

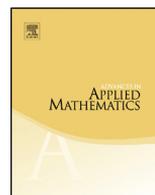


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Friezes, weak friezes, and T-paths

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ABSTRACT

Frieze patterns form a nexus between algebra, combinatorics, and geometry. T -paths with respect to triangulations of surfaces have been used to obtain expansion formulae for cluster variables.

This paper will introduce the concepts of weak friezes and T -paths with respect to dissections of polygons. Our main result is that weak friezes are characterised by satisfying an expansion formula which we call the T -path formula.

We also show that weak friezes can be glued together, and that the resulting weak frieze is a frieze if and only if so was each of the weak friezes being glued.

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0. Introduction

This paper considers dissections of polygons as central objects and introduces weak friezes associated with them. Our main result shows that they may be characterised by an expansion formula which we call the T -path formula, see Definitions 0.1, 0.2 and Theorem A.

Weak friezes are strongly related to the generalised frieze patterns defined in [2, sec. 5] and to the frieze patterns of [4, sec. 1], which form a nexus between algebra, combinatorics, and geometry; see [11] for a recent survey. T -paths with respect to dissections of polygons are a generalisation of T -paths with respect to triangulations of polygons as defined in [14, sec. 1.2].

In preparation for the proof of Theorem A, we show in Theorem B that weak friezes can be glued together. There is also a notion of frieze, and Theorem C shows that when weak friezes are glued together, the result is a frieze if and only if each of the weak friezes being glued is a frieze.

Note that T -paths with respect to triangulations of polygons and general surfaces were used in [1, thm. 2.10], [7, thm. 3.8], [12, thm. 1.1], [14, thm. 1.2], [15, thm. 3.1], and [16, thm. 3.5] to obtain expansion formulae for cluster variables. This permitted the resolution of the positivity conjecture for cluster algebras arising from surfaces, stating that all Laurent polynomials for cluster variables have positive coefficients. Cluster algebras and the positivity conjecture were introduced in [5, def. 2.3] and [5, sec. 1], and the conjecture was later resolved in general, see [6, cor. 0.4] and [10, thm. 1.1].

(0.i). *Friezes and weak friezes*

Throughout the paper, a polygon P is a finite set V of three or more vertices with a cyclic order. We often imagine it realised as a convex n -angle in the Euclidean plane. If $\alpha \neq \beta$ are vertices of P , then there is a diagonal $\{\alpha, \beta\}$, and the set of diagonals of P is denoted $\text{diag}(P)$. A dissection D of P is a set of pairwise non-crossing diagonals between non-neighbouring vertices, and the empty dissection $D = \emptyset$ is allowed.

Throughout the paper, $(K, +, \cdot)$ is a fixed semifield, see Definition 1.1. An important feature is that K has no “subtraction” in the sense of an inverse operation of $+$. Examples are $(\mathbb{R}_{>0}, +, \cdot)$, the positive real numbers with addition and multiplication, and $(\mathbb{Z}, \max, +)$, the so-called tropical semifield. In the latter case, the two binary operations are indeed given by maximum and addition in \mathbb{Z} .

Definition 0.1 (*Friezes and weak friezes*). Let P be a polygon, $f : \text{diag}(P) \rightarrow K$ a map. Write $f(\alpha, \beta) := f(\{\alpha, \beta\})$.

(i) f is called a *frieze* if it satisfies the *Ptolemy relation*

$$f(\alpha, \beta)f(\gamma, \delta) = f(\alpha, \gamma)f(\beta, \delta) + f(\alpha, \delta)f(\beta, \gamma) \quad (0.1)$$

when $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are crossing diagonals.

- (ii) If D is a dissection of P , then f is called a *weak frieze with respect to D* if it satisfies the Ptolemy relation (0.1) when $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are crossing diagonals satisfying $\{\gamma, \delta\} \in D$.

The relation between (weak) friezes and (generalised) frieze patterns will be explained in Section (0.iv). Note that friezes with values in the tropical semifield $(\mathbb{Z}, \max, +)$ were defined in [8, sec. 2.1]; they are known as tropical friezes.

For a fixed polygon P , any map $\text{diag}(P) \rightarrow K$ is a weak frieze with respect to the empty dissection. For example, a *trivial weak frieze* is a map on diagonals with constant value 1_K . Gluing together three trivial weak friezes with values in $\mathbb{R}_{>0}$, two of them on 4-gons, the third on a pentagon, gives the weak frieze f on a 9-gon shown in Fig. 1. Weak friezes can be glued by Theorem B, and the edges along which we glue become internal diagonals which are adjoined to the dissection D .

(0.ii). T-paths

The notion of T -path with respect to a triangulation of a polygon was defined in [14, sec. 1.2], and we generalise it to T -path with respect to a dissection as follows.

Definition 0.2 (*T-paths and the T-path formula*). Let P be a polygon, D a dissection of P . If $\pi_1 \neq \pi_p$ are vertices, then a *T-path from π_1 to π_p with respect to D* is an ordered tuple $\pi = (\pi_1, \dots, \pi_\ell)$ of vertices satisfying the following.

- (i) $\{\pi_1, \pi_2\}, \{\pi_2, \pi_3\}, \dots, \{\pi_{\ell-1}, \pi_\ell\}$ are pairwise different diagonals and $\pi_\ell = \pi_p$.
- (ii) No diagonal $\{\pi_i, \pi_{i+1}\}$ crosses a diagonal in D .
- (iii) The diagonals $\{\pi_{2j}, \pi_{2j+1}\}$ are in D , and cross the diagonal $\{\pi_1, \pi_p\}$ at pairwise different points which progress monotonically in the direction from π_1 to π_p .

The set of T -paths from π_1 to π_p with respect to D is denoted $\mathcal{P}(D, \pi_1, \pi_p)$. A map $f : \text{diag}(P) \rightarrow K$ is said to *satisfy the T-path formula with respect to D* if

$$f(\alpha, \beta) = \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)} f(\pi) \tag{0.2}$$

for all vertices $\alpha \neq \beta$ of P , where

$$f(\pi) := \frac{\prod_{i \text{ odd}} f(\pi_i, \pi_{i+1})}{\prod_{j \text{ even}} f(\pi_j, \pi_{j+1})} \tag{0.3}$$

for a T -path π .

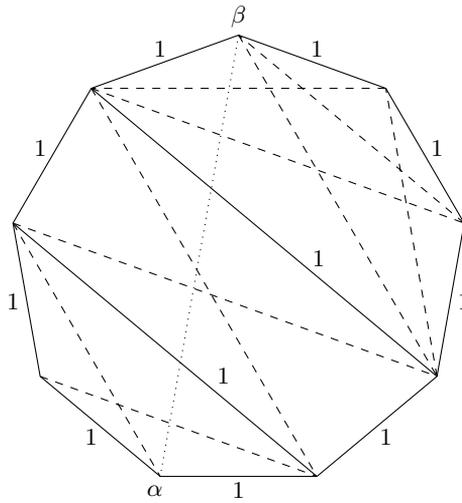


Fig. 1. A weak frieze f with respect to the dissection D consisting of the two unbroken diagonals. It has values in $\mathbb{R}_{>0}$ and is equal to 1 on each edge, unbroken diagonal, and dashed diagonal (some of these values are shown). It has been obtained by gluing together trivial weak friezes on the two squares and the pentagon created by D , see Theorem B. The value $f(\alpha, \beta)$ on the dotted diagonal $\{\alpha, \beta\}$ is 4, see Section (0.ii).

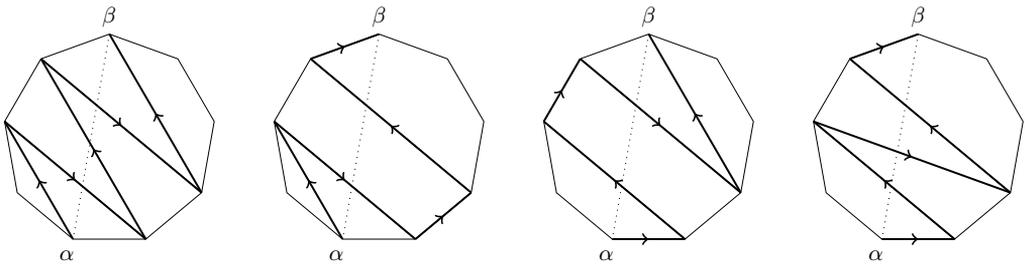


Fig. 2. In Fig. 1, these are the T -paths from α to β with respect to the dissection D .

For example, in the situation of Fig. 1, the T -paths from α to β with respect to D are shown in Fig. 2. Note that T -paths from π_1 to π_p do not necessarily have the same length, see Fig. 3 for an example.

Our first main result is the following, which is inspired by the expansion formula for cluster algebras of type A given in [14, thm. 1.2], and has that result as a special case.

Theorem A (Weak friezes and T -paths). *Let P be a polygon with a dissection D . A map $f : \text{diag}(P) \rightarrow K$ is a weak frieze with respect to D if and only if it satisfies the T -path formula with respect to D , see Definition 0.2.*

For example, the weak frieze f in Fig. 1 has value 1 on each of the four T -paths in Fig. 2, so $f(\alpha, \beta) = 1 + 1 + 1 + 1 = 4$ by the T -path formula.

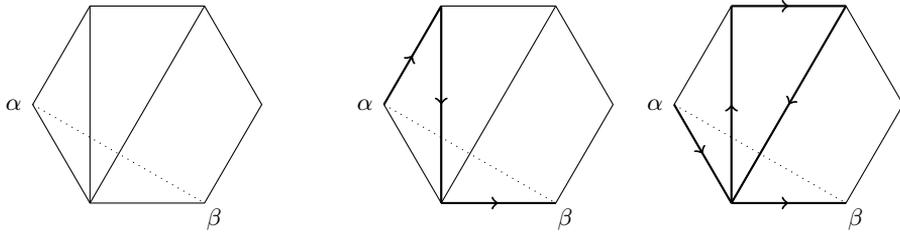


Fig. 3. A dissection of a hexagon (left) and two T -paths from α to β that are of different length (middle and right).

(0.iii). *Gluing*

The proof of Theorem A relies on our second main result.

Theorem B (*Gluing weak friezes*). Let P be a polygon, let d_1, \dots, d_m be pairwise non-crossing internal diagonals dividing P into subpolygons P_1, \dots, P_{m+1} , and let D_i be a dissection of P_i for each i . The disjoint union $D = \{d_1, \dots, d_m\} \cup D_1 \cup \dots \cup D_{m+1}$ is a dissection of P .

Let $f_i : \text{diag}(P_i) \rightarrow K$ be a weak frieze with respect to D_i for each i . Assume that if P_i and P_j share a diagonal d , then $f_i(d) = f_j(d)$ (note that such a d must be an edge of both P_i and P_j).

Then there is a unique weak frieze $f : \text{diag}(P) \rightarrow K$ with respect to D such that $f|_{\text{diag}(P_i)} = f_i$ for each i .

Our third main result concerns friezes.

Theorem C (*Gluing friezes*). In Theorem B, the weak frieze f is a frieze if and only if each f_i is a frieze.

(0.iv). *Frieze patterns*

We end the introduction by explaining how (weak) friezes are related to (generalised) frieze patterns.

Let $n \geq 3$ be an integer and consider the coordinate system on a horizontal strip in Fig. 4. The coordinates are elements of \mathbb{Z}/n , and taking a step right adds 1 to each coordinate. The first coordinate is constant when ascending diagonally, then second when descending diagonally. A (weak) frieze f on an n -gon P is turned into a pattern on the strip by labelling the vertices of P by \mathbb{Z}/n and placing the value $f(i, j)$ at position (i, j) in the coordinate system. Note that since $f(i, j) = f(j, i)$, the pattern has a glide symmetry.

First, it is classic that if f is a frieze with values in $\mathbb{R}_{>0}$ satisfying $f(i, i + 1) = 1$ for each $i \in \mathbb{Z}/n$, then the corresponding pattern is a frieze pattern as defined in [4, sec.

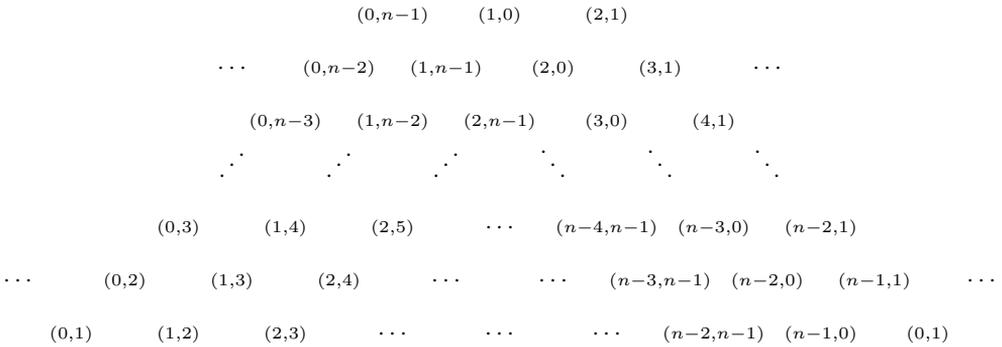


Fig. 4. Coordinate system on a horizontal strip used for turning (weak) friezes on the n -gon into (generalised) frieze patterns.

$$\begin{array}{ccc}
 & f(i, j + 1) & \\
 f(i, j) & & f(i + 1, j + 1) \\
 & f(i + 1, j) &
 \end{array}$$

Fig. 5. If f is a frieze with values in $\mathbb{R}_{>0}$ satisfying $f(i, i + 1) = 1$, then f satisfies $f(i, j)f(i + 1, j + 1) - f(i, j + 1)f(i + 1, j) = 1$, the unimodular equation. The four values of f in the equation constitute a “diamond” in the corresponding frieze pattern as shown.

1]. Namely, if $i, j \in \mathbb{Z}/n$ are non-neighbouring vertices, then $\{i, j\}$ and $\{i + 1, j + 1\}$ are crossing diagonals and Equation (0.1) implies

$$f(i, j)f(i + 1, j + 1) - f(i, j + 1)f(i + 1, j) = 1.$$

This is the unimodular equation from [4, sec. 1], see Fig. 5. Every frieze pattern in the sense of [4, sec. 1] arises like this by [9, thm. 2.5].

Secondly, if f is any frieze with values in $\mathbb{R}_{>0}$, then the corresponding pattern is a frieze pattern with coefficients as defined in [13, sec. 3], see [3, def. 2.1]. Equation (0.1) becomes the equation in [3, def. 2.1(iii)]. Every frieze pattern with coefficients which has positive entries arises like this by [3, thm. 3.3].

Thirdly, if f is a weak frieze with values in $\mathbb{N} = \{1, 2, \dots\}$ satisfying $f(i, i + 1) = 1$ for each $i \in \mathbb{Z}/n$, then the corresponding pattern is a generalised frieze pattern as defined in [2, sec. 5]. For instance, the weak frieze f in Fig. 1 gives the generalised frieze pattern in Fig. 6. The entries in the first and last row are 1 because they are the values $f(i, i + 1)$. The entry $f(\alpha, \beta) = 4$ is shown in red.

The paper is organised as follows: Section 1 collects some definitions, Section 2 proves Theorems B and C, Section 3 establishes some properties of T -paths, and Section 4 proves Theorem A.

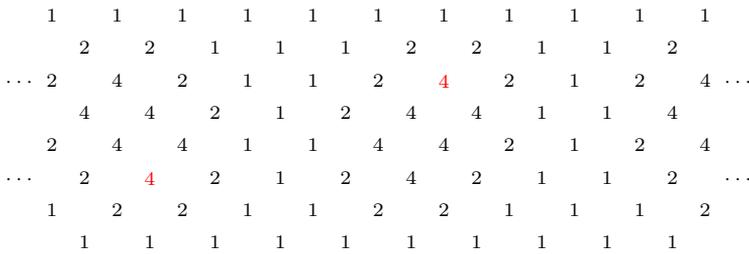


Fig. 6. The generalised frieze pattern corresponding to the weak frieze in Fig. 1. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

1. Semifields and polygons

Recall that throughout the paper, $(K, +, \cdot)$ is a fixed semifield in the following sense.

Definition 1.1. A *semifield* is a triple $(K, +, \cdot)$, where K is a set and $+$ and \cdot are binary operations, satisfying the following.

- The operation $+$ is associative and commutative.
- The operation \cdot turns K into a commutative group. The unit element is denoted 1_K , the inverse of x by x^{-1} .
- The operation \cdot distributes over $+$.

We abbreviate $(K, +, \cdot)$ to K and write $xy := x \cdot y$ and $\frac{x}{y} := x \cdot y^{-1}$.

Definition 1.2. A *polygon* P is a finite set V of three or more *vertices* with a cyclic order.

The predecessor and successor of $\alpha \in V$ are denoted α^- and α^+ .

A *diagonal* of P is a two-element subset of V . The set of diagonals is denoted $\text{diag}(P)$.

The diagonal $\{\alpha, \beta\}$ has *end points* α and β . Diagonals of the form $\{\alpha, \alpha^+\}$ are called *edges*. The remaining diagonals are called *internal diagonals*.

The diagonals $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ *cross* if $\alpha, \beta, \gamma, \delta$ are four distinct vertices which appear in the order $\alpha, \gamma, \beta, \delta$ or $\alpha, \delta, \beta, \gamma$ in V .

A *subpolygon* is a subset of V of three or more vertices equipped with the induced cyclic order.

A *dissection* of P is a set D of pairwise non-crossing internal diagonals. If D has m elements, then it divides P into $m + 1$ subpolygons. Note that D can be empty.

If V has n elements, then P is called an *n-gon*, and we often imagine it realised as a convex n -angle in the Euclidean plane. This means that we can make sense of Definition 0.2(iii), which says that the diagonals $\{\pi_{2j}, \pi_{2j+1}\}$ cross the diagonal $\{\pi_1, \pi_p\}$ “at pairwise different points which progress monotonically in the direction from π_1 to π_p ”. In our figures of n -angles, the positive direction of cyclically ordered vertices is anticlockwise.

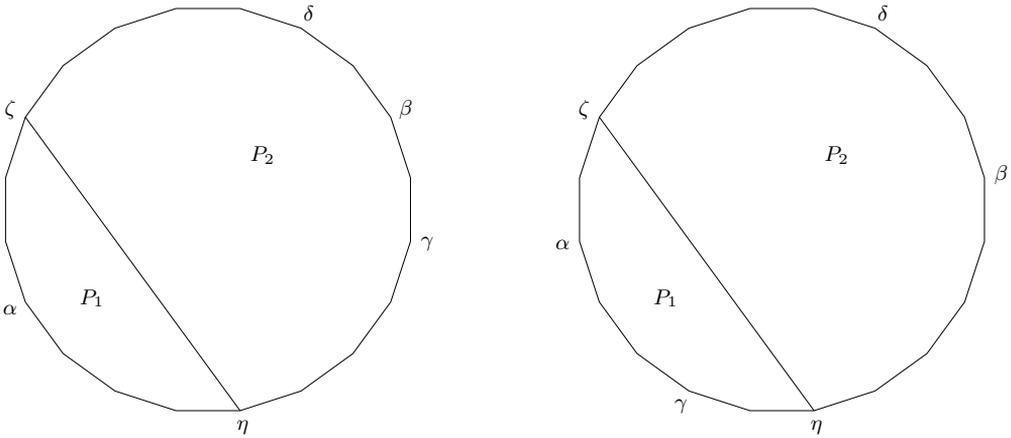


Fig. 7. The configurations in the proofs of Theorems B and C.

Definition 1.3 (*T-paths II*). We extend the notation of Definition 0.2 as follows: If ρ_1, \dots, ρ_t are vertices of P , then we set

$$\mathcal{P}(D, \pi_1, \pi_p)^{(\rho_1, \dots, \rho_t)} = \{ \pi \in \mathcal{P}(D, \pi_1, \pi_p) \mid \pi_1 = \rho_1, \dots, \pi_t = \rho_t \}.$$

The notation is further extended by permitting the sign \neq , for instance,

$$\mathcal{P}(D, \pi_1, \pi_p)^{(\rho_1, \neq \rho_2)} = \{ \pi \in \mathcal{P}(D, \pi_1, \pi_p) \mid \pi_1 = \rho_1, \pi_2 \neq \rho_2 \}.$$

2. Proofs of Theorems B and C

In the proofs of Theorems B and C we will assume $m = 1$, that is, there is one diagonal d_1 dividing P into subpolygons P_1 and P_2 . This implies the general case by an easy induction. Note that the proofs do not use subtraction, which is unavailable in the semifield K .

Let P have the set of vertices V , denote d_1 by d , and pick $\zeta, \eta \in V$ such that $d = \{\zeta, \eta\}$ while the sets of vertices of P_1 and P_2 are $V_1 = \{ \varepsilon \in V \mid \zeta \leq \varepsilon \leq \eta \}$ and $V_2 = \{ \varepsilon \in V \mid \eta \leq \varepsilon \leq \zeta \}$, see Fig. 7. Set

$$U_1 = \{ \varepsilon \in V \mid \zeta < \varepsilon < \eta \} , \quad U_2 = \{ \varepsilon \in V \mid \eta < \varepsilon < \zeta \}.$$

There are disjoint unions $D = D_1 \cup \{d\} \cup D_2$ and

$$V = U_1 \cup \{\zeta, \eta\} \cup U_2.$$

Proof of Theorem B. Since $d = \{\zeta, \eta\}$ is in D , by Definition 0.1(ii) a weak frieze f with respect to D must satisfy

$$f(\zeta, \eta)f(\alpha, \beta) = f(\zeta, \alpha)f(\eta, \beta) + f(\zeta, \beta)f(\eta, \alpha)$$

when $\{\zeta, \eta\}$ and $\{\alpha, \beta\}$ are crossing diagonals of P . If f also satisfies $f \Big|_{\text{diag}(P_i)} = f_i$ for $i \in \{1, 2\}$ and $x := f_1(\zeta, \eta) = f_2(\zeta, \eta)$, then $f(\alpha, \beta)$ must be given by the entries of the following table, according to whether α and β are in U_1, U_2 , or $\{\zeta, \eta\}$.

| $\alpha \backslash \beta$ | U_1 | $\{\zeta, \eta\}$ | U_2 |
|---------------------------|---|---|---|
| U_1 | $f_1(\alpha, \beta)$ | $f_1(\alpha, \beta)$ | $x^{-1}[f_1(\zeta, \alpha)f_2(\eta, \beta) + f_2(\zeta, \beta)f_1(\eta, \alpha)]$ |
| $\{\zeta, \eta\}$ | $f_1(\alpha, \beta)$ | $f_1(\alpha, \beta) = f_2(\alpha, \beta)$ | $f_2(\alpha, \beta)$ |
| U_2 | $x^{-1}[f_1(\zeta, \beta)f_2(\eta, \alpha) + f_2(\zeta, \alpha)f_1(\eta, \beta)]$ | $f_2(\alpha, \beta)$ | $f_2(\alpha, \beta)$ |

This shows uniqueness of the weak frieze f claimed in Theorem B.

To show existence of f , let $f(\alpha, \beta)$ be defined by the table. We must show that f is a weak frieze with respect to D ; that is, for crossing diagonals $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ with $\{\gamma, \delta\} \in D$, Equation (0.1) holds. It is necessary to treat a number of cases. We leave most of them to the reader, but show the computation for the case

$$\zeta < \alpha < \eta < \gamma < \beta < \delta < \zeta,$$

see Fig. 7 (left), in which

$$\alpha \in U_1, \tag{2.1}$$

$$\beta, \gamma, \delta \in U_2. \tag{2.2}$$

Equation (2.2) implies that $\{\gamma, \delta\}, \{\zeta, \beta\}, \{\eta, \beta\}$ are diagonals of P_2 , so in particular $\{\gamma, \delta\} \in D_2$. Since $\{\gamma, \delta\}$ crosses each of $\{\zeta, \beta\}$ and $\{\eta, \beta\}$ while f_2 is a weak frieze with respect to D_2 , we get the following equations.

$$f_2(\zeta, \beta)f_2(\gamma, \delta) = f_2(\zeta, \gamma)f_2(\beta, \delta) + f_2(\zeta, \delta)f_2(\beta, \gamma) \tag{2.3}$$

$$f_2(\eta, \beta)f_2(\gamma, \delta) = f_2(\eta, \gamma)f_2(\beta, \delta) + f_2(\eta, \delta)f_2(\beta, \gamma) \tag{2.4}$$

The definition of f and Equations (2.1) and (2.2) give

$$f(\alpha, \beta) = x^{-1}[f_1(\zeta, \alpha)f_2(\eta, \beta) + f_2(\zeta, \beta)f_1(\eta, \alpha)] \tag{2.5}$$

and analogous equations where β is replaced with γ or δ . Hence

$$\begin{aligned} f(\alpha, \beta)f(\gamma, \delta) &\stackrel{(a)}{=} x^{-1}[f_1(\zeta, \alpha)f_2(\eta, \beta) + f_2(\zeta, \beta)f_1(\eta, \alpha)]f_2(\gamma, \delta) \\ &= x^{-1}f_1(\zeta, \alpha)f_2(\eta, \beta)f_2(\gamma, \delta) \\ &\quad + x^{-1}f_1(\eta, \alpha)f_2(\zeta, \beta)f_2(\gamma, \delta) \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(b)}{=} x^{-1}f_1(\zeta, \alpha)[f_2(\eta, \gamma)f_2(\beta, \delta) + f_2(\eta, \delta)f_2(\beta, \gamma)] \\
 &\quad + x^{-1}f_1(\eta, \alpha)[f_2(\zeta, \gamma)f_2(\beta, \delta) + f_2(\zeta, \delta)f_2(\beta, \gamma)] \\
 &= x^{-1}[f_1(\zeta, \alpha)f_2(\eta, \delta) + f_2(\zeta, \delta)f_1(\eta, \alpha)]f_2(\beta, \gamma) \\
 &\quad + x^{-1}[f_1(\zeta, \alpha)f_2(\eta, \gamma) + f_2(\zeta, \gamma)f_1(\eta, \alpha)]f_2(\beta, \delta) \\
 &\stackrel{(c)}{=} f(\alpha, \delta)f(\beta, \gamma) + f(\alpha, \gamma)f(\beta, \delta)
 \end{aligned}$$

as desired, where each red factor is replaced in the subsequent step (coloured factors available online). For (a) and (c), use Equation (2.5) and analogous equations where β is replaced with γ or δ . For (b), use Equations (2.3) and (2.4). The remaining equalities are simple computations. \square

Proof of Theorem C. “Only if” is clear.

“If”: Assume that f_1 and f_2 are friezes. We must show that f is a frieze; that is, for crossing diagonals $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$, Equation (0.1) holds. It is necessary to treat a number of cases. We leave most of them to the reader, but show the computation for the case

$$\zeta < \alpha < \gamma < \eta < \beta < \delta < \zeta,$$

see Fig. 7 (right), in which

$$\alpha, \gamma \in U_1, \tag{2.6}$$

$$\beta, \delta \in U_2. \tag{2.7}$$

Equation (2.6) implies that $\{\zeta, \gamma\}, \{\eta, \alpha\}$ are diagonals of P_1 . Since they cross while f_1 is a frieze, we get the first of the following equations, and the second follows by an analogous argument.

$$f_1(\zeta, \gamma)f_1(\eta, \alpha) = f_1(\zeta, \eta)f_1(\gamma, \alpha) + f_1(\zeta, \alpha)f_1(\gamma, \eta) \tag{2.8}$$

$$f_2(\zeta, \beta)f_2(\eta, \delta) = f_2(\zeta, \eta)f_2(\beta, \delta) + f_2(\zeta, \delta)f_2(\beta, \eta) \tag{2.9}$$

Equations (2.6) and (2.7) mean that Equation (2.5) is still valid, and so are the analogous equations where the pair (α, β) is replaced by one of $(\alpha, \delta), (\gamma, \beta), (\gamma, \delta)$. Hence

$$\begin{aligned}
 &x^2 f(\alpha, \beta)f(\gamma, \delta) \\
 &\stackrel{(a)}{=} [f_1(\zeta, \alpha)f_2(\eta, \beta) + f_2(\zeta, \beta)f_1(\eta, \alpha)][f_1(\zeta, \gamma)f_2(\eta, \delta) + f_2(\zeta, \delta)f_1(\eta, \gamma)] \\
 &= f_1(\zeta, \alpha)f_2(\eta, \beta)f_1(\zeta, \gamma)f_2(\eta, \delta) + f_1(\zeta, \alpha)f_2(\eta, \beta)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &\quad + f_2(\zeta, \beta)f_1(\eta, \alpha)f_1(\zeta, \gamma)f_2(\eta, \delta) + f_2(\zeta, \beta)f_1(\eta, \alpha)f_2(\zeta, \delta)f_1(\eta, \gamma)
 \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(b)}{=} f_1(\zeta, \alpha)f_2(\eta, \beta)f_1(\zeta, \gamma)f_2(\eta, \delta) + f_1(\zeta, \alpha)f_2(\eta, \beta)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &\quad + [f_2(\zeta, \eta)f_2(\beta, \delta) + f_2(\zeta, \delta)f_2(\beta, \eta)]f_1(\eta, \alpha)f_1(\zeta, \gamma) \\
 &\quad + f_2(\zeta, \beta)f_1(\eta, \alpha)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &= f_1(\zeta, \alpha)f_2(\eta, \beta)f_1(\zeta, \gamma)f_2(\eta, \delta) + f_1(\zeta, \alpha)f_2(\eta, \beta)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &\quad + f_2(\zeta, \eta)f_2(\beta, \delta)f_1(\eta, \alpha)f_1(\zeta, \gamma) + f_2(\zeta, \delta)f_2(\beta, \eta)f_1(\eta, \alpha)f_1(\zeta, \gamma) \\
 &\quad + f_2(\zeta, \beta)f_1(\eta, \alpha)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &\stackrel{(c)}{=} f_1(\zeta, \alpha)f_2(\eta, \beta)f_1(\zeta, \gamma)f_2(\eta, \delta) + f_1(\zeta, \alpha)f_2(\eta, \beta)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &\quad + f_2(\zeta, \eta)f_2(\beta, \delta)[f_1(\zeta, \eta)f_1(\gamma, \alpha) + f_1(\zeta, \alpha)f_1(\gamma, \eta)] \\
 &\quad + f_2(\zeta, \delta)f_2(\beta, \eta)f_1(\eta, \alpha)f_1(\zeta, \gamma) + f_2(\zeta, \beta)f_1(\eta, \alpha)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &= f_1(\zeta, \alpha)f_2(\eta, \beta)f_1(\zeta, \gamma)f_2(\eta, \delta) + f_2(\zeta, \eta)f_2(\beta, \delta)f_1(\zeta, \eta)f_1(\gamma, \alpha) \\
 &\quad + f_2(\zeta, \delta)f_2(\beta, \eta)f_1(\eta, \alpha)f_1(\zeta, \gamma) + f_2(\zeta, \beta)f_1(\eta, \alpha)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &\quad + f_1(\zeta, \alpha)f_1(\eta, \gamma)[f_2(\eta, \beta)f_2(\zeta, \delta) + f_2(\zeta, \eta)f_2(\beta, \delta)] \\
 &\stackrel{(d)}{=} f_1(\zeta, \alpha)f_2(\eta, \beta)f_1(\zeta, \gamma)f_2(\eta, \delta) + f_2(\zeta, \eta)f_2(\beta, \delta)f_1(\zeta, \eta)f_1(\gamma, \alpha) \\
 &\quad + f_2(\zeta, \delta)f_2(\beta, \eta)f_1(\eta, \alpha)f_1(\zeta, \gamma) + f_2(\zeta, \beta)f_1(\eta, \alpha)f_2(\zeta, \delta)f_1(\eta, \gamma) \\
 &\quad + f_1(\zeta, \alpha)f_1(\eta, \gamma)f_2(\zeta, \beta)f_2(\eta, \delta) \\
 &= x^2 f(\gamma, \alpha)f_2(\beta, \delta) \\
 &\quad + [f_1(\zeta, \alpha)f_2(\eta, \delta) + f_2(\zeta, \delta)f_1(\eta, \alpha)][f_1(\zeta, \gamma)f_2(\eta, \beta) + f_2(\zeta, \beta)f_1(\eta, \gamma)] \\
 &\stackrel{(e)}{=} x^2 f(\alpha, \gamma)f(\beta, \delta) + x^2 f(\alpha, \delta)f(\beta, \gamma).
 \end{aligned}$$

Multiplying by x^{-2} gives Equation (0.1) as desired. Each red factor is replaced in the subsequent step (coloured factors available online). To get (a) and (e), use Equation (2.5) and analogous equations where the pair (α, β) is replaced by one of (α, δ) , (γ, β) , (γ, δ) . To get (b) and (d), use Equation (2.9). To get (c), use Equation (2.8). The remaining equalities are simple computations. \square

3. Lemmas on T -paths

Recall that T -paths, the notation $\mathcal{P}(D, \alpha, \beta)$, and the T -path formula were introduced in Definition 0.2. Additional notation was introduced in Definition 1.3, and we will use this material without further comment.

Lemma 3.1. *Let P be a polygon with a dissection D and let $\alpha \neq \beta$ be vertices of P .*

- (i) *There is a bijection $\mathcal{P}(D, \alpha, \beta) \rightarrow \mathcal{P}(D, \beta, \alpha)$ given by $(\pi_1, \dots, \pi_p) \mapsto (\pi_p, \dots, \pi_1)$.*
- (ii) *If $f : \text{diag}(P) \rightarrow K$ is a map, then*

$$\sum_{\pi \in \mathcal{P}(D, \alpha, \beta)} f(\pi) = \sum_{\rho \in \mathcal{P}(D, \beta, \alpha)} f(\rho).$$

Proof. If $(\pi_1, \dots, \pi_p) \in \mathcal{P}(D, \alpha, \beta)$, then $\{\pi_{p-1}, \pi_p\}$ shares an end point with $\{\alpha, \beta\} = \{\pi_1, \pi_p\}$. It follows that these diagonals do not cross, so $p - 1$ is odd by Definition 0.2(iii). Hence p is even, which makes it easy to check part (i), and implies that Equation (0.3) gives $f((\pi_1, \dots, \pi_p)) = f((\pi_p, \dots, \pi_1))$. Combining this equation with part (i) proves part (ii). \square

Lemma 3.2. *Let P be a polygon with a dissection D and assume that the diagonal $\{\alpha, \beta\}$ crosses no diagonal in D . Then:*

- (i) $\mathcal{P}(D, \alpha, \beta) = \{(\alpha, \beta)\}$.
- (ii) *If $f : \text{diag}(P) \rightarrow K$ is a map, then Equation (0.2) holds.*

Proof. (i): If $\pi = (\pi_1, \dots, \pi_p) \in \mathcal{P}(D, \alpha, \beta)$ then by Definition 0.2(iii) there is no diagonal which could appear as $\{\pi_2, \pi_3\}$, since $\{\alpha, \beta\}$ crosses no diagonal in D . Hence $p = 2$ and the only possibility is $\pi = (\alpha, \beta)$, which is indeed in $\mathcal{P}(D, \alpha, \beta)$.

(ii): Equation (0.2) holds because the sum on the right hand side has only the term $f(\alpha, \beta)$ by part (i) of the lemma. \square

Setup 3.3. Let P be a polygon, V the set of vertices of P , and let D be a non-empty dissection of P . It is easy to see that we can accomplish the following setup, which is sketched in Fig. 8 and will be assumed for the rest of Section 3:

The diagonal $d = \{\zeta, \eta\}$ in D divides P into subpolygons P_1 and P_2 such that $D = \{d\} \cup D_2$ for a dissection D_2 of P_2 . The sets of vertices of P_1 and P_2 are

$$V_1 = \{\varepsilon \in V \mid \zeta \leq \varepsilon \leq \eta\} \ , \ V_2 = \{\varepsilon \in V \mid \eta \leq \varepsilon \leq \zeta\},$$

and we set

$$U_1 = \{\varepsilon \in V \mid \zeta < \varepsilon < \eta\} \ , \ U_2 = \{\varepsilon \in V \mid \eta < \varepsilon < \zeta\}.$$

There are disjoint unions

$$V = V_1 \cup U_2 = U_1 \cup V_2 = U_1 \cup \{\zeta, \eta\} \cup U_2.$$

Observe that no diagonal in D has an end point in U_1 . In particular, D contains no internal diagonals of P_1 . Accordingly, we say that P_1 is an *ear*.

Lemma 3.4. *Assume that $\alpha, \beta \in V_2$. Then $\mathcal{P}(D, \alpha, \beta) = \mathcal{P}(D_2, \alpha, \beta)$.*

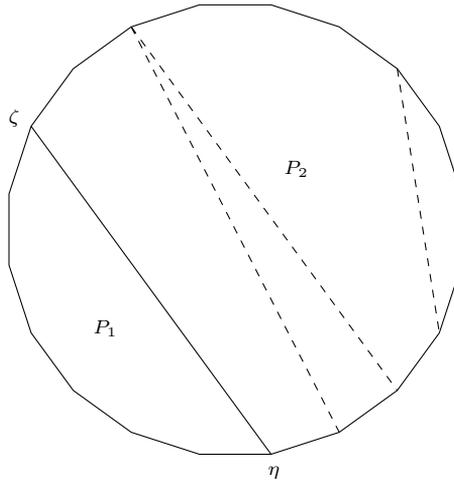


Fig. 8. The polygon P is divided into subpolygons P_1 and P_2 by the diagonal $d = \{\zeta, \eta\}$. The dissection D of P has the form $D = \{d\} \cup D_2$, where D_2 (dashed) is a dissection of P_2 .

Proof. It is easy to show that the inclusion \supseteq holds, and that for the inclusion \subseteq we only need to show that each $\pi = (\pi_1, \dots, \pi_p) \in \mathcal{P}(D, \alpha, \beta)$ satisfies $\pi_i \in V_2$ for each i . Assume the opposite and pick $\pi = (\pi_1, \dots, \pi_p) \in \mathcal{P}(D, \alpha, \beta)$ with $\pi_j \in U_1$ for some j . Observe that $2 \leq j \leq p - 1$ because $\pi_1 = \alpha$ and $\pi_p = \beta$ are in V_2 by assumption. However, no diagonal in D has an end point in U_1 , so the diagonals $\{\pi_{j-1}, \pi_j\}$ and $\{\pi_j, \pi_{j+1}\}$ are not in D , contradicting Definition 0.2(iii). \square

Lemma 3.4 has the following immediate consequence.

Lemma 3.5. *If $f : \text{diag}(P) \rightarrow K$ satisfies the T-path formula with respect to D , then $f|_{\text{diag}(P_2)}$ satisfies the T-path formula with respect to D_2 , see Definition 0.2.*

Lemma 3.6. *Assume that $\alpha \in U_1$, $\beta \in U_2$ and let $\pi = (\pi_1, \dots, \pi_p) \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \neq \eta)}$ be given. Then $\eta < \pi_i \leq \zeta$ for $i \geq 2$.*

Proof. Observe that $p \geq 3$. By Definition 0.2(iii) the diagonal $\{\pi_2, \pi_3\}$ is in D , so both end points are in V_2 . Since $\pi_2 = \zeta$ and $\pi_3 \neq \eta$, it follows that $\eta < \pi_3 < \zeta$, see Fig. 9.

By Definition 0.2(iii), the diagonals $\{\pi_{2j}, \pi_{2j+1}\}$ are in D , and they cross the diagonal $\{\alpha, \beta\}$ at pairwise different points which progress monotonically from α to β . It follows that each $\{\pi_{2j}, \pi_{2j+1}\}$ has at least one end point, which we will denote γ , satisfying $\pi_3 \leq \gamma \leq \pi_2$ whence $\eta < \gamma \leq \zeta$, see Fig. 9. It is enough to show that the other end point of $\{\pi_{2j}, \pi_{2j+1}\}$, which we will denote δ , also satisfies $\pi_3 \leq \delta \leq \pi_2$.

Assuming the opposite, we have $\pi_2 < \delta < \pi_3$. If we had $\pi_3 < \gamma < \pi_2$, then $\{\pi_{2j}, \pi_{2j+1}\} = \{\gamma, \delta\}$ would cross $\{\pi_2, \pi_3\}$ which is in D , contradicting Definition 0.2(ii).

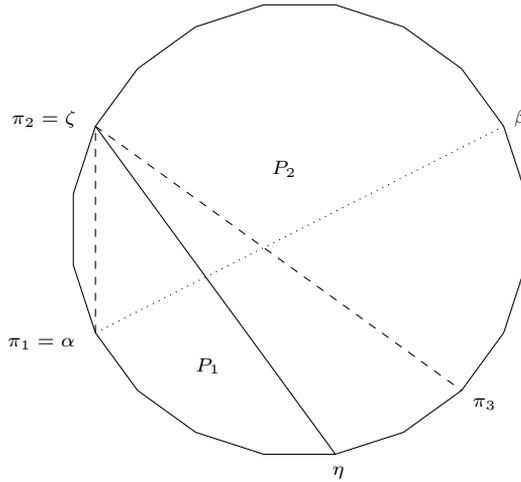


Fig. 9. The T -path π in the proof of Lemma 3.6 starts as shown (dashed).

If we had γ equal to π_3 or π_2 , then $\{\pi_{2j}, \pi_{2j+1}\} = \{\gamma, \delta\}$ would cross $\{\alpha, \beta\}$ closer to α than $\{\pi_2, \pi_3\}$ does, contradicting Definition 0.2(iii). \square

Lemma 3.7. *Assume that $\alpha \in U_1, \beta \in U_2$ and let $\pi = (\pi_1, \dots, \pi_p) \in \mathcal{P}(D, \alpha, \beta)$ be given. Then $\pi_2 \in \{\zeta, \eta\}$.*

Proof. Since $\pi_1 = \alpha \in U_1$, we must have $\pi_2 \in V_1$, since the alternative $\pi_2 \in U_2$ would imply that $\{\pi_1, \pi_2\}$ crossed $\{\zeta, \eta\}$, contradicting Definition 0.2(ii) since $\{\zeta, \eta\}$ is in D . Note that $\pi_2 \in V_1$ implies $\pi_2 \neq \beta$ so $p \geq 3$. Assuming $\pi_2 \notin \{\zeta, \eta\}$, we have $\pi_2 \in U_1$ whence $\{\pi_2, \pi_3\}$ is not in D , contradicting Definition 0.2(iii). \square

Lemma 3.8. *Assume that $\alpha \in U_1, \beta \in U_2$ and let $f : \text{diag}(P) \rightarrow K$ be a map.*

(i) *There is a bijection*

$$\mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \neq \eta)} \xrightarrow{R} \mathcal{P}(D, \eta, \beta)^{(\eta, \zeta)}$$

given by $R(\alpha, \zeta, \pi_3, \dots, \pi_p) = (\eta, \zeta, \pi_3, \dots, \pi_p)$, which satisfies

$$\frac{f(\alpha, \zeta)}{f(\eta, \zeta)} f(R(\pi)) = f(\pi). \tag{3.1}$$

(ii) *There is a bijection*

$$\mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \eta)} \xrightarrow{S} \mathcal{P}(D, \eta, \beta)^{(\eta, \neq \zeta)}$$

given by $S(\alpha, \zeta, \eta, \pi_4, \dots, \pi_p) = (\eta, \pi_4, \dots, \pi_p)$, which satisfies

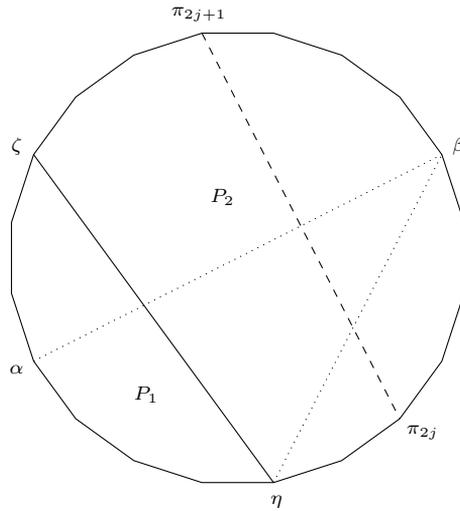


Fig. 10. The figure concerns Lemma 3.8. In Step 1 of the proof the diagonals $\{\pi_{2j}, \pi_{2j+1}\}$ cross $\{\alpha, \beta\}$, and since $\eta < \pi_i \leq \zeta$ for $i \geq 2$ they also cross $\{\eta, \beta\}$. In Step 2 of the proof the diagonals $\{\rho_{2j}, \rho_{2j+1}\}$ cross $\{\eta, \beta\}$, and since $\eta \leq \rho_i \leq \zeta$ for $i \geq 1$ they also cross $\{\alpha, \beta\}$.

$$\frac{f(\alpha, \zeta)}{f(\zeta, \eta)} f(S(\pi)) = f(\pi). \tag{3.2}$$

Proof. Parts (i) and (ii) of the lemma can be proved by similar methods, and we only show the proof of (i).

Step 1: R maps into $\mathcal{P}(D, \eta, \beta)^{(\eta, \zeta)}$: Let $\pi = (\alpha, \zeta, \pi_3, \dots, \pi_p) \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \neq \eta)}$ be given. We must show that $\rho := R(\pi) = (\eta, \zeta, \pi_3, \dots, \pi_p)$ is in $\mathcal{P}(D, \eta, \beta)^{(\eta, \zeta)}$.

The first two vertices of ρ are indeed $\rho_1 = \eta, \rho_2 = \zeta$. It is hence enough to prove that ρ satisfies Definition 0.2, parts (i)-(iii). Part (ii) for ρ is clear from part (ii) for π combined with $\{\eta, \zeta\} \in D$.

Definition 0.2(i) holds for ρ because it holds for π while $\rho_i = \pi_i \neq \eta$ for $i \geq 3$. Indeed, Lemma 3.6 says

$$\eta < \pi_i \leq \zeta \text{ for } i \geq 2.$$

This inequality also implies Definition 0.2(iii) for ρ , because it means that since the diagonals $\{\rho_{2j}, \rho_{2j+1}\} = \{\pi_{2j}, \pi_{2j+1}\}$ cross $\{\alpha, \beta\}$ at pairwise different points which progress monotonically in the direction from α to β , they cross $\{\eta, \beta\}$ at pairwise different points which progress monotonically in the direction from η to β , see Fig. 10.

Step 2: R is surjective: Let $\rho = (\eta, \zeta, \rho_3, \dots, \rho_p) \in \mathcal{P}(D, \eta, \beta)^{(\eta, \zeta)}$ be given. We will show that $\pi := (\alpha, \zeta, \rho_3, \dots, \rho_p) \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \neq \eta)}$, whence $\rho = R(\pi)$ is clear.

The first three vertices of π are indeed $\pi_1 = \alpha, \pi_2 = \zeta, \pi_3 \neq \eta$, since $\pi_3 = \rho_3 \neq \eta$ because $\rho \neq (\eta, \zeta, \eta, \dots)$ by Definition 0.2(i). It is hence enough to prove that π satisfies Definition 0.2, parts (i)-(iii). Part (ii) for π is clear from part (ii) for ρ combined with the observation that the internal diagonal $\{\alpha, \zeta\}$ of P_1 does not cross any diagonal in D .

Definition 0.2(i) holds for π because it holds for ρ while $\pi_i = \rho_i \neq \alpha$ for $i \geq 3$. Indeed, Lemma 3.4 implies

$$\eta \leq \rho_i \leq \zeta \text{ for } i \geq 1.$$

This inequality also implies Definition 0.2(iii) for π , because it means that since the diagonals $\{\pi_{2j}, \pi_{2j+1}\} = \{\rho_{2j}, \rho_{2j+1}\}$ cross $\{\eta, \beta\}$ at pairwise different points which progress monotonically from η to β , they cross $\{\alpha, \beta\}$ at pairwise different points which progress monotonically from α to β , see Fig. 10.

Step 3: R is injective: This is clear from the formula defining R .

Step 4: Equation (3.1): This follows immediately by combining Equation (0.3) with the definition of R . \square

Lemma 3.9. *Assume that $\alpha \in U_1, \beta \in U_2$ and let $f : \text{diag}(P) \rightarrow K$ be a map such that $f|_{\text{diag}(P_2)}$ satisfies the T -path formula with respect to D_2 , see Definition 0.2. Then*

$$f(\zeta, \eta)^{-1} [f(\alpha, \zeta)f(\eta, \beta) + f(\alpha, \eta)f(\zeta, \beta)] = \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)} f(\pi). \tag{3.3}$$

Proof. Lemma 3.4 gives the first of the following equalities, and Lemma 3.8(i) gives the second and third equalities.

$$\begin{aligned} \frac{f(\alpha, \zeta)}{f(\eta, \zeta)} \sum_{\rho \in \mathcal{P}(D_2, \eta, \beta)^{(\eta, \zeta)}} f(\rho) &= \frac{f(\alpha, \zeta)}{f(\eta, \zeta)} \sum_{\rho \in \mathcal{P}(D, \eta, \beta)^{(\eta, \zeta)}} f(\rho) \\ &= \frac{f(\alpha, \zeta)}{f(\eta, \zeta)} \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \neq \eta)}} f(R(\pi)) \\ &= \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \neq \eta)}} f(\pi). \end{aligned} \tag{3.4}$$

Similarly, Lemmas 3.4 and 3.8(ii) give the following equalities.

$$\begin{aligned} \frac{f(\alpha, \zeta)}{f(\zeta, \eta)} \sum_{\sigma \in \mathcal{P}(D_2, \eta, \beta)^{(\eta, \neq \zeta)}} f(\sigma) &= \frac{f(\alpha, \zeta)}{f(\zeta, \eta)} \sum_{\sigma \in \mathcal{P}(D, \eta, \beta)^{(\eta, \neq \zeta)}} f(\sigma) \\ &= \frac{f(\alpha, \zeta)}{f(\zeta, \eta)} \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \eta)}} f(S(\pi)) \\ &= \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta, \eta)}} f(\pi). \end{aligned} \tag{3.5}$$

The sum of Equations (3.4) and (3.5) is

$$\frac{f(\alpha, \zeta)}{f(\zeta, \eta)} \sum_{\tau \in \mathcal{P}(D_2, \eta, \beta)} f(\tau) = \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta)}} f(\pi).$$

Since $f|_{\text{diag}(P_2)}$ satisfies the T -path formula with respect to D_2 , this reads

$$\frac{f(\alpha, \zeta)}{f(\zeta, \eta)} f(\eta, \beta) = \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \zeta)}} f(\pi). \tag{3.6}$$

By symmetry,

$$\frac{f(\alpha, \eta)}{f(\eta, \zeta)} f(\zeta, \beta) = \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)^{(\alpha, \eta)}} f(\pi). \tag{3.7}$$

Lemma 3.7 implies that the sum of Equations (3.6) and (3.7) is Equation (3.3). \square

4. Proof of Theorem A

In this section we provide a proof of Theorem A. Recall that the theorem states the following: For D a dissection of a polygon P , a map $f : \text{diag}(P) \rightarrow K$ is a weak frieze with respect to D if and only if it satisfies the T -path formula with respect to D .

Proof of Theorem A. Recall that the T -path formula was introduced in Definition 0.2. It will be invoked a number of times in the proof.

“If”: Assume that f satisfies the T -path formula with respect to D . We will prove by induction on $\ell := |D|$ that f is a weak frieze with respect to D .

If $\ell = 0$ then $D = \emptyset$ and it is immediate from Definition 0.1(ii) that f is a weak frieze with respect to D .

If $\ell \geq 1$, then we can assume to be in the situation of Setup 3.3 and use the notation introduced there. Setting $d_1 = d$, $D_1 = \emptyset$ then puts us in the situation of Theorem B where $m = 1$.

It is immediate from Definition 0.1(ii) that $f|_{\text{diag}(P_1)}$ is a weak frieze with respect to $D_1 = \emptyset$. Lemma 3.5 says that $f|_{\text{diag}(P_2)}$ satisfies the T -path formula with respect to D_2 . Since $D = \{d\} \cup D_2$ we have $|D_2| = \ell - 1$, so by induction $f|_{\text{diag}(P_2)}$ is a weak frieze with respect to D_2 .

By Theorem B there exists a unique weak frieze $\tilde{f} : \text{diag}(P) \rightarrow K$ with respect to D such that

$$\tilde{f}|_{\text{diag}(P_i)} = f|_{\text{diag}(P_i)} \tag{4.1}$$

for $i \in \{1, 2\}$. It is sufficient to show $\tilde{f} = f$, that is

$$\tilde{f}(\alpha, \beta) = f(\alpha, \beta) \tag{4.2}$$

for vertices $\alpha \neq \beta$ of P . There are four cases to consider.

Case 1: $\alpha, \beta \in V_1$. Then Equation (4.2) follows from Equation (4.1).

Case 2: $\alpha, \beta \in V_2$. Same argument as in Case 1.

Case 3: $\alpha \in U_1, \beta \in U_2$. Since $f|_{\text{diag}(P_2)}$ satisfies the T -path formula with respect to D_2 , Lemma 3.9 gives Equation (3.3). Since f satisfies the T -path formula with respect to D , the equation reads

$$f(\zeta, \eta)^{-1} [f(\alpha, \zeta)f(\eta, \beta) + f(\alpha, \eta)f(\zeta, \beta)] = f(\alpha, \beta).$$

Equation (4.1) implies that this reads

$$\tilde{f}(\zeta, \eta)^{-1} [\tilde{f}(\alpha, \zeta)\tilde{f}(\eta, \beta) + \tilde{f}(\alpha, \eta)\tilde{f}(\zeta, \beta)] = f(\alpha, \beta). \tag{4.3}$$

But \tilde{f} is a weak frieze with respect to D and $d = \{\zeta, \eta\}$ is in D , so Equation (4.3) becomes Equation (4.2).

Case 4: $\alpha \in U_2, \beta \in U_1$. This reduces to Case 3 because each side of Equation (4.2) is symmetric in α and β since f and \tilde{f} are defined on $\text{diag}(P)$.

“Only if”: Assume that f is a weak frieze with respect to D . We will prove by induction on $\ell := |D|$ that f satisfies the T -path formula with respect to D .

If $\ell = 0$ then $D = \emptyset$. It follows that if $\alpha \neq \beta$ are vertices of P , then $\{\alpha, \beta\}$ crosses no diagonal in D , so Equation (0.2) holds by Lemma 3.2(ii). That is, f satisfies the T -path formula with respect to D .

If $\ell \geq 1$, then we can assume to be in the situation of Setup 3.3 and use the notation introduced there. It is clear from Definition 0.1(ii) that $f|_{\text{diag}(P_2)}$ is a weak frieze with respect to D_2 . Since $D = \{d\} \cup D_2$ we have $|D_2| = \ell - 1$, so by induction $f|_{\text{diag}(P_2)}$ satisfies the T -path formula with respect to D_2 . To show that f satisfies the T -path formula with respect to D , let $\alpha \neq \beta$ be vertices of P . We must show that Equation (0.2) holds, and there are four cases to consider.

Case 1: $\alpha, \beta \in V_1$. Then the diagonal $\{\alpha, \beta\}$ crosses no diagonal in D , so Equation (0.2) holds by Lemma 3.2(ii).

Case 2: $\alpha, \beta \in V_2$. Then

$$f(\alpha, \beta) = \sum_{\pi \in \mathcal{P}(D_2, \alpha, \beta)} f(\pi) = \sum_{\pi \in \mathcal{P}(D, \alpha, \beta)} f(\pi)$$

as desired. The first equality holds because $f|_{\text{diag}(P_2)}$ satisfies the T -path formula with respect to D_2 , and the second equality is by Lemma 3.4.

Case 3: $\alpha \in U_1, \beta \in U_2$. Since $f|_{\text{diag}(P_2)}$ satisfies the T -path formula with respect to D_2 , Lemma 3.9 applies and gives Equation (3.3). Since $d = \{\zeta, \eta\}$ is in D and crosses

$\{\alpha, \beta\}$ while f is a weak frieze with respect to D , Equation (3.3) becomes Equation (0.2) as desired.

Case 4: $\alpha \in U_2, \beta \in U_1$. This reduces to Case 3 because each side of Equation (0.2) is symmetric in α and β . The left hand side is symmetric since f is defined on $\text{diag}(P)$, and the right hand side is symmetric by Lemma 3.1(ii). \square

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