in the following theorem.

Theorem 1.5. *Suppose Conditions 1.1–1.4. Then the eigenvalues of H are absent above E_0, i.e. $\sigma_{\mathrm{pp}}(H) \cap (E_0, \infty) = \emptyset$.*

Various of our conditions are optimal for exclusion of embedded eigenvalues. It is well known in Schrödinger operator theory that the von Neumann Wigner potential, see for example [FH] or [RS, Section XIII.3], provides an example of a positive eigenvalue for a decaying potential $O(r^{-1})$, $r = |x|$. Whence the conclusion of Theorem 1.5 is in general false if the first condition of (1.5) is relaxed as $\limsup_{r \to \infty} r|V_1| < \infty$. An example of a Laplace-Beltrami operator having an embedded eigenvalue is constructed in [Ku1]. This is for a hyperbolic metric, and the example shows similarly that the conclusion of Theorem 1.5 in general is false if the first condition of (1.4) is relaxed as $\limsup_{r \to \infty} |\rho_1| < \infty$. (Actually Kumura uses the von Neumann Wigner potential in his construction.)

The proof of Theorem 1.5 follows the scheme of [FHH2O, FH, DeGé, MS] employing in particular a Mourre-type commutator estimate and exponential decay estimates of a priori eigenstates. In our geometric setting the "Mourre commutator" can be very singular (in particular not bounded relatively to H in any usual sense). Consequently we only have a weak (however sufficient) version of the commutator estimate, see Corollary 3.2.

We use throughout the paper the standard notation $\langle \sigma \rangle = (1 + |\sigma|^2)^{1/2}$ and (as above) d for exterior differentiation (acting on functions on M). Note that in local

coordinates $p := -\mathrm{id}$ takes the form $p = (p_1, \ldots, p_d)$. We shall slightly abuse notation writing for example $p\psi \in \mathcal{H} = L^2(M)$ for $\psi \in C_c^\infty(M)$ even though the correct meaning here is a section of the (complexified) cotangent bundle, i.e. $p\psi \in \Gamma(T^*M)$. Note at this point that $\|p\psi\| := \|p\psi\|_{\Gamma(T^*M)} = \|\,|p\psi|\,\|_{\mathcal{H}}$. If A is an operator on $\mathcal{H}$ and $\psi \in \mathcal{D}(A)$ we denote the expectation $\langle \psi, A\psi \rangle$ by $\langle A \rangle_\psi$. Unimportant positive constants are denoted by C, in particular C may vary from occurrence to occurrence. The dependence on other variables is sometimes indicated by subscripts such as C_ν.

2. Discussion and examples

In this section we investigate how general our conditions are by looking at several examples.

2.1. **Global conditions.** We recall some general criteria for self-adjointness and the unique continuation property.

Proposition 2.1. *Let (M, g) be a complete Riemannian manifold of dimension $d \geq 1$. Then the free Schrödinger operator H_0 is essentially self-adjoint on $C_c^\infty(M)$. Suppose V is real-valued, measurable, bounded outside a compact set and in addition: $V \in L_{\mathrm{loc}}^2(M)$ for $d = 1, 2, 3$, $V \in L_{\mathrm{loc}}^p(M)$ for some $p > 2$ if $d = 4$ while $V \in L_{\mathrm{loc}}^{d/2}(M)$ for $d \geq 5$. Then V is relatively compact. In particular H is essentially self-adjoint on $C_c^\infty(M)$.*

We refer to [Ch] and [RS, Theorems X.20 and X.21]. We can generalize the class of potentials to the Stummel class, see e.g. [DoGa].

As for the unique continuation property, Condition 1.4, there is an extensive literature although mostly for Schrödinger operator theory, see e.g. [JK]. For general connected manifolds we refer to [Wo] and references therein, quoting here the following sufficient conditions supplementing connectivity and the conditions in Proposition 2.1: 1) $d = 2, 3, 4$ and V is globally bounded, or 2) $d \geq 5$. One could (of course) add 3) $d = 1$.

2.2. **Conditions inside an end.** In the sequel we consider a connected and complete (M, g) of dimension $d \geq 2$ and take (for simplicity) $V = 0$. We shall examine the meaning of Condition 1.1 in the case where, in addition, (M, g) has the following explicit *end* structure: There exists an open subset $E \subset M$ such that isometrically the closure $\bar{E} \cong [0, \infty) \times S$ for some $(d-1)$-dimensional manifold S, and that

$$g = \mathrm{d}r \otimes \mathrm{d}r + g_{\alpha\beta}(r, \sigma)\,\mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta; \quad g_{rr} = 1, \ g_{r\alpha} = g_{\alpha r} = 0, \tag{2.1}$$

where $(r, \sigma) \in [0, \infty) \times S$ denotes local coordinates and the Greek indices run over $2, \ldots, d$. Whence actually r is globally defined in E and it is a smooth distance function (here given as the distance to $\{0\} \times S$). In particular we have $|\mathrm{d}r| = 1$ which obviously implies (1.1) and (1.3). Notice here that Condition 1.1 involves only the part of the function r at large values, so in agreement with Condition 1.1 we can cut and extend it to a smooth function on M obeying $r \geq 1$. This is tacitly understood below. To examine the remaining statements (1.2) and (1.4) of Condition 1.1 we compute

$$\nabla^2 r^2 = 2\,\mathrm{d}r \otimes \mathrm{d}r + r(\partial_r g_{\alpha\beta})\,\mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta, \tag{2.2a}$$

$$\triangle r^2 = g^{ij}(\nabla^2 r^2)_{ij} = 2 + r g^{\alpha\beta}(\partial_r g_{\alpha\beta}). \tag{2.2b}$$

2.2.1. *End of warped product type.* If we consider the *warped product* case where $g_{\alpha\beta}(r,\sigma) = f(r)h_{\alpha\beta}(\sigma)$ we obtain, using (2.2a) and (2.2b), the following examples fulfilling also (1.2) and (1.4) of Condition 1.1.

Examples 2.2.
(1) Let $f = r^{2p}$ with $p > 0$. Then (1.2) and (1.4) hold with $c = \min\{2, 2p\}$ and $\rho_1 = 0$ respectively, and the critical energy $E_0 = 0$.
(2) Let $f = \exp(\kappa r^q)$ with $\kappa > 0$ and $q \in (0,1)$. Then (1.2) and (1.4) hold with $c = 2$ and $\rho_1 = 0$ respectively, and $E_0 = 0$.
(3) Let $f = \exp(2\kappa r)$ with $\kappa > 0$. Then (1.2) and (1.4) hold with $c = 2$ and $\rho_1 = 0$ respectively, and $E_0 = \kappa^2(d-1)^2/8$.

Remarks 2.3.
1) For all of these examples it is easy to compute that the essential spectrum $\sigma_{\text{ess}}(H) \supseteq [E_0, \infty)$. If in addition $M \setminus E$ and S are compact then we have $\sigma_{\text{ess}}(H) = [E_0, \infty)$. Whence indeed the absence of eigenvalues in (E_0, ∞) as stated in Theorem 1.5 is optimal under these additional conditions for the above examples (except possibly that the threshold energy $E = E_0$ in a concrete situation might not be an eigenvalue neither).
2) A metric obtained by taking $p = 1$ in (1) (and assuming also $M \setminus E$ and S compact), and possibly perturb it, is dubbed a "scattering metric" in [Me, MZ]. As shown by Melrose absence of positive eigenvalues holds for scattering metrics. Since it is not required in Condition 1.1 that r is an exact distance function we may still have this condition fulfilled in perturbed situations (letting r be the unperturbed distance function). In this spirit Donnelly [Do] studied perturbations of the Euclidean metric (corresponding to $p = 1$ in (1)) using a certain function of this type (i.e. not an exact distance function), and he proved absence of positive eigenvalues for such model. More generally, but roughly still in the framework of perturbations of (1), absence of embedded eigenvalues was obtained in [Ku2], and for hyperbolic models (roughly for perturbations of (3)) it was done in [Ku1]. However Kumura's results are stated in terms of an exact distance function and parts of his results involve conditions on the radial curvature. Whence his framework is seemingly somewhat different. It turns out, however, that his conditions imply properties that are stronger than our conditions. We will discuss an example of this point in Corollary 2.4 and Remark 2.5 2).
3) Under the condition of warped product metrics growth rates between $f = r^{2p}$ with $p > 1/2$ and $f = \exp(\kappa r^q)$ with $\kappa > 0$ and $q \in (0, 1/2)$ define a class of metrics for which the scattering theory [IS] applies. More generally Conditions 1.1–1.4 are weaker than the conditions used in [IS].

2.2.2. *Volume growth and curvature.* Here let us relate the critical energy E_0 to geometric quantities. We continue to assume (2.1) in the end E although without warped product structure. In the coordinates $(r, \sigma) \in [0, \infty) \times S$ used in (2.1) we have

$$\triangle r^2 = 2 + 2r\triangle r, \quad \triangle r = \partial_r \ln \sqrt{\det g},$$

so that we can measure the volume growth in the radial direction in terms of the function $\triangle r$. By (2.2a) the inequality (1.2), necessarily with $c \leq 2$, is equivalent to

$$(r\partial_r g_{\alpha\beta} - cg_{\alpha\beta})_{\alpha,\beta} \geq 0 \text{ for } r \geq r_0. \tag{2.3}$$

In particular the induced metric on the angular manifold $S_{\tilde{r}} = \{x \in \bar{E} \mid r = \tilde{r}\}$ grows as a function of $\tilde{r}$. By taking the trace of (2.3) assuming here and henceforth $c = 2$ and $\tilde{c} = 1$ in (1.2) and (1.3), respectively, we obtain

$$r \triangle r \geq (d-1) \text{ for } r \geq r_0.$$

Consider the special case of "asymptotic volume growth rate"

$$\triangle r = \rho_+ + o(\tfrac{1}{r}); \ \rho_+ > 0. \tag{2.4}$$

Then, setting $\rho_2 = 2 + 2r\rho_+$ and $\rho_1 = \triangle r^2 - \rho_2 = o(1)$ in (1.4), we can write E_0 in terms of the volume growth rate

$$E_0 = \rho_+^2/8. \tag{2.5}$$

Next, noting that the radial curvatures R_{rad} can control the second fundamental form (by a standard comparison argument, see e.g. [IS, Remark 1.13] for a reference) we recover a result from [Ku1] (here slightly extended).

Corollary 2.4. *Suppose (M, g) is connected and complete having an end E with metric of the form (2.1). Suppose there exists $\kappa > 0$ such that the radial curvature R_{rad} satisfies*

$$R_{\mathrm{rad}} = -\big(\kappa^2 + o(\tfrac{1}{r})\big)g \ \text{ on } S_r \ (\text{uniformly in } x \in E),$$

and there exists $r_1 \geq 0$ such that

$$R_{\mathrm{rad}} \leq 0 \ \text{ on } S_{\tilde{r}} \ \text{ for all } \tilde{r} \geq r_1 \ \text{ and } \nabla^2 r \geq 0 \ \text{ on } S_{r_1}.$$

Then $\sigma_{\mathrm{pp}}(H_0) \cap (\kappa^2(d-1)^2/8, \infty) = \emptyset$.

Proof. We have, cf. [Ku1, Proposition 2.2],

$$\nabla^2 r_{|S_r} = (\kappa + o(\tfrac{1}{r}))(g - \mathrm{d}r \otimes \mathrm{d}r), \tag{2.6}$$

and thus (2.4) holds with $\rho_+ = \kappa(d-1)$. Indeed we have (1.2) with $c = 2$, and $E_0 = \kappa^2(d-1)^2/8$ by (2.5). The result follows from Theorem 1.5. $\square$

Remarks 2.5.
1) The radial curvatures R_{rad} and K_{rad} of [IS] and [Ku1], respectively, are different objects but they contain equivalent information.
2) The inequalities (1.2) and (1.4) may be viewed as bounds on the minimal and the mean curvatures (including the differential of the latter) of S_r, respectively, whereas (2.6) certainly is a uniform asymptotic result for all the principal curvatures.

3. Mourre-type commutator

Suppose from this point Conditions 1.1–1.4. As a preliminary step in the proof of Theorem 1.5 we show in this section a version of the so-called Mourre estimate. We shall use the Mourre-type commutator with respect to the "conjugate operator"

$$A = \mathrm{i}[H_0, r^2] = \tfrac{1}{2}\{(\partial_i r^2)g^{ij}p_j + p_i^* g^{ij}(\partial_j r^2)\} = rp^r + (p^r)^* r; \quad p^r = -\mathrm{i}\partial^r.$$

While not necessarily being self-adjoint this operator is certainly symmetric as defined on $C_c^\infty(M)$, and that suffices for our applications.

Lemma 3.1. *As a quadratic form on $C_c^\infty(M)$,*

$$\mathrm{i}[H, A] = p_i^*(\nabla^2 r^2 - \tfrac{1}{2}\rho_1 g)^{ij} p_j + \tfrac{1}{2}(\rho_1 H_0 + H_0 \rho_1) + \mathrm{i}\alpha^i p_i - \mathrm{i} p_i^* \alpha^i + \beta;$$
$$\alpha_i = \tfrac{1}{4}(\partial_i \rho_2) + V_1(\partial_i r^2),$$
$$\beta = (\triangle r^2)V_1 - 2r\partial^r V_2.$$

Proof. We note the commutator formulas, valid for any $\phi \in C^\infty(M)$,

$$-[H_0, [H_0, \phi]] = p_i^*(\nabla^2 \phi)^{ij} p_j - \tfrac{1}{4}(\triangle^2 \phi), \tag{3.1a}$$
$$p_i^* \phi g^{ij} p_j = \phi H_0 + H_0 \phi + \tfrac{1}{2}(\triangle \phi). \tag{3.1b}$$

As for (3.1a) we refer to [Do, Lemma 2.5] or [IS, Corollary 4.2]. The lemma follows by first using (3.1a) with $\phi = r^2$ and then (3.1b) with $\phi = \tfrac{1}{2}\rho_1$. $\square$

We introduce for $\sigma \geq 0$

$$H_\sigma = H - \tfrac{\sigma^2}{2}|\mathrm{d}r|^2. \tag{3.2}$$

We shall consider H_σ and as an operator defined on $C_c^\infty(M)$ only. We recall the definitions of χ_ν and E_0, (1.6) and (1.7), respectively.

Corollary 3.2. *Let $E \in (E_0, \infty)$. There exist $\gamma > 0$ and $C > 0$ such that, if $\nu \geq 1$ is large, then for any $\sigma \geq 0$, as quadratic forms on $C_c^\infty(M)$,*

$$\chi_\nu \mathrm{i}[H_\sigma, A]\chi_\nu \geq \gamma \chi_\nu^2 - C\chi_\nu(H_\sigma - E)^2 \chi_\nu.$$

Proof. We shall use Lemma 3.1 and in particular the functions α and β appearing there. Choose constants $c' \in (0, \tilde{c})$ and $\gamma > 0$ such that for all large enough $r \geq 1$

$$r\partial^r |\mathrm{d}r|^2 \geq -\tfrac{2c'+\rho_1}{2}|\mathrm{d}r|^2 \text{ and } \quad E - V - \tfrac{\alpha^2}{2(c-\tilde{c})c'} \geq \gamma/c'. \tag{3.3}$$

Noting $|\triangle r^2| \leq Cr$ for large r, cf. (1.4), we have for all large $r \geq 1$

$$\nabla^2 r^2 - \tfrac{1}{2}\rho_1 g \geq (c + c' - \tilde{c})g, \tag{3.4a}$$
$$\beta - \rho_1 V + \rho_1 E \geq -\tfrac{\gamma}{2}, \tag{3.4b}$$
$$(c' + \tfrac{1}{2}\rho_1)^2 \leq \tilde{c}^2. \tag{3.4c}$$

Then by using (3.4a) and the Cauchy Schwarz inequality we obtain for all large $\nu \geq 1$

$$\chi_\nu \mathrm{i}[H_\sigma, A]\chi_\nu \geq \chi_\nu \Big\{ (c' + \tfrac{1}{2}\rho_1)(H_\sigma - E) + (H_\sigma - E)(c' + \tfrac{1}{2}\rho_1) - \tfrac{\alpha^2}{(c-\tilde{c})}$$
$$- (2c' + \rho_1)V + (2c' + \rho_1)(\tfrac{1}{2}\sigma^2 |\mathrm{d}r|^2 + E) + \beta + \sigma^2 r\partial^r |\mathrm{d}r|^2 \Big\} \chi_\nu. \tag{3.5}$$

By using in turn (3.3), (3.4b) and (3.4c) we obtain with $C := 2\tilde{c}^2/\gamma$

$$\chi_\nu \mathrm{i}[H_\sigma, A]\chi_\nu \geq \chi_\nu \Big\{ 2c'E - 2c'V - \tfrac{\alpha^2}{(c-\tilde{c})} - (c' + \tfrac{1}{2}\rho_1)^2/C - C(H_\sigma - E)^2 - \tfrac{\gamma}{2} \Big\} \chi_\nu$$
$$\geq \chi_\nu \Big\{ 2\gamma - \tfrac{\gamma}{2} - C(H_\sigma - E)^2 - \tfrac{\gamma}{2} \Big\} \chi_\nu,$$

and whence the assertion. $\square$

4. Exponential decay of eigenstates

The proof of Theorem 1.5, given in this section, depends on the following exponential decay estimate which in turn will be proved in Section 5.

Proposition 4.1. *Let* $E \in \sigma_{\mathrm{pp}}(H) \cap (E_0, \infty)$ *and suppose* $\phi \in \mathcal{D}(H)$ *satisfies* $H\phi = E\phi$. *Then for any* $\sigma \geq 0$ *one has* $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$.

To implement Condition 1.3 efficiently we need to strengthen the stated approximation property under some additional conditions (fulfilled for eigenstates due to Proposition 4.1).

Lemma 4.2. *Let* $\psi \in \mathcal{D}(H)$. *There exists* $\nu_0 \geq 1$ *such that for* $\nu \geq \nu_0$ *and for any* $\sigma \geq 0$ *such that* $\mathrm{e}^{\sigma r}\psi, \mathrm{e}^{\sigma r}H\psi \in \mathcal{H}$ *the following properties hold: The states* $\chi_\nu \mathrm{e}^{\sigma r}p\psi, \mathrm{e}^{\sigma r}p\chi_\nu\psi \in \mathcal{H}$ *and there exists a sequence* $\psi_n \in C_{\mathrm{c}}^\infty(M)$ *(possibly depending on* σ) *such that as* $n \to \infty$

$$\|\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_n)\| + \|\chi_\nu \mathrm{e}^{\sigma r}(p\psi - p\psi_n)\| + \|\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_n)\| \to 0. \qquad (4.1)$$

Proof. Step I. Note the distributional identity

$$\chi_\nu \mathrm{e}^{\sigma r}p\psi = \mathrm{e}^{\sigma r}p\chi_\nu\psi + \mathrm{i}\mathrm{e}^{\sigma r}\psi\chi_\nu'\mathrm{d}r.$$

Applied to the given ψ we see that $\chi_\nu \mathrm{e}^{\sigma r}p\psi \in \mathcal{H}$ if and only if $\mathrm{e}^{\sigma r}p\chi_\nu\psi \in \mathcal{H}$.

Step II. We claim that there exists $C > 0$ such that, if $\nu \geq 1$ is large, then for any $\psi \in C_{\mathrm{c}}^\infty(M)$ and $\sigma \geq 0$

$$\|\chi_\nu \mathrm{e}^{\sigma r}|p\psi|\|^2 \leq \|\chi_\nu \mathrm{e}^{\sigma r}H\psi\|^2 + C\langle\sigma\rangle^2\|\chi_{\nu/2}\mathrm{e}^{\sigma r}\psi\|^2. \qquad (4.2)$$

In fact by (3.1b)

$$\|\chi_\nu \mathrm{e}^{\sigma r}|p\psi|\|^2 = 2\operatorname{Re}\langle\chi_\nu \mathrm{e}^{\sigma r}\psi, \chi_\nu \mathrm{e}^{\sigma r}H\psi\rangle + \tfrac{1}{2}\langle\psi, (\triangle\chi_\nu^2 \mathrm{e}^{2\sigma r})\psi\rangle - 2\langle\chi_\nu \mathrm{e}^{\sigma r}\psi, V\chi_\nu \mathrm{e}^{\sigma r}\psi\rangle$$

$$\leq \|\chi_\nu \mathrm{e}^{\sigma r}H\psi\|^2 + C\langle\sigma\rangle^2\|\chi_{\nu/2}\mathrm{e}^{\sigma r}\psi\|^2.$$

Here we used Condition 1.1 and the following consequence

$$|\triangle r| = \tfrac{1}{2r}|(\triangle r^2) - 2|\mathrm{d}r|^2| \leq C \text{ for } r = r(x) \text{ large.} \qquad (4.3)$$

Step III. We consider the case $\sigma = 0$, and hence suppose only $\psi \in \mathcal{D}(H)$. Let $\psi_n \in C_{\mathrm{c}}^\infty(M)$ and large $\nu \geq 1$ be as in Condition 1.3. Then, regarding (4.1), it suffices to consider the middle term. By (4.2) we have

$$\|\chi_\nu(p\psi_n - p\psi_{n'})\|^2 \leq C\big(\|\chi_\nu(H\psi_n - H\psi_{n'})\|^2 + \|\chi_{\nu/2}(\psi_n - \psi_{n'})\|^2\big).$$

This implies $\chi_\nu p\psi_n$ converges strongly. Since also $\chi_\nu p\psi_n$ converges in distributional sense to $\chi_\nu p\psi$, we obtain that the limit $\chi_\nu p\psi \in \mathcal{H}$ and then in turn, by letting $n' \to \infty$ above, (4.1) for $\sigma = 0$.

Step IV. We let $\sigma > 0$ and suppose $\mathrm{e}^{\sigma r}\psi, \mathrm{e}^{\sigma r}H\psi \in \mathcal{H}$. Choose $\psi_n \in C_{\mathrm{c}}^\infty(M)$ and large $\nu \geq 1$ as in Condition 1.3, again. As for the first and the third terms of (4.1), we compute as follows: Put $\psi_{n,\nu'} = \bar\chi_{\nu'}\psi_n$ for $\nu' \geq 2\nu$ and with $\bar\chi_{\nu'} := 1 - \chi_{\nu'}$. Then we decompose

$$\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_{n,\nu'}) = \bar\chi_{\nu'}\mathrm{e}^{\sigma r}\chi_\nu(\psi - \psi_n) + \chi_{\nu'}\mathrm{e}^{\sigma r}\psi. \qquad (4.4)$$

We put

$$R_{\nu'} = \mathrm{i}[H, \chi_{\nu'}] = \tfrac{1}{2}(\chi_{\nu'}'p^r + (p^r)^*\chi_{\nu'}') = \chi_{\nu'}'p^r - \tfrac{\mathrm{i}}{2}\big(\chi_{\nu'}''|\mathrm{d}r|^2 + \chi_{\nu'}'\triangle r\big), \qquad (4.5)$$

and decompose similarly

$$
\begin{aligned}
&\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_{n,\nu'}) \\
&= \bar\chi_{\nu'}\mathrm{e}^{\sigma r}\chi_\nu(H\psi - H\psi_n) + \chi_{\nu'}\mathrm{e}^{\sigma r}H\psi + \mathrm{i}\mathrm{e}^{\sigma r}R_{\nu'}(\psi - \psi_n) - \mathrm{i}\mathrm{e}^{\sigma r}R_{\nu'}\psi.
\end{aligned}
\tag{4.6}
$$

The norm of the right-hand side of (4.4) can be arbitrarily small by first letting ν' be large and then n large accordingly (using that $\bar\chi_{\nu'}\mathrm{e}^{\sigma r}$ is bounded). Similarly the norm of first three terms on the right-hand side of (4.6) can be arbitrarily small by first letting ν' be large and then n large accordingly (for the third term we use Step III, i.e. (4.1) with $\sigma = 0$). It remains to consider the last term on the right-hand side of (4.6). We claim that

$$
\|\mathrm{e}^{\sigma r}R_{\nu'}\psi\| \leq C/\nu'. \tag{4.7}
$$

To show this we use again Step III to write

$$
\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi\|^2 = \lim_{m\to\infty}\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi_m\|^2.
$$

On the other hand by the derivation of (4.2)

$$
\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi_m\|^2 \leq C\big(\|\chi'_{\nu'}\mathrm{e}^{\sigma r}H\psi_m\|^2 + \big(\tfrac{\langle\sigma\rangle}{\nu'}\big)^2\|\chi_{\nu/2}\bar\chi_{2\nu'}\mathrm{e}^{\sigma r}\psi_m\|^2\big),
$$

and hence we conclude by taking the limit that

$$
\begin{aligned}
\|\chi'_{\nu'}\mathrm{e}^{\sigma r}p\psi\|^2 &\leq \big(\tfrac{C_\sigma}{\nu'}\big)^2\big(\|\chi_\nu\bar\chi_{2\nu'}\mathrm{e}^{\sigma r}H\psi\|^2 + \|\chi_{\nu/2}\bar\chi_{2\nu'}\mathrm{e}^{\sigma r}\psi\|^2\big) \\
&\leq \big(\tfrac{C_\sigma}{\nu'}\big)^2\big(\|\mathrm{e}^{\sigma r}H\psi\|^2 + \|\mathrm{e}^{\sigma r}\psi\|^2\big).
\end{aligned}
\tag{4.8}
$$

A consequence of (4.8) is indeed (4.7), and whence in turn also the last term on the right-hand side of (4.6) is small for ν' sufficiently large.

We conclude that there exists a sequence of indices $(\nu'(m), n(m))$ so that with $\psi_m := \psi_{n(m),\nu'(m)}$ (here and henceforth slightly abusing notation)

$$
\|\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_m)\| + \|\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_m)\| \to 0.
$$

In particular, using here (4.2), the right-hand side of

$$
\|\chi_{2\nu}\mathrm{e}^{\sigma r}p(\psi_n - \psi_{n'})\|^2 \leq C\big(\|\chi_{2\nu}\mathrm{e}^{\sigma r}H(\psi_n - \psi_{n'})\|^2 + \|\chi_\nu\mathrm{e}^{\sigma r}(\psi_n - \psi_{n'})\|^2\big)
$$

is small for $n, n' \to \infty$. We can from this point mimic the last part of Step III. $\square$

Proof of Theorem 1.5. Suppose $E \in \sigma_{\mathrm{pp}}(H) \cap (E_0, \infty)$ and let ϕ be any corresponding eigenstate. Then, by Proposition 4.1, for any $\nu \geq 1$ and $\sigma \geq 0$

$$
\phi_\sigma = \phi_{\sigma,\nu} := \chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\phi \in \mathcal{H}. \tag{4.9}
$$

We will choose $\nu \geq 1$ large in agreement with Lemma 4.2 with $\psi = \phi$. In the following computations we actually have to first choose an approximate sequence for ϕ from $C_\mathrm{c}^\infty(M)$ and then take the limits. This can be done by using Lemma 4.2 and the closedness of H, but since the verification is rather straightforward we shall not elaborate on this point.

We compute, putting $R_\nu = \mathrm{i}[H_0, \chi_\nu] = \mathrm{Re}\,(\chi'_\nu p^r)$ as in (4.5),

$$
H\phi_\sigma = E\phi_\sigma + \tfrac{\sigma^2}{2}|\mathrm{d}r|^2\phi_\sigma - \mathrm{i}\sigma(\mathrm{Re}\,p^r)\phi_\sigma - \mathrm{i}\mathrm{e}^{\sigma(r-4\nu)}R_\nu\phi. \tag{4.10}
$$

In particular indeed $\phi_\sigma \in \mathcal{D}(H)$. Take inner product with ϕ_σ and compute

$$
\langle H\rangle_{\phi_\sigma} = \mathrm{Re}\,\langle H\rangle_{\phi_\sigma} = \langle E + \tfrac{\sigma^2}{2}|\mathrm{d}r|^2\rangle_{\phi_\sigma} + \tfrac{\mathrm{i}}{2}\langle[R_\nu, \chi_\nu\mathrm{e}^{2\sigma(r-4\nu)}]\rangle_\phi.
$$

Whence
$$\langle H\rangle_{\phi_\sigma} \geq \langle E + \tfrac{\sigma^2}{2}|\mathrm{d}r|^2\rangle_{\phi_\sigma} - C\langle\sigma\rangle\|\phi\|^2,$$
where $C > 0$ does not depend on ν or σ because $r \leq 2\nu$ on supp χ_ν'. On the other hand if $c' \in (0, \tilde{c})$ and $\nu \geq 1$ is large then, cf. (3.5) with $\sigma = 0$,
$$2c'\langle H\rangle_{\phi_\sigma} \leq \langle\mathrm{i}[H, A]\rangle_{\phi_\sigma} - \mathrm{Re}\,\langle\rho_1 H\rangle_{\phi_\sigma} + C\|\phi_\sigma\|^2.$$
We fix such c' assuming in addition (for a later application)
$$\liminf_{r\to\infty}(r\partial^r|\mathrm{d}r|^2 + c'|\mathrm{d}r|^2) > 0. \tag{4.11}$$

We compute the first and the second terms on the right-hand side. By (4.10) again
$$\begin{aligned}
&\langle\mathrm{i}[H, A]\rangle_{\phi_\sigma}\\
&\quad = \sigma^2\,\mathrm{Im}\,\langle A|\mathrm{d}r|^2\rangle_{\phi_\sigma} - 2\sigma\,\mathrm{Re}\,\langle(\mathrm{Re}\,p^r)A\rangle_{\phi_\sigma} - 2\,\mathrm{Re}\,\langle R_\nu \mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi,
\end{aligned} \tag{4.12}$$
while
$$\begin{aligned}
&-\mathrm{Re}\,\langle\rho_1 H\rangle_{\phi_\sigma}\\
&\quad = -E\langle\rho_1\rangle_{\phi_\sigma} - \tfrac{\sigma^2}{2}\langle\rho_1|\mathrm{d}r|^2\rangle_{\phi_\sigma} - \sigma\,\mathrm{Im}\,\langle\rho_1\,\mathrm{Re}\,p^r\rangle_{\phi_\sigma} - \mathrm{Im}\,\langle\rho_1\chi_\nu \mathrm{e}^{2\sigma(r-4\nu)}R_\nu\rangle_\phi
\end{aligned} \tag{4.13}$$
The first and the second terms of (4.12) are estimated using
$$\mathrm{Im}\,(A|\mathrm{d}r|^2) = -r(\partial^r|\mathrm{d}r|^2),$$
$$-2\,\mathrm{Re}\,((\mathrm{Re}\,p^r)A) = -(\mathrm{Re}\,p^r)(2r(\mathrm{Re}\,p^r) - \mathrm{i}|\mathrm{d}r|^2) + \text{h.c.} \leq (\partial^r|\mathrm{d}r|^2).$$
As for the third term of (4.12) we estimate (recall the notation $\bar{\chi}_\nu = 1 - \chi_\nu$)
$$\begin{aligned}
&-2\,\mathrm{Re}\,\langle R_\nu \mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi\\
&\quad \leq \|\mathrm{e}^{\sigma(r-4\nu)}R_\nu\phi\|^2 + \|\bar{\chi}_{2\nu}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\phi\|^2\\
&\quad \leq \left\{\|\chi_\nu' \mathrm{e}^{\sigma(r-4\nu)}p^r\phi\| + \tfrac{1}{2}\|(\chi_\nu''|\mathrm{d}r|^2 + \chi_\nu'(\triangle r))\mathrm{e}^{\sigma(r-4\nu)}\phi\|\right\}^2\\
&\qquad + \left\{\|2r\bar{\chi}_{2\nu}\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}p^r\phi\|\right.\\
&\qquad\qquad + \left.\|\bar{\chi}_{2\nu}(2r|\mathrm{d}r|^2\chi_\nu' + 2\sigma r\chi_\nu|\mathrm{d}r|^2 + \tfrac{1}{2}(\triangle r^2)\chi_\nu)\mathrm{e}^{\sigma(r-4\nu)}\phi\|\right\}^2\\
&\quad \leq C\nu^2\||\chi_{\nu/2}|p\phi|\|^2 + C\nu^2\langle\sigma\rangle^2\|\phi\|^2,
\end{aligned}$$
where we have used (4.3). By using (4.1) and (4.2) (both with $\sigma = 0$) we then conclude
$$-2\,\mathrm{Re}\,\langle R_\nu \mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi \leq C\nu^2\langle\sigma\rangle^2\|\phi\|^2.$$
Next, we compute the third and fourth terms of (4.13). Note that we can not differentiate ρ_1. But by the support property of χ_ν' (the one used before) the fourth term is estimated similarly to the third term of (4.12), and we obtain
$$-\mathrm{Im}\,\langle\rho_1\chi_\nu \mathrm{e}^{2\sigma(r-4\nu)}R_\nu\rangle_\phi \leq C\langle\sigma\rangle^2\|\phi\|^2.$$
We proceed for the third term of (4.13):
$$\begin{aligned}
&-\mathrm{Im}\,\langle\rho_1\,\mathrm{Re}\,p^r\rangle_{\phi_\sigma}\\
&\quad = -\mathrm{Im}\,\langle\rho_1 p^r\rangle_{\phi_\sigma} + \tfrac{1}{2}\langle\rho_1(\triangle r)\rangle_{\phi_\sigma}\\
&\quad \leq -\mathrm{Im}\,\langle\phi_\sigma, \rho_1\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}p^r\phi\rangle + C\|\phi\|^2 + C\sigma\langle|\rho_1|\rangle_{\phi_\sigma} + C\|\phi_\sigma\|^2\\
&\quad \leq C\big(\sup\chi_{\nu/2}|\rho_1|\big)\langle\sigma\rangle^{-1}\||\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}|p\phi|\|^2 + C\|\phi\|^2 + C\langle\sigma\rangle\langle|\rho_1|\rangle_{\phi_\sigma} + C\|\phi_\sigma\|^2.
\end{aligned}$$

We apply (4.1) and (4.2) to the first term on the right-hand side yielding

$$- \operatorname{Im} \langle \rho_1 \operatorname{Re} p^r \rangle_{\phi_\sigma} \leq C \big(\sup \chi_{\nu/2} |\rho_1| \big) \langle \sigma \rangle \|\phi_\sigma\|^2 + C \langle \sigma \rangle \|\phi\|^2 + C \|\phi_\sigma\|^2.$$

We summarize

$$\sigma^2 \big\langle r(\partial^r |\mathrm{d}r|^2) + c' |\mathrm{d}r|^2 - C \big(\sup \chi_{\nu/2} |\rho_1| \big) \big\rangle_{\phi_\sigma} - C \langle \sigma \rangle \|\phi_\sigma\|^2 \leq C \nu^2 \langle \sigma \rangle^2 \|\phi\|^2. \quad (4.14)$$

We shall apply (4.14) to a fixed $\nu \geq 1$ chosen so large that the quantity $r(\partial^r |\mathrm{d}r|^2) + c' |\mathrm{d}r|^2 - C \big(\sup \chi_{\nu/2} |\rho_1| \big)$ is greater than some positive constant on $\operatorname{supp} \chi_\nu$. Note that this in turn is doable since we have assumed (4.11).

Now assume $\chi_{5\nu} \phi \not\equiv 0$. After division by $\langle \sigma \rangle^2$ on both sides of (4.14) the left-hand side grows exponentially as $\sigma \to \infty$ whereas the right-hand side is bounded, and hence we obtain a contradiction. Thus $\chi_{5\nu} \phi \equiv 0$, and then by Condition 1.4 we conclude that $\phi(x) = 0$ in M. $\square$

5. Auxiliary operators

In this section we give the proof of Proposition 4.1. We introduce regularized weights

$$\theta_m(r) = r(1 + \tfrac{r}{m})^{-1}, \quad m \geq 1,$$

and denote the derivatives in r by $\theta_m^{(k)}(r)$, e.g.,

$$\theta'_m(r) = \theta_m^{(1)}(r) = (1 + \tfrac{r}{m})^{-2}.$$

We introduce furthermore

$$\Theta_m(r) = \Theta_m^{\sigma, \delta}(r) = \sigma r + \delta \theta_m(r), \quad \sigma, \delta \geq 0,$$

and denote the derivatives by $\Theta_m^{(k)}(r)$ as above. Now we define some observables:

$$B = \mathrm{i}[H_0, r] = \tfrac{1}{2}(p^r + (p^r)^*) = p^r + \tfrac{1}{2\mathrm{i}}(\triangle r),$$

$$B_m = \mathrm{i}[H_0, \Theta_m] = \tfrac{1}{2}\big(\Theta'_m p^r + (p^r)^* \Theta'_m\big) = \Theta'_m p^r + \tfrac{1}{2\mathrm{i}}\{(\triangle r)\Theta'_m + |\mathrm{d}r|^2 \Theta''_m\},$$

$$R_\nu = \mathrm{i}[H_0, \chi_\nu] = \tfrac{1}{2}(\chi'_\nu p^r + (p^r)^* \chi'_\nu), \quad \nu \geq 1.$$

Then we have the properties:

$$A = 2Br - \tfrac{1}{\mathrm{i}} |\mathrm{d}r|^2 = 2rB + \tfrac{1}{\mathrm{i}} |\mathrm{d}r|^2 \quad (5.1\mathrm{a})$$

$$B_m = B\Theta'_m - \tfrac{1}{2\mathrm{i}} |\mathrm{d}r|^2 \Theta''_m = \Theta'_m B + \tfrac{1}{2\mathrm{i}} |\mathrm{d}r|^2 \Theta''_m, \quad (5.1\mathrm{b})$$

$$\begin{aligned}(B_m)^2 &= B(\Theta'_m)^2 B - \tfrac{1}{2}(\partial^r |\mathrm{d}r|^2)\Theta'_m \Theta''_m - \tfrac{1}{2}|\mathrm{d}r|^4 \Theta'_m \Theta'''_m - \tfrac{1}{4}|\mathrm{d}r|^4 (\Theta''_m)^2 \\ &\leq B(\Theta'_m)^2 B + C\delta(\sigma + \delta),\end{aligned} \quad (5.1\mathrm{c})$$

where the last inequality is for large r. We set for $\nu' \geq 2\nu$ and $\psi \in C_c^\infty(M)$

$$\psi_m = \psi_{m, \nu, \nu'} = \chi_{\nu, \nu'} \mathrm{e}^{\Theta_m} \psi; \quad \chi_{\nu, \nu'} = \chi_\nu \bar{\chi}_{\nu'}, \quad \bar{\chi}_{\nu'} = 1 - \chi_{\nu'},$$

not to be mixed up with ψ_n in Lemma 4.2. We recall the notation (3.2). A computation shows, cf. (4.10), that

$$\begin{aligned}&\mathrm{i}(H_\sigma - E)\psi_m \\ &= \mathrm{i}\chi_{\nu, \nu'} \mathrm{e}^{\Theta_m}(H - E)\psi + \big\{B_m - \tfrac{1}{2\mathrm{i}}((\Theta'_m)^2 - \sigma^2)|\mathrm{d}r|^2\big\} \psi_m + \mathrm{e}^{\Theta_m}(R_\nu - R_{\nu'})\psi.\end{aligned} \quad (5.2)$$

Lemma 5.1. *Let $\sigma_0 \geq 0$ be fixed.*

(i) *Let $\epsilon > 0$. Then there exists $C > 0$ such that, if $\nu \geq 1$ is large, for any $m \geq 1$, $0 \leq \delta \leq 1$ and $0 \leq \sigma \leq \sigma_0$, as quadratic forms on $C_c^\infty(M)$,*

$$\chi_\nu \operatorname{Re}(AB_m)\chi_\nu \geq 2\chi_\nu Br\Theta'_m B\chi_\nu - (\epsilon + C\delta)\chi_\nu^2.$$

(ii) *Let $\epsilon' > 0$. Then there exists $C > 0$ such that, if $\nu \geq 1$ is large, for any $\nu' \geq 2\nu$, $m \geq 1$, $0 \leq \delta \leq 1$, $0 \leq \sigma \leq \sigma_0$, $E \in \mathbb{R}$ and $\psi \in C_c^\infty(M)$*

$$
\begin{aligned}
&\|(H_\sigma - E)\psi_m\|^2 \\
&\quad \leq 5\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + \epsilon'\langle Br\Theta'_m B\rangle_{\psi_m} + C\delta\|\psi_m\|^2 \\
&\quad\quad + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C(\nu')^{-2}(\|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}\psi\|^2 + \|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}p\psi\|^2).
\end{aligned}
$$

Proof. (i) By (5.1a) and (5.1b)

$$
\begin{aligned}
\operatorname{Re}(AB_m) &= \tfrac{1}{2}(2Br - \tfrac{1}{\mathrm{i}}|dr|^2)(\Theta'_m B + \tfrac{1}{2\mathrm{i}}|dr|^2\Theta''_m) + \text{h.c.} \\
&= Br\Theta'_m B + \tfrac{1}{2\mathrm{i}}Br|dr|^2\Theta''_m - \tfrac{1}{2\mathrm{i}}|dr|^2\Theta'_m B + \tfrac{1}{4}|dr|^4\Theta''_m + \text{h.c.} \\
&= 2Br\Theta'_m B - \tfrac{1}{2}\big\{(\partial^r|dr|^2)(\Theta'_m + r\Theta''_m) + |dr|^4(\Theta''_m + r\Theta'''_m)\big\}.
\end{aligned}
$$

Then by (1.1) and (1.3) the assertion follows.

(ii) By (5.2), (5.1c), (1.1) and (4.3)

$$
\begin{aligned}
&\|(H_\sigma - E)\psi_m\|^2 \\
&\quad \leq 5\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + 5\langle(B_m)^2\rangle_{\psi_m} + \tfrac{5}{4}\|((\Theta'_m)^2 - \sigma^2)|dr|^2\psi_m\|^2 \\
&\quad\quad + 5\|\mathrm{e}^{\Theta_m}R_\nu\psi\|^2 + 5\|\mathrm{e}^{\Theta_m}R_{\nu'}\psi\|^2 \\
&\quad \leq 5\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + 5\langle B(\Theta'_m)^2 B\rangle_{\psi_m} + C\delta\|\psi_m\|^2 \\
&\quad\quad + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C(\nu')^{-2}(\|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}\psi\|^2 + \|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}p\psi\|^2).
\end{aligned}
$$

Now choose $\nu \geq 1$ large enough so that $5\Theta'_m \leq 5(\sigma_0 + 1) \leq \epsilon'r$ on $\operatorname{supp}\chi_\nu$, and we are done. $\square$

Proof of Proposition 4.1. We let E and ϕ be as in the proposition. Set

$$\sigma_0 = \sup\{\sigma \geq 0 | \mathrm{e}^{\sigma r}\phi \in \mathcal{H}\},$$

and assume $\sigma_0 < \infty$. If $\sigma_0 > 0$ we choose $\sigma \in [0, \sigma_0)$ and a small $\delta > 0$ such that $\sigma + \delta > \sigma_0$. If $\sigma_0 = 0$ we set $\sigma = 0$ and choose a small $\delta > 0$. These numbers will be determined more precisely in the following arguments. In any case we have $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$. We indicate below the dependence of constants using subscripts.

Due to Corollary 3.2, for any $\psi \in C_c^\infty(M)$

$$\|\psi_m\|^2 \leq \gamma^{-1}\langle\mathrm{i}[H_\sigma, A]\rangle_{\psi_m} + C_0\|(H_\sigma - E)\psi_m\|^2; \quad C_0 = C/\gamma. \tag{5.3}$$

We estimate the right-hand side using Lemma 5.1. For the first term of (5.3) we use (5.2) and Lemma 5.1(i) with $\epsilon = \frac{\gamma}{3}$ estimating

$$
\begin{aligned}
\langle \mathrm{i}[H_\sigma, A]\rangle_{\psi_m} \\
= {}& - \langle \mathrm{i}(H_\sigma - E)\psi_m, A\psi_m\rangle + \text{h.c.} \\
= {}& - \langle \mathrm{i}\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H - E)\psi, A\psi_m\rangle - \langle B_m\psi_m, A\psi_m\rangle \\
& + \langle \tfrac{1}{2\mathrm{i}}|\mathrm{d}r|^2((\Theta_m')^2 - \sigma^2)\psi_m, A\psi_m\rangle - \langle \mathrm{e}^{\Theta_m}(R_\nu - R_{\nu'})\psi, A\psi_m\rangle + \text{h.c.} \\
\leq {}& 2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H - E)\psi\|\|A\psi_m\| - 2\operatorname{Re}\langle AB_m\rangle_{\psi_m} - \langle (r\partial^r|\mathrm{d}r|^2)((\Theta_m')^2 - \sigma^2)\rangle_{\psi_m} \\
& - \langle 2r|\mathrm{d}r|^4\Theta_m'\Theta_m''\rangle_{\psi_m} + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) \\
& + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2) \\
\leq {}& C(\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H - E)\psi\|^2 - 4\langle Br\Theta_m'B\rangle_{\psi_m} + (\tfrac{2\gamma}{3} + C_1\delta)\|\psi_m\|^2 \\
& + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 \\
& + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2),
\end{aligned}
$$

where we used that $r/\nu' \leq 2\sqrt{r/\nu'}$ on $\operatorname{supp}\chi_{\nu,2\nu'}$ to estimate $(\nu')^{-2}\|A\psi_m\|^2$.

On the other hand, for the second term of (5.3), let us choose $\epsilon' = \frac{4}{\gamma C_0}$ in Lemma 5.1(ii). Then (5.3) is estimated as

$$
\begin{aligned}
\|\psi_m\|^2 \leq {}& C(\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H - E)\psi\|^2 + \left(\tfrac{2}{3} + (\tfrac{C_1}{\gamma} + C_2)\delta\right)\|\psi_m\|^2 \\
& + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2).
\end{aligned}
$$

Now fix $\nu \geq 1$ sufficiently large (so that the above estimates hold), and let σ and δ be such that $\frac{2}{3} + (\frac{C_1}{\gamma} + C_2)\delta \leq \frac{3}{4}$ and $\sigma + \delta > \sigma_0$. Then

$$
\begin{aligned}
\tfrac{1}{4}\|\psi_m\|^2 \leq {}& C(\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H - E)\psi\|^2 + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) \\
& + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2).
\end{aligned}
\tag{5.4}
$$

By Lemma 4.2 we can replace ψ of (5.4) by ϕ. This makes the first term on the right-hand side disappear. Next let $\nu' \to \infty$ invoking Lebesgue's dominated convergence theorem. Note that the third term disappears, and consequently we are left with the bound

$$
\|\chi_\nu \mathrm{e}^{\Theta_m}\phi\|^2 \leq 4C_\nu(\|\chi_{\nu/2}\phi\|^2 + \|\chi_{\nu/2}p\phi\|^2).
\tag{5.5}
$$

By letting $m \to \infty$ in (5.5) invoking Lebesgue's monotone convergence theorem we conclude that $\chi_\nu \mathrm{e}^{(\sigma+\delta)r}\phi \in \mathcal{H}$. This is a contradiction since $\sigma + \delta > \sigma_0$. $\qquad\square$

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