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Abstract

We discuss a stochastic differential equation, as a modelling framework for the turbulent velocity field, that is capable of capturing basic stylized facts of the statistics of velocity increments. In particular, we focus on the evolution of the probability density of velocity increments characterized by a normal inverse Gaussian shape with heavy tails for small scales and aggregational Gaussianity for large scales. In addition, we show that the proposed model is in accordance with Kolmogorov's refined similarity hypotheses.

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1 Introduction

Since the pioneering work of Kolmogorov (1962) and Obukhov (1962), intermittency of the turbulent velocity field is of major interest in turbulence research. From a probabilistic point of view, intermittency refers, in particular, to the increase of the non-Gaussian behaviour of the probability density function (pdf) of velocity increments with decreasing scale. A typical scenario is characterized by a Gaussian shape for the large scales (larger than scales at the so-called inertial range), turning to exponential tails within the inertial range and stretched exponential tails for dissipation scales (below the inertial range) [3, 4].

It was reported in [5] that the evolution of the pdf of velocity increments for all amplitudes and all scales can be described within one class of tractable distributions, the normal inverse Gaussian (NIG) distributions. Furthermore, the subsequent analysis of the observed parameters of the NIG distributions revealed that the pdf's of different data sets with different Reynolds numbers (ranging from $R_\lambda = 80$ up to $R_\lambda = 17000$) all collapse after applying a scale transformation that is related to one of the parameters of the estimated NIG distributions. As a consequence, the collapse of pdf's immediately resulted in a broader and more general reformulation of the concept of Extended Self Similarity [6] in terms of a stochastic equivalence class.

The analysis in [5] is to a large extent based on purely empirical footings without providing a theoretical model for the turbulent velocity field. In view of the significance of the derived results, a theoretical basis is clearly asked for. This is one goal we want to achieve in this paper. We present a general spatio-temporal framework for modelling the turbulent velocity field, leading to a stochastic differential equation that is able to reproduce the observed evolution of the pdf of turbulent velocity increments.

The second goal we want to achieve in this paper is to present our model as a class of stochastic processes that are, to a large extent, in accordance with Kolmogorov's refined similarity hypotheses (K62) [1]. The first hypothesis states that the pdf of the stochastic variable

$$V_r = \frac{\Delta u(r)}{(r\varepsilon_r)^{1/3}} \tag{1.1}$$

depends, for $r \ll L$, only on the local Reynolds number $\mathrm{Re}_r = r(r\varepsilon_r)^{1/3}/\nu$. Here, $\Delta u(r)$ denotes the velocity increment at scale r, ν the viscosity, L the integral scale and $r\varepsilon_r$ is the integrated energy dissipation over a domain of linear size r. The second hypothesis states that, for $\mathrm{Re}_r \gg 1$, the pdf of V_r does not depend on Re_r, either, and is therefore universal. Although, for small r, an additional r dependence of the pdf of V_r has been observed [7], the validity of several aspects of K62 has been verified experimentally and by numerical simulation of turbulence [7, 8, 9].

The outline of the paper is as follows. In Section 2 we present our stochastic framework for the turbulent velocity field in its full generality. Section 3 provides the necessary mathematical background on quadratic variation, infinitely divisible distributions and Lévy processes. Infinitely divisible distributions and Lévy processes are the main building blocks of the model. Quadratic variation will be proposed as a natural substitute for the usual definition of the integrated energy dissipation which is not applicable for non-differentiable stochastic processes. As a special case of the general spatio-temporal framework for the turbulent velocity field we discuss in Section 4 a one-dimensional model that allows to gain analytical insight into the dynamics of the velocity field. The focus is on the evolution of the pdf of velocity increments across scales and statistics related to K62. The theoretical results are illustrated and supplemented in Section 5 through simulations. Section 6 concludes with an outlook.

2 Modelling framework

We propose to study a modelling framework for the turbulent velocity field based on the concept of stochastic differential equations. In full generality the framework specifies the velocity vector $u_t(\sigma)$ at time t and position σ as a stationary process, defined as a stochastic integral

$$u_t(\sigma) = \bar{u} + \int_{-\infty}^{t} \int_S g(t-s; |\rho - \sigma|) \, \mathrm{d}M_s(\mathrm{d}\rho) \tag{2.1}$$

where $\bar{u}$ is the mean velocity, g is a deterministic kernel, and M is a random measure on $\mathbb{R} \times S$, S denoting the space of possible locations.

In the present paper we limit discussion to the main component of the velocity vector and we shall only consider the dynamics at a single fixed location. We therefore drop reference to the location in the notation and let

$$u_t = \bar{u} + \int_{-\infty}^{t} g(t-s) \, \mathrm{d}M_s. \tag{2.2}$$

The aim is to show that suitable choices of g and M can reproduce key stylized features of the time-wise behaviour of the velocity. Without loss of generality we assume that $g(0) = 1$.

We choose the process M to satisfy a stochastic differential equation

$$\mathrm{d}M_t = \beta \varepsilon_t \, \mathrm{d}t + \sqrt{\varepsilon_t} \, \mathrm{d}W_t \tag{2.3}$$

where ε denotes a positive stationary process and W is a Brownian motion independent of ε. This type of process M is often encountered in other areas of application, in particular financial econometrics.

Combining (2.2) and (2.3) we get

$$u_t = \bar{u} + \beta \int_{-\infty}^{t} g(t-s) \varepsilon_s \, \mathrm{d}s + \int_{-\infty}^{t} g(t-s) \sqrt{\varepsilon_s} \, \mathrm{d}W_s. \tag{2.4}$$

In the present context of turbulence we conceive of ε as expressing the time varying intermittency while W generates independent innovation impulses.

The strength of the modelling framework (2.4) lies in the fact that the intermittency generating term ε and the function g can, to a large extent, be chosen arbitrarily. In the next Section we identify ε with the local energy dissipation. Therefore, the model (2.4) establishes a framework that derives the model for the velocity field directly from the presumed model for the local energy dissipation.

The calculations in Section 4 show that a considerable part of the statistics of the velocity field are independent of the specific choice of the model for the energy dissipation. In particular, the evolution of the pdf of velocity increments from heavy tails to a Gaussian shape with increasing scale and the statistics related to K62 are predominantly mediated by the structure of (2.4).

3 Mathematical background

This Section outlines the mathematical tools we require for modelling the turbulent velocity field as a spatio-temporal stochastic process. The basic notions are semimartingales, Lévy processes and quadratic variation. For later purposes we also provide the definitions and basic properties of normal inverse Gaussian distributions and inverse Gaussian distributions. While the former approximates the distribution of velocity increments for all scales and all amplitudes, the latter will be used to explicitely model the intermittency of the velocity field.

The stochastic processes we propose as a model for the velocity field are nowhere differentiable, thus the definition of the energy-dissipation as the square of velocity derivatives does not make sense in this context. As an alternative definition of the energy-dissipation for the more general case of stochastic processes we propose to use the concept of quadratic variation, as outlined below.

3.1 Semimartingales and quadratic variation

In the language of stochastic analysis the process u, as given by (2.4), is a Brownian semimartingale. A key result of stochastic analysis states that for any semimartingale u, whether Brownian or not, the limit

$$[u]_t = \lim_{n\to\infty} \sum_{j=1}^{n} \left(u_{tj/n} - u_{t(j-1)/n} \right)^2 \tag{3.1}$$

exists, as a limit in probability. The derived process $[u]$ expresses the cumulative variation exhibited by u and is called the *quadratic variation (*QV). The monograph [11] is a lucid and comprehensive account of the basic parts of stochastic analysis. For further properties, see [12].

We may calculate $[u]$ from (2.4) using Ito algebra. Specifically, the differential of u is

$$\mathrm{d}u_t = a_t\,\mathrm{d}t + \sqrt{\varepsilon_t}\,\mathrm{d}W_t \tag{3.2}$$

where

$$a_t = \beta\varepsilon_t + \beta \int_{-\infty}^{t} g'(t-s)\,(\varepsilon_s\,\mathrm{d}s + \sqrt{\varepsilon_s}\,\mathrm{d}W_s) \tag{3.3}$$

is of finite variation. Thus

$$(\mathrm{d}u_t)^2 = a_t^2(\mathrm{d}t)^2 + 2a_t\sqrt{\varepsilon_t}\,\mathrm{d}t\,\mathrm{d}W_t + \varepsilon_t(\mathrm{d}W_t)^2. \tag{3.4}$$

By Ito algebra $(\mathrm{d}t)^2 = \mathrm{d}t\,\mathrm{d}W_t = 0$ while $(\mathrm{d}W_t)^2 = \mathrm{d}t$. All in all, we obtain

$$(\mathrm{d}u_t)^2 = \varepsilon_t\,\mathrm{d}t \tag{3.5}$$

and

$$[u]_t = \int_0^t (\mathrm{d}u_t)^2 \,\mathrm{d}s = \int_0^t \varepsilon_s \,\mathrm{d}s. \tag{3.6}$$

In the setting of stochastic differential equations of the Brownian semimartingale type the quantity $(\mathrm{d}u_t)^2/\mathrm{d}t$ is the natural analogue of the squared first order derivative of the velocity which in the classical formulation is taken to express the local energy dissipation. Consequently, the quadratic variation $[u]_t$ is the stochastic analogue for the integrated energy dissipation and ε_t can be identified with the local energy dissipation.

It is to note that the quadratic variation $[u]_t$ is independent of β, i.e. independent of the second term in (2.4). That term is responsible for the skewness of velocity increments. The skewness of the distribution of $u_t - u_0$ has a relatively complicated expression, and in this paper we restrict attention to the infinitesimal skewness $\mathrm{E}\{(\mathrm{d}u_t)^3\}$, noting that $\mathrm{E}\{\mathrm{d}u_t\} = 0$ due to the stationarity of u_t. Here, $\mathrm{E}\{\}$ denotes the expectation. From the differential of u (3.2) we get, using the independence of ε and W

$$\mathrm{E}\{(\mathrm{d}u_t)^3\} = 3\beta \left(\mathrm{E}\{\varepsilon_0^2\} + \int_0^\infty g'(t)\mathrm{E}\{\varepsilon_0\varepsilon_t\} \,\mathrm{d}t \right) (\mathrm{d}t)^2. \tag{3.7}$$

Under the additional simplifying (weak) assumptions

$$\int_0^\infty |g'(t)| \mathrm{d}t = 1 \tag{3.8}$$

and

$$\mathrm{E}\{\varepsilon_0^2\} - \mathrm{E}\{\varepsilon_0\varepsilon_t\} > 0 \tag{3.9}$$

and g monotonically decreasing, we finally get

$$\mathrm{E}\{(\mathrm{d}u_t)^3\} = 3\beta(\mathrm{d}t)^2 \int_0^\infty |g'(t)| \left(\mathrm{E}\{\varepsilon_0^2\} - \mathrm{E}\{\varepsilon_0\varepsilon_t\}\right) \,\mathrm{d}t > 0. \tag{3.10}$$

This result is in accordance with the positive skewness of temporal turbulent velocity increments as follows from the famous 4/5-law of Kolmogorov [13], invoking Taylor's Frozen Flow Hypothesis [14]. In our stochastic framework (2.4), the positive skewness of temporal velocity increments is directly related to the positive autocorrelation (3.9) of the local energy dissipation.

3.2 Lévy processes and OU processes

Besides the Brownian semimartingales two other basic types of semimartingales are Lévy processes and Ornstein-Uhlenbeck (OU) processes. These are also central to our general modelling approach.

A Lévy process is a stochastic process with independent and identically distributed increments. The Poisson process and the stable processes (Lévy flights) as well as Brownian motion are of this type. But the class of Lévy processes is much wider than this, the inverse Gaussian and the normal inverse Gaussian Lévy processes being important examples; these are Lévy processes for which the laws of the increments are, respectively, inverse Gaussian and normal inverse Gaussian (For definition and properties of these laws, see subsection 3.3 below).

The Lévy processes, other than Brownian motion, enter our modelling framework only indirectly via the concept of OU processes. An OU process is a stationary process Z satisfying a stochastic differential equation of the form

$$\mathrm{d}Z_t = -\lambda Z_t \,\mathrm{d}t + \mathrm{d}L_t \tag{3.11}$$

where L is a Lévy processes, called the background driving Lévy process (BDLP). This equation has a stationary solution for any Lévy process L such that $\mathrm{E}\left\{\log(1+|L_1|)\right\} < \infty$. In particular, taking L to be an inverse Gaussian Lévy process we obtain as solution Z the so called OU-IG process, which will be applied in the sequel, as a model for the intermittency.

Integrated with respect to Brownian motion, OU-IG processes have the property to show a pronounced NIG-shape with heavy tails for the increments at small scales. For the large scales, the pdf of increments tends to a Gaussian-like shape. This is the property we want to model for the turbulent velocity field.

3.3 Normal inverse Gaussian and inverse Gaussian distributions

The normal inverse Gaussian law, with parameters α, β, μ and δ, is the distribution on the real axis $\mathbf{R}$ having probability density function

$$p(x;\alpha,\beta,\mu,\delta) = a(\alpha,\beta,\mu,\delta) q\left(\frac{x-\mu}{\delta}\right)^{-1} K_1\left\{\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right\} e^{\beta x} \tag{3.12}$$

where $q(x) = \sqrt{1+x^2}$ and

$$a(\alpha,\beta,\mu,\delta) = \pi^{-1}\alpha \exp\left\{\delta\sqrt{\alpha^2-\beta^2} - \beta\mu\right\} \tag{3.13}$$

and where K_1 is the modified Bessel function of the third kind and index 1. The domain of variation of the parameters is given by $\mu \in \mathbf{R}$, $\delta \in \mathbf{R}_+$, and $0 \leq |\beta| < \alpha$. The distribution is denoted by $\mathrm{NIG}(\alpha,\beta,\mu,\delta)$.

The standardised third and fourth order cumulants are

$$\begin{aligned}\bar{c}_3 &= \frac{c_3}{c_2^{3/2}} = 3\frac{\rho}{\{\delta\alpha(1-\rho^2)^{1/2}\}^{1/2}} \\ \bar{c}_4 &= \frac{c_4}{c_2^2} = 3\frac{1+4\rho^2}{\delta\alpha(1-\rho^2)^{1/2}}\end{aligned} \tag{3.14}$$

where $\rho = \beta/\alpha$. We note that the NIG distribution (3.12) has semiheavy tails; specifically,

$$p(x;\alpha,\beta,\mu,\delta) \sim \text{const.}\, |x|^{-3/2} \exp\left(-\alpha|x| + \beta x\right), \ x \to \pm\infty. \tag{3.15}$$

NIG shape triangle For some purposes it is useful, instead of the classical skewness and kurtosis quantities (3.14), to work with the alternative asymmetry and steepness parameters χ and ξ defined by

$$\chi = \rho\xi \tag{3.16}$$

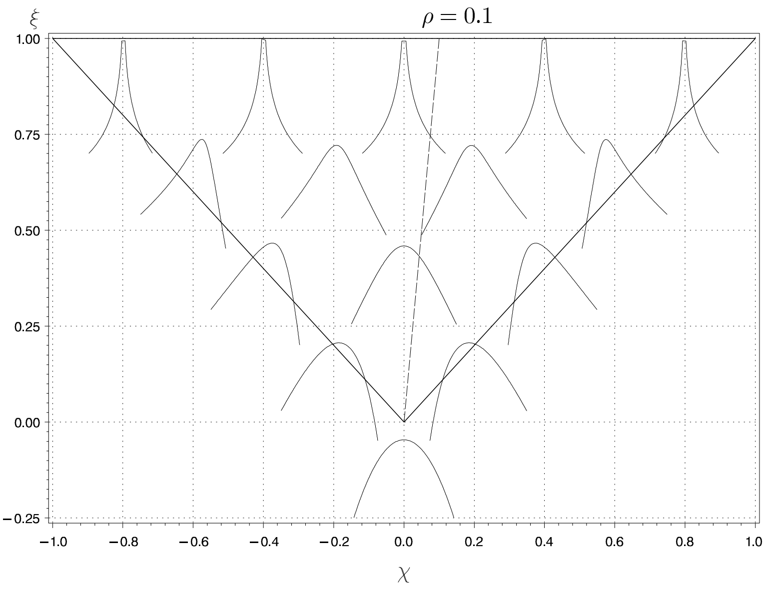

Figure 1: The shape triangle of the NIG distributions with the log density functions of the standardized distributions, i.e. with mean 0 and variance 1, corresponding to the values $(\chi, \xi) = (\pm 0.8, 0.999)$, $(\pm 0.4, 0.999)$, $(0.0, 0.999)$, $(\pm 0.6, 0.75)$, $(\pm 0.2, 0.75)$, $(\pm 0.4, 0.5)$, $(0.0, 0.5)$, $(\pm 0.2, 0.25)$ and $(0.0, 0.0)$. The coordinate system of the log densities is placed at the corresponding value of (χ, ξ). Furthermore, the line corresponding to $\rho = 0.1$, i.e. $\chi = 0.1\xi$, is shown.

and

$$\xi = (1 + \bar{\gamma})^{-1/2} \tag{3.17}$$

where $\bar{\gamma} = \delta\sqrt{\alpha^2 - \beta^2}$. Like $\bar{c}_3$ and $\bar{c}_4$, these parameters are invariant under location-scale changes and the domain of variation for (χ, ξ) is the *normal inverse Gaussian shape triangle*

$$\{(\chi, \xi) : -1 < \chi < 1, 0 < \xi < 1\}. \tag{3.18}$$

The distributions with $\chi = 0$ are symmetric, and the normal law occurs as limiting case for (χ, ξ) near to $(0, 0)$. Figure 1 gives an impression of the shape of the NIG distributions for various values of (χ, ξ). The dashed line in Figure 1 corresponds to $\rho = 0.1$ and represents the location of the pdf of turbulent velocity increments as reported in [5].

As discussed in the papers cited in [5], the class of NIG distributions and processes have been found to provide accurate modelling of a great variety of empirical findings in the physical sciences and in financial econometrics.

As a second infinitely divisible distribution we need the inverse Gaussian distribution (IG). This distribution will be used to model the intermittency of the velocity field. The inverse Gaussian law, with parameters δ and ψ, is the distribution on the positive real axis $\mathbf{R}_+$ having probability density function

$$p(x; \delta, \psi) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\psi} x^{-3/2} \exp\{-(\delta^2 x^{-1} + \psi^2 x)/2\} \tag{3.19}$$

where the parameters δ and ψ satisfy $\delta > 0$ and $\psi \geq 0$.

4 A temporal model for the turbulent velocity field

Section 2 introduced the modelling framework for the turbulent velocity field in its full generality. Here, we focus on two specific properties of the turbulent velocity field, namely the evolution of the pdf of velocity increments across scales, and statistics related to K62. In order to keep the mathematics as simple as possible, we restrict the discussion of the model to the one-dimensional, purely temporal set-up (2.4). The restriction to temporal dynamics fully covers the standard experimental situation where only a time series of one component of the velocity at a fixed spatial position is accessible.

We also neglect the skewness of the velocity field, setting $\beta = 0$ in (2.4) for mathematical convenience. The skewness of the velocity field is not essential for the evolution of the pdf of velocity increments from heavy tails at small scales to a Gaussian shape at large scales. We also expect that neglecting the skewness of the velocity field does not alter the basic statistical properties of the Kolmogorov variable (1.1), in particular its conditional distributions. A more detailed discussion of the influence of the skewness term will be given elsewhere.

4.1 Evolution of the pdf of velocity increments

We discuss the pdf of velocity increments $u_t - u_0$, where $t > 0$, in terms of cumulants. In our non-skewed set-up (2.4) with $\beta = 0$, the third order cumulant is zero for all scales t. Therefore, the fourth order cumulant is the first order that distinguishes between a Gaussian shape for the large scales and heavy tails for small scales. Without specifying the function g and the local energy dissipation ε_t in detail, the large scale limit of $u_t - u_0$ approaches a Gaussian shape. In addition, we are able to show analytically, for specific choices of g and ε_t, that the small scale limit has pronounced heavy tails.

We shall denote the m-th order cumulant of an arbitrary random variable u by $c_m(u)$ and write the cumulant function of u as

$$\mathrm{C}\{\zeta \ddagger u\} = \log \mathrm{E}\left\{e^{i\zeta u}\right\}. \tag{4.1}$$

Furthermore, for any positive random variable X we define the *kumulant function* $\bar{\mathrm{K}}$ of X by

$$\bar{\mathrm{K}}\{\theta \ddagger X\} = \log \mathrm{E}\left\{e^{-\theta X}\right\}. \tag{4.2}$$

To get some insight into the statistical properties of the stationary increments $u_t - u_0$, we first calculate the cumulant function. We have

$$u_t - u_0 = \int_{-\infty}^{t} \left(g(t-s) - \mathbf{1}_{(-\infty,0]}(s) g(-s)\right) \sqrt{\varepsilon_s}\, \mathrm{d}W_s, \tag{4.3}$$

where $\mathbf{1}_{[a,b]}$ denotes the indicator function on $[a, b]$. Since, conditionally on ε, the process u is Gaussian, we get for the cumulant function of $u_t - u_0$ the form

$$\mathrm{C}\{\zeta \ddagger u_t - u_0\} = \bar{\mathrm{K}}\{\tfrac{1}{2}\zeta^2 \ddagger Q(t)\} \tag{4.4}$$

where

$$Q(t) = c_2\left(u_t - u_0 | \varepsilon\right) = \int_{-\infty}^{t} \left(g(t-s) - \mathbf{1}_{(-\infty,0]}(s) g(-s)\right)^2 \varepsilon_s \, \mathrm{d}s \tag{4.5}$$

is the conditional variance of $u_t - u_0$ given ε.

Differentiating the cumulant function (4.4) gives

$$c_2(u_t - u_0) = \mathrm{E}\left\{Q(t)\right\} = c_1(\varepsilon_0) G(t) \tag{4.6}$$

where

$$G(t) = \int_{-\infty}^{t} \left(g(t-s) - \mathbf{1}_{(-\infty,0]}(s) g(-s)\right)^2 \, \mathrm{d}s. \tag{4.7}$$

Furthermore,

$$\tfrac{1}{3} c_4(u_t - u_0) = c_2\left(Q(t)\right) = c_2\left(\varepsilon_0\right) \langle g, \tau \rangle (t) \tag{4.8}$$

where

$$\langle g, \tau \rangle (t) = \int_{-\infty}^{t} \int_{-\infty}^{t} h(t,s) h(t,s') \tau(|s - s'|) \, \mathrm{d}s \, \mathrm{d}s', \tag{4.9}$$

$$h(t,s) = \left(g(t-s) - \mathbf{1}_{(-\infty,0]}(s) g(-s)\right)^2 \tag{4.10}$$

and where τ is the autocorrelation function of ε.

It follows that

$$\bar{c}_4(u_t - u_0) = \frac{c_4(u_t - u_0)}{c_2(u_t - u_0)^2} = 3 \frac{c_2(\varepsilon_0) \langle g, \tau \rangle (t)}{\left(c_1(\varepsilon_0) G(t)\right)^2}. \tag{4.11}$$

For the large scale limit we get from (4.7) and (4.9)

$$\lim_{t \to \infty} \langle g, \tau \rangle (t) = 4 \int_{0}^{\infty} \int_{0}^{\infty} g^2(s) g^2(s') \tau(|s - s'|) \, \mathrm{d}s \, \mathrm{d}s' \tag{4.12}$$

and

$$\lim_{t \to \infty} \bar{c}_4(u_t - u_0) = 3 \frac{c_2(\varepsilon_0)}{c_1(\varepsilon_0)^2} \frac{\int_0^\infty \int_0^\infty g^2(s) g^2(s') \tau(|s - s'|) \, \mathrm{d}s \, \mathrm{d}s'}{\left(\int_0^\infty g(s)^2 \, \mathrm{d}s\right)^2}. \tag{4.13}$$

We therefore have the upper bound (noting $\tau(t) \leq 1$)

$$\lim_{t \to \infty} \bar{c}_4(u_t - u_0) \leq 3 \frac{c_2(\varepsilon_0)}{c_1(\varepsilon_0)^2}. \tag{4.14}$$

Consequently, for small $c_2(\varepsilon_0)/c_1(\varepsilon_0)^2$ we may expect the law of $u_t - u_0$ to be close to Gaussian. In this case, if the law of $u_t - u_0$ has heavy tails for small to moderate t, our model (2.4) will show the evolution of the pdf of velocity increments from heavy tails at small scales to a Gaussian shape at large scales.

The result (4.14) is a rough upper bound for the large scale limit of $\bar{c}_4(u_t - u_0)$. For a more accurate statement, including the small scale limit $t \to 0$, one has to specify the function g and the local energy dissipation ε_t. A simple example is given by assuming the process ε_t to be of OU-IG-type (3.11)

$$\varepsilon_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} \, \mathrm{d}L_s \tag{4.15}$$

and setting

$$g(t) = e^{-\gamma t}. \tag{4.16}$$

The parameter λ controls the correlations of the local energy dissipation as follows from $\tau(t) = e^{-\lambda t}$. The parameter γ controls the correlations of the velocity field since $\bar{\tau}(t) = e^{-\gamma t}$, where $\bar{\tau}$ denotes the autocorrelation function of u. In this case we obtain after a straightforward calculation of $G(t)$ and $\langle g, \tau\rangle(t)$ in (4.11)

$$\lim_{t\to\infty} \overline{c}_4(u_t - u_0) = \frac{3c_2(L_1 - L_0)}{2c_1(L_1 - L_0)^2} \frac{\gamma\lambda}{2\gamma + \lambda}. \tag{4.17}$$

and

$$\lim_{t\to 0} \overline{c}_4(u_t - u_0) = \frac{3c_2(L_1 - L_0)}{2c_1(L_1 - L_0)^2}\lambda. \tag{4.18}$$

The heaviness of the tails of the pdf of velocity increments increases with increasing λ, i.e. with a faster decrease of correlations of the local energy dissipation. Qualitatively, the same behaviour is observed for turbulent flows where the heaviness of the tails of the pdf of velocity increments increases with increasing Reynolds number and with increasing intermittency exponent μ [15], defined as $\mathrm{E}\{\varepsilon_0\varepsilon_t\} \sim t^{-\mu}$. (Due to this power-law behaviour, the assumption of ε following an OU-IG process, for which $\mathrm{E}\{\varepsilon_0\varepsilon_t\} = c_2(L_1 - L_0)(2\lambda)^{-1}e^{-\lambda t} + c_1(L_1 - L_0)^2\lambda^{-2}$, is not a realistic approach for modelling the local energy dissipation. We come back to this point in the concluding Section.)

Combining the two limits (4.17) and (4.18) yields

$$\frac{\lim_{t\to 0} \overline{c}_4(u_t - u_0)}{\lim_{t\to\infty} \overline{c}_4(u_t - u_0)} = 2 + \frac{\lambda}{\gamma}. \tag{4.19}$$

It is the ratio of the exponents λ and γ that spans the range for the evolution of the density of velocity increments $u_t - u_0$. Depending on the choice of the parameters λ and γ, we can model arbitrary small and large scale limits of $\bar{c}_4(u_t - u_0)$.

4.2 Statistics of the Kolmogorov variable

We now turn to the discussion of the statistics of the Kolmogorov variable (1.1) within our stochastic framework (2.4) with $\beta = 0$. In particular, we show that the Kolmogorov variable V can be represented as the product of two independent variates, namely a standard normal random variable and a process that completely contains the dependence of V on the integrated energy dissipation. Based on this decomposition, some analytical results concerning the conditional pdf of V for the small and large scale limit can be derived.

Following the discussion in Section 3.1 we replace the integrated energy dissipation in (1.1) by the quadratic variation and define the stochastic analogue of the classical Kolmogorov variable as

$$V_t = \frac{u_t - u_0}{\left(\bar{u}\,[u]_t\right)^{1/3}}. \tag{4.20}$$

The introduction of the mean velocity $\bar{u}$ turns V_t into a non-dimensional stochastic process.

To reveal the basic statistical properties of the process V_t we note that (4.20) may be rewritten as

$$V_t = \frac{u_t - u_0}{Q(t)^{1/2}} \frac{Q(t)^{1/2}}{\left(\bar{u}\,[u]_t\right)^{1/3}} = UR_t \tag{4.21}$$

where

$$U = \frac{u_t - u_0}{Q(t)^{1/2}} \tag{4.22}$$

and

$$R_t = \frac{Q(t)^{1/2}}{\left(\bar{u}\,[u]_t\right)^{1/3}}. \tag{4.23}$$

The variable U is a standard normal random variable and independent of R_t. The dependence of V_t on $[u]_t$ - or, equivalently, ε_t - is thus completely contained in the process R_t.

To proceed further, we specify the function g to be of the form (4.16). We gain some insight into the properties of the process R_t for $t \to 0$ noting the decomposition of the conditional variance of velocity increments

$$Q(t) = \left(1 - e^{-\gamma t}\right)^2 \int_{-\infty}^{0} e^{2\gamma s} \varepsilon_s \,\mathrm{d}s + \left(\int_{0}^{t} e^{-2\gamma(t-s)} \varepsilon_s \,\mathrm{d}s - [u]_t \right) + [u]_t. \tag{4.24}$$

Focusing on the first term on the right hand side of (4.24) we get in leading order for $t \to 0$

$$\mathrm{E}\left\{ \left(1 - e^{-\gamma t}\right)^2 \int_{-\infty}^{0} e^{2\gamma s} \varepsilon_s \,\mathrm{d}s \right\} = c_1(\varepsilon_0)(2\gamma)^{-1} \left(1 - e^{-\gamma t}\right)^2 \sim \frac{c_1(\varepsilon_0)\gamma}{2} t^2. \tag{4.25}$$

For the second term in (4.24) we have, by (3.6)

$$\int_{0}^{t} e^{-2\gamma(t-s)} \varepsilon_s \,\mathrm{d}s - [u]_t = -\int_{0}^{t} \left(1 - e^{-2\gamma(t-s)}\right) \varepsilon_s \,\mathrm{d}s \tag{4.26}$$

and in the limit $t \to 0$, to leading order

$$\mathrm{E}\left\{ \int_{0}^{t} \left(1 - e^{-2\gamma(t-s)}\right) \varepsilon_s \,\mathrm{d}s \right\} = c_1(\varepsilon_0) \left(t - (2\gamma)^{-1} \left(1 - e^{-2\gamma t}\right) \right) \sim 2c_1(\varepsilon_0)\gamma t^2. \tag{4.27}$$

Since the first term in (4.24) is strictly positive and the second one is strictly negative we conclude that they are both predominantly of order t^2 for small t. Therefore, since the mean of $[u]_t$ is linear in t, we conclude that the quadratic variation dominates in (4.24) for small t and consequently

$$V_t \sim U[u]_t^{1/6}. \tag{4.28}$$

The small scale dependence of V_t on the integrated energy dissipation is in conformity with the corresponding result for the turbulent velocity field that follows from kinematic considerations at scales smaller than dissipation scales.

We can also draw a conclusion for the large scale limit $t \to \infty$. If we assume the intermittency process ε_t to be ergodic, we get $[u]_t \sim tc_1(\varepsilon_0)$. Furthermore, since

$$\mathrm{E}\left\{Q(t)\right\} = c_1(\varepsilon_0)\gamma^{-1} \left(1 - e^{-\gamma t}\right) \tag{4.29}$$

we get for $t \to \infty$

$$\mathrm{E}\{|V_t|\} \sim t^{-1/3}. \tag{4.30}$$

The behaviour $\mathrm{E}\{|V_t|\} \propto t^{-0.4}$ is reported for high Reynolds number atmospheric data in [7]. In their analysis the range of t where the exponent 0.4 holds is small. For larger t an exponent of 1/3 seems to better fit their data.

The small scale limit (4.28) and the large scale limit (4.30) are both in accordance with the corresponding experimental results. For the time being we are not able to analytically treat the case of moderate t which is the most interesting in view of K62. For these scales we have to refer to the simulations in the next Section.

5 Simulation

The analytical results in the last Section mainly concern the statistics of velocity increments and the statistics of the Kolmogorov variable for the small and large scale limits. The corresponding results for moderate scales are only accessible through numerical simulation.

For the simulations we use a discretised version of the non-skewed model (2.4) with $\beta = 0$. For the weight function we set

$$g(t) = e^{-\gamma t}\mathbf{1}_{[0,T]}. \tag{5.1}$$

where γ and T are positive numbers. The introduction of T associates a finite decorrelation time to the velocity field u. We further specify the process ε as a truncated OU-IG process, i.e.

$$\varepsilon_t = \int_{t-\bar{T}}^{t} e^{-\lambda(t-s)}\, \mathrm{d}L_s, \tag{5.2}$$

where L is an IG(δ, ψ)-Lévy process. The assumption (5.2) coincides for $\bar{T} \to \infty$ with the definition of an ordinary OU-IG process.

The values for the parameters of the simulation of u are $\bar{u} = 1$, $\lambda = 1$, $\bar{T} = 100$, $\gamma = 0.1$, $\delta = 1$, $\psi = 1$ and $T = 40$ and we discretised all stochastic integrals with a finite step size $\Delta t = 1$. Hence we simulated, for $t = 0, 1, \ldots, N$ with $N = 2 \cdot 10^6$,

$$\varepsilon_t = \sum_{j=t-\bar{T}}^{t-1} e^{-\lambda(t-j)}\left(L_{j+1} - L_j\right) \tag{5.3}$$

and

$$u_t = \sum_{j=t-T}^{t-1} e^{-\gamma(t-j)}\sqrt{\varepsilon_j}\left(W_{j+1} - W_j\right). \tag{5.4}$$

For the quadratic variation we used the approximation

$$[u]_t = \sum_{j=0}^{t-1} \left(u_{j+1} - u_j\right)^2 \tag{5.5}$$

which coincides with the usual definition of the energy dissipation for the temporal resolution $\Delta t = 1$.

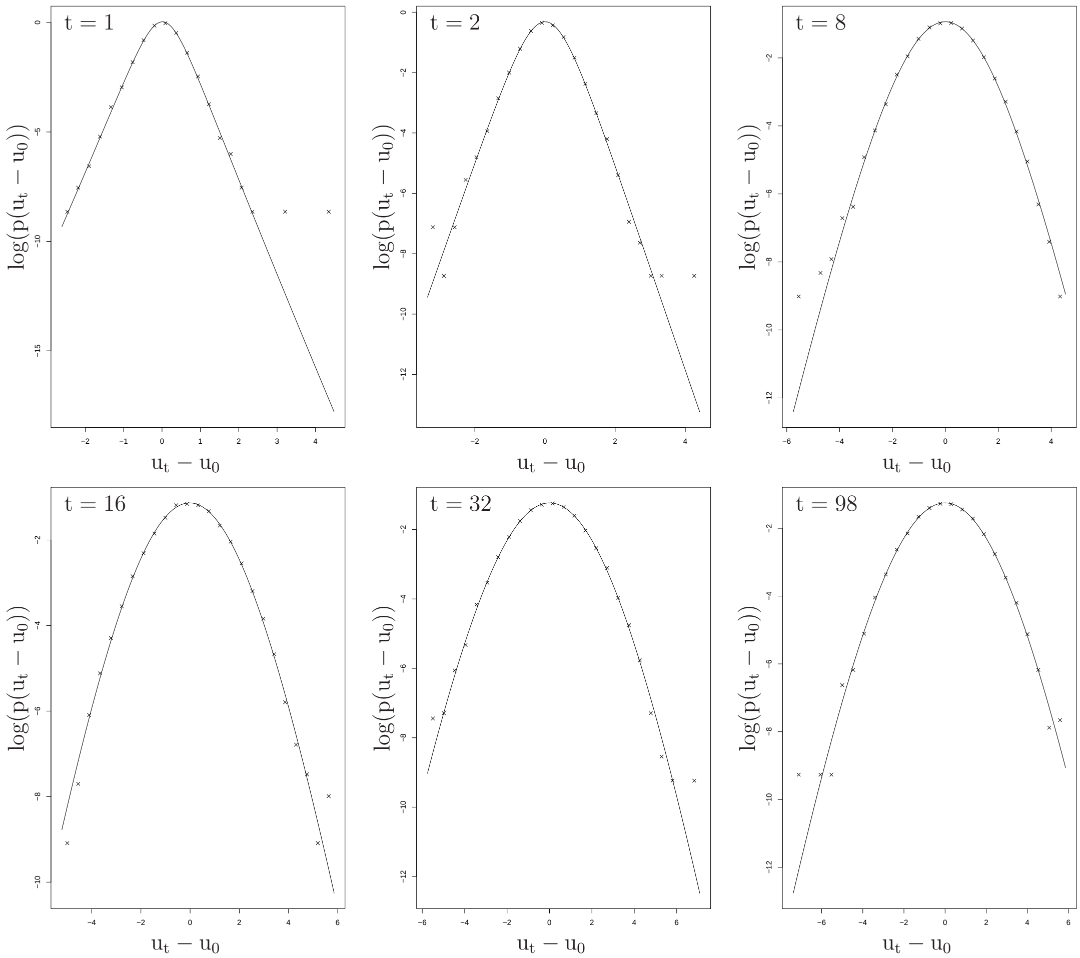

Figure 2: Logarithm of the probability densities of the simulated increments $u_t - u_0$ with $t = 1, 2, 8, 16, 32, 98$. The solid lines denote the approximation within the class of NIG distributions (fitting by maximum likelihood).

Figure 2 shows the evolution of the probability densities of the simulated increments $u_t - u_0$ for various scales t. We clearly observe heavy tails for the small scales and an approximately Gaussian shape for the large scales. The solid lines denote the approximation of the densities within the class of NIG-distributions. The densities of $u_t - u_0$ qualitatively display the empirical findings about the evolution across scales of turbulent velocity increments reported in [5].

We further substantiate the scale dependence in Figure 3 which shows the NIG shapes for the densities as displayed in Figure 2. The parameter χ is zero for all scales reflecting the symmetry of the densities. The steepness parameter ξ decreases with increasing scale. Noting the expression $\xi = (1 + 3/\overline{c}_4)^{-1/2}$ for symmetric NIG-distributions, Figure 3 visualizes the evolution from heavy tails (large ξ) to an approximately Gaussian shape (in the limit $(\xi, \chi) \to (0, 0)$). These findings are very similar to the corresponding results for the turbulent velocity field as reported in [5] (see also Figure 1).

We now turn to the investigation of the Kolmogorov variable V_t. Figure 4 shows the unconditional densities of V_t. We first note that the unconditional densities at moderate and large scales are approximately Gaussian, in accordance with the

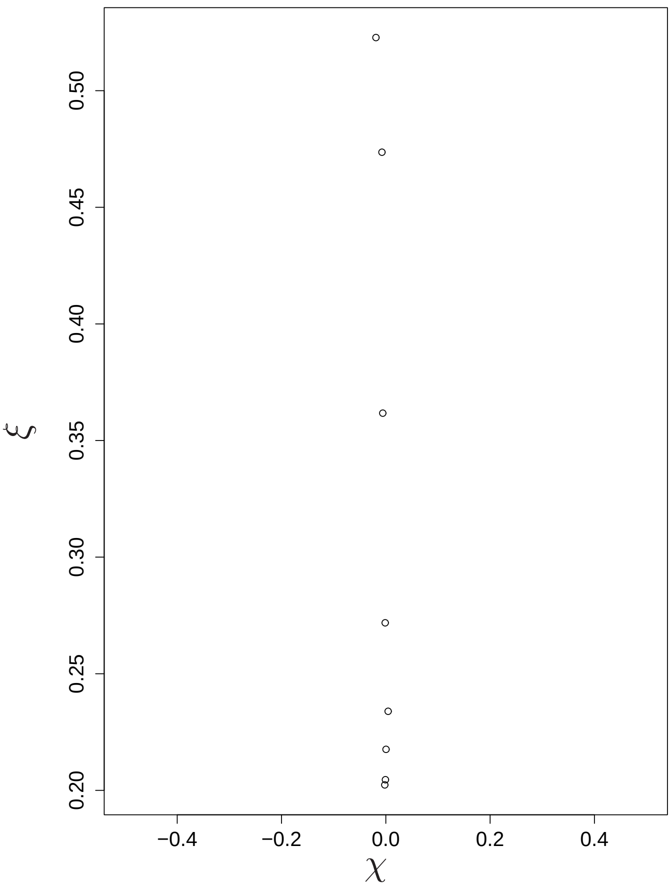

Figure 3: NIG shapes for the densities of the simulated increments $u_t - u_0$, with $t = 1, 2, 4, 8, 16, 32, 64, 98$ (from top to bottom).

findings in [10, 7, 8, 9]. For not too large scales, the densities collapse for small amplitudes while for large amplitudes, the densities are scale dependent. For the very small scales, a bimodal distribution is observed. The bimodality is related to the heavy tails of the pdf of velocity increments at small scales [16].

Comparable results are reported in [8]. The authors discuss the evolution of the second order empirical moments of V_t across scales, showing an increase with increasing scale, reaching a plateau at intermediate scales and finally a decrease with further increasing scale. The same behaviour holds for the simulation of our model. Figure 5 shows the second order moments of V_t as a function of scale t.

Figures 6–8 show the conditional densities $p(V_t|[u]_t^{1/3})$ for various scales t and various values of $[u]_t^{1/3}$. For small t, the conditional densities strongly depend on $[u]_t^{1/3}$. With decreasing values of $[u]_t^{1/3}$, the dependence gets smaller and for large enough t ($t \approx 16$ in our simulation), the conditional densities do not depend on $[u]_t^{1/3}$. This independence also holds for the larger scales $16 \leq t < \bar{T}$ (not shown here). These findings agree well with results reported for the turbulent velocity field [10, 7, 8] and reveal the gist of K62.

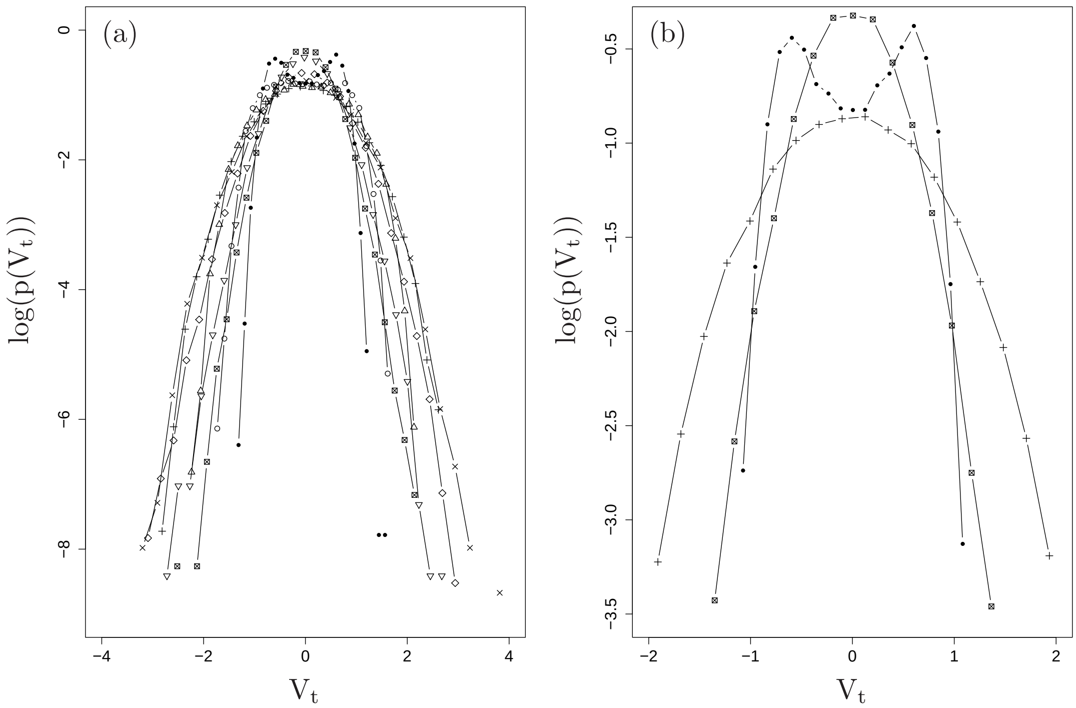

Figure 4: (a) Logarithm of the simulated unconditional densities $p(V_t)$ of the Kolmogorov variable V_t for $t = 1$ (●), $t = 2$ (○), $t = 4$ (△), $t = 8$ (+), $t = 16$ (×), $t = 32$ (◇), $t = 64$ (▽) and $t = 98$ (⊠). (b) Amplification of the relevant part of (a) for $t = 1$ (●), $t = 8$ (+) and $t = 98$ (⊠).

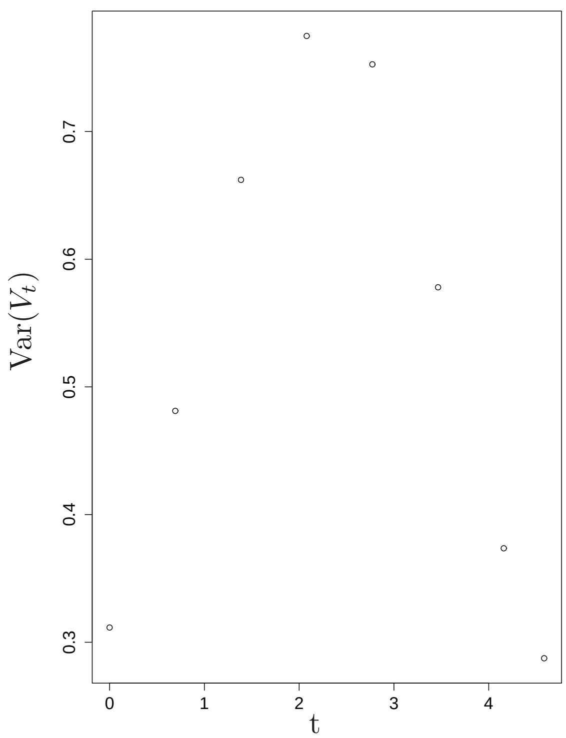

Figure 5: Estimated variance $\mathrm{Var}(V_t)$ of the simulated Kolmogorov variable V_t.

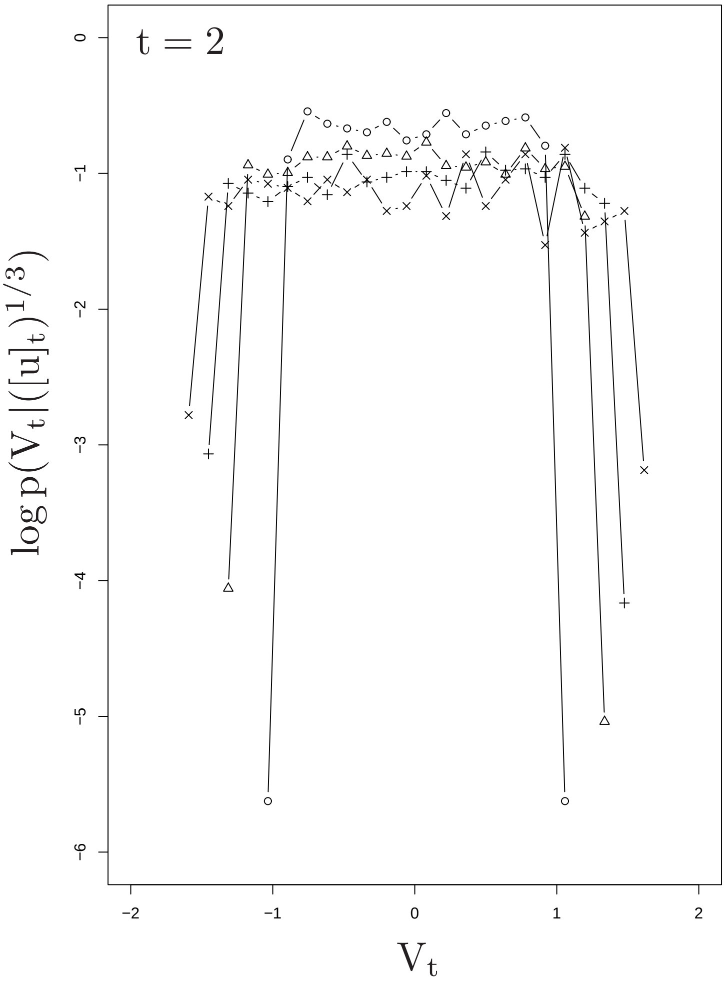

Figure 6: Logarithm of the conditional densities $p(V_t|[u]_t^{1/3})$ of the simulated Kolmogorov variable V_t for $t = 2$ with $[u]_t^{1/3} = 0.45$ ($\circ$), $[u]_t^{1/3} = 0.77$ ($\triangle$), $[u]_t^{1/3} = 0.99$ ($+$) and $[u]_t^{1/3} = 1.20$ ($\times$).

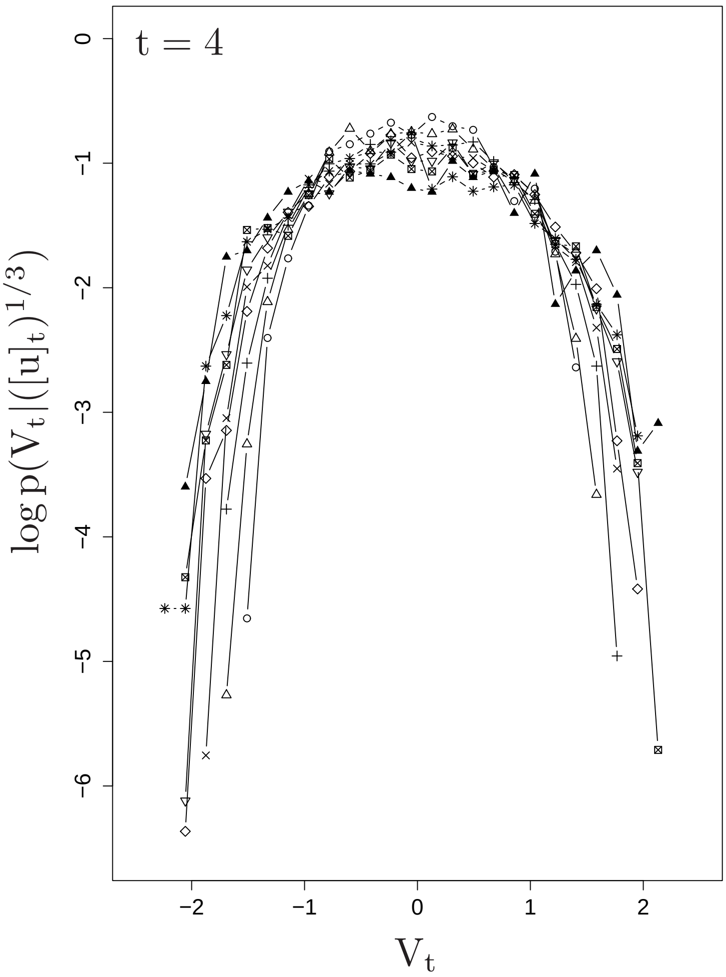

Figure 7: Logarithm of the conditional density $p(V_t|[u]_t^{1/3})$ of the simulated Kolmogorov variable V_t for $t = 4$ with $[u]_t^{1/3} = 0.55$ (○), $[u]_t^{1/3} = 0.66$ (△), $[u]_t^{1/3} = 0.76$ (+), $[u]_t^{1/3} = 0.86$ (×), $[u]_t^{1/3} = 0.96$ (◇), $[u]_t^{1/3} = 1.07$ (▽), $[u]_t^{1/3} = 1.17$ (⊠), $[u]_t^{1/3} = 1.27$ (∗) and $[u]_t^{1/3} = 1.38$ (▲).

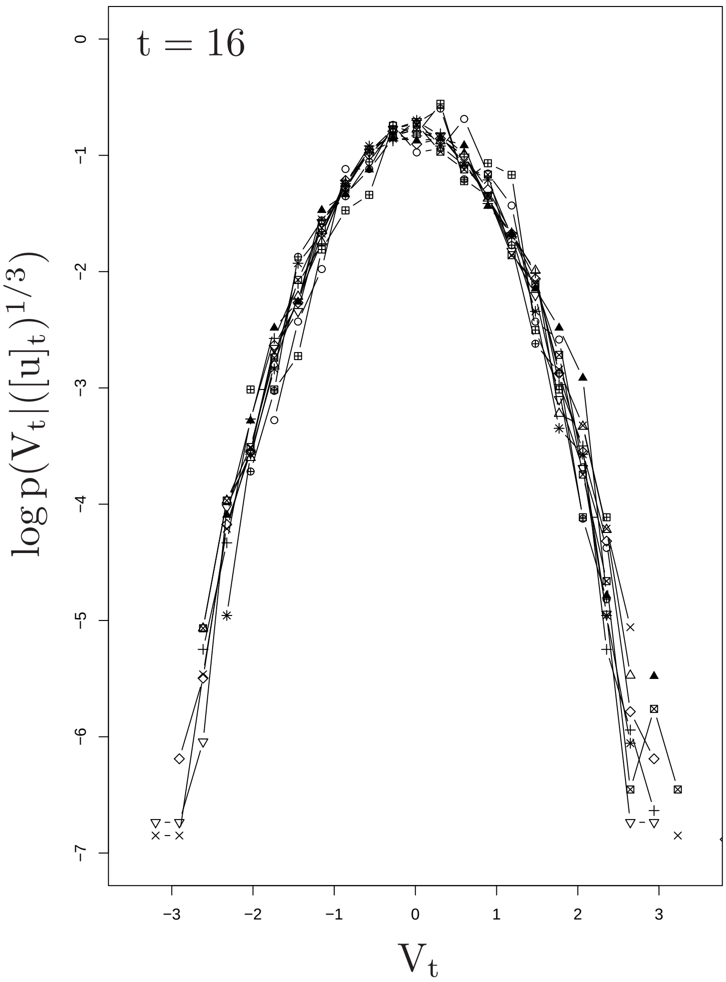

Figure 8: Logarithm of the conditional density $p(V_t|[u]_t^{1/3})$ of the simulated Kolmogorov variable V_t for $t = 16$ with $[u]_t^{1/3} = 0.98$ (○), $[u]_t^{1/3} = 1.16$ (△), $[u]_t^{1/3} = 1.26$ (+), $[u]_t^{1/3} = 1.35$ (×), $[u]_t^{1/3} = 1.44$ (◇), $[u]_t^{1/3} = 1.53$ (▽), $[u]_t^{1/3} = 1.63$ (⊠), $[u]_t^{1/3} = 1.72$ (∗), $[u]_t^{1/3} = 1.81$ (▲), $[u]_t^{1/3} = 1.9$ (⊕) and $[u]_t^{1/3} = 2.0$ (⊞).

6 Conclusion

Summarizing the main results, we state that our proposed semimartingale framework allows modelling in conformity with the observed evolution of the pdf of velocity increments across scales and with the experimental verification of Kolmogorov's refined similarity hypotheses. The relation between general stochastic processes and K62 is also discussed in [10]. The authors propose fractional Brownian motion (fBm) as a stochastic process that diplays the main properties of K62. However, the use of fBm there is accompanied with a mathematical inconsistency, connected to the fact that for fBm (except Brownian motion itself) the quadratic variation is either identically 0 or ∞. (This, incidentally, implies that fBm is not a semimartingale.) Furthermore, fBm is a non-stationary Gaussian process and does not capture the heavy tails for the pdf of velocity increments at small scales. Thus, to our knowledge, the model (2.4) seems to be the first approach to the turbulent velocity field that comprises both, the evolution of the density of velocity increments across scales and the statistics of the Kolmogorov variable V.

For the simulation we restricted to a very simple form of the intermittency term ε_t as an OU-IG process which is easy to implement but not realistic for the turbulent energy dissipation field. A realistic approach would be to use a more advanced model for the energy dissipation. In particular we think of Lévy based models that allow to explicitely control the correlation structure of the energy dissipation field [17, 18, 19, 20]. Controlling the correlation structure of the energy dissipation seems to enable to model the evolution of the density of velocity increments in a way that displays the detailed behaviour reported in [5].

The fact that using an OU-IG process for ε_t works so surprisingly well indicates that models of the form (2.4) are the appropriate framework in the turbulence context. In particular, the calculations in Section 3.1 and Section 4 show that main parts of the turbulence statistics can be reproduced without specifying the intermittency terms ε_t and weight functions g. In this respect, only a more detailed modelling of the correlation structure of the intermittency term can narrow these degrees of freedom.

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