Mediated interactions between ions in quantum degenerate gases

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We explore the interaction between two trapped ions mediated by a surrounding quantum degenerate Bose or Fermi gas. Using perturbation theory valid for weak atom-ion interaction, we show analytically that the interaction mediated by a Bose gas has a power-law behaviour for large distances whereas it has a Yukawa form for intermediate distances. For a Fermi gas, the mediated interaction is given by a power-law for large density and by a Ruderman-Kittel-Kasuya-Yosida form for low density. For strong atom-ion interactions, we use a diagrammatic theory to demonstrate that the mediated interaction can be a significant addition to the bare Coulomb interaction between the ions, when an atom-ion bound state is close to threshold. Finally, we show that the induced interaction leads to substantial and observable shifts in the ion phonon frequencies.

Mediated interactions play a crucial role for our understanding of nature. They are central to Landau’s quasiparticle theory, which provides a highly successful effective description of many-body systems used across a wide range of energy scales [1], and all interactions are mediated by gauge bosons at a fundamental level [2]. In condensed matter physics, conventional superconductivity is caused by an attractive electron-electron interaction mediated by lattice phonons, and interactions mediated by spin fluctuations are conjectured to be the origin of high Tc superconductivity [3]. One has furthermore observed an effective interaction between bosons mediated by fermions in an atomic gas [4], and recent experimental results are consistent with its presence between individual bosons forming Fermi polarons [5].

The exquisite control of trapped ions combined with the great flexibility of cold atomic gases makes hybrid atom-ion systems a promising new platform for quantum simulation [6]. Remarkable experimental progress has been reported regarding atom-ion collisions, sympathetic cooling, molecular physics [7–13], Rydberg atom-ion mixtures [14, 15], and mobile ions in a Bose-Einstein condensate (BEC) [16–19]. Recently, the first observation of atom-ion Feshbach resonances was reported, opening up the exciting possibility to tune the interaction strength like in neutral gases [20]. Theoretically, an early study explored the properties of a static ion in a BEC [21], and mobile ions have been predicted to form quasiparticle states when they are immersed in a BEC [22–24] or a degenerate Fermi gas [25], which are charged analogues of neutral Bose and Fermi polarons [5, 26–34].

Here, we investigate the interaction between two ions mediated by a BEC or a degenerate Fermi gas. For weak atom-ion interactions, it follows from perturbation theory that the interaction mediated by a BEC is given by a power-law and a Yukawa form for large and intermediate distances, respectively, whereas it has a power-law or a Ruderman-Kittel-Kasuya-Yosida (RKKY) form for a Fermi gas with large and small density. Generalising our theory to strong interactions, we show that the mediated interaction can be sizeable compared to the Coulomb interaction when the atom-ion interaction is close to threshold for supporting a new bound state, i.e., near a Feshbach resonance. We finally discuss how the mediated interaction leads to observable changes in the phonon frequencies of trapped ions.

Two ions in an atomic gas.— We consider two static ions separated by $\mathbf{R}$ in a quantum degenerate gas consisting of identical bosonic or fermionic atoms of mass $m$ and density $n$ at zero temperature. The Hamiltonian is

$$\hat{H} = \sum_{\mathbf{k}} \frac{k^2}{2m} \hat{c}^\dagger_{\mathbf{k}} \hat{c}_{\mathbf{k}} + \frac{g}{2} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \hat{c}^\dagger_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{c}_{\mathbf{k}'} \hat{c}_{\mathbf{k}'} + \sum_{\mathbf{k},\mathbf{q}} V_q \hat{c}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{c}_{\mathbf{k}} \left( 1 + e^{-i\mathbf{q}\cdot\mathbf{R}} \right),$$

(1)

where $\hat{c}^\dagger_{\mathbf{k}}$ creates an atom with momentum $\mathbf{k}$, $V_q$ is the Fourier transform of the atom-ion interaction potential. Throughout the paper, we set $\hbar$ and the system volume to be unity. We define $g = 4\pi a_{BB}/m$ with $a_{BB}$ the atom-atom scattering length as the strength of the atom-atom short range interaction. For bosons, we assume $n^{1/3}a_{BB} \ll 1$ so that interaction effects can be described using Bogoliubov theory, whereas it plays no role for fermionic atoms due to the Pauli exclusion principle.

The electric field from the ions polarises the atoms and gives rise to a long range atom-ion interaction of the form $-\alpha/r^4$, where $\alpha$ is proportional to the atomic polarizability [6]. The corresponding characteristic length scale $r_{\text{ion}} = \sqrt{2\alpha n} c$ of this interaction can easily be of the same order of magnitude as the average interparticle distance in atomic gases, so it is crucial to include the long range $1/r^4$ tail explicitly in our theory [21]. To include the short range repulsion due to the overlap between the atom and
Having analysed the atom-ion scattering, we now include these effects in the induced interaction between two ions in a BEC, considering nances in real atom-ion systems [20].

For a BEC, the static density-density correlation function is \( \chi(q,0) = -4\pi m/(q^2 + 2/\xi^2) \) to leading order in \( n^{1/3}a_{BB} \), where \( \xi = 1/\sqrt{8\pi na_{BB}} \) is the healing length of the BEC. Using this in Eq. (3), one finds [36]

\[
V_{\text{ind}}(R) = \frac{mV_q v_0}{2\pi a_{BB} R^4}
\]

for the physically relevant limit \( R \gg b, c, \xi \). Thus, the long range induced interaction is proportional to \( 1/R^4 \) like the bare atom-ion interaction with a magnitude given by \( (m_{\text{ion}}a_{BB})^{-1}(r_{\text{ion}}/R)^2 \) where we have used \( V_q = \alpha \pi^2/r_{\text{ion}} \). The long range interaction is determined by the \( q \to 0 \) behaviour of the BEC density-density correlation function determining the compressibility of the BEC. As a result, Eq. (4) is independent of the density \( n \) and inversely proportional to the Bose-Bose scattering length \( a_{BB} \), reflecting that a more compressible BEC leads to a stronger induced interaction.

For shorter distance with \( b, c \ll R \ll \xi \), Eq. (3) can also be evaluated analytically giving [36]

\[
V_{\text{ind}}(R) = -\frac{3nm\alpha^2}{b^2 (b^2 - c^2)^2} \frac{1}{R} e^{-\sqrt{2}R/\xi}.
\]

This has the same functional form as the Yukawa interaction obtained for neutral impurities in a BEC [37, 38].

**BEC and strong interaction.** We now turn to the case of a strong atom-ion interaction and the presence of bound states. The scattering of an atom on a static ion is described by the scattering matrix, which obeys

\[
\mathcal{T}(p', p; \omega) = V_{p' - p} + \sum_k \mathcal{T}(p', k; \omega)G(k, \omega)V_{k - p}.
\]

in the ladder approximation, where \( p' \) (\( p \)) is the momentum of the in-coming (out-going) atom with energy \( \omega \) and \( G(k, \omega) \) is the atom Green’s function. Equation (6) is illustrated diagrammatically in Fig. 2(a) and is exact in the case of vacuum scattering. The scattering matrix has poles at the bound state energies of an atom in the potential of the ion, and \( \mathcal{T}(0, 0; 0) \) diverges every time a new bound state appears. It is easy to see that the scattering matrix for the ion at position \( R \) is given by

\[
\mathcal{T}_R(p', p; \omega) = \mathcal{T}(p', p; \omega) \exp[-i(p' - p) \cdot R].
\]

Having analysed the atom-ion scattering, we now include these effects in the induced interaction between

\[
E_2 = \sum_q V_q^2 [1 + \cos(q \cdot R)] \chi(q,0) = \tilde{E}_2 + V_{\text{ind}}(R)
\]
FIG. 2. (a) Diagrams for the atom-ion scattering matrix. The ion is indicated by a *, the wavy line is the atom-ion interaction, and the blue line is the atom Green’s function. (b) The induced interaction mediated by a Bogoliubov mode (double line) in a BEC. Dashed lines indicate condensate atoms. (c) The induced interaction mediated by a particle-hole excitations in a Fermi gas.

the two ions. For a BEC, we use the Bogoliubov Green’s function $G_{11}(k, \omega) = u_{k}^{2} / (\omega - E_{k}) - u_{0}^{2} / (\omega + E_{k})$ for $G(k, \omega)$ in Eq. (6), where $E_{k} = \sqrt{\epsilon_{k}^{2} + 2n g \epsilon_{k}}$ is the excitation spectrum and $v_{k}^{2} = u_{k}^{2} - 1 = [(\epsilon_{k} + n g) / E_{k} - 1] / 2$. The dominant contribution to the induced interaction between the two ions is the exchange of a sound mode in the BEC as described by the diagram in Fig. 2(b). Evaluating this diagram gives [36]

$$V_{\text{ind}}(R) = \sum_{k} T(k, 0; 0)^{2} \chi(k, 0) \cos(k \cdot R) \tag{7}$$

where we have used the symmetry $T(k, k'; \omega) = T(k', k; \omega)$. Comparing to Eq. (3), we see that the strong coupling result is obtained by substituting the bare atom-ion interaction with the scattering matrix. It follows that Eq. (7) reduces to the weak coupling result for $b / r_{\text{ion}} \gtrsim 1$.

One can moreover derive from Eq. (7) that [36]

$$V_{\text{ind}}(R) = \frac{m T(0, 0; 0)}{2 \pi a_{BB}} \frac{\alpha}{R^{4}} \tag{8}$$

when $R \gg [b, c, \xi, m T(0, 0; 0)]$. Hence, the induced interaction is proportional to $1/R^{4}$ as for the weak interaction case, when $R$ is much larger than the characteristic length scale $r_{\text{ion}}$ of the atom-ion interaction and spatial size of the bound state that emerges at resonance. It follows from Eq. (8) that the interaction is very strong close to resonance where a new atom-ion dimer state becomes stable and $T(0, 0; 0)$ diverges. In addition, Eq. (8) shows that the sign of $V_{\text{ind}}(R)$ is determined by $T(0, 0; 0)$.

In Fig. 1, the induced interaction between two ions in a BEC with density $n r_{\text{ion}}^{3} = 1$, healing length $\xi / r_{\text{ion}} = 2$, and different values of $b$ is plotted. It increases with increasing depth of the atom-ion interaction (decreasing $b$), becoming very large as the resonance value $b \simeq 0.5168 r_{\text{ion}}$ is approached and a bound atom-ion state emerges [36]. Interestingly, $V_{\text{ind}}(R)$ has a node when $b / r_{\text{ion}} < 0.5168$ so that it is repulsive in long range limit. This is because $T(0, 0; 0)$ changes sign when a bound state enters the potential, see Eq. (8). We furthermore see that the analytical results for the long range part of the interaction given by Eqs. (4) and (8) agree well with the numerical results. In the inset, we plot the induced interaction for a large healing length $\xi = 100 r_{\text{ion}}$ and weak atom-ion interaction to illustrate that it is given by Eq. (4) and Eq. (5) for $R \gg \xi$ and $R \lesssim \xi$, respectively. Figure 1 clearly shows how the strength and sign of the induced interaction depend critically on the shape of the long-range atom-ion interaction and the presence of the atom-ion bound states.

**Fermi gas and weak interaction.**—We now turn to the case of two ions in a Fermi gas exploring first a weak atom-ion interaction so that Eq. (3) is valid. The density-density correlation function of a Fermi gas is [39] $\chi(q, 0) = \sum_{k} (f_{k} - f_{k+q}) / (\xi_{k} - \xi_{k+q})$, where $f_{k} = \exp(\beta \xi_{k}) + 1]^{-1}$ is the Fermi function and $\xi_{k} = k^{2} / 2m - \epsilon_{F}$ with $\epsilon_{F} = k_{F}^{2} / 2m = (6 \pi^{2} n)^{2 / 3} / 2m$ the Fermi energy. Using this in Eq. (3), we can derive [36]

$$V_{\text{ind}}(R) = \frac{m k_{F}}{2 \pi F_{k} R^{4} \cos(2k_{F} R)} \frac{b c}{R^{4}} \tag{9}$$

with $\gamma = \pi m a^{2} / 3 / 16 \theta^{2} (b^{2} - c^{2})^{2}$ in the physically relevant limit $R \gg b, c$. Equation (9) shows that when the density is so low that the typical interparticle spacing is far larger than the length scale $r_{\text{ion}}$, the atom-ion interaction can be treated as a contact interaction and the induced interaction has the characteristic RKKY form for a degenerate Fermi gas [40–42]. In the high density regime on the other hand, the interaction has the same $1/R^{4}$ dependence as for a BEC, Eq. (4), although the strength is reduced by a factor $2 k_{F} a_{BB} / \pi \ll 1$. This reflects that the Fermi gas is much less compressible than the BEC.

**Fermi gas and strong interaction.**—For the case of strong atom-ion interaction, we return to Eq. (6) and calculate the scattering matrix using the Fermi Green’s function $G(k, \omega) = 1 / (\omega - k^{2} / 2m + \epsilon_{F})$. The induced interaction is mediated by particle-hole excitations in the Fermi gas and the leading term is given by the diagram shown in Fig. 2(c). This yields [36]

$$V_{\text{ind}}(R) = \sum_{q,k} \left[ 2 \theta(k_{F} - k) \frac{\text{Re}[T^{2}(k + q, k, \xi_{k})]}{\xi_{k} - \xi_{k+q}} \right. \left. - \int_{-\epsilon_{F}}^{0} d \omega \frac{\text{Im}[T^{2}(k + q, \omega + \eta)]}{\pi} \cos(q \cdot R) \right] \tag{10}$$

where we have taken zero temperature and assumed zero population of any bound states.

In Fig. 3, the induced ion-ion interaction obtained from Eq. (10) is plotted for the density $k_{F} r_{\text{ion}} = 0.1$ and various values of $b$. Only the s-wave channel of the atom-ion scattering contributes for this low density, which simplifies the numerics significantly. We see that the interaction increases as the atom-ion potential deepens with decreasing $b$ towards $b = 0.5748 r_{\text{ion}}$ where a bound state
appears at the Fermi surface. Friedel oscillation characteristic of an interaction mediated by a Fermi sea is also clearly visible. As the bound state energy decreases below zero with decreasing $b < 0.5748 r_{\text{ion}}$, the interaction again decreases reflecting that it becomes off-resonant.

**Experimental probing.**—We now show that trapped ion experiments are in fact very well suited to explore induced interactions via observable shifts in their phonon spectrum. Consider two ions in a linear rf trap with single ion trapping frequencies $\omega_x, \omega_z \gg \omega_z$ such that the ions will align along the $x$-axis. The slow dynamics in the assumed rf-field free $x$-direction is determined by [43]

$$U = \frac{\kappa}{2}(x_2^2 + x_1^2) + \frac{Z^2 e^2}{4 \pi \epsilon_0} \frac{1}{x_2 - x_1} + V_{\text{ind}}(x_2 - x_1),$$

where $x_j$ is the $x$-coordinate of ion $j$ with $x_2 > x_1$ and $Z e$ is the charge of the ions. The first term in Eq. (11) represents the electrostatic trapping potential in terms of the force constant $\kappa$, the second is the Coulomb interaction, and the third the mediated interaction between two ions. Even though the equilibrium distance between the ions can be affected significantly by the induced interaction [36], it is typically difficult to measure without dramatically perturbing the ion-atom system.

A more promising approach is to measure the phonon frequencies corresponding to small oscillations of the ions around their equilibrium positions. Such frequencies can experimentally be measured with a high accuracy through the application of a single blue sideband excitation to a long-lived electronic state by a narrow bandwidth laser, followed by detecting fluorescence addressing a different fast decaying transition. A similar approach has previously been used to determine the mass of an unknown molecular ion [44].

The frequency of the Kohn mode where the two ions oscillate in-phase is $\omega_x = \sqrt{\kappa / M}$, independent of any interaction between the ions. Note that the effective mass $M$ can be different from that of the bare ions due to the dressing of the ions by the surrounding BEC [23, 24]. This dressing effect is distinct from those coming from the induced interaction, and it can be determined by measuring the oscillation frequency $\omega_x$ of a single ion in a BEC – a procedure that has already been pursued successfully for a neutral impurity in a Fermi gas [47].

Contrary to the Kohn mode, the mode where the two ions oscillate out-of-phase (See Fig. 1) is affected by the induced interaction. In Fig. 4, we plot its frequency obtained from Eq. (11) by evaluating the relevant second derivatives of the induced interaction around the equilibrium positions numerically, compared to its value $\omega_0 = \sqrt{3} \omega_x$ in the absence of the induced interaction. We use parameters $m, M = m_{\text{ion}},$ and $\alpha$ appropriate for a $^7\text{Li}^{138}\text{Ba}^+$ mixture. The density of the BEC is $n = 3 \times 10^{14} \text{cm}^{-3}$, which with $r_{\text{ion}} = 76.8 \mu\text{m}$ corresponds to $n_{\text{ion}}^3 = 0.136$. Also, $\omega/r_{\text{ion}} = 2.2$ corresponding to $a_{BB} = a_0$. This is indeed an approximate value for the $^7\text{Li} - ^7\text{Li}$ scattering length in a wide range of magnetic fields [45, 46], where one should expect several $^7\text{Li}^{138}\text{Ba}^+$ Feshbach resonances based on recent experimental results [20]. We see that the relative frequency shift increases with increasing trapping frequency reflecting the increased relative strength of the induced interaction compared to the Coulomb repulsion for a shorter distance. The shift is negative corresponding to an attractive induced interaction counteracting the repulsive Coulomb interaction between the ions, and it has the largest magnitude close to the resonance value $r_{\text{ion}}/a_B = 0$, where a new bound atom-ion state appears. In the supplemental material, we show frequency shifts for the case of a $^{87}\text{Rb} - ^{87}\text{Rb}^+$ (or $^{87}\text{Sr}^+$) mixture [36].

To explore how close one has to be resonance for an observation of the induced interaction, Fig. 5 shows the frequency shift as a function of the atom-ion scattering length $a_B$ using the same parameters corresponding to a $^7\text{Li}^{138}\text{Ba}^+$ mixture as in Fig. 4. The trapping frequency is $20 \times 2 \pi \times 10^9 \text{Hz}$. Since the values of the relative
frequency shift shown are all above a realistic experimental resolution $\Delta \omega / \omega \lesssim 10^{-4}$. Fig. 5 shows how large the atom-ion scattering length has to be for the mediated interaction to be observable. Assuming that one can realise very large scattering lengths of the same order of magnitude as in neutral atomic gases [45], this indicates that detecting the frequency shifts due to the induced interaction is within experimental reach. Very close to resonance, the induced interaction seems to exceed the Coulomb repulsion between the ions and lead to a collapse as indicated by the grey region in Fig. 5. Such a collapse is however unphysical and will be halted by short range physics not included in our model.

Conclusions and outlook.— We analysed the interaction between two ions mediated by a BEC or a Fermi gas. For weak atom-ion interactions, we derived several analytical results, and our theory was then generalised to strong atom-ion interactions. Finally, we discussed trapped ions as a promising platform for probing mediated interactions in a quantum many-body system with unprecedented control and precision.

Our results motivate future work in several directions. Throughout, we have assumed that the bound states of the atom-ion potential are populated corresponding to considering the so-called repulsive branch, which has been observed experimentally for neutral impurities in Fermi and Bose gases [5, 26, 27, 31, 32, 34]. It would be interesting to explore the influence of populating the bound states, which may give rise to an additional attractive interaction for short distances [48]. Another interesting question concerns temperature effects on the mediated interaction and in particular how it changes when the BEC melts. Finally, it would be useful to apply alternative approaches such as numerical Monte-Carlo calculations to explore the highly challenging strongly interacting regime [49].

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Supplemental Material: Mediated interactions between ions in quantum degenerate gases

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WEAK INTERACTIONS

The energy shift due to two ions separated by a distance \( R \) is to second order in the interaction

\[
E_2 = \sum_{M \neq G} \langle G| \sum_{k,q} c_{k+q}^\dagger c_k V_q \left( 1 + e^{i\mathbf{q} \cdot \mathbf{R}} \right) |M\rangle \langle M| \sum_{k,q} c_{k+q}^\dagger c_k V_q \left( 1 + e^{-i\mathbf{q} \cdot \mathbf{R}} \right) |G\rangle \frac{E_G - E_M}{E_G - E_M}
\]

(1)

where \(|G\rangle\) and \(|M\rangle\) are respectively the ground state and the excited state of the surrounding bath with energies \( E_G \) and \( E_M \). Evaluating this gives

\[
E_2 = \sum_{\mathbf{q}} \chi(\mathbf{q}, 0) V_q^2 [1 + \cos(\mathbf{q} \cdot \mathbf{R})]
\]

(2)

where

\[
\chi(\mathbf{q}, 0) = \sum_{M \neq G} \frac{2 |\langle G| \mathbf{q} |M\rangle|^2}{E_G - E_M}
\]

(3)

with \( \rho_\mathbf{q} = \sum_k c_{k+\mathbf{q}}^\dagger c_{-\mathbf{q} k} \) is the static density-density correlation function of the bath.

Two ions in a BEC

Using that \( \chi(\mathbf{q}, 0) = -2n_0/(\epsilon_\mathbf{q} + 2n_0 \alpha) \) for a weakly interacting BEC, the \( R \)-dependent part of Eq. (2) yields

\[
V_{\text{ind}}(R) = -\frac{2\pi^2 n_0 m \alpha^2}{b^2 (b^2 - c^2)^4} \int \left[ 4bc^2 e^{-cq} + e^{-bq} (-4bc^2 - b^4q + c^4q) \right]^2 \frac{1}{q^2 + 2/\xi^2} \sin qR dq
\]

(4)

for the induced interaction in second order perturbation theory.

When \( R \gg b, c, \xi \), \( V_{\text{ind}}(R) \) can be calculated approximately with integration by parts \( \int_a^b u(x) v'(x) dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x) dx \) using

\[
u_0(q) = \left[ 4bc^2 e^{-cq} + e^{-bq} (-4bc^2 - b^4q + c^4q) \right]^2 \frac{1}{q^2 + 2/\xi^2} \frac{1}{qR}
\]

and

\[
u'_{0}(q) = \sin qR.
\]

(5)

This gives

\[
V_{\text{ind}}(R) = \frac{2\pi^2 n_0 m \alpha^2}{b^2 (b^2 - c^2)^4} \int dq u_0(q) v_0(q).
\]

(6)

Another partial integration with \( u_1(q) = u_0'(q)/R \) and \( v_1'(q) = -\cos qR \) yields

\[
V_{\text{ind}}(R) = -\frac{2\pi^2 n_0 m \alpha^2}{b^2 (b^2 - c^2)^4} \int dq u_1'(q) v_1(q).
\]

(7)
A final partial integration with $u_2(q) = u'_1(q)/R$ and $v'_2(q) = -\sin qR$ gives
\[ V_{\text{ind}}(R) \approx -\frac{\pi m\alpha^2}{2a_{BB}b} \frac{b^2 + 2bc - c^2}{(b + c)^2} \frac{1}{R^4} = \frac{mV_{q=0}}{2\pi a_{BB} R^4} \alpha. \] (8)

When $b, c \ll R \lesssim \xi$, the exponential factor in $V_q$ can be taken as constant 1 in Eq. (4) because for $q\xi, qR \sim 1$, $qb, qc \ll 1$. This gives
\[ V_{\text{ind}}(R) \approx -2\pi^2 n_0 m\alpha^2 \frac{(b^4 - c^4)^2}{b^2 (b^2 - c^2)^4} \int \frac{q^2}{q^2 + 2/\xi^2} \frac{\sin qR}{qR} dq = -2\pi^2 n_0 m\alpha^2 \frac{(b^2 + c^2)^2}{b^2 (b^2 - c^2)^2} \frac{\pi}{2R} e^{-\sqrt{2R/\xi}}. \] (9)

### Two ions in a Fermi gas

For a one-component Fermi gas, the static density-density correlation function is
\[ \chi(q, 0) = \sum_k \frac{f_k - f_{k+q}}{\epsilon_k - \xi_{k+q}} = -\frac{2}{(2\pi)^2} \frac{m}{q} \frac{1}{8} \left[ 4k_F q + (4k_F^2 - q^2) \ln \left( \frac{2k_F + q}{2k_F - q} \right) \right] \] (10)
where $\xi_k = \epsilon_k - \mu$ with the chemical potential $\mu = \epsilon_{k_F}$. Using this in Eq. (2) gives
\[ V_{\text{ind}}(R) = -\frac{m}{2(2\pi)^2} \int dqqV_q \frac{\sin(qR)}{R} \left[ 4k_F q + (4k_F^2 - q^2) \ln \left( \frac{2k_F + q}{2k_F - q} \right) \right]. \] (11)

When $k_F b \gg 1$, the main contribution is from $q \ll k_F$ due to the exponential decay of the potential $V_q$. As a result, we can expand the factor in square bracket of Eq. (11) in $\tilde{q} = q/k_F$, which yields
\[ V_{\text{ind}}(R) \approx -\frac{mk_F^2}{2(2\pi)^2} \int \tilde{q} V_q \frac{\sin(qR)}{R} 8\tilde{q} \left[ \frac{4b}{R} \left( \frac{(b - c)^2}{(4b^2 + R^2)^2} + \frac{2c^2(b^4 - c^4)}{4b^2 + R^2} + \frac{-2b^4 c^2 + 2c^6}{(b + c)^2 + R^2} \right) \right. \]
\[ + \left. 4bc^4 \left( \text{ArcCot} \left( \frac{2b}{R} \right) + \text{ArcCot} \left( \frac{2c}{R} \right) - 2\text{ArcTan} \left( \frac{R}{b + c} \right) \right) \right] \] (12)

The leading term for $R \gg b, c$ is
\[ -\frac{mk_F^2}{b(b + c)^2} (b^2 + 2bc - c^2) \frac{1}{R^4} = \frac{mk_F}{\pi^2} V_{q=0} \frac{\alpha}{R^4}. \] (13)

When $k_F b \ll 1$, $k_F c \ll 1$ and $R \gg b, c$, the exponential function in $V_q$ can be approximated by 1 in Eq. (11) because for $q \sim k_F$ or $qR \sim 1$, $qb, qc \ll 1$. We then get
\[ V_{\text{ind}}(R) \approx -\frac{ma^2 k_F^4}{25} \left[ \frac{b^2 + c^2}{b(b^2 - c^2)} \right]^2 \int \tilde{q} \frac{\sin \left( \tilde{q} \tilde{R} \right)}{\tilde{R}} \left[ 4\tilde{q} + (4 - \tilde{q}^2) \ln \left( \frac{2 + \tilde{q}}{2 - \tilde{q}} \right) \right] \] (14)
\[ = -\frac{ma^2}{2^5} \left[ \frac{b^2 + c^2}{b(b^2 - c^2)} \right]^2 \frac{2\pi}{R^4} \left[ -2k_F R \cos(2k_F R) + \sin(2k_F R) \right] \]
where $\tilde{q} = q/k_F$ and $\tilde{R} = k_F R$.

### SCATTERING BETWEEN STATIC ION AND ATOM

The atom-ion scattering length can be extracted from the zero energy/momentum scattering matrix as $a_{BB} = mT(0, 0; 0)/2\pi$. When $a_{BB}$ diverges there is a bound state at threshold. In Fig. 1, we plot $T(0, 0; 0)$ as a function of the parameter $b$ in the model potential $V_q$ for different BEC healing lengths $\xi$. We clearly see the emergence of bound states as $b$ decreases.
FIG. 1. $\mathcal{T}(0,0,0)$ vs $b$ for $c = 0.0023 r_{\text{ion}}$ and different $\xi$.

**STRONG ATOM-ION INTERACTIONS IN A BEC**

Diagram Fig. 2b in the main text for two ions in a BEC corresponds to the summation of the four diagrams in Fig. 2. This gives

$$V_{\text{ind}}(R) = \sum_q \exp(i\mathbf{q} \cdot \mathbf{R}) n_0 \left[ \mathcal{T} (\mathbf{q}, 0, 0) G_{11} (\mathbf{q}, 0) \mathcal{T} (0, \mathbf{q}, 0) + \mathcal{T} (0, -\mathbf{q}, 0) G_{22} (\mathbf{q}, 0) \mathcal{T} (-\mathbf{q}, 0, 0) \right]$$

$$+ \mathcal{T} (\mathbf{q}, 0, 0) G_{21} (\mathbf{q}, 0) \mathcal{T} (-\mathbf{q}, 0, 0) + \mathcal{T} (0, -\mathbf{q}, 0) G_{12} (\mathbf{q}, 0) \mathcal{T} (0, \mathbf{q}, 0) \right]$$

where the BEC Green’s functions are

$$G_{22}(\mathbf{k}, i\omega) = G_{11}(\mathbf{k}, -i\omega) = \frac{u_k^2}{-i\omega - E_k} - \frac{v_k^2}{i\omega + E_k} \quad \text{and} \quad G_{12}(\mathbf{k}, i\omega) = G_{21}(\mathbf{k}, i\omega) = \frac{u_k v_k}{i\omega + E_k} - \frac{u_k v_k}{i\omega - E_k}.$$  \hspace{1cm} (16a)

Since $V_{p' - p}$ is a symmetric matrix, we have $\mathcal{T}(\mathbf{q}, 0, 0) = \mathcal{T}(0, \mathbf{q}, 0)$. Using this and $\mathcal{T}(-\mathbf{q}, 0, 0) = \mathcal{T} (\mathbf{q}, 0, 0)$ gives

$$V_{\text{ind}}(R) = \sum_q 2 \cos (\mathbf{q} \cdot \mathbf{R}) \mathcal{T} (\mathbf{q}, 0, 0)^2 n_0 [G_{11} (\mathbf{q}, 0) + G_{22} (\mathbf{q}, 0)] = \sum_q \cos (\mathbf{q} \cdot \mathbf{R}) \mathcal{T} (\mathbf{q}, 0, 0)^2 \chi (\mathbf{q}, 0).$$  \hspace{1cm} (17)

Plugging in the static density-density correlation function of a BEC gives

$$V_{\text{ind}}(R) = -\frac{2n_0 m\alpha^2}{\pi^2 r_{\text{ion}}^3} \int_{0}^{\infty} \tilde{\mathcal{T}} (\tilde{q}, 0, 0) \frac{1}{\tilde{q}^2 + 2 \left(1/\tilde{\xi}\right)^2} \frac{\sin \tilde{q} \tilde{R}}{\tilde{q} \tilde{R}} \tilde{q}^2 d\tilde{q}$$

where $\tilde{q} = qr_{\text{ion}}, \tilde{R} = R/r_{\text{ion}}, \tilde{\xi} = \xi/r_{\text{ion}}$ and $\tilde{\mathcal{T}} = \mathcal{T} r_{\text{ion}}/\alpha$.  \hspace{1cm} (18)
Given that \( T(q,0,0) \) is smooth, we can analyze the large \( R \) behaviour following the same steps as for the weak interaction case above. Using that \( T(q,0,0) \) and all its derivatives go to 0 for \( q \to \infty \) except on resonance, one can derive the leading term for large \( R \) as

\[
V_{\text{ind}}(R) = \frac{2n_0\alpha a^2}{\pi^2 r_{\text{ion}}^3} \left[ \frac{2\xi^2 \hat{T}(0,0,0) \lim_{\hat{q} \to 0} \left[ \hat{T}'(\hat{q},0,0) \right]}{R^4} \right] = \frac{2}{\pi^2} \frac{T(0,0,0)}{g} \lim_{q \to 0} \frac{\left[ T'(q,0,0) \right]}{R^4}.
\]

(19)

The derivative of the scattering matrix is from Eq. (6) in the main manuscript

\[
\lim_{q \to 0} T'(q,0,0) = \lim_{q \to 0} V_q + \frac{1}{2\pi^2} \int_0^\infty k^2 dk V_k \left( -\frac{u_k^2 + v_k^2}{E_k} \right) \lim_{q \to 0} V'_0 (q,k) + \sum_{k_1} V_{k_1} \left( \frac{u_{k_1}^2 + v_{k_1}^2}{E_{k_1}} \right) \frac{1}{2\pi^2} \int_0^\infty k_{k_2}^2 dk_{k_2} V_{k_2-k_1} \left( -\frac{u_{k_2}^2 + v_{k_2}^2}{E_{k_2}} \right) \lim_{q \to 0} V'_0 (q,k_2) + \ldots
\]

(20)

where \( V_0(p',p) = (1/2) \int_0^\infty V(p'-p)d\theta \) is the angular average of the interaction. We have \( \lim_{q \to 0} V(q) = \alpha \pi \). Moreover, since \( \lim_{q \to 0} V_0(q,k) = \alpha \pi \theta(q-k) \) it follows that the second term in Eq. (20) vanishes for \( q \to 0 \). One can show that also all higher order terms vanish. As a result, \( \lim_{q \to 0} T'(q,0,0) = \alpha \pi \) and consequently,

\[
V_{\text{ind}}(R) = \frac{2T(0,0,0)}{g} \frac{\alpha}{R^4}. \quad (21)
\]

**STRONG ATOM-ION INTERACTIONS IN A FERMI GAS**

Diagram Fig. 2c in the main text for two ions in a Fermi gas corresponds to

\[
V_{\text{ind}}(R) = \sum_q \sum_k \frac{1}{\beta} \sum_{i\omega} T(k + q,k,i\omega) G(k + q,i\omega) T(k,k + q,i\omega) e^{iqR} = \sum_q \sum_k \frac{1}{\beta} \sum_{i\omega} T^2(k + q,k,i\omega) G(k + q,i\omega) G(k,i\omega)e^{iqR},
\]

(22)

where \( i\omega = i(2n + 1)\pi/\beta \) with \( n \) an integer and \( \beta = 1/k_B T \) is a fermionic Matsubara frequency. In the second line, we have used that \( V(p' - p) \) is a symmetric matrix so that \( T(k + q,k,i\omega) = T(k,k + q,i\omega) \).

Evaluating the Matsubara sum gives \( V_{\text{ind}}(R) = V_{\text{ind}}^{12}(R) + V_{\text{ind}}^{33}(R) \) with

\[
V_{\text{ind}}^{12}(R) = \sum_q \sum_{k \leq k_F} \left\{ 2\Re\left[ T^2(k + q,k,\xi_k) \right] \frac{1}{\xi_k - \xi_k + q} \right\} e^{iqR}, \quad (23)
\]

and

\[
V_{\text{ind}}^{33}(R) = \sum_q \sum_k \left\{ -\frac{1}{\pi} \int_{-\mu}^{0} d\omega \Im\left[ T^2(k + q,k,\omega + i\eta) \right] \frac{1}{\omega - \xi_k + q} \right\} e^{iqR}, \quad (24)
\]

in the zero temperature limit, where the integral over \( \omega \) starts from \(-\mu \) because \( \Im\left[ T^2(k + q,k,\omega + i\eta) \right] = 0 \) for \( \omega < -\mu \). Here, we have assumed that bound states are not populated.

**TWO IONS IN A LINEAR RF TRAP**

\( ^7\text{Li}-^{138}\text{Ba}^+ \) mixture

We show in Figure 3 the equilibrium distance between two ions in a linear rf trap with an induced interaction using parameters appropriate for a \( ^7\text{Li}-^{138}\text{Ba}^+ \) mixture.
FIG. 3. The ratio of the equilibrium distance between two ions with \( (R_0) \) and without \( (l) \) the induced interaction as a function of the trapping frequency \( \omega_x \) for a \(^7\)Li-\(^{138}\)Ba\(^+\) mixture. The inset shows the equilibrium distance \( R_0 \) as a function of \( \omega_x \).

\(^{87}\)Rb-\(^{87}\)Rb\(^+\) (or \(^{87}\)Sr\(^+\)) mixture

Figure 4 shows the frequency shift of the out-of-phase phonon mode with the induced interaction relative to without the induced interaction using parameters appropriate for a \(^{87}\)Rb-\(^{87}\)Rb\(^+\) (or \(^{87}\)Sr\(^+\)) mixture. Figure 5 shows the corresponding equilibrium distance between the two ions.

FIG. 4. The relative frequency shift of the out-of-phase phonon mode for a \(^{87}\)Rb-\(^{87}\)Rb\(^+\) (or \(^{87}\)Sr\(^+\)) mixture.

FIG. 5. Same as in Fig. 3 but for parameters corresponding to a \(^{87}\)Rb-\(^{87}\)Rb\(^+\) (or \(^{87}\)Sr\(^+\)) mixture.