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Ole E. Barndorff-Nielsen and Jan Pedersen

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Ole E. Barndorff-Nielsen and Jan Pedersen

Department of Mathematical Sciences, University of Aarhus, Denmark.

E-mails: oebn@imf.au.dk, jan@imf.au.dk.

Abstract

The problem of defining subordination of a homogeneous Lévy basis by a non-negative homogeneous Lévy basis is discussed. An explicit construction, generalizing the usual one-dimensional case, is given. This construction involves certain random meta-time changes.

Keywords: Subordination; Lévy bases and sheets; meta-time change.
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1 Introduction

In recent years the fundamental concept of subordination of a Lévy process by a subordinator has been generalized in many directions; see e.g. [3, 4, 8]. Related to this, Barndorff-Nielsen [2] considered the following problem: Given an $\mathbb{R}^d$-valued homogeneous Lévy basis $\Lambda_X = \{\Lambda_X(A) : A \in \mathcal{B}_b(\mathbb{R}^k)\}$, and an independent $\mathbb{R}_+$-valued homogeneous Lévy basis $\Lambda_T = \{\Lambda_T(A) : A \in \mathcal{B}_b(\mathbb{R}^k)\}$ how should one define subordination of Λ_X by Λ_T?

Let us first consider the case $k = 1$. There are Lévy processes $X = \{X_t : t \geq 0\}$ and $T = \{T_t : t \geq 0\}$ associated with Λ_X and Λ_T in the sense that

$$X_t = \Lambda_X((0,t]) \quad \text{and} \quad T_t = \Lambda_T((0,t]) \tag{1.1}$$

for $t \geq 0$. Thus, we can simply define a subordinated process $Y = \{Y_t : t \geq 0\}$ in the usual way as $Y_t = X_{T_t}$. However, when $k \geq 2$ there is no immediate analogue. To see this, note that there are so-called Lévy sheets $X = \{X_t : t \in \mathbb{R}_+^k\}$ and $T = \{T_t : t \in \mathbb{R}_+^k\}$ associated with Λ_X and Λ_T, and these are defined as in (1.1), where $(0,t]$ now is an interval in $\mathbb{R}^k$. But T_t is one-dimensional while t is k-dimensional, thus excluding the possibility of defining Y_t as X_{T_t} when $k \geq 2$. Barndorff-Nielsen argued that one should not construct a subordinated process; rather, the appropriate concept is a subordinated random measure $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}^k)\}$ defined such that conditional on Λ_T, $M(A_1), \ldots, M(A_n)$ are independent for all disjoint $A_1, \ldots, A_n$, and the distribution of $M(A)$ for $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ is $\mu^{\Lambda_T(A)}$ where $\mu = \mathcal{L}(\Lambda_X((0,e]))$ and $e = (1, \ldots, 1) \in \mathbb{R}_+^k$ is the vector of ones.

In the present paper we give an explicit construction in terms of Λ_X and Λ_T of Barndorff-Nielsen's subordinated measure M. For notational convenience, instead of considering M and Λ_X as Lévy bases on $\mathbb{R}^k$ we look at the restriction to $\mathbb{R}_+^k$; the general case follows trivially from this. Specifically, we argue that a natural definition of $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ is $M(A) = \Lambda_X(\phi_T^{-1}(A))$ where $\phi_T : \mathbb{R}_+^k \to \mathbb{R}^k$ is a (random) mapping satisfying that $\mathrm{Leb}(\phi_T^{-1}(A)) = \Lambda_T(A)$ for $A \in \mathcal{B}_b(\mathbb{R}_+^k)$. We also use the notation $\mathbf{T} = \phi_T^{-1}$ for the inverse image of ϕ_T. In a sense one can think of $\mathbf{T}$ as a kind of *meta-time change*, replacing *time changes* in the case $k = 1$. We show that this definition generalizes the case $k = 1$ in a natural way and, in particular, that M is a homogeneous Lévy basis.

This construction gives emphasis to the viewpoint that in the multiparameter case $k \geq 2$ the right concept is a subordinated measure instead of subordinated process.

In Section 2 we recall the definitions of homogeneous Lévy sheets and bases and show that these are in one-to-one correspondence. To pave the way for the analysis of M we state in Section 3 a lemma showing that it is possible to define a mapping ϕ_T with the above properties. Section 4 is about meta-time changes, meaning that we consider the measure M for a fixed outcome of T. Although meta-times exist for any T they are in full generality somewhat involved to define explicitly. However, in practice the most important case is when the measure induced by T is the sum of a constant times Lebesgue measure and a discrete measure. In this case alternative useful representations of M are given. In the last section it is shown that M is a homogeneous Lévy basis.

2 Homogeneous Lévy sheets and Lévy bases

Let d and k denote positive integers. For $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^d)$ in $\mathbb{R}^d$ let $\langle x, y\rangle$ denote their inner product and $|x|$ be the corresponding norm. Let $D = \{x \in \mathbb{R}^d : |x| \leq 1\}$. Throughout the paper all random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{L}(X)$ denote the law of a random vector X. For a set S and two families $\{X_t : t \in S\}$ and $\{Y_t : t \in S\}$ of random vectors with X_t and Y_t in $\mathbb{R}^d$ write $\{X_t : t \in S\} \overset{\mathscr{D}}{=} \{Y_t : t \in S\}$ if the finite dimensional marginals are the same. We say that $\{X_t : t \in S\}$ is a modification of $\{Y_t : t \in S\}$ if $X_t = Y_t$ a.s. for all $t \in S$. Let $\widehat{\mu}$ denote the characteristic function of a distribution μ on $\mathbb{R}^d$, $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{\mathrm{i}\langle z,x\rangle}\mu(\mathrm{d}x)$ for $z \in \mathbb{R}^d$. Let $\mathrm{ID}(\mathbb{R}^d)$ denote the class of d-dimensional infinitely divisible distributions. Recall that a distribution μ on $\mathbb{R}^d$ is in $\mathrm{ID}(\mathbb{R}^d)$ if and only if $\widehat{\mu}$ is given by $\widehat{\mu}(z) = \exp\left[-\frac{1}{2}z\Sigma z^\top + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d} g(z,x)\,\nu(\mathrm{d}x)\right]$, $z \in \mathbb{R}^d$, where $g(z,x) = e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_D(x)$, $\top$ denotes the transpose, and (Σ, ν, γ) is the characteristic triplet of μ; that is, Σ is a $d \times d$ non-negative definite matrix, ν is a Lévy measure on $\mathbb{R}^d$ and $\gamma \in \mathbb{R}^d$. Denote the entries of Σ by Σ^{ij} and the coordinates of γ by γ^j for $i, j = 1, \ldots, d$. For $t \geq 0$ and $\mu \in \mathrm{ID}(\mathbb{R}^d)$, μ^t denotes the distribution in $\mathrm{ID}(\mathbb{R}^d)$ with $\widehat{\mu^t} = \widehat{\mu}^t$.

For $a = (a^1, \ldots, a^k) \in \mathbb{R}_+^k$ and $b = (b^1, \ldots, b^k) \in \mathbb{R}_+^k$ write $a \leq b$ if $a^j \leq b^j$ for all j and $a < b$ if $a^j < b^j$ for all j, and define the half-open interval $]a, b]$ as $]a, b] = \{t \in \mathbb{R}_+^k : a < t \leq b\}$. Let $[a, b] = \{t \in \mathbb{R}_+^k : a \leq t \leq b\}$.

For $F = \{F_t : t \in \mathbb{R}_+^k\}$ with $F_t \in \mathbb{R}^d$ and $a \leq b$ define the increment of F

over $]a,b]$, $\Delta_a^b F$, as

$$\Delta_a^b F = \sum_{\epsilon_1=0}^{1} \cdots \sum_{\epsilon_k=0}^{1} (-1)^{k-(\epsilon_1+\cdots+\epsilon_k)} F_{(c^1(\epsilon_1),\ldots,c^k(\epsilon_k))},$$

where $c^j(1) = b^j$ and $c^j(0) = a^j$. For example, if $k = 1$ we have $\Delta_a^b F = F_b - F_a$ and when $k = 2$ then $\Delta_a^b F = F_{(b^1,b^2)} + F_{(a^1,a^2)} - F_{(a^1,b^2)} - F_{(b^1,a^2)}$. Let $\mathcal{A} = \{t \in \mathbb{R}_+^k : kt^j = 0 \text{ for some } j\}$. For $\mathcal{R} = (R_1, \ldots, R_k)$ where R_j is either $\leq$ or $>$ write $a\mathcal{R}b$ if $a^j R_j b^j$ for all j.

We say that $F = \{F_t : t \in \mathbb{R}_+^k\}$ is *lamp* if the following three conditions are satisfied: (i) for $t \in \mathbb{R}_+^k$ the limit $F(t,\mathcal{R}) = \lim_{u\to t, t\mathcal{R}u} F_u$ exists for each of the 2^k relations $\mathcal{R} = (R_1, \ldots, R_k)$ where R_j is either $\leq$ or $>$; here we let $F(t,\mathcal{R}) = F_t$ if there is no u with $t\mathcal{R}u$. (ii) $F_t = F(t,\mathcal{R})$ for $\mathcal{R} = (\leq, \ldots, \leq)$. (iii) $F_t = 0$ for $t \in \mathcal{A}$. Here lamp stands for *limits along monotone paths*. This is the multiparameter analogue of being càdlàg. See Adler et al. [1] for references to the literature on lamp trajectories. When F is lamp and $t \in \mathbb{R}_+^k \setminus \mathcal{A}$ define $\Delta_t F = \lim_{n\to\infty} \Delta_{t_n}^t F$ where t_n is any sequence with $t_n \to t$ and $t_n < t$. If F is continuous at the point t then $\Delta_t F = 0$ but the converse is not true, that is, we can have $\Delta_t F = 0$ without F being continuous at t.

Definition 2.1. Let $X = \{X_t : t \in \mathbb{R}_+^k\}$ be a family of random vectors in $\mathbb{R}^d$. We say that X has *independent increments* if $X_t = 0$ for all $t \in \mathcal{A}$ a.s. and $\Delta_{a_1}^{b_1} X, \ldots, \Delta_{a_n}^{b_n} X$ are independent whenever $n \geq 2$ and $]a_1, b_1], \ldots,]a_n, b_n]$ are disjoint; if in addition X is continuous in probability and $\Delta_{t+a}^{t+b} X \stackrel{\mathscr{D}}{=} \Delta_a^b X$ for all $a, b, t \in \mathbb{R}_+^k$ with $a \leq b$, then X is called an $\mathbb{R}^d$*-valued homogeneous Lévy sheet in law on* $\mathbb{R}_+^k$, and if also almost all sample paths are lamp then X is called an $\mathbb{R}^d$*-valued homogeneous Lévy sheet on* $\mathbb{R}_+^k$.

A homogeneous Lévy sheet is a special case of the additive processes considered by Adler et al. [1], p. 5, and of the Lévy sheets considered by Dalang and Walsh [5] (in the case $k = 2$). In fact, a process satisfying all the above conditions except the homogeneity condition $\Delta_{t+a}^{t+b} X \stackrel{\mathscr{D}}{=} \Delta_a^b X$ would be called a Lévy sheet by Dalang and Walsh. It follows e.g. from [1], Proposition 4.1, that any homogeneous Lévy sheet in law has a modification which is a homogeneous Lévy sheet. It is easily seen that if $X = \{X_t : t \in \mathbb{R}_+^k\}$ is a homogeneous Lévy sheet in law then $X_t = \Delta_0^t X$ a.s. for all $t \in \mathbb{R}_+^k$; moreover $\mathcal{L}(\Delta_a^b X) \in \mathrm{ID}(\mathbb{R}^d)$ for all $a, b \in \mathbb{R}_+^k$ with $a \leq b$ and there is a $\mu \in \mathrm{ID}(\mathbb{R}^d)$ such that $\mathcal{L}(\Delta_a^b X) = \mu^{\mathrm{Leb}(]a,b])}$ for all such a and b, where Leb denotes Lebesgue measure on $\mathbb{R}^k$. We say that X *is associated with* μ or with the characteristic triplet of μ.

Definition 2.2. Let $\Lambda = \{\Lambda(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$, where $\mathcal{B}_b(\mathbb{R}_+^k)$ is the set of bounded Borel sets in $\mathbb{R}_+^k$, denote a family of random vectors in $\mathbb{R}^d$. We call Λ an $\mathbb{R}^d$*-valued homogeneous Lévy basis on* $\mathbb{R}_+^k$ if the following conditions are satisfied: (i) $\Lambda(A_1), \ldots, \Lambda(A_n)$ are independent whenever $A_1, \ldots, A_n \in \mathcal{B}_b(\mathbb{R}_+^k)$ are disjoint. (ii) $\Lambda(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Lambda(A_n)$ a.s. whenever $A_1, A_2, \ldots \in \mathcal{B}_b(\mathbb{R}_+^k)$ are disjoint with $\cup_{n=1}^\infty A_n \in \mathcal{B}_b(\mathbb{R}_+^k)$. Here the series converges almost surely. (iii) For all $t \in \mathbb{R}_+^k$ and $A \in \mathcal{B}(\mathbb{R}_+^k)$ we have $\Lambda(A) \stackrel{\mathscr{D}}{=} \Lambda(t + A)$.

If Λ is a homogeneous Lévy basis basis then $\mathcal{L}(\Lambda(A)) \in \mathrm{ID}(\mathbb{R}^d)$ for all $A \in \mathcal{B}_b(\mathbb{R}^k_+)$. Moreover, there is a $\mu \in \mathrm{ID}(\mathbb{R}^d)$ such that $\mathcal{L}(\Lambda(A)) = \mu^{\mathrm{Leb}(A)}$ for all $A \in \mathcal{B}_b(\mathbb{R}^k_+)$. We say that the *homogeneous Lévy basis is associated with μ or its characteristic triplet*. Finally, recall that Rajput and Rosiński [9] call $\Lambda = \{\Lambda(A) : A \in \mathcal{B}_b(\mathbb{R}^k_+)\}$ an independently scattered ID random measure if it satisfies (i) and (ii) of Definition 2.2 and $\mathcal{L}(\Lambda(A)) \in \mathrm{ID}(\mathbb{R}^d)$ for all $A \in \mathcal{B}_b(\mathbb{R}^k_+)$. For simplicity we refer to this as a Lévy basis.

The following shows that, not surprisingly, there is a one-to-one correspondence between homogeneous Lévy sheets (in law) and homogeneous Lévy bases.

Theorem 2.3. *Let $X = \{X_t : t \in \mathbb{R}^k_+\}$ be a homogeneous Lévy sheet associated with $\mu \in \mathrm{ID}(\mathbb{R}^d)$ with characteristic triplet (Σ, ν, γ). Let*

$$J(C) = \#\{(t, \Delta_t X) : t \in \mathbb{R}^k_+ \setminus \mathcal{A},\ (t, \Delta_t X) \in C \text{ and } \Delta_t X \neq 0\}$$

for $C \in \mathcal{B}(\mathbb{R}^k_+ \times \mathbb{R}^d)$.

Then we have the following.

(1) $J = \{J(C) : C \in \mathcal{B}(\mathbb{R}^k_+ \times \mathbb{R}^d)\}$ *is a Poisson random measure with intensity measure* Leb $\times \nu$.

(2) *Let $\nu^1(B) = \nu(B \cap D)$ and $\nu^2(B) = \nu(B \cap D^c)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Define*

$$X_t^1 = \int_{[0,t]\times\mathbb{R}^d} y 1_D(y)\,(J - \mathrm{Leb}\times\nu)(\mathrm{d}(s,y)),$$
$$X_t^2 = \int_{[0,t]\times\mathbb{R}^d} y 1_{D^c}(y)\,J(\mathrm{d}(s,y)).$$

We then have that $X_t = X_t^1 + X_t^2 + X_t^{\mathrm{g}} + t\gamma$, where $\{X_t^{\mathrm{g}} : t \in \mathbb{R}^k_+\}$, $\{X_t^1 : t \in \mathbb{R}^k_+\}$ and $\{X_t^2 : t \in \mathbb{R}^k_+\}$ are independent, $\{X_t^{\mathrm{g}} : t \in \mathbb{R}^k_+\}$ is a homogeneous Lévy sheet associated with $(\Sigma, 0, 0)$ and $\{X_t^i : t \in \mathbb{R}^k_+\}$ is a homogeneous Lévy sheet associated with $(0, \nu^i, 0)$ for $i = 1, 2$.

(3) *There exists one and up to modification only one homogeneous Lévy basis $\Lambda = \{\Lambda(A) : A \in \mathcal{B}_b(\mathbb{R}^k_+)\}$ satisfying $\Lambda([0,t]) = X_t$ a.s. for $t \in \mathbb{R}^k_+$. In addition, Λ is given by*

$$\Lambda(A) = \int_{A\times\mathbb{R}^d} y 1_D(y)\,(J - \mathrm{Leb}\times\nu)(\mathrm{d}(t,y)) + \int_{A\times\mathbb{R}^d} y 1_{D^c}(y)\,J(\mathrm{d}(t,y)) + \int_A \mathrm{d}X_t^{\mathrm{g}} + \gamma\,\mathrm{Leb}(A) \quad a.s. \tag{2.1}$$

for $A \in \mathcal{B}_b(\mathbb{R}^k_+)$.

Theorem 2.3(1)–(2) are essentially contained in [1], Theorem 4.6. The only difference is that J above is a Poisson random measure on $\mathbb{R}^k_+ \times \mathbb{R}^d$ while Theorem 4.6 of [1] is formulated in terms of Poisson random measures on $\mathbb{R}^d$. The proofs are essentially the same and hence we omit the proof of Theorem 2.3(1)–(2). See also [5] in the case $k = 2$.

For $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ define $\int_A \mathrm{d}X_t^{\mathrm{g}} = \int 1_A(t)\,\mathrm{d}X_t^{\mathrm{g}}$ where we recall that $\int f(t)\,\mathrm{d}X_t^{\mathrm{g}}$ (a random vector in $\mathbb{R}^d$) is definable by approximation by step functions in the usual way for all measurable $f : \mathbb{R}_+^k \to \mathbb{R}$ satisfying $\int (f(t))^2\,\mathrm{d}t < \infty$. Moreover, we have $\mathcal{L}(\int f(t)\,\mathrm{d}X_t^{\mathrm{g}}) = N_d(0, \Sigma(f))$, where $\Sigma^{ij}(f) = \Sigma^{ij} \int (f(t))^2\,\mathrm{d}t$. The result in Theorem 2.3(3) is immediate from fundamental properties of integrals with respect to (compensated) Poisson random measures cf. e.g. [6]. In the case $k = 2$, Theorem 2.3(3) can also be found in [5], Theorem 2.6.

We call the process $X^{\mathrm{g}} = \{X_t^{\mathrm{g}} : t \in \mathbb{R}_+^k\}$ above the *Gaussian part of* X and the measure J the *jump measure of* X. We also denote it by J_X. Finally, we call Λ above the *homogeneous Lévy basis induced by* X, also to be denoted by Λ_X.

Proposition 2.4. *Let $\Lambda = \{\Lambda(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ be a homogeneous Lévy basis. Let $\widetilde{X}_t = \Lambda([0,t])$. We then have the following results: For $a \leq b$, $\Delta_a^b \widetilde{X} = \Lambda(]a,b])$ a.s. In particular $\widetilde{X} = \{\widetilde{X}_t : t \in \mathbb{R}_+^k\}$ is a homogeneous Lévy sheet in law. Let $X = \{X_t : t \in \mathbb{R}_+^k\}$ be a homogeneous Lévy sheet which is a modification of $\widetilde{X}$. Then for $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ we have $\Lambda(A) = \Lambda_X(A)$ a.s., where Λ_X is the Lévy basis generated by X.*

Proof. It is easily seen that $\Delta_a^b \widetilde{X} = \Lambda(]a,b])$ a.s. The uniqueness part of Theorem 2.3(3) implies $\Lambda = \Lambda_X$. □

Remark 2.5. (1) Theorem 2.3(3) shows that a homogeneous Lévy sheet generates a homogeneous Lévy basis by (2.1) and Proposition 2.4 shows conversely that any homogeneous Lévy basis is generated in this way. We call (2.1) *the Lévy-Itô decomposition of* Λ and call J in that equation the jump measure of Λ. We refer to [5] (for the case $k = 2$) and [7] for the Lévy-Itô decomposition of non-homogeneous Lévy bases.

(2) Let Λ be an $\mathbb{R}^d$-valued homogeneous Lévy basis on $\mathbb{R}_+^k$ associated with the characteristic triplet (Σ, ν, γ). Assume that $\int_{\mathbb{R}^d} (1 \wedge |x|)\,\nu(\mathrm{d}x) < \infty$. Then for $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ the representation (2.1) simplifies to

$$\Lambda(A) = \int_{A \times \mathbb{R}^d} y\, J(\mathrm{d}(t,y)) + \int_A \mathrm{d}X_t^{\mathrm{g}} + \gamma_0 \,\mathrm{Leb}(A) \quad a.s.$$

where the first integral is defined pointwise almost surely and where $\gamma_0 = \gamma - \int_A y 1_D(y)\,\nu(\mathrm{d}y)$. Here *pointwise almost surely* signifies that, for almost all ω, the integral $\int_{A\times\mathbb{R}^d} y\, J(\mathrm{d}(t,y))(\omega)$ is a usual Lebesgue integral. Thus if in addition $\nu(\mathbb{R}^d \setminus \mathbb{R}_+^d) = 0$, $\Sigma = 0$ and $\gamma_0 \in \mathbb{R}_+^d$ then we can extend Λ such that $\Lambda(A)$ is defined for all A in $\mathcal{B}(\mathbb{R}_+^k)$ rather than $\mathcal{B}_b(\mathbb{R}_+^k)$; however, some of the coordinates of $\Lambda(A)$ may be equal to ∞. In addition, almost surely all coordinates of Λ are non-negative measures.

3 Meta-times

The purpose of this section is to state a result showing that any measure m on $\mathbb{R}_+^k$ which is finite on compacts is the image measure of Leb under some mapping ϕ. This result is essentially well known, at least when m is finite, so in the next lemma we simply state a version of it which suits our purposes well.

Lemma 3.1. *Let $m = \{m(A) : A \in \mathcal{B}(\mathbb{R}^k_+)\}$ be a non-negative measure on $\mathbb{R}^k_+$ satisfying $m(\mathcal{A}) = 0$ and $m(A) < \infty$ for all $A \in \mathcal{B}_b(\mathbb{R}^k_+)$. Then there exists a measurable mapping $\phi : \mathbb{R}^k_+ \to \mathbb{R}^k$ such that*

$$m(A) = \mathrm{Leb}(\phi^{-1}(A)) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^k_+) \tag{3.1}$$

and $\phi^{-1}(A)$ is a bounded set for all $A \in \mathcal{B}_b(\mathbb{R}^k_+)$.

Remark 3.2. We refer to the inverse image ϕ^{-1} as a *meta-time* associated with m and we often denote it by $\mathbf{T}$. By the above lemma and properties of inverse images we can regard $\mathbf{T}$ as a mapping $\mathbf{T} : \mathcal{B}_b(\mathbb{R}^k_+) \to \mathcal{B}_b(\mathbb{R}^k_+)$ satisfying: (i) $\mathbf{T}(A)$ and $\mathbf{T}(B)$ are disjoint whenever $A, B \in \mathcal{B}_b(\mathbb{R}^k_+)$ are disjoint. (ii) $\mathbf{T}(\cup_{n=1}^\infty A_n) = \cup_{n=1}^\infty \mathbf{T}(A_n)$ whenever $A_1, A_2, \ldots$ are in $\mathcal{B}_b(\mathbb{R}^k_+)$ and $\cup_{n=1}^\infty A_n$ is in $\mathcal{B}_b(\mathbb{R}^k_+)$. (iii) $m(A) = \mathrm{Leb}(\mathbf{T}(A))$ for all $A \in \mathcal{B}_b(\mathbb{R}^k_+)$.

Proof. Let $u \in \mathbb{R}^k \setminus \mathbb{R}^k_+$ be arbitrary.

(1) First assume that $m(\mathbb{R}^k_+) < \infty$. Take an interval $[a, b]$ in $\mathbb{R}^k_+$ with $\mathrm{Leb}([a, b]) = m(\mathbb{R}^k_+)$. Then m is the image measure of Lebesgue measure on $[a, b]$ under some mapping $\psi : [a, b] \to \mathbb{R}^k_+$. That is, $m(A) = \mathrm{Leb}(\psi^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R}^k_+)$. Indeed, this is essentially the well known result (cf. e.g. [10]) that any distribution on $\mathbb{R}^k_+$ can be generated from k independent and uniformly distributed random variables. Letting

$$\phi(t) = \begin{cases} \psi(t) & t \in [a, b] \\ u & t \in \mathbb{R}^k_+ \setminus [a, b] \end{cases}$$

one sees that ϕ has the required properties.

(2) If instead $m(\mathbb{R}^k_+) = \infty$ we can take a sequence A_n, $n = 1, 2, \ldots$, of disjoint bounded Borel sets in $\mathbb{R}^k_+$ covering $\mathbb{R}^k_+$ and satisfying that for all $t \in \mathbb{R}^k_+$ the interval $[0, t]$ is contained in the finite union of some of the A_n's. Define, for all $n \geq 1$, $m_n = m(\cdot \cap A_n)$. Since the m_n's are finite measures there is a sequence of disjoint intervals $[a_1, b_1], [a_2, b_2], \ldots$ in $\mathbb{R}^k_+$ and measurable mappings $\psi_n : [a_n, b_n] \to A_n$ such that $m_n(A) = \mathrm{Leb}(\psi_n^{-1}(A))$ for all $A \in \mathcal{B}(A_n)$. Since $m = \sum_{n\geq 1} m_n$ we can define

$$\phi(t) = \begin{cases} \psi_n(t) & t \in [a_n, b_n] \text{ for some } n \\ u & t \in \mathbb{R}^k_+ \setminus (\cup_{n=1}^\infty [a_n, b_n]). \end{cases}$$

Clearly, since for any $t \in \mathbb{R}^k_+$ the interval $[0, t]$ is contained in the union of a finite number of A_n's it follows that $\phi^{-1}([0, t])$ is contained in the union of a finite number of intervals $[a_n, b_n]$. □

Example 3.3. Let m be as in the lemma above. In many cases of interest, the mapping ϕ in the lemma has a very simple expression, as the following shows.

(1) Assume m is concentrated on a set $\mathcal{T} = \{t_n\}_{n=1}^\infty \subseteq \mathbb{R}^k_+ \setminus \mathcal{A}$. Take a disjoint sequence $R_1, R_2, \ldots$ of bounded Borel sets in $\mathbb{R}^k_+$ such that $\mathrm{Leb}(R_n) = m(\{t_n\})$ for all n. Define $\phi(t) = t_n$ when $t \in R_n$ for some n and let $\phi(t) = u$ for $t \in \mathbb{R}^k_+ \setminus (\cup_{n=1}^\infty R_n)$, where $u \in \mathbb{R}^k \setminus \mathbb{R}^k_+$ is arbitrary. The sets R_n can be chosen arbitrarily, showing in particular that ϕ is not at all uniquely determined.

(2) If $m = \mathrm{Leb}/c$ for some $c > 0$ we can use $\phi(t) = ct$.

(3) The case when $m = m_1 + m_2$ where $m_1 = \mathrm{Leb}/c$ and m_2 is concentrated on $\{t_n\}_{n=1}^{\infty} \subseteq \mathbb{R}_+^k \setminus \mathcal{A}$ can be handled as follows. Let the sets R_n above be subsets of

$$\{s = (s^1, \dots, s^k) \in \mathbb{R}_+^k : 0 \le s^j \le 1 \text{ for all } j = 1, \dots, k\}.$$

Let $e = (1, \dots, 1) \in \mathbb{R}_+^k$ be the vector of ones. By defining ϕ as

$$\phi(t) = \begin{cases} t_n & \text{if } t \in R_n \text{ for some } n \\ u & \text{if } t \in \{s = (s^1, \dots, s^k) \in \mathbb{R}_+^k : s^j \in [0,1]\} \setminus (\cup_{n=1}^{\infty} R_n) \\ c(t - e) & \text{if } t \in \{s = (s^1, \dots, s^k) \in \mathbb{R}_+^k : s^j > 1\}, \end{cases}$$

equation (3.1) is easily verified.

(4) Assume $k = 1$ and let $T_t = m([0,t])$ for all $t \ge 0$. Define $\phi : \mathbb{R}_+ \to \mathbb{R}$ as

$$\phi(y) = \inf\{t \ge 0 : T(t) \ge y\}.$$

where $\inf \emptyset = u \in \mathbb{R} \setminus \mathbb{R}_+$. Then

$$[0, T_t] = \mathbf{T}([0,t]) \quad \text{for all } t \ge 0 \tag{3.2}$$

and hence $m(A) = \mathrm{Leb}(\mathbf{T}(A))$ for all $A \in \mathcal{B}(\mathbb{R}_+)$.

4 Meta-time changes

In the one-dimensional case $k = 1$ one uses increasing functions to model a *time change* as in (4.3) below. The purpose of the present section is to show that in the case $k \ge 2$ certain *meta-time changes* give similar results. In fact, we show that the appropriate generalization of the process Y in (4.3) is the random measure M in (4.1) where in the latter equation $\mathbf{T}$ is a meta-time as defined in Section 3.

Let $X = \{X_t : t \in \mathbb{R}_+^k\}$ be an $\mathbb{R}^d$-valued homogeneous Lévy sheet on $\mathbb{R}_+^k$ associated with $\mu \in \mathrm{ID}(\mathbb{R}^d)$. Denote the corresponding homogeneous Lévy basis by Λ_X. Let $m = \{m(A) : A \in \mathcal{B}(\mathbb{R}_+^k)\}$ be a non-negative measure on $\mathbb{R}_+^k$ satisfying $m(\mathcal{A}) = 0$ and $m(A) < \infty$ for all $A \in \mathcal{B}_b(\mathbb{R}_+^k)$. Set $T_t = m([0,t])$ for all $t \in \mathbb{R}_+^k$ and let $\phi : \mathbb{R}_+^k \to \mathbb{R}^k$ be given as in Lemma 3.1. Let $\mathbf{T} = \phi^{-1}$ be the corresponding meta-time associated with m.

Define $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ as

$$M(A) = \Lambda_X(\mathbf{T}(A)) \quad \text{for } A \in \mathcal{B}_b(\mathbb{R}_+^k). \tag{4.1}$$

Using the properties of $\mathbf{T}$ in Remark 3.2 one sees that M is a (non-homogeneous) Lévy basis. Since in addition Λ_X is a homogeneous Lévy basis associated with μ it follows that

$$\mathcal{L}(M(A)) = \mu^{m(A)} \quad \text{for } A \in \mathcal{B}_b(\mathbb{R}_+^k). \tag{4.2}$$

In particular, for $t \in \mathbb{R}_+^k$, $M((0,t])$ has characteristic triplet $T_t(\Sigma, \nu, \gamma)$ where (Σ, ν, γ) is the characteristic triplet of μ. We say that M *is defined from* Λ_X *by meta-time change with* $\mathbf{T}$.

Remark 4.1. Let $k = 1$ and let us show that in this case the above definition generalizes the usual concept of a time change in a natural way. For this purpose, define a process $Y = \{Y_t : t \geq 0\}$ by time changing X with T:

$$Y_t = X_{T_t}. \tag{4.3}$$

Hence, Y is a càdlàg process with independent increments and it is an additive process (i.e. also continuous in probability) if T is continuous. The distribution of an increment is given as

$$\mathcal{L}(Y_t - Y_s) = \mu^{T_t - T_s} \quad \text{for } 0 \leq s < t. \tag{4.4}$$

Let ϕ be given as in Example 3.3(4). Using (3.2) rewrite Y_t in terms of the Lévy basis Λ_X as

$$Y_t = \Lambda_X([0, T_t]) = \Lambda_X(\mathbf{T}([0, t])) \quad \text{for } t \geq 0. \tag{4.5}$$

This shows that (4.1) provides a natural generalization of (4.3) and (4.5) since we simply replace $[0, t]$ by an arbitrary bounded Borel set; in return we get a measure M instead of a process Y. Similarly, (4.4) is generalized by (4.2).

Remark 4.2. There are many alternative representations of M and in the following we consider some of them. Let $\{L_t : t \geq 0\}$ denote an $\mathbb{R}^d$-valued Lévy process with $\mu = \mathcal{L}(L_1)$. Thus, in the language of [2], $\{L_t : t \geq 0\}$ is a Lévy seed associated with μ.

(1) If $A_1, \dots, A_r$ are disjoint bounded Borel sets then

$$(M(A_1), \dots, M(A_r)) \stackrel{\mathscr{D}}{=} (L^{A_1}_{m(A_1)}, \dots, L^{A_r}_{m(A_r)}),$$

where $\{L^{A_j}_t : t \geq 0\}$, for $j = 1, \dots, r$, are independent copies of $\{L_t : t \geq 0\}$. This follows since $\mathcal{L}(L^{A_j}_{m(A_j)}) = \mu^{m(A_j)}$. If instead $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r$ then

$$(M(A_1), \dots, M(A_r)) \stackrel{\mathscr{D}}{=} (L_{m(A_1)}, \dots, L_{m(A_r)}).$$

(2) Consider the case where m is given as in Example 3.3(1); that is, m is concentrated on $\mathcal{T} = \{t_n\}_{n=1}^\infty \subseteq \mathbb{R}^k_+ \setminus \mathcal{A}$. For $A \in \mathcal{B}_b(\mathbb{R}^k_+)$ we then have

$$\begin{aligned} M(A) &= \Lambda_X(\mathbf{T}(A \cap \mathcal{T}^c)) + \Lambda_X(\mathbf{T}(A \cap \mathcal{T})) \\ &= \Lambda_X(\mathbf{T}(A \cap \mathcal{T})) = \sum_{n : t_n \in A} \Lambda_X(\mathbf{T}(\{t_n\})) \quad a.s. \end{aligned} \tag{4.6}$$

where the series converges almost surely and the first term on the right-hand side of (4.6) vanishes since $\mathrm{Leb}(\mathbf{T}(A \cap \mathcal{T}^c)) = 0$ by (3.1). Since, by the same equation, $\mathrm{Leb}(\mathbf{T}(\{t_n\})) = m(\{t_n\})$ we have $\Lambda_X(\mathbf{T}(\{t_n\})) \stackrel{\mathscr{D}}{=} L_{m(\{t_n\})}$. Taking a sequence $\{L^n_t : t \geq 0\}$, $n = 1, 2, \dots$, of independent copies of $\{L_t : t \geq 0\}$ we thus have for all $r \geq 1$ and $A_1, \dots, A_r \in \mathcal{B}_b(\mathbb{R}^k_+)$ that

$$(M(A_1), \dots, M(A_r)) \stackrel{\mathscr{D}}{=} \Big(\sum_{n : t_n \in A_1} L^n_{m(\{t_n\})}, \dots, \sum_{n : t_n \in A_r} L^n_{m(\{t_n\})} \Big). \tag{4.7}$$

If $\mu = N_d(\gamma, I)$ (where I is the $d \times d$ identity matrix) this simplifies as follows. Let $\epsilon_1, \epsilon_2, \dots$ denote a sequence of independent and identically distributed random vectors with law $N_d(0, I)$. Then (4.7) is equivalent to

$$(M(A_1), \dots, M(A_r)) \stackrel{\mathscr{D}}{=} \Big(\sum_{n:t_n \in A_1} \gamma m(\{t_n\}) + [m(\{t_n\})]^{1/2} \epsilon_n, \dots, \sum_{n:t_n \in A_r} \gamma m(\{t_n\}) + [m(\{t_n\})]^{1/2} \epsilon_n \Big).$$

(3) Finally consider the case $m = m_1 + m_2$ as in Example 3.3(3) where $m_1 = \mathrm{Leb}/c$ and m_2 is concentrated on $\mathcal{T}$. Then $M = M_1 + M_2$ where $M_i = \{M_i(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ for $i = 1, 2$ are independent and given by

$$M_1(A) = \Lambda_X(\mathbf{T}(A \cap \mathcal{T}^c)) \quad \text{and} \quad M_2(A) = \Lambda_X(\mathbf{T}(A \cap \mathcal{T})) \quad \text{for } A \in \mathcal{B}_b(\mathbb{R}_+^k).$$

The measure M_1 is a homogeneous Lévy basis associated with $\mu^{1/c}$ and M_2 can be represented as M in (2).

Remark 4.3. From the Lévy-Itô decomposition (2.1) of Λ_X we have, a.s. for $A \in \mathcal{B}_b(\mathbb{R}_+^k)$,

$$M(A) = \int_{\mathbf{T}(A) \times \mathbb{R}^d} y 1_D(y)\, (J_X - \mathrm{Leb} \times \nu)(\mathrm{d}(t,y)) + \int_{\mathbf{T}(A) \times \mathbb{R}^d} y 1_{D^c}(y)\, J_X(\mathrm{d}(t,y)) + \int_{\mathbf{T}(A)} \mathrm{d}X_t^{\mathrm{g}} + \gamma\, \mathrm{Leb}(\mathbf{T}(A)).$$

Applying the transformation rule on the first two integrals we get the following Lévy-Itô type representation of M:

$$M(A) = \int_{A \times \mathbb{R}^d} y 1_D(y)\, (\widetilde{J}_X - m \times \nu)(\mathrm{d}(t,y)) + \int_{A \times \mathbb{R}^d} y 1_{D^c}(y)\, \widetilde{J}_X(\mathrm{d}(t,y)) + \int_{\mathbf{T}(A)} \mathrm{d}X_t^{\mathrm{g}} + \gamma m(A) \quad a.s. \quad \text{for } A \in \mathcal{B}_b(\mathbb{R}_+^k),$$

where $\widetilde{J}_X = \{\widetilde{J}_X(C) : C \in \mathcal{B}(\mathbb{R}_+^k \times \mathbb{R}^d)\}$ is the Poisson random measure given by $\widetilde{J}_X(A \times B) = J_X(\mathbf{T}(A) \times B)$ for all $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$.

5 Extended subordination

Let $X = \{X_t : t \in \mathbb{R}_+^k\}$, $\Lambda_X = \{\Lambda_X(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$, μ and (Σ, ν, γ) be given as in the previous section. That is, X is an $\mathbb{R}^d$-valued homogeneous Lévy sheet on $\mathbb{R}_+^k$ associated with μ, which has characteristic triplet (Σ, ν, γ), and Λ_X is the homogeneous Lévy basis induced by X. Let $T = \{T_t : t \in \mathbb{R}_+^k\}$ be an $\mathbb{R}_+$-valued homogeneous Lévy sheet associated with a distribution $\lambda \in \mathrm{ID}(\mathbb{R})$. Let λ have Lévy measure ρ and drift $\beta \in \mathbb{R}_+$; that is, $\rho(\mathbb{R}_-) = 0$, $\int_{\mathbb{R}_+} (1 \wedge x)\, \rho(\mathrm{d}x) < \infty$ and

$$\widehat{\lambda}(u) = \exp[\mathrm{i}\beta u + \int_{\mathbb{R}_+} (e^{\mathrm{i}ux} - 1)\, \rho(\mathrm{d}x)] \quad \text{for } u \in \mathbb{R}.$$

Let $\Lambda_T = \{\Lambda_T(A) : A \in \mathcal{B}(\mathbb{R}_+^k)\}$ be the non-negative homogeneous Lévy basis induced by $T = \{T_t : t \in \mathbb{R}_+^k\}$. By removing a null set if necessary it follows from Remark 2.5(2) that Λ_T has the pointwise representation

$$\begin{aligned}\Lambda_T(A)(\omega) &= \int_{A\times\mathbb{R}_+} y\, J_T(\mathrm{d}(t,y))(\omega) + \beta\,\mathrm{Leb}_1(A) \\ &= \sum_{t\in A} \Lambda_T(\{t\})(\omega) + \beta\,\mathrm{Leb}_1(A) \quad \text{for } \omega \in \Omega \text{ and } A \in \mathcal{B}(\mathbb{R}_+^k), \end{aligned} \tag{5.1}$$

where Leb_1 denotes Lebesgue measure on $\mathbb{R}_+$ and the series converges for all $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ and $\omega \in \Omega$. Let $\mathcal{F}^T = \sigma(\Lambda_T(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k))$ be the sigma-field generated by Λ_T.

Pointwise the measure $A \to \Lambda_T(A)(\omega)$ is the sum of a discrete measure and a constant times Lebesgue measure. By the construction in Example 3.3(3) there is an $(\mathcal{F}^T \times \mathcal{B}(\mathbb{R}_+^k), \mathcal{B}(\mathbb{R}^k))$-measurable mapping $\phi_T : \Omega \times \mathbb{R}_+^k \to \mathbb{R}^k$ such that for all $\omega \in \Omega$ and $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ the set $\mathbf{T}(A)(\omega)$, given by $\mathbf{T}(A)(\omega) = \{x \in \mathbb{R}_+^k : \phi_T(\omega, x) \in A\}$, is bounded, and

$$\Lambda_T(A)(\omega) = \mathrm{Leb}(\mathbf{T}(A)(\omega)). \tag{5.2}$$

That is, for each ω, $\mathbf{T}(\cdot)(\omega)$ is a meta-time associated with $\Lambda_T(\cdot)(\omega)$.

Define $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ as

$$M(A) = \Lambda_X(\mathbf{T}(A)) \quad \text{for } A \in \mathcal{B}_b(\mathbb{R}_+^k) \tag{5.3}$$

where as usual we suppress ω on both sides. We say that M appears by *extended subordination* of Λ_X by Λ_T or of X by T; and we write $M = \Lambda_X \curlywedge \Lambda_T$ or $M = X \curlywedge T$.

In practice the meta-time $\mathbf{T}$ can be hard to work with directly. Therefore it is important to note that if we condition on T then, by (5.1), the useful representations of M in Remark 4.2 apply. For example, if λ above is a Poisson or negative binomial distribution then almost surely Λ_T is concentrated on a finite number of points on compacts. If λ is a gamma or an inverse Gaussian distribution then almost surely Λ_T is concentrated on a dense subset of $\mathbb{R}_+^k$. In this case we can approximate Λ_T pointwise in ω by a random measure which is concentrated on a finite number of points, for instance by removing all jumps of magnitude less than ϵ for some small ϵ; this also gives a pointwise approximation to the meta-time $\mathbf{T}$.

The following corresponds to the theorem in Section 3.1 of [2].

Theorem 5.1. *Assume $M = \Lambda_X \curlywedge \Lambda_T$ as above. Then $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ is a homogeneous Lévy basis associated with the measure $\mu^\# \in \mathrm{ID}(\mathbb{R}^d)$ with characteristic triplet $(\Sigma^\#, \nu^\#, \gamma^\#)$, where*

$$\begin{aligned}\Sigma^\# &= \beta\Sigma, \\ \nu^\#(B) &= \beta\nu(B) + \int_0^\infty \mu^s(B)\,\rho(\mathrm{d}s), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \\ \gamma^\# &= \beta\gamma + \int_0^\infty \int_{|x|\le 1} x\,\mu^s(\mathrm{d}x)\,\rho(\mathrm{d}s).\end{aligned}$$

Proof. Conditional on $\mathcal{F}^T$, and hence also unconditionally, M satisfies the σ-additivity condition in Definition 2.2(ii).

Let $n \geq 1$, and $A_1, \ldots, A_n \in \mathcal{B}_b(\mathbb{R}_+^k)$ be disjoint. Conditional on $\mathcal{F}^T$ we are in the setting of the previous section. That is, $M(A_1), \ldots, M(A_n)$ are independent conditional on $\mathcal{F}^T$ and $\mathcal{L}(M(A_j)|\mathcal{F}^T) = \mu^{\Lambda_T(A_j)}$. Therefore, for arbitrary $z_1, \ldots, z_n \in \mathbb{R}^d$ we have

$$E\Big[\prod_{j=1}^n e^{\mathrm{i}\langle z_j, M(A_j)\rangle}\Big|\mathcal{F}^T\Big] = \prod_{j=1}^n \widehat{\mu}(z_j)^{\Lambda_T(A_j)}.$$

Since $\Lambda_T(A_1), \ldots, \Lambda_T(A_n)$ are independent it thus follows that

$$E\Big[\prod_{j=1}^n e^{\mathrm{i}\langle z_j, M(A_j)\rangle}\Big] = \prod_{j=1}^n E\big[\widehat{\mu}(z_j)^{\Lambda_T(A_j)}\big],$$

showing that $M(A_1), \ldots, M(A_n)$ are independent. Since moreover $\mathcal{L}(\Lambda_T(A)) = \mathcal{L}(\Lambda_T(t+A))$ for all $t \in \mathbb{R}_+^k$ and $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ it follows that $\mathcal{L}(M(A)) = \mathcal{L}(M(t+A))$. Thus, M is a homogeneous Lévy basis.

Choose an arbitrary set $A \in \mathcal{B}_b(\mathbb{R}_+^k)$ with $\mathrm{Leb}(A) = 1$. Then $\mu^\# = \mathcal{L}(M(A))$ and by the above we have for $z \in \mathbb{R}^d$ that

$$\widehat{\mu^\#}(z) = E[\widehat{\mu}(z)^{\Lambda_T(A)}].$$

Let $\{L_t : t \geq 0\}$ be a Lévy process with $\mathcal{L}(L_1) = \mu$ and $\{H_t : t \geq 0\}$ be a subordinator independent of L with $\mathcal{L}(H_1) = \lambda$. It is easily seen that

$$E[e^{\mathrm{i}\langle z, L_{H_1}\rangle}] = E[\widehat{\mu}(z)^{H_1}] = \widehat{\mu^\#}(z).$$

In other words $\mathcal{L}(L_{H_1}) = \mu^\#$, which means that $\mu^\#$ appears as the law of a subordinated process in the usual sense. It is therefore well known, e.g. from [11], Theorem 30.1, that the characteristic triplet of $\mu^\#$ is as indicated. □

Remark 5.2. Above we assumed that Λ_T is a non-negative homogeneous Lévy basis; however, it is possible to define $M = \Lambda_X \curlywedge \Lambda_T$ in a much more general context. For example, assume that $\Lambda_T = \{\Lambda_T(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ is stationary in the sense that $\{\Lambda_T(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\} \overset{\mathscr{D}}{=} \{\Lambda_T(t + A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ for all $t \in \mathbb{R}_+^k$ and that for all ω $A \mapsto \Lambda_T(A)(\omega)$ is a non-negative measure on $\mathbb{R}_+^k$. Assume Λ_T and Λ_X are independent and note that we do no longer assume that Λ_T is a Lévy basis. Using Lemma 3.1 define $\phi_T : \Omega \times \mathbb{R}_+^k \to \mathbb{R}^k$ such that we have (5.2) and let $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ be given as in (5.3). Then $M = \{M(A) : A \in \mathcal{B}_n(\mathbb{R}_+^k)\}$ is a homogeneous random measure in the sense that for any disjoint sequence $A_1, A_2, \ldots$ with $A = \cup_{n=1}^\infty A_n \in \mathcal{B}_b(\mathbb{R}_+^k)$ we have $M(A) = \sum_{n=1}^\infty M(A_n)$ a.s. Moreover, by a slight modification of the above proof it follows that M is stationary. In general M is no longer a Lévy basis. But conditionally on T it is.

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