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Abstract

In this chapter we present an overview of the main ideas and methods in the fractional integration and cointegration literature. We do not attempt to give a complete survey of this enormous literature, but rather a more introductory treatment suitable for a researcher or graduate student wishing to learn about this exciting field of research. With this aim, we have surely overlooked many relevant references for which we apologize in advance. Knowledge of standard time series methods, and in particular methods related to nonstationary time series, at the level of a standard graduate course or advanced undergraduate course is assumed.

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1 Introduction

Fractional time series constitute a general class of models which are able to capture a wide range of stationary and nonstationary behaviours and can display the so-called long memory property, where a single parameter (known as memory) characterizes a fundamental part of the persistence of the time series. In the classical sense, a purely stochastic scalar time series ζ_t , $t \in \mathbb{Z} = \{t : t = 0, \pm 1, \dots\}$, is integrated of order d , denoted $\zeta_t \in I(d)$, if it can be represented as a stationary and invertible autoregressive-moving average (ARMA) process after differencing it d times. Usually, the parameter d has been assumed to be either 0, 1, or 2, but the class of fractional time series models is characterized by a non-integer value of d . Earlier surveys of this phenomenon include Ballie (1996), Robinson (2003), and Gil-Alana and Hualde (2009), and a recent monograph treatment is given by Hassler (2018).

Undoubtedly, the key aspect of the definition of (fractional) integration is the concept of an $I(0)$ process, which in popular terms has been referred to as a “short memory”, “weakly

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dependent”, “short-range dependent”, or “weakly autocorrelated” process. The $I(0)$ concept has been given different, although relatively closely related, meanings in the literature. Define the spectral density of ζ_t as

$$f_\zeta(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_\zeta(j) e^{ij\lambda},$$

where $\gamma_\zeta(j)$ represents the lag j autocovariance of the process ζ_t . Given the spectral density $f_\zeta(\lambda)$, one definition of $I(0)$ is the following. A zero-mean scalar covariance stationary process ζ_t , $t \in \mathbb{Z}$, with spectral density $f_\zeta(\lambda)$ is integrated of order zero, denoted $\zeta_t \in I(0)$, if

$$0 < f_\zeta(\lambda) < \infty \text{ for all } \lambda \in (-\pi, \pi]. \quad (1)$$

This definition strengthens the original idea of an $I(0)$ process (which was based on stationarity and invertibility and related only to frequency zero behaviour), and avoids complications due to poles or zeros of the spectral density outside frequency zero which may have important consequences.

An alternative definition of $I(0)$ is to include all processes ζ_t that satisfy a functional central limit theorem (FCLT) with a Brownian motion limit process; that is, with $[\cdot]$ denoting integer part and T the sample size, for $r \in [0, 1]$,

$$T^{-1/2} \sum_{t=1}^{[Tr]} \zeta_t \Rightarrow W(r) \text{ as } T \rightarrow \infty,$$

where “ \Rightarrow ” denotes weak convergence of the associated probability measures and $W(r)$ denotes a Brownian motion with variance $\sigma^2 = \lim_{T \rightarrow \infty} T^{-1} E((\sum_{t=1}^T \zeta_t)^2) > 0$. In any case, the distinction between these different definitions of $I(0)$ is not that relevant because proper additional conditions are usually imposed so that an invariance principle holds.

Letting $u_t \in I(0)$, we can define an $I(1)$ process X_t as

$$\Delta X_t = u_t, \quad t \in \mathbb{Z}, \quad (2)$$

where $\Delta = 1 - L$ and L are the difference and the lag operators, respectively. Inverting the Δ operator, it is immediate to show that

$$X_t = X_0 + \sum_{j=1}^t u_j = X_0 + \sum_{j=0}^{t-1} u_{t-j}, \quad t \geq 1, \quad (3)$$

where $\sum_{j=0}^{t-1} L^j$ is an integration (or cumulation or partial summation) operator. Note that the integration operator in (3) ends at $j = t - 1$ because the summation $\sum_{j=0}^{\infty} u_{t-j}$ is not well-defined in the mean-square sense. The choice of starting point, $t = 0$, in (3) is arbitrary, and as such nothing is said about X_t for $t < 0$, which may be taken to be zero. It can be shown that the stochastic difference equation (2) does not have a stationary solution (e.g., Brockwell and Davis, 1991, p. 86), so, in this sense, we say that process X_t is nonstationary. Similar arguments and conclusions can be made for $\Delta^2 X_t = u_t$, etc.

In a multivariate context, the concepts of nonstationarity and integration lead naturally to that of cointegration. An early definition of cointegration from Engle and Granger (1987), see

also Granger (1981), is that the p -dimensional vector X_t is cointegrated if all the components of X_t are $I(d)$ and there exists a $\beta \neq 0$ such that $\beta'X_t \in I(d - b)$. This definition has commonly been applied, especially in regression setups with $d = b = 1$, such that the components of X_t are $I(1)$ but $\beta'X_t \in I(0)$. A problem with this simple definition is that it is not invariant to nonsingular linear transformations (for example a transformation involving $\beta'X_t$ will no longer be $I(1)$ by this definition). Thus, more general definitions have been provided, starting with a definition of $I(0)$ for multivariate processes. We next give a general definition of $I(0)$ and $I(d)$ inspired by that of Johansen (1995).

Definition 1 (Integration order) *Let X_t be a p -dimensional process such that*

$$X_t - E(X_t) = \sum_{j=0}^{\infty} C_j \varepsilon_{t-j},$$

where ε_t is a p -dimensional zero mean independent and identically distributed (i.i.d.) sequence with $\text{Var}(\varepsilon_t) = \Omega$, and $C(z) = \sum_{j=0}^{\infty} C_j z^j$ is convergent for $|z| \leq 1 + \delta$ for some $\delta > 0$. We say that $X_t \in I(0)$ if $C(1) \neq 0$. Furthermore, we say that $X_t \in I(d)$ if $\Delta^d X_t \in I(0)$.

For example, for $d = 1$ the definition of a multivariate $I(1)$ process in Definition 1 coincides with that in (3). Unlike the simple definition from Engle and Granger (1987), Definition 1 implies that the integration order of an $I(d)$ vector ($d = 0, 1$) is invariant to nonsingular linear transformations; that is, if the p -dimensional vector $X_t \in I(d)$, then $AX_t \in I(d)$ for any nonsingular $p \times p$ matrix A .

Definition 2 (Cointegration) *The p -dimensional process X_t is cointegrated if $X_t \in I(d)$ and there exists a $\beta \neq 0$ such that $\beta'X_t \in I(d - b)$ for $b > 0$.*

The concept of cointegration captures the idea of long-run equilibrium among the observables. When $d = b = 1$, this is understood in the sense that these are themselves nonstationary, or $I(1)$, so that the variables have no natural “level”. However, there exists an (economic) equilibrium relation given by β , such that the equilibrium relation $\beta'X_t$ is stationary, or $I(0)$, possibly around a non-zero mean/level. In other words, while shocks to the variables X_t themselves have a permanent effect, any deviation from the equilibrium $\beta'X_t$ has only a transitory effect. The success of nonstationary time series analysis and the concept of cointegration seems largely due to this interpretation of cointegration as a long-run equilibrium among the (economic) variables.

The definitions of integration and cointegration given above are valid not only for integer values of d and b , but also for real values. Non-integer values of d and/or b lead to the concepts of fractional integration and fractional cointegration. Importantly, the interpretation of cointegration as an (economic) equilibrium concept still applies in a more general form in the fractional setting.

In this chapter we present an overview of the concepts discussed above when d and/or b are not integers. In particular, we present and discuss the main ideas and methods in the fractional integration and cointegration literature, as related to definitions, estimation, and inference. We deliberately do not attempt to give a complete survey of this enormous literature, but rather a more introductory treatment. The aim is to provide a treatment that is useful for a researcher or graduate student wishing to learn about this exciting field of research

or possibly to apply some of the methods in practice. For these reasons, we have surely overlooked many relevant references for which we apologize in advance. We will assume knowledge of standard time series methods, and in particular methods related to nonstationary time series, at the level of a standard graduate course or advanced undergraduate course.

The remainder of this chapter is laid out as follows. In the next section we present and discuss definitions and general issues related to fractional integration. In Section 3 we discuss inference in fractional models, and in Section 4 we discuss fractional cointegration. In both sections we present both semiparametric and parametric methods as well as hypothesis testing approaches. Section 5 discusses applications of fractional integration and cointegration. However, we keep this section relatively brief in view of the survey of applications by Henry and Zaffaroni (2003). Finally, in Section 6 we conclude the chapter by briefly mentioning some additional related topics with further references to relevant work.

2 Fractional integration

For a number of years, increased interest has developed in a wider framework which takes into account that $I(0)$, and also $I(1)$, $I(2)$, ..., are very specific types of stationary and nonstationary processes, respectively. Thus, as a direct consequence of the definition of integrated process given above, one could think about a process which is $I(0)$ after d -differencing, where d is not necessarily an integer. Following early contributions by Granger and Joyeux (1980) and Hosking (1981), we can define this operation by means of the binomial expansion

$$\Delta^{-d} = (1 - L)^{-d} = \sum_{j=0}^{\infty} \pi_j(d) L^j, \tag{4}$$

where $\pi_0(d) = 1$ for all d ,

$$\pi_j(d) = \frac{\Gamma(j + d)}{\Gamma(d)\Gamma(j + 1)}, \quad j > 0,$$

and $\Gamma(\cdot)$ denotes the Gamma function with the convention $\Gamma(d) = \infty$ for $d = 0, -1, -2, \dots$

Letting $u_t \in I(0)$ with $E(u_t) = 0$, we can say that $X_t \in I(d)$ or $X_t \in F(d)$ (that is, X_t is a fractional process of order d) if

$$\Delta^d X_t = u_t, \quad t \in \mathbb{Z}. \tag{5}$$

The summation on the left-hand side of (5) is well-defined in the mean-square sense if $d > -1/2$. This property is sometimes referred to as invertibility. Furthermore, if $d < 1/2$, it can be shown that (5) has a unique stationary solution given by

$$X_t = \Delta^{-d} u_t = \sum_{j=0}^{\infty} \pi_j(d) u_{t-j}, \tag{6}$$

where the summation is well defined in this case. Thus, for (5) and (6) to make sense, it would usually be assumed that $d \in (-1/2, 1/2)$. Note that, by Stirling's approximation,

$$\pi_j(d) \sim \frac{1}{\Gamma(d)} j^{d-1} \text{ as } j \rightarrow \infty, \tag{7}$$

where “ \sim ” means that the ratio of the left- and right-hand sides tends to one. This shows how (5) and (6) are well-defined in the mean-square sense when $d > -1/2$ and $d < 1/2$, respectively. For example, (6) is well-defined when $d < 1/2$ because

$$\sum_{j=0}^{\infty} \pi_j^2(d) \leq K \sum_{j=1}^{\infty} j^{2(d-1)} < \infty,$$

where throughout K represents an arbitrarily large, finite constant.

In the particular case where u_t is a white noise process with $\text{Var}(u_t) = \sigma^2$, we say that $X_t = \Delta^{-d}u_t$ is a fractionally integrated noise. The impulse response function then takes the simple form

$$\frac{\partial X_t}{\partial u_{t-j}} = \pi_j(d), \quad j = 1, 2, \dots$$

In view of (7) and $d \in (-1/2, 1/2)$, it follows that shocks are transitory and their impact show a slow hyperbolic decay, which stands in contrast with the much faster exponential decay in stationary and invertible (finite-order) ARMA settings. Furthermore, it can be shown that the lag j autocovariance of the process X_t is given by

$$\gamma_X(j) = \frac{\sigma^2 \Gamma(1-2d) \Gamma(j+d)}{\Gamma(d) \Gamma(1-d) \Gamma(1+j-d)}, \quad j \geq 0,$$

where by Stirling’s approximation,

$$\gamma_X(j) \sim \frac{\sigma^2 \Gamma(1-2d)}{\Gamma(d) \Gamma(1-d)} j^{2d-1} \text{ as } j \rightarrow \infty. \quad (8)$$

Thus, $\gamma_X(j)$ also shows slow hyperbolic decay and, in fact, is not summable when $d > 0$. The spectral density of X_t in this case is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} |1 - e^{i\lambda}|^{-2d} = \frac{\sigma^2}{2\pi} (2 \sin(\lambda/2))^{-2d} \sim \frac{\sigma^2}{2\pi} \lambda^{-2d} \text{ as } \lambda \rightarrow 0, \quad (9)$$

which is unbounded at the origin when $d > 0$.

The behavior of the impulse response, autocovariance, and spectral density functions have given rise to the (now standard) terminology that X_t has long memory, short memory, or negative memory depending on whether $d > 0$, $d = 0$, or $d < 0$, respectively.

For a general $I(0)$ process u_t , Granger and Joyeux (1980) and Hosking (1981) showed that, under certain additional regularity conditions,

$$\gamma_X(j) \sim K j^{2d-1} \text{ as } j \rightarrow \infty, \quad (10)$$

where K is a constant depending only on the parameters, and

$$f_X(\lambda) = |1 - e^{i\lambda}|^{-2d} f_u(\lambda) \sim f_u(0) \lambda^{-2d} \text{ as } \lambda \rightarrow 0. \quad (11)$$

Thus, essentially the same results given for the fractionally integrated noise apply. The behavior of the spectral density also relates directly to the concept of an $I(0)$ process as implied by Robinson’s (1993) definition of a covariance stationary $I(d)$ process, which he defined as one with spectral density

$$g(\lambda) = |1 - e^{i\lambda}|^{-2d} \bar{g}(\lambda),$$

where $0 < \bar{g}(0) < \infty$. This implied definition of an $I(0)$ process also appears in Robinson (1994a), Marinucci and Robinson (2001), and Robinson and Yajima (2002).

When $d > 0$ we note that

$$\sum_{j=0}^{\infty} \pi_j(-d) = 0.$$

This implies that the process X_t is covariance stationary, but with zero spectrum at the origin, see also (9). It also implies that, for any constant μ and $d > 0$, the fractional difference $\Delta^d \mu = \mu \sum_{j=0}^{\infty} \pi_j(-d) = 0$, thus generalizing in a natural way the fact that $\Delta \mu = 0$.

2.1 Nonstationary fractional integration

The above discussion focused on the stationary case. However, it is both empirically and theoretically relevant to consider nonstationary situations, preferably including the celebrated $I(1)$ process as a special case. In such situations, the treatment of initial conditions (pre-sample observations) may be critical and, in fact, affects not only the choice of a convention regarding initial observations from which to obtain parameter estimates, but also the underlying process under consideration.

Two different definitions have been applied in the literature. The first of these (Type I) uses the idea in (3), while the second (Type II) uses truncation. These definitions mirror different definitions of fractional Brownian motions (also denoted Type I and Type II) to which the suitably normalized fractionally integrated processes converge; see Marinucci and Robinson (1999) for a very detailed analysis of the different types of convergence and Section 2.3 below for an overview.

Definition 3 (Type I fractionally integrated process) *Let k be an integer and d a real number such that $-1/2 < d - k < 1/2$. Given a scalar $I(0)$ process u_t with mean zero, the process X_t is Type I fractionally integrated of order d , denoted $X_t \in F_1(d)$, if*

$$\Delta^k X_t = \Delta^{-(d-k)} u_t.$$

We note that the differencing operators in Definition 3 cannot be combined. Specifically, the integer-order difference Δ^k involves only a finite summation, and hence always exists, whereas the fractional difference $\Delta^{-(d-k)}$ exists because it is assumed that $-1/2 < d - k < 1/2$. Thus, the integer-differenced series can be defined as a fractional process, which is then cumulated back. This interpretation has given rise to the notion of ‘difference-and-cumulate-back’.

As an example, when $1/2 < d < 3/2$, the process ΔX_t is an $I(d - 1)$ covariance stationary process, but X_t itself is nonstationary. Inverting the Δ -operator we find the following generalization of (3),

$$X_t = X_0 + \sum_{j=1}^t \Delta^{-(d-k)} u_j, \quad t \geq 1, \tag{12}$$

so that when $d = 1$ Definition 3 implies the standard definition of an $I(1)$ process known from the integer-integration framework. The same considerations apply for larger integer orders.

In particular, the choice of starting point $t = 0$ in (12) is arbitrary, as in (3), and for larger values of k the choice $t = 1 - k$ would result in a very similar equation and interpretation. As in (3), the values of X_t for $t < 1 - k$ may then be taken to be zero.

Some early works using Type I fractionally integrated processes include Sowell (1990), Hurvich and Ray (1995), Chan and Terrin (1995), Jeganathan (1999), Velasco (1999a,b), among others.

An important limitation of the Type I process in Definition 3 is the (necessary) requirement that $-1/2 < d - k < 1/2$. This implies that, in principle, a different definition is given for each value of the integer k . Perhaps more importantly, the definition does not cover the values $d = \pm 1/2, \pm 3/2$, etc. To make the parameter space for d compact in an estimation setting, it would then be necessary to remove open neighborhoods of these values from the parameter space. This is not only mathematically displeasing, but has important implications for hypothesis testing. The value $d = 1/2$, for example, is exactly the boundary between stationarity and nonstationarity and is therefore an obvious value to impose under the null hypothesis so it is unfortunate if it is not in the parameter space.

Definition 4 (Type II fractionally integrated process) *Let d be any real number. Given a scalar $I(0)$ process u_t with mean zero, the process X_t is Type II fractionally integrated of order d , denoted $X_t \in F_2(d)$, if*

$$X_t = \sum_{j=0}^{t-1} \pi_j(d) u_{t-j} = \Delta_+^{-d} u_t \tag{13}$$

with the obvious definition of the truncated operator Δ_+^{-d} .

This definition has different implications from those of Definition 3. For example, in the case $d < 1/2$ and $d \neq 0$, the Type II process is nonstationary while the Type I process is stationary. However, as shown in Lemma 3.4 of Robinson and Marinucci (2001), under relatively mild conditions,

$$\lim_{t \rightarrow \infty} |\text{Cov}(\Delta^{-k} \Delta^{-(d-k)} u_t, \Delta^{-k} \Delta^{-(d-k)} u_{t+k}) - \text{Cov}(\Delta_+^{-d} u_t, \Delta_+^{-d} u_{t+k})| = 0 \text{ for all } k \geq 0. \tag{14}$$

Thus, in this case, the Type II process $\Delta_+^{-d} u_t$ could be considered “asymptotically stationary” because the nonstationarity is due only to the truncation on the right-hand side of (13). For $d \geq 1/2$, $\Delta_+^{-d} u_t$ is purely nonstationary and the truncation in (13) ensures that the Type II process is well-defined in the mean-square sense. Robinson (2005a) provides additional bounds for differences between the two fractionally integrated processes. Finally, we note that the linear convolution in (13) can be computed very efficiently by the use of the fast Fourier transform; see Jensen and Nielsen (2014).

The Type II definition has been applied by Marinucci and Robinson (2000, 2001), Robinson and Marinucci (2001), Robinson and Hualde (2003), Shimotsu and Phillips (2005), Johansen and Nielsen (2010, 2012a), among others.

Note that both Definitions 3 and 4 imply that $E(X_t) = 0$. However, it is straightforward to extend the definitions to cover processes with a nonzero deterministic component by defining $Y_t = \mu_t + X_t$, where X_t is defined in either of Definitions 3 or 4 and μ_t is an arbitrary deterministic term.

In the remainder of this chapter, the type of fractional integration will in most cases either be clear from the context or the distinction is not relevant, and we will then use the generic notation $I(d)$. In cases where the distinction is important we will clearly indicate either Type I or Type II in the text.

Note that Definitions 3 and 4 are identical for $d = 0$ and for positive integers. Also note that they both impose zero initial conditions for the nonstationary range of the memory parameter d . However, this can easily be relaxed because, focusing on (5),

$$\Delta^d X_t = \sum_{j=0}^{\infty} \pi_j(-d) X_{t-j} = \sum_{j=0}^{t-1} \pi_j(-d) X_{t-j} + \sum_{j=t}^{\infty} \pi_j(-d) X_{t-j} = \Delta_+^d X_t + \Delta_-^d X_t, \quad (15)$$

using the obvious definitions for the operators Δ_+^d and Δ_-^d . Re-arranging and using (5),

$$\Delta_+^d X_t = u_t - \Delta_-^d X_t.$$

As justified by Johansen (2008) and Johansen and Nielsen (2010, 2012a), Δ_+^d is an invertible operator for any d (because it involves only a finite summation). Consequently,

$$X_t = \Delta_+^{-d} u_t - \Delta_+^{-d} \Delta_-^d X_t = \sum_{j=0}^{t-1} \pi_j(d) u_{t-j} + \mu_t, \quad (16)$$

where μ_t is an initial condition term which depends only on X_{-j} , $j \geq 0$.

Clearly, (16) generalizes (3), (12), and (13) to allow for a more general initial condition if conditions are imposed such that μ_t exists. In the special case where $d = 1$ it holds that $\mu_t = X_0$ which results in (3). The most common assumption in the literature (either implicitly or explicitly) is that $X_t = 0$ for $t \leq 0$, which results in $\mu_t = 0$ and hence the Type II definition with zero initial condition. More generally, Johansen and Nielsen (2010, 2012a, 2016) give several different conditions on X_{-j} , $j \geq 0$, that ensure existence of μ_t . For example, setting $X_{-j} = 0$ but only for all $j \geq N_0$, where N_0 could be an arbitrarily large but finite integer.

2.2 ARFIMA processes

The autoregressive fractionally integrated moving-average (ARFIMA, or sometimes FARIMA) class is the most popular family of parametric models displaying fractional integration.

Definition 5 (ARFIMA process) *Let d be such that $-1/2 < d < 1/2$ and let ε_t be a zero mean white noise process with $\text{Var}(\varepsilon_t) = \sigma^2$. Then X_t , $t \in \mathbb{Z}$, is an ARFIMA(p, d, q) process if*

$$\phi(L)\Delta^d X_t = \theta(L)\varepsilon_t, \quad (17)$$

where

$$\begin{aligned} \phi(L) &= 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p, \\ \theta(L) &= 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q, \end{aligned}$$

and $\phi(z)$, $\theta(z)$ have no common zeroes and satisfy $\phi(z)\theta(z) \neq 0$ for $|z| \leq 1$.

Under these conditions, the AR and MA polynomials, $\phi(z)$ and $\theta(z)$, imply both stationarity and invertibility. Hence, (17) has the unique stationary solution

$$X_t = \phi^{-1}(L)\theta(L)\Delta^{-d}\varepsilon_t, \quad (18)$$

and X_t is a stationary and invertible process. Furthermore, X_t has mean zero, spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} |\phi(e^{i\lambda})|^{-2} |\theta(e^{i\lambda})|^2 |1 - e^{i\lambda}|^{-2d} \sim \frac{\sigma^2 \theta^2(1)}{2\pi \phi^2(1)} \lambda^{-2d} \text{ as } \lambda \rightarrow 0,$$

and autocovariances that satisfy (10). Interestingly, for this process the memory parameter d drives the long-run behavior, whereas the ARMA parameters in $\phi(z)$ and $\theta(z)$ control the short-run dynamics.

For $d \leq -1/2$, the ARFIMA(p, d, q) process $X_t, t \in \mathbb{Z}$, can be defined as

$$\phi(L)X_t = \theta(L)\Delta^{-d}\varepsilon_t, \tag{19}$$

imposing that $\phi(z), \theta(z)$ have no common zeroes and that $\phi(z)\theta(z) \neq 0$ for $|z| \leq 1$. In this case, the unique stationary solution to (19) is given by (18), although this solution is not invertible.

Nonstationarity can be accommodated within the ARFIMA framework using two different strategies. One possibility in the spirit of the Type I process is to keep (17), but relax the conditions on $\phi(z)$ so as to allow for unit roots in the AR polynomial. This alternative was pursued by, e.g., Jeganathan (1999), and it implies that, depending on the number of unit roots, X_t could be $I(d+1), I(d+2)$, etc. Another possibility is to define the observable process as a Type II fractionally integrated process and maintain the conditions on $\phi(z)$. That is, instead of (17), defining

$$\phi(L)\Delta_+^d X_t = \theta(L)\varepsilon_t,$$

using the truncated filter in (13).

2.3 Fractional Brownian motion and FCLT

Marinucci and Robinson (1999) describe the two alternative forms of fractional Brownian motion that have been employed in the literature. These two forms are linked to the different types of fractionally integrated processes described before, in the sense that the two types of fractional Brownian motions are the limits of such processes in corresponding functional limit theorems. First, the Type I fractional Brownian motion was introduced by Mandelbrot and Van Ness (1968). Letting $B(r)$ be a standard Brownian motion, for $-1/2 < d < 1/2$, one version of this process is

$$B_d(r) = \begin{cases} \frac{1}{A(d)} \left(\int_{-\infty}^0 ((r-s)^d - (-s)^d) dB(s) + \int_0^r (r-s)^d dB(s) \right) & \text{for } r \geq 0, \\ \frac{1}{A(d)} \left(\int_{-\infty}^r ((r-s)^d - (-s)^d) dB(s) - \int_r^0 (-s)^d dB(s) \right) & \text{for } r < 0, \end{cases}$$

where

$$A(d) = \left(\frac{1}{2d+1} + \int_0^\infty ((1+s)^d - s^d)^2 ds \right)^{1/2}.$$

It can be shown that $E(B_d(r)) = 0$ for $r \in \mathbb{R}$,

$$E(B_d(r_1)B_d(r_2)) = \frac{1}{2} \left(|r_1|^{2d+1} + |r_2|^{2d+1} - |r_2 - r_1|^{2d+1} \right) \text{ for } r_1, r_2 \in \mathbb{R}, \tag{20}$$

and also that, for $j \in \mathbb{Z}$, the sequence of increments $B_d(j + 1) - B_d(j)$ is identically $N(0, 1)$ distributed. However, unlike in the short memory case, where $d = 0$ (and $B_0(r) = B(r)$), this sequence is correlated if $d \neq 0$ with autocovariances decaying at rate j^{2d-1} . It can be shown that if X_t is a mean zero Type I fractionally integrated process with $-1/2 < d < 1/2$, then

$$\frac{1}{T^{d+1/2}} \sum_{t=1}^{\lfloor Tr \rfloor} X_t \Rightarrow c_d B_d(r) \text{ for } r \in [0, 1], \tag{21}$$

where c_d is a constant that is proportional to the long-run variance of $\Delta^d X_t$.

Alternatively, for any $d > -1/2$, the Type II fractional Brownian motion $W_d(r)$ can be defined as

$$W_d(r) = \begin{cases} (2d + 1)^{1/2} \int_0^r (r - s)^d dB(s) & \text{for } r \geq 0, \\ -(2d + 1)^{1/2} \int_r^0 (s - r)^d dB(s) & \text{for } r < 0, \end{cases} \tag{22}$$

where, as for $B_d(r)$, $W_0(r) = B(r)$. It can be shown that $E(W_d(r)) = 0$ for $r \in \mathbb{R}$ and

$$E(W_d^2(r)) = |r|^{2d+1},$$

so that $E(W_d^2(r)) = E(B_d^2(r))$. However, as explained in Marinucci and Robinson (1999), the covariance structure of $W_d(r)$ is substantially different from (20), noting that the increments $W_d(r_2) - W_d(r_1)$ at equally-spaced intervals are nonstationary (although asymptotically stationary in a sense similar to (14)). Additionally, for $j = 0, 1, \dots$, the increments $W_d(j + 1) - W_d(j)$ are also nonstationary, having zero mean but a non-constant variance and time-dependent covariances. However, $W_d(r)$ is self-similar of degree $d + 1/2$, meaning that the distribution of $W_d(kr)$ is identical to that of $k^{d+1/2}W_d(r)$. This self-similarity property is also shared by $B_d(r)$. As in (21), it can be shown that if X_t is a mean zero Type II fractionally integrated process of order d with $d > -1/2$, then

$$\frac{1}{T^{d+1/2}} \sum_{t=1}^{\lfloor Tr \rfloor} X_t \Rightarrow k_d W_d(r) \text{ for } r \in [0, 1], \tag{23}$$

where, again, k_d is a constant that is proportional to the long-run variance of $\Delta_+^d X_t$.

The result (21) has been established under various conditions; see Davydov (1970), Taqqu (1975), and others. The result (23) has been established by Akonom and Gouriéroux (1987) and Marinucci and Robinson (2000). Assuming that u_t in (13) is a linear processes generated by a zero-mean white noise process ε_t , the most stringent condition is the requirement $E|\varepsilon_t|^q < \infty$ for some $q > \max\{2, 1/(d + 1/2)\}$, so that a large number of moments might be required if d is close to $-1/2$ (of course, Gaussianity suffices). Johansen and Nielsen (2012b) showed that this moment condition is necessary for the functional central limit theorem.

2.4 Sources of long memory

Several (economic) mechanisms have been proposed that generate long memory in observed data. These include aggregation of heterogeneous units, the error duration model, models of learning with rational agents, and marginalization in large systems. The earliest and most well-known mechanism seems to be that the aggregation of heterogeneous units can produce processes which display long memory properties, so we will discuss this in more detail.

The conceptual framework for the aggregation idea is the random coefficients AR(1) process,

$$X_{i,t} = \alpha_i X_{i,t-1} + \varepsilon_{i,t}, \quad t \in \mathbb{Z}, \quad i = 1, \dots, N, \quad (24)$$

where $\varepsilon_{i,t}$ is a zero mean white noise process independent of the random coefficients α_i , which are i.i.d. Beta with shape parameters $p > 1$ and $q > 1$. Robinson (1978) explored the properties of this process for a more general distribution function for α_i , establishing necessary and sufficient conditions for the continuity of the spectral density of $X_{i,t}$, exemplifying also his results to the Beta distribution case.

Granger (1980) aggregated $X_{i,t}$ in (24) over units, defining $X_t = \sum_{i=1}^N X_{i,t}$, and then showed that, as $N \rightarrow \infty$, the aggregated series X_t has autocovariances that decay at a hyperbolic rate like a long memory process. More specifically, the autocovariances of X_t behave like (10) with memory parameter $d = 1 - q/2$. Hence, the parameter q , which determines the slope of the density function of α_i near one, determines the amount of memory in X_t . Similar ideas have been explored by Zaffaroni (2004), Davidson and Sibbertsen (2005), and others, and aggregation as a source of long memory has been widely used in empirical works; e.g., Gadea and Mayoral (2006) for inflation series or Byers, Davidson, and Peel (2007) for series of political support.

Haldrup and Vera Valdés (2017) justified that the aggregated series, although displaying the usual long memory properties, does not satisfy an ARFIMA specification. In particular, they derived the properties of a fractionally differenced (with the adequate memory parameter) cross-sectional aggregated series and show that its autocovariances, while summable, do not correspond to those of an ARMA. This result introduces a word of caution when modelling aggregated series with a parametric model.

An alternative source of long memory is the error duration model proposed by Parke (1999). In this model, the i.i.d. shock ε_t has a stochastic duration n_s , after which it disappears from the observed process. Letting $g_{s,t} = \mathbb{I}(t \leq s + n_s)$, where $\mathbb{I}(A)$ denotes the indicator function of the event A , the observed process is $y_t = \sum_{s=-\infty}^t g_{s,t} \varepsilon_s$. Parke (1999) showed that, depending on the values of the survival parameters $p_k = P(g_{s,s+k} = 1)$, y_t can exhibit both long memory and fractional integration. For example, if $p_k = \Gamma(k+d)\Gamma(2-d)/(\Gamma(k+2-d)\Gamma(d))$ for $d \in [0, 1]$ then y_t is fractionally integrated noise with memory parameter d .

More recently, Chevillon and Mavroeidis (2017) have proposed a model of learning that can generate long memory endogenously, without any persistence in the exogenous shocks. The learning mechanism is embedded in a prototypical representative-agent forward-looking model in which agents' beliefs are updated using linear learning algorithms. Depending on the weights that agents place on past observations when they update their beliefs, and on the magnitude of the feedback from expectations to the endogenous variable, this model and learning mechanism can generate long memory in the observed series. This is distinctly different from the case of rational expectations, where the memory of the endogenous variable is determined exogenously.

Also very recently, Chevillon, Hecq, and Laurent (2018) have shown how a high-dimensional vector autoregressive (VAR) model of finite order can generate fractional integration in the marginalized univariate series. That is, when an observed univariate series is in fact generated by a high-dimensional VAR with cross-section dependence, this leads to long memory behavior in the univariate series. An example is given where the final equation representa-

tion of a VAR(1) leads to univariate fractional noises.

2.5 Long memory and structural breaks

An important early literature showed that structural change and unit roots can be confused (e.g. Perron, 1989, and Stock, 1994). This phenomenon generalizes to long memory in the sense that processes like regime-switching can display typical long memory (or fractionally integrated) behaviour. This issue was anticipated in the pioneering work of Klemes (1974) who justified that the Hurst phenomenon could be caused by two features which arise in hydrologic process: nonstationarity in the mean and random walks with one absorbing barrier. In a similar vein, Lobato and Savin (1997) pointed out nonstationarity in mean as a reason for rejecting the weak dependence hypothesis in favor of the long memory alternative. Granger and Teräsvirta (1999) showed that a similar behaviour could occur within a stationary setting. Specifically, they analyzed the nonlinear autoregressive model

$$y_t = \text{sign}(y_{t-1}) + \varepsilon_t,$$

where $\text{sign}(\cdot) = \pm 1$ with the same sign as the argument and ε_t is i.i.d. $N(0, \sigma^2)$. This process has zero mean and it is stationary with k 'th order autocorrelation given by $(1 - 2p)^k$, where $p = \Pr(\varepsilon_t > 1)$. Although the autocorrelation does not exhibit hyperbolic decay as in (8), it is clear that, as p decreases, switches are more infrequent so the process has more memory. Looking at the linear properties of the data through estimated correlations or the periodogram, it can well be concluded that the behavior of y_t is closer to that of an $I(d)$ process rather than short memory.

Diebold and Inoue (2001) justified how stochastic regime-switching models can display typical long memory behaviour. More specifically, they first studied the properties of a mixture model y_t given by

$$y_t = \begin{cases} 0 & \text{with probability } 1 - p_T, \\ \varepsilon_t & \text{with probability } p_T, \end{cases}$$

where ε_t is i.i.d. $N(0, \sigma^2)$. Here, it is easily shown that, letting the mixture probability p_T tend to zero at an appropriate rate, the variance of the partial sum of y_t behaves like that of an $I(d)$ process. Extensions of this simple mixture model present very similar patterns. Additionally, they studied the behaviour of a particular stochastic permanent break model given by

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, \\ \mu_t &= \mu_{t-1} + \frac{\varepsilon_{t-1}^2}{\gamma_T + \varepsilon_{t-1}^2} \varepsilon_{t-1}, \end{aligned}$$

where ε_t is i.i.d. $N(0, \sigma^2)$ and $\gamma_T > 0$ is varying with T . This model is an approximation of a mean-plus-noise extension of the mixture model presented before, so it is expected that a similar long memory pattern appears, which is also the case when γ_T grows with the sample size appropriately. Finally, they discussed the properties of the Markov-switching model of Hamilton (1989). Again, the variance of the partial sum of this process matches the behaviour of an $I(d)$ process when transition probabilities tend to one at an appropriate rate. In any case, Diebold and Inoue (2001) warn against the temptation to draw potentially naïve conclusions about the specific character of $I(d)$ processes. In their view, at least in

the settings described in their paper, structural change and $I(d)$ are two different labels for the same phenomenon, so there is little value in assigning labels like “true” or “spurious” to one or the other.

Similar works include Gourieroux and Jasiak (2001), who provided further evidence on the connection between infrequent breaks and long memory by analyzing the estimated correlogram of a regime-switching model, and Granger and Hyung (2004), who (in contemporaneous and independent work from Diebold and Inoue, 2001), studied a similar occasional break process. Specifically, they proposed

$$\begin{aligned} y_t &= m_t + \varepsilon_t, \\ m_t &= m_{t-1} + q_t \eta_t, \end{aligned}$$

where ε_t and η_t are white noises,

$$q_t = \begin{cases} 0 & \text{with probability } 1 - p_T, \\ 1 & \text{with probability } p_T, \end{cases}$$

and p_T converges slowly to zero with T such that Tp_T tends to a non-zero finite constant. In this setting, y_t displays some of the properties of an $I(d)$ process.

3 Inference in (univariate) fractionally integrated models

3.1 Semiparametric estimation of fractional integration

Semiparametric estimation of the memory parameter has a long tradition in the time series literature. These methods are based on the following local approximation of the spectral density of a stationary long memory process X_t ,

$$f_X(\lambda) \sim G\lambda^{-2d} \text{ as } \lambda \rightarrow \infty, \quad (25)$$

where G is a positive constant and $-1/2 < d < 1/2$; see also (9). Here, two classes of frequency domain estimators have become very popular: the log-periodogram approach proposed by Geweke and Porter-Hudak (1983)¹ whose theoretical properties were first analyzed by Robinson (1995a), and the local Whittle (also known as Gaussian semiparametric) estimator, proposed by Künsch (1987) and analyzed by Robinson (1995b). See Velasco (2006) for a very detailed review on these inference methods and their extensions.

The main reason why these methods have become so popular is that, unlike fully parametric approaches, semiparametric methods remain agnostic about the short-run structure of the model. The obvious advantage of this strategy is that it limits considerably the risk of misspecification. Additionally, given that these procedures rely on conditions of the spectral density $f_X(\lambda)$ near zero frequency, the estimation methods are robust to a nonstandard behaviour of $f_X(\lambda)$ at higher frequencies, like that implied by poles or zeros. Therefore, the methods are valid for a very wide range of processes. As will be seen, most of these methods are invariant to the presence of a non-zero mean. This is a very attractive feature of the semiparametric approach because in a stationary long memory context the mean is estimated potentially with a very slow rate (this is particularly the case if d is close to $1/2$); see

¹On the occasion of its 35th anniversary, this famous article was celebrated with a special issue of the *Journal of Time Series Analysis*; see Nielsen and Hualde (2019).

(21). Of course, this advantage is not exclusive of semiparametric methods, but, given that these procedures are nearly always expressed in the frequency domain, dealing with deterministic components is typically easier in this context.

The main price to pay for the greater generality that semiparametric estimation allows is a slower rate of convergence compared with that achieved by parametric alternatives. In particular, given a user-chosen parameter m (known as bandwidth) such that $m/T \rightarrow 0$ as $T \rightarrow \infty$, the typical rate achieved by semiparametric methods is \sqrt{m} which is slower than the usual \sqrt{T} parametric rate. Methods to improve this rate in a semiparametric framework have been proposed, but these always rely on imposing stronger smoothness conditions on $f_X(\lambda)$, so there is a trade-off between generality and rate of convergence. This discussion prompts another important issue which always arises in semiparametric estimation, namely bandwidth choice.

The log-periodogram estimator is based on taking logs in (25) and approximating the log-spectral density by the log-periodogram. Define the Fourier frequencies as $\lambda_j = 2\pi j/T$ for $j = 0, 1, \dots, T$, the discrete Fourier transform of an arbitrary vector process ζ_t as

$$w_\zeta(\lambda) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^T \zeta_t e^{it\lambda},$$

and for another vector process ξ_t (possibly the same one), the (cross-)periodogram as

$$I_{\zeta\xi}(\lambda) = w_\zeta(\lambda)w'_\xi(-\lambda), \quad I_\zeta(\lambda) = I_{\zeta\zeta}(\lambda), \tag{26}$$

with prime denoting transposition. Then, for an observable scalar process X_t with spectral density satisfying (25), the following regression model is proposed

$$\log I_X(\lambda_j) = \log G - 2d \log \lambda_j + v_j, \tag{27}$$

where v_j is an error term. Because (25) holds for $\lambda \rightarrow 0$, the approximation (27) is expected to work well for low frequencies λ_j , and the log-periodogram estimator of d , denoted \widehat{d}_{LP} , is derived from estimating (27) by ordinary least squares (OLS) using observations corresponding to $j = l, \dots, m$. Here, l is a trimming parameter, so the lowest Fourier frequencies $\lambda_0, \dots, \lambda_{l-1}$ are not considered in the estimation. By the properties of the complex exponential, removal of the components associated with λ_0 makes the estimation procedure invariant to a non-zero mean.

The limiting properties of \widehat{d}_{LP} were justified by Robinson (1995a) (actually, his estimator was more general, including possible pooling of adjacent Fourier frequencies). The assumptions included X_t being Gaussian and a smoothness condition for $f_X(\lambda)$, namely

$$f_X(\lambda) = G\lambda^{-2d}(1 + O(\lambda^\beta)) \text{ as } \lambda \rightarrow \infty, \quad \beta \in (0, 2]. \tag{28}$$

Thus, the larger β , the closer is $f(\lambda)$ to $G\lambda^{-2d}$ locally (for an ARFIMA process $\beta = 2$). Imposing the bandwidth condition

$$\frac{\sqrt{m} \log m}{l} + \frac{l (\log T)^2}{m} + \frac{m^{1+2\beta}}{T^{2\beta}} \rightarrow 0 \text{ as } T \rightarrow \infty, \tag{29}$$

and using very technical arguments to deal with the nonlinearity in $\log I_X(\lambda_j)$, Robinson (1995a) justified that

$$\sqrt{m}(\widehat{d}_{LP} - d) \rightarrow_d N(0, \pi^2/24) \text{ as } T \rightarrow \infty. \tag{30}$$

Nicely, as long as (29) is satisfied, (30) does not depend on the trimming parameter l , which was introduced because removal of the frequencies closest to zero allowed for controlling adequately an inherent bias problem in the estimation. From the theoretical viewpoint, some important improvements over Robinson's (1995a) results have been introduced, like relaxing the Gaussian assumption (Velasco, 2000) and avoiding trimming and using frequencies $\lambda_1, \dots, \lambda_m$ in the estimation (Hurvich, Deo and Brodsky, 1998).

An alternative semiparametric approach is based on maximizing a local approximation to the frequency domain Whittle likelihood. The parametric Whittle log-likelihood is given by

$$Q_T(\theta) = \frac{1}{T} \sum_{j=1}^T \left(\log f_X(\lambda_j; \theta) + \frac{I_X(\lambda_j)}{f_X(\lambda_j; \theta)} \right),$$

where the spectral density of X_t is considered to be a known function, $f_X(\lambda_j; \theta)$, up to a vector of unknown parameters, θ . Künsch (1987) suggested to use just m frequencies, with $m/T \rightarrow 0$ as $T \rightarrow \infty$, replacing also $f_X(\lambda_j; \theta)$ by the local approximation (25), to obtain the local Whittle log-likelihood

$$Q_m(G, d) = \frac{1}{m} \sum_{j=1}^m \left(\log(G\lambda_j^{-2d}) + \frac{I_X(\lambda_j)}{G\lambda_j^{-2d}} \right). \tag{31}$$

Concentrating $Q_m(G, d)$ with respect to G , we define the local Whittle estimator of d as

$$\hat{d}_{LW} = \arg \min_{d \in D} R_m(d),$$

where

$$R_m(d) = \log \hat{G}(d) - \frac{2d}{m} \sum_{j=1}^m \log \lambda_j, \quad \hat{G}(d) = \frac{1}{m} \sum_{j=1}^m \lambda_j^{2d} I_X(\lambda_j),$$

and D is a compact subset of $(-1/2, 1/2)$. The local Whittle estimator is attractive because of the likelihood interpretation and because it is more efficient than \hat{d}_{LP} under weaker conditions. On the other hand, while the log-periodogram estimator has an explicit form, the local Whittle is defined only implicitly and hence requires numerical optimization. Robinson (1995b) justified the limiting properties of \hat{d}_{LW} under certain conditions, including the smoothness condition (28),

$$X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} c_j^2 < \infty, \tag{32}$$

the ε_t in (32) being stationary and ergodic with finite fourth moment, $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$, $E(\varepsilon_t^2 | \mathcal{F}_{t-1}) = 1$ a.s., where \mathcal{F}_t is the σ -field of events generated by ε_s , $s \leq t$, and conditional (on \mathcal{F}_{t-1}) third and fourth moments of ε_t equal the corresponding unconditional moments. Under the bandwidth condition

$$\frac{1}{m} + \frac{m^{1+2\beta}(\log m)^2}{T^{2\beta}} \rightarrow 0 \text{ as } T \rightarrow \infty,$$

and following the usual strategy for dealing with implicitly defined estimators, Robinson (1995b) first proved that \hat{d}_{LW} is consistent (in fact, under weaker conditions than those above), and then

$$\sqrt{m}(\hat{d}_{LW} - d) \rightarrow_d N(0, 1/4) \text{ as } T \rightarrow \infty. \tag{33}$$

Compared with (30), (33) implies that \widehat{d}_{LW} is more efficient than \widehat{d}_{LP} . Indeed, the asymptotic variance of \widehat{d}_{LP} is approximately 1.6 times that of \widehat{d}_{LW} .

The finite sample performance of both estimators have been thoroughly illustrated by means of many Monte Carlo experiments. A particular phenomenon worth mentioning is the large bias reported in finite samples; see, e.g., Agiakloglou, Newbold, and Wohar (1993) or Nielsen and Frederiksen (2005). The classical example of this problem is the behaviour of the semiparametric estimators when the data is generated by a stationary ARMA(1,0) process with autoregressive parameter ϕ_1 close to (but below) 1. In this case, the spectral density of the process is obviously bounded, but in finite samples it is difficult to discern from that of a long memory process (with a spectral pole at zero frequency). As a result, semiparametric estimators typically overestimate the true memory parameter, $d = 0$. This finite-sample distortion can be very noticeable and, in fact, this is an illustration of a well-known identification problem: the ARFIMA(1,0,0) model with $\phi_1 = 1$ is identical to the ARFIMA(0,1,0) model.

As with most finite-sample issues, the problem is alleviated as T increases, although in some cases very slowly. For any given sample size, there is a trade-off between bias and variance in the choice of bandwidth, m , where a higher bandwidth tends to increase bias and reduce variance. Asymptotically, the mean-squared-error minimizing bandwidth choice is $m = O(T^{4/5})$, see Hurvich, Deo, and Brodsky (1998), but in practice this often results in very large bias rendering the estimator quite useless. One practical way to deal with the bias issue and choice of bandwidth is to perform the estimation with a range of bandwidth parameters and check sensitivity to bandwidth choice. Excessive sensitivity of the estimates to bandwidth choice is a clear signal that the bias problem is affecting estimation. At a theoretical level, Andrews and Guggenberger (2003) and Andrews and Sun (2004) have extended the log-periodogram and local Whittle approaches, respectively, by including local polynomial approximations to the short-memory component instead of the constant G . They have shown that this reduces the bias by an order of magnitude, while only increasing variance by a multiplicative constant.

The original work on semiparametric estimation has been extended in several directions. First, dealing with nonstationarity is a crucial issue in practice. In the spirit of the Type I fractionally integrated process, a simple possibility is to apply the “differencing and adding back” strategy, although this requires some a priori knowledge about the value of d . For example, if we know that $d \in (1/2, 3/2)$, we could estimate $d - 1$ from ΔX_t , obtaining $\widehat{d - 1}$ and then define $\widehat{d} = \widehat{d - 1} + 1$. Higher integration orders can be dealt with by taking further integer differences. An alternative solution was proposed by Velasco (1999a,b), who generalized the log-periodogram and local Whittle estimators to cover nonstationarity by means of tapering; that is, a type of periodogram smoothing. The two main drawbacks of this strategy are the following. First, the order of the needed taper depends on the memory of the process, hence, again, some a priori knowledge of d is needed. Second, tapering inflates the asymptotic variance of the estimators. On the positive side, tapering is a way of dealing with the possible presence of a polynomial deterministic structure, because an adequate taper periodogram is invariant to such deterministic components. In a similar vein, for a Type II fractionally integrated process, Shimotsu and Phillips (2005) proposed the exact local Whittle (ELW) estimator. This is a frequency domain estimator, but instead of “whitening”

the periodogram (like previous proposals), it applies fractional differences to the observable process. Nicely, this strategy permits consideration of any value of d as long as it is contained in a closed interval of width no larger than $9/2$ (although this requirement could possibly be relaxed by an alternative proof method), and it retains the semiparametric efficiency result (33). However, an important drawback of this estimator is in dealing with deterministic components. Shimotsu (2010) studied the sense in which an unknown mean could have undesirable effects on ELW estimation, and proposed modified estimators to accommodate an unknown mean and a polynomial trend. Finally, for a Type I fractionally integrated process, Abadir, Distaso, and Giraitis (2007) proposed the extended local Whittle estimator, which relies on the fully extended discrete Fourier transform and periodogram as an alternative way to deal with the nonstationarity. This estimator does not require estimation of the mean and achieves the result (33) for $d \in (-3/2, \infty)$, although excluding values $d = 1/2, 3/2, \dots$.

An important source of bias is structural breaks and other low-frequency contaminations. In fact, there is a broad research topic that analyzes several distinct issues which could fall under the general label of long memory and structural breaks; see Section 2.5. This literature has been developed in both parametric and semiparametric settings, but the latter appears to be more attractive in practice since any parametric choice for a short-memory component is complicated due to the possible presence of breaks. We will briefly comment on some of the most significant works. Hidalgo and Robinson (1996) proposed a test for a unique structural change at a known time point based on OLS estimation in a regression model with a long memory error whose autocorrelations are modeled semiparametrically. Relatedly, Ohanissian, Russell, and Tsay (2008) introduced a testing procedure for the null of stationary long memory against the alternative of spurious long memory (originating, e.g., from random level shift processes or other regime-switching-type processes). Their procedure takes advantage of the invariance of the long memory property under temporal aggregation, and it is based on the log-periodogram estimator. Focusing on the same problem, Qu (2011) proposed an alternative testing strategy based on the local Whittle likelihood.

Other works attempt to analyze the properties of long memory estimators when applied to processes with contaminations such as infrequent level shifts. Along the evidence presented in Section 2.5, it is well known that in this case the log-periodogram finds long memory. For example, Smith (2005) characterized the bias of the log-periodogram estimator and proposed a modified log-periodogram which alleviates the bias problem. Similar extensions and modifications are proposed and analyzed by McCloskey and Perron (2013) and Hou and Perron (2014). With a similar motivation, Iacone, Nielsen, and Taylor (2021) extended Lobato and Robinson's (1998) semiparametric test for the null hypothesis of $I(0)$ against fractionally integrated alternatives to test the more general null hypothesis of $I(d)$ for $d \in (-1/2, 1/2)$ while being robust to multiple level shifts at unknown points in time. Their semiparametric approach extends the parametric one of Iacone, Leybourne, and Taylor (2019) who allowed for the possibility of a single break at an unknown date.

Another important research area is the extension of semiparametric methods to deal with perturbed fractional processes, where the observable series is composed of a long memory process contaminated by an additive noise term. These processes are clearly related to the structural breaks contaminations, but the perturbed fractional processes are directly motivated by long memory stochastic volatility models, which have important applications in finance. Works here include Breidt, Crato, and de Lima (1998), Deo and Hurvich (2001),

Hurvich and Ray (2003), Sun and Phillips (2003), Arteché (2004, 2006), Hurvich, Moulines, and Soulier (2005), Haldrup and Nielsen (2007), Frederiksen and Nielsen (2008), Perron and Qu (2010), Frederiksen, Nielsen, and Nielsen (2012), McCloskey and Perron (2013), and Hou and Perron (2014).

Finally, other extensions have focused on improving the asymptotic performance of estimators in the semiparametric setting. The standard estimators have convergence rates no better than $T^{2/5}$ (see Giraitis, Robinson, and Samarov, 1997), although this rate can be substantially slower if the spectral density of the process, f , is not sufficiently smooth around frequency zero. In this sense, the possible smoothness of f around zero can be exploited by means of bias-reducing techniques in order to improve the semiparametric rate to the parametric rate, \sqrt{T} . As mentioned previously, Andrews and Guggenberger (2003) and Andrews and Sun (2004) exploited smoothness by extending the log periodogram and the local Whittle approaches, respectively, by local polynomials. Similarly, Robinson and Henry (2003) proposed a general M-estimator which, in particular, nests the log-periodogram and local Whittle approaches and uses higher-order kernels. These approaches have attractive theoretical properties, but a somewhat disappointing behaviour in finite samples (see, e.g., Nielsen and Frederiksen, 2005, and García-Enríquez and Hualde, 2019). A similar idea is to use global smoothness conditions on f outside frequency zero to obtain improvements by means of a broadband approach instead of focusing on a local-to-zero band of frequencies. This latter approach is analyzed by Moulines and Soulier (1999) and Hurvich and Brodsky (2001).

3.2 Parametric estimation of fractional integration

Let X_t , $t \in \mathbb{Z}$, be a Gaussian covariance stationary and invertible process with $E(X_t) = \mu$ and spectral density $f_X(\lambda_j; \theta)$ that is known up to a vector of unknown parameters, θ . Given a sample X_1, \dots, X_T , it is immediate to construct the log-likelihood. Let $\tilde{X} = (X_1, \dots, X_T)'$, $\Sigma(\theta) = \text{Var}(\tilde{X})$, and denote by $\tilde{\mathbf{1}}$ a $T \times 1$ vector of ones. Then, ignoring constants, the log-likelihood is

$$Q_{1T}(\theta, \mu) = \frac{1}{T} \log |\Sigma(\theta)| + \frac{1}{T} (\tilde{X} - \mu \tilde{\mathbf{1}})' \Sigma^{-1}(\theta) (\tilde{X} - \mu \tilde{\mathbf{1}}). \quad (34)$$

Letting $\bar{X} = T^{-1} \sum_{t=1}^T X_t$ and $X_t^{dem} = X_t - \bar{X}$, and using Θ to denote the parameter space for θ , the following estimators (at least) of θ have been considered in the literature,

$$\begin{aligned} \hat{\theta}_1 &= \arg \min_{\theta \in \Theta} Q_{1T}(\theta, \bar{X}), \\ (\hat{\theta}_2, \hat{\mu}) &= \arg \min_{\theta \in \Theta, \mu \in \mathbb{R}} Q_{1T}(\theta, \mu), \\ \hat{\theta}_3 &= \arg \min_{\theta \in \Theta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log f_X(\lambda; \theta) d\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_{X^{dem}}(\lambda)}{f_X(\lambda; \theta)} d\lambda \right), \\ \hat{\theta}_4 &= \arg \min_{\theta \in \Theta} \left(\frac{1}{T} \sum_{j=1}^{T-1} \log f_X(\lambda_j; \theta) + \frac{1}{T} \sum_{j=1}^{T-1} \frac{I_X(\lambda_j)}{f_X(\lambda_j; \theta)} \right). \end{aligned}$$

Here, $\hat{\theta}_1$ is a simplification over $\hat{\theta}_2$, while $\hat{\theta}_3$ and $\hat{\theta}_4$ optimize an approximation to the log-likelihood function called Whittle's (1953) approximation (in continuous and discrete versions). From a computational point of view, the frequency domain estimators $\hat{\theta}_3$ and $\hat{\theta}_4$ are simpler because there is no need to calculate $\Sigma^{-1}(\theta)$, which is complicated; see Sowell (1992)

for a discussion of exact maximum likelihood estimation of a stationary Gaussian zero-mean ARFIMA process (so, using (34) with $\mu = 0$), focusing on computational issues. Also, between $\widehat{\theta}_3$ and $\widehat{\theta}_4$, the latter seems preferable, because it is simpler computationally (taking advantage of the fast Fourier transform), and because it is invariant to a non-zero mean since, by the properties of the complex exponential, $I_{X^{dem}}(\lambda_j) = I_X(\lambda_j)$ for $j = 1, \dots, T - 1$.

These estimators have been well studied in the time series literature; for a comparison, see Robinson (1994a). The earliest works showed that, under regularity conditions (requiring at least that $f(\lambda; \theta_0)$ is continuous in λ , where subscript zero denotes true value), all estimators are consistent, asymptotically normal, and asymptotically efficient (e.g. Hannan, 1973). Obviously, the continuity condition rules out fractionally integrated X_t with $d_0 > 0$. Subsequently, Fox and Taqqu (1986) analyzed $\widehat{\theta}_3$ for a possibly long-range dependent stationary Gaussian sequence X_t and for a special parametrization of $f_X(\lambda; \theta) = \sigma^2 g_X(\lambda; \theta)$, where $\int_{-\pi}^{\pi} \log g_X(\lambda; \theta) d\lambda = 0$ (so that X_t/σ has one-step prediction error independent of θ). They showed consistency and asymptotic normality with rate of convergence \sqrt{T} (although this was not justified for the estimator of σ^2). Dahlhaus (1989) improved these results, justifying the consistency, asymptotic normality, and asymptotic efficiency of $\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3$, and suggesting it also for $\widehat{\theta}_4$. Nicely, his results hold without the need of the special parametrization imposed by Fox and Taqqu (1986). Giraitis and Surgailis (1990) and Hosoya (1996), among others, obtained analogous results without the need of the Gaussianity assumption.

For the nonstationary case, Velasco and Robinson (2000) proposed a tapered version of $\widehat{\theta}_4$, obtaining \sqrt{T} -consistency and asymptotic normality, but with an inflated asymptotic variance (due to tapering). With the exception of this work, the previous ones consider only (a subset of) the stationary region, which is clearly too restrictive for most practical purposes.

An estimation method that overcomes this problem and is valid for any value of d is the conditional (or truncated) sum-of-squares method. We exemplify this estimator for a zero-mean Type II fractionally integrated process. Let the observable X_t be given by

$$X_t = \Delta_+^{-d} u_t, \tag{35}$$

$$u_t = \theta(L; \varphi) \varepsilon_t, \quad \theta(s; \varphi) = \sum_{j=0}^{\infty} \theta_j(\varphi) s^j, \tag{36}$$

where ε_t is a zero-mean white noise with $\text{Var}(\varepsilon_t) = \sigma^2$ and d is a real number lying on a closed interval $[\nabla_1, \nabla_2]$ with $\nabla_1 < \nabla_2$. Here, $\theta(s; \varphi)$ covers for example ARMA processes or the exponential-spectrum model (Bloomfield, 1973), so φ is in general a $p \times 1$ dimensional vector of parametric short memory parameters. The aim is to estimate $\tau_0 = (d_0, \varphi'_0)'$ from observables $X_t, t = 1, \dots, T$. For $\tau = (d, \varphi)'$, define the residual,

$$\varepsilon_t(\tau) = \Delta^d \theta^{-1}(L; \varphi) X_t,$$

and the conditional sum-of-squares estimator,

$$\widetilde{\tau} = \arg \min_{\tau \in \mathcal{T}} \frac{1}{T} \sum_{t=1}^T \varepsilon_t^2(\tau), \tag{37}$$

where $\mathcal{T} = [\nabla_1, \nabla_2] \times \Psi$ and Ψ is a compact subset of \mathbb{R}^p . In (37) it is critical that $X_t = 0$ for $t \leq 0$, although this could be relaxed to allow for a finite number of non-zero (but

bounded) initial values as in Johansen and Nielsen (2010, 2012a, 2016). The loss function in (37) corresponds to the conditional log-likelihood function (concentrated with respect to σ^2), which has been applied for example by Box and Jenkins (1970) for the case where d is a known integer.

The conditional sum-of-squares estimator enjoys very attractive features. It has the same limit distribution as that derived by Fox and Taqqu (1986) and Dahlhaus (1989) for the parametric Whittle estimator, but without assuming Gaussianity, and it is asymptotically efficient under Gaussianity. Also, it is computationally very simple because it does not require inversion of a large $T \times T$ matrix. However, the main advantage over other parametric methods is that these results hold for any value of d , as long as it lies on an arbitrarily large compact interval, so that both the nonstationary and noninvertible range is covered by the theory. The conditional sum-of-squares estimator has been intensely studied by the literature. Li and McLeod (1986) proposed this method for stationary ARFIMA models with $0 < d < 1/2$ and Robinson (2006) analyzed it for this range of d . The first analysis of the estimator in nonstationary situations was Beran (1995), who applied it to a potentially nonstationary ARFIMA model, although the proof of consistency, which is a necessary preliminary step for establishing the limit distribution, is not rigorous due to a circular argument. Tanaka (1999) and Nielsen (2004a) gave local consistency proofs, while Hualde and Robinson (2011) and Nielsen (2015) provided rigorous global consistency proofs, so that asymptotic results for $\tilde{\tau}$ were finally formally established. These results were extended by Cavaliere, Nielsen, and Taylor (2015, 2017, 2020) to cover general forms of conditional and unconditional heteroskedasticity. The technical difficulty with the consistency proof arises when $[\nabla_1, \nabla_2]$ is of length greater than $1/2$ because the behaviour of the loss function is completely different depending on whether $d_0 - d < 1/2$ (where $\varepsilon_t(\tau)$ is asymptotically stationary) or $d_0 - d > 1/2$ (where $\varepsilon_t(\tau)$ is purely nonstationary), and the case where $d_0 - d$ is close to $1/2$ requires a very specific and detailed treatment.

From a theoretical perspective, the conditional sum-of-squares estimator appears to be preferable. However, it can exhibit noticeable finite-sample bias as documented by Nielsen and Frederiksen (2005). For example, this occurs for the ARFIMA(1, d ,0) model with a positive autoregressive parameter, which is an issue that is also present in the semiparametric setting. Nielsen and Frederiksen (2005) showed that the exact maximum likelihood estimator (MLE) suffers from the same problem, but also that it is somewhat alleviated for the Whittle estimator, although the latter covers only the stationary region.

The second limitation is dealing with deterministic components. This is a very relevant issue because empirical evidence suggests that observed economic time series are formed by both stochastic and deterministic components (see, e.g., Stock and Watson, 1988, Johansen and Juselius, 1990, Johansen, 1995). In fact, most of the previous parametric approaches allowed for an unknown and possibly non-zero mean, but this issue is complicated to tackle with a conditional sum-of-squares approach when allowing also for nonstationarity. In fact, there is no simple solution to this problem and it can be shown that apparently sensible simple strategies like, for example, “differencing and adding back” to eliminate a drift and then estimate $d - 1$ from ΔX_t do not work for all values of d ; see Hualde and Nielsen (2021). Given (35) and (36), Hualde and Nielsen (2020) analyze the model

$$Z_t = \mu t_+^\gamma + X_t, \tag{38}$$

where $t_+^\gamma = t^\gamma \mathbb{I}(t \geq 1)$. This is related to models of Robinson (2005b) and Robinson and Iacone (2005), although these authors included more deterministic terms and assumed that power law parameters (like γ) were known. Hualde and Nielsen (2020) considered conditional sum-of-squares estimation of the parameters in (38), that is $(\mu, \gamma, d, \varphi, \sigma^2)$, where d and γ were allowed to belong to arbitrarily large compact sets. Their main results were that parameter estimates related to the deterministic component are consistent and asymptotically normal only for parts of the parameter space depending on the relative strength of the stochastic and deterministic components, whereas consistency, asymptotic normality, and efficiency of parameter estimates related to the stochastic component was established for the entire parameter space.

3.3 Testing the unit root hypothesis against fractional alternatives

Testing the unit root (or $I(1)$) hypothesis has attracted great theoretical and empirical interest. In this context, there are several different approaches. The most common approach considers Dickey-Fuller or Phillips-Perron-type tests, which, within an autoregressive framework, test whether there is a unit root in the autoregressive polynomial. These tests are designed against stationary (specifically, $I(0)$) alternatives, and have been shown to have poor power against fractionally integrated alternatives (Diebold and Rudebusch, 1991a). The fractionally integrated model allows for a different strategy; that is, to test $H_0 : d = 1$ versus the alternative $H_1 : d \neq 1$ (of course, this alternative could be one-sided). The previously established theory in semiparametric or parametric contexts can be used to derive Wald, likelihood ratio, or Lagrange multiplier tests with standard asymptotic null distributions.

The main caveat here must be to ensure adequate model fit. That is, to ensure that the estimated model fits the data prior to conducting the statistical test, or in other words to test whether the assumptions underlying the asymptotic theory are likely to be satisfied for the case at hand. This is usually done by analysis of the model residuals by testing, for example, whether these are serially uncorrelated, (conditionally) homoskedastic, etc. This can be challenging with a purely LM-based approach, e.g. Robinson (1994b), where the model has not been estimated and no residuals are available upon which to base such model diagnostics. However, with a Wald or likelihood ratio approach, conducting such standard model diagnostics prior to testing the hypothesis of interest should be straightforward.

3.4 Dickey-Fuller-type tests with fractionally integrated errors

The first important contribution here is Sowell (1990), who analyzed the asymptotic behaviour of the OLS estimator of ϕ , say $\hat{\phi}$, in the model

$$X_t = \phi X_{t-1} + \Delta^{-d} u_t,$$

where d is a known value such that $-1/2 < d < 1/2$, $u_t \in I(0)$, and $\phi = 1$ so there is a unit root in the autoregressive polynomial. Specifically, Sowell (1990) proved that, under mild regularity conditions,

$$T^{2d\mathbb{I}(d < 0) + 1} (\hat{\phi} - 1) \rightarrow_d A(d), \tag{39}$$

where $A(d)$ is a random variable which depends on d ($A(0)$ is the well known Dickey-Fuller distribution). Result (39) is the basis to analyze the limiting distribution of the Dickey-Fuller t -statistic. It is noticeable that in (39) the convergence rate (and, in fact, the form of $A(d)$) depends heavily on whether $d < 0$ or $d \geq 0$. In the former case, the rate can be very

slow if d is close to $-1/2$. In the latter case, the usual T -consistency is obtained but the form of $A(d)$ changes substantially depending on whether $d > 0$ or $d = 0$.

Instead of considering the OLS estimator of ϕ , Ling and Li (2001) considered the Gaussian MLE, which (for i.i.d. u_t) is the OLS estimator of $\Delta^d X_t$ on $\Delta^{d-1} X_{t-1}$. This estimator has the usual Dickey-Fuller distribution regardless of the value of d , as is seen by (equivalently) considering $Y_t = \Delta^d X_t$ instead of X_t . Relatedly, Chan and Terrin (1995) developed asymptotic theory for the OLS estimator in an autoregressive process with fractionally integrated innovations.

In a similar fashion, Dolado, Gonzalo, and Mayoral (2002) introduced the model

$$\alpha(L)\Delta X_t = \phi\Delta^{d_1} X_{t-1} + \varepsilon_t, \tag{40}$$

where ε_t is white noise, $\alpha(z)$ is a polynomial with roots outside the unit circle, and d_1 is fixed or pre-estimated. They proposed to test $H_0 : \phi = 0$, in which case $X_t \in I(1)$, against $H_1 : \phi < 0$, in which case it is claimed that $X_t \in I(d_1)$. Indeed, $X_t \in I(0)$ if $d_1 = 0$, this being the well-known result from the standard Dickey-Fuller approach. However, in general, determining the integration order of X_t under H_1 would require conditions on the parameters under which all roots of the polynomial $\pi(z) = \alpha(z)(1-z)^{1-d_1} - \phi z$ are outside the unit circle. The conditions given by Dolado, Gonzalo, and Mayoral (2002) are insufficient, as shown by the counter-example in footnote 3 of Johansen and Nielsen (2010). Indeed, general conditions appear impossible to derive (see Johansen, 2008), so the integration order of X_t under the alternative is unknown. In this sense the model (40) is ill-posed. Similar issues arise in a related model that was proposed and analyzed by Lobato and Velasco (2006).

Finally, Johansen and Nielsen (2010) introduced a fractional autoregressive model where a unit root test can be implemented. The idea behind their model is the following. Consider the usual autoregressive model in error correction mechanism form

$$\Delta Y_t = \pi Y_{t-1} + \sum_{i=1}^k \phi_i \Delta Y_{t-i} + \varepsilon_t, \tag{41}$$

where ε_t is an i.i.d. sequence with $E(\varepsilon_t) = 0$ and $\text{Var}(\varepsilon_t) = \sigma^2$. Replace in (41) the usual difference and lag operators, Δ and $L = 1 - \Delta$, by the corresponding fractional operators Δ^b and $L_b = 1 - \Delta^b$, respectively, obtaining

$$\Delta^b Y_t = \pi L_b Y_t + \sum_{i=1}^k \phi_i \Delta^b L_b^i Y_t + \varepsilon_t. \tag{42}$$

Defining $X_t = \Delta^{b-d} Y_t$, it is immediate to obtain

$$\Delta^d X_t = \pi L_b \Delta^{d-b} X_t + \sum_{i=1}^k \phi_i L_b^i \Delta^d X_t + \varepsilon_t, \tag{43}$$

which is the fractional autoregressive model employed by Johansen and Nielsen (2010).

Model (43) has interesting properties that are very similar in essence to those of the standard autoregressive process (41). Note that (43) can be expressed as

$$\zeta(L_b)\Delta^{d-b} X_t = \varepsilon_t,$$

where

$$\zeta(z) = 1 - z - \pi z - \sum_{i=1}^k \phi_i z^i (1 - z), \quad (44)$$

so $\zeta(z)$ is the usual $(k + 1)$ -order autoregressive polynomial from (41). Thus, by performing the change of variables $z = 1 - (1 - v)^b$, the properties of the process (43) depend on the roots of $\zeta(z)$. The main representation result for this model is that if all roots of $\zeta(z)$ are outside a modified unit circle (the modification having to do with the change of variables), then $X_t \in I(d - b)$. Alternatively, if $\pi = 0$ and all remaining roots are outside the modified unit circle, then $X_t \in I(d)$. Consequently, this model can be the basis for a unit root test, where $H_0 : \pi = 0$, in which case $X_t \in I(d)$, versus $H_1 : \pi \neq 0$, in which case $X_t \in I(d - b)$. This generalizes the usual unit root tests, which are obtained in the special case $d = b = 1$. As in Sowell (1990) or Ling and Li (2001), d is not restricted to be 1, but unlike earlier papers, the gap b between integration orders under H_0 and H_1 is also not restricted to be 1.

Johansen and Nielsen (2010, 2012a) proved consistency of the MLE of $(d, b, \pi, \phi_1, \dots, \phi_k)$. They also showed that, when $b_0 < 1/2$, the estimators are jointly asymptotically Gaussian, and when $b_0 > 1/2$, the estimator of $(d, b, \phi_1, \dots, \phi_k)$ are asymptotically Gaussian whereas that of π is a functional of Type II fractional Brownian motion depending on b_0 and is asymptotically independent of the other estimator. Finally, they derived the asymptotic distribution of the likelihood ratio (unit root) test for $\pi = 0$, which is another functional of Type II fractional Brownian motion that depends only the fractional gap b_0 , and not d_0 . Thus, this distribution is not pivotal, but a plug-in approach can be used to derive critical values using the estimator of b_0 (see MacKinnon and Nielsen, 2014).

4 Fractional cointegration

Once the concept of fractional integration has been introduced, that of fractional cointegration is a natural step forward. In fact, the definition of cointegration given in Definition 2 can be directly applied to a fractional situation. In the fractional context, however, there are many more possibilities of cointegration allowed under Definitions 1 and 2 compared with the $I(1)/I(0)$ paradigm, especially when the observables have different orders of fractional integration.

For example, denoting by d_{\max} the highest integration order of the observables, Definitions 1 and 2 imply that there is cointegration whenever a linear combination of the observables has integration order smaller than d_{\max} . Thus, if the i 'th variable has fractional integration order smaller than d_{\max} , then cointegration arises trivially with the unit vector, e_i , as a cointegration vector. Consequently, Flôres and Szafarz (1996) strengthened slightly this idea, requiring also that the linear combination must involve nontrivially an $I(d_{\max})$ observable. An alternative (and much stronger) definition is given by Robinson and Marinucci (2003), which required the linear combination of the observables to have an integration order smaller than d_{\min} , where d_{\min} is the smallest integration order of the observables. Robinson and Yajima (2002) grouped the vector of observables into subvectors of variables with the same integration order, and define cointegration whenever there exists at least one of these subvectors that is cointegrated in Engle and Granger's (1987) sense. They also exemplified the differences among the various definitions. In a bivariate situation these definitions are equivalent. For any of these alternative definitions of cointegration, a crucial concept is the

cointegrating rank; that is, the number of linearly independent cointegrating vectors. The space spanned by these cointegrating vectors is the cointegrating space.

Based on the different definitions of (fractional) cointegration, many different models displaying cointegration have been proposed. To demonstrate these ideas, we now discuss a rather general model. Let u_t , $t \in \mathbb{Z}$, be a p -dimensional covariance-stationary unobservable process with zero mean and spectral density, $f_u(\lambda)$, given by

$$E(u_0 u'_j) = \int_{-\pi}^{\pi} e^{ij\lambda} f_u(\lambda) d\lambda, \tag{45}$$

which is assumed to be nonsingular and continuous at all frequencies, c.f. (1). As will be clarified below, the treatment of $f_u(\lambda)$ is what distinguishes semiparametric and parametric inference methods. In the former, $f_u(\lambda)$ is considered a nonparametric function, whereas in the latter, $f_u(\lambda)$ is assumed known up to a finite-dimensional vector of unknown parameters. Also, for real numbers d_i , $i = 1, \dots, p$, such that

$$d_1 \leq d_2 \leq \dots \leq d_p, \tag{46}$$

and a $p \times p$ nonsingular matrix Υ , we define the p -dimensional vector of observables X_t , $t \in \mathbb{Z}$, as

$$\Upsilon X_t = \text{diag} \left(\Delta_+^{-d_1}, \Delta_+^{-d_2}, \dots, \Delta_+^{-d_p} \right) u_t, \tag{47}$$

noting that there is no loss of generality in (46). The cointegrating properties in the general system (47) depend on Υ and the possible presence of strict inequalities in (46). For example, if all elements in the last column of Υ^{-1} are non-zero, then all individual components of X_t are $I(d_p)$. This is standard in the traditional cointegrating setting where $d_p = 1$. If, for some $1 \leq r < p$, we set

$$d_r < d_{r+1} = \dots = d_p,$$

then (47) is a system with cointegrating rank r , where the first r rows of Υ are the cointegrating vectors. More general cointegrating possibilities can be captured by (47), and a general specification for Υ and (46) which allows for the possibility of multiple cointegrating subspaces is given in Hualde and Robinson (2010). These subspaces appear naturally in a fractional setting and are characterized by particular directions in the cointegrating space that lead to larger reductions in the integration orders.

To illustrate some interesting characteristics of a cointegrated system, we next discuss a special case of (46)–(47). For $r < p$ we set

$$\Upsilon = \begin{pmatrix} I_r & -\beta' \\ 0_{p-r,r} & I_{p-r} \end{pmatrix} \tag{48}$$

and

$$d_1 = \dots = d_r < d_{r+1} = \dots = d_p, \tag{49}$$

where I_s is the s -dimensional identity matrix, $0_{i,j}$ is a $i \times j$ matrix of zeros and β is a $(p-r) \times r$ unrestricted matrix. This particular case has been heavily stressed in theory and practice and, as explained in Hualde and Robinson (2010), identification restrictions such as those imposed in (48)–(49) are always possible whenever the cointegrating rank among the observables is r and there is only one cointegrating subspace. However, given this situation,

the ordering of the variables in X_t is not innocuous, although this ordering can be inferred from data (Hualde, 2008). Letting $X_t = (X'_{1t}, X'_{2t})'$ and $u_t = (u'_{1t}, u'_{2t})'$, where X_{1t} and u_{1t} are $r \times 1$ vectors while X_{2t} and u_{2t} are $(p - r) \times 1$ vectors, (47)–(49) represent a fractional extension of Phillips' (1991a) triangular system,

$$X_{1t} = \beta' X_{2t} + \Delta_+^{-(d-b)} u_{1t}, \tag{50}$$

$$X_{2t} = \Delta_+^{-d} u_{2t}, \tag{51}$$

where for some $b > 0$ we use the simplifying notation $d - b = d_1$ and $d = d_{r+1}$. In (50)–(51) all individual components of X_{2t} are $I(d)$, whereas all cointegrating errors are $I(d - b)$. The individual components of X_{1t} are in general also $I(d)$, although they could be $I(d - b)$ if the corresponding row of β' only contains zeros. In the latter case, such a cointegrating relation will be denoted as trivial and just indicates that a particular observable has an integration order smaller than d , so that there exists a unit cointegrating vector. Note also that, apart from the restriction in (49), both integration orders are unrestricted, so many different cointegrating possibilities are allowed by (50)–(51), including equilibrium relations among nonstationary observables that are not $I(1)$ or even among stationary ones (the so-called stationary cointegration). It also permits a slower convergence to equilibrium due to long memory cointegrating errors. As denoted by Phillips and Loretan (1991), (50)–(51) with $d = b = 1$, represents “a typical cointegrated system” in structural form. In particular, (50) could be regarded as a stochastic version of the partial equilibrium relationships $X_{1t} - \beta' X_{2t}$, with $\Delta_+^{-(d-b)} u_{1t}$ representing deviations from this equilibrium, whereas (51) is a reduced form equation.

The Type II nature of (50)–(51) accommodates integration orders in both stationary or nonstationary ranges, but it implies that, irrespective of the values taken by d and b , X_t is nonstationary as is the differenced process $\Delta_+^d X_t$ (although it is asymptotically stationary). It is interesting to describe some of the properties of the stationary version of $\Delta_+^d X_t$, that is $x_t = (u'_{2t}\beta + \Delta^b u'_{1t}, u'_{2t})'$, because they illustrate some key features of cointegrated models that form the basis of many inferential procedures. Noting (45), the spectral density of x_t is

$$f_x(\lambda) = \begin{pmatrix} (1 - e^{i\lambda})^b I_r & \beta' \\ 0_{p-r,r} & I_{p-r} \end{pmatrix} f_u(\lambda) \begin{pmatrix} (1 - e^{-i\lambda})^b I_r & 0_{r,p-r} \\ \beta & I_{p-r} \end{pmatrix}.$$

Using the approximations $f_u(\lambda) = (2\pi)^{-1}\Omega(1+O(\lambda^2))$ and $|1 - e^{i\lambda}|^{2b} = 2^b(1 - \cos \lambda)^b = O(\lambda^{2b})$ as $\lambda \rightarrow 0$, it holds that

$$f_x(\lambda) = \begin{pmatrix} \beta' \Omega_{22} \beta & \beta' \Omega_{22} \\ \Omega_{22} \beta & \Omega_{22} \end{pmatrix} (1 + O(\lambda^b) + O(\lambda^2)) \text{ as } \lambda \rightarrow 0, \tag{52}$$

where Ω_{22} is the $(p - r) \times (p - r)$ lower right block of $2\pi f_u(0)$, i.e. the so-called long-run variance of u_{2t} . For $d < 1/2$, (52) motivates a multivariate version of the local approximation (11) given by

$$f_x(\lambda) \sim G \lambda^{-2d} \text{ as } \lambda \rightarrow 0, \tag{53}$$

where “ \sim ” is taken elementwise for real and imaginary parts separately and

$$G = \begin{pmatrix} \beta' \Omega_{22} \beta & \beta' \Omega_{22} \\ \Omega_{22} \beta & \Omega_{22} \end{pmatrix} = (\beta, I_{p-r})' \Omega_{22} (\beta, I_{p-r}) \tag{54}$$

is a $p \times p$ matrix with rank $p-r$. Additionally, it can be shown that, as $\lambda \rightarrow 0$, the r smallest eigenvalues of $f_x(\lambda)$ are $O(\lambda^b)$, so they approach zero, whereas the $p-r$ largest eigenvalues are bounded away from zero in the limit; see also Velasco (2003a) and Nielsen (2004b) for further details.

In the following subsections we will present some testing and estimation procedures for fractional cointegration. These methods can be classified in several different ways. One possibility is to distinguish between semiparametric and parametric methods, depending on whether the parametric structure of the $I(0)$ error input process that generates the observables and cointegrating errors is known or unknown. As previously done in Section 3, we will focus on this method classification, but within each category we also distinguish between what Jeganathan (1997) denotes as first and second stage procedures.

According to Jeganathan (1997), limiting distributions of first stage procedures are non-standard and unsuitable for use in statistical inference, whereas second stage procedures imply estimators belonging to the locally asymptotically mixed normal family. This class of estimators enjoy several attractive features. They are symmetrically distributed, median unbiased, and an optimal theory of inference applies under Gaussian assumptions (Phillips, 1991a, and Saikkonen, 1991). Also, they lead to Wald and likelihood ratio test statistics with standard χ^2 null limit distributions. In the context of standard cointegration with unit root observables and $I(0)$ cointegrating errors, Jeganathan (1997) suggested that first stage procedures could be used to test for the presence of unit roots in a given model, and then, by second stage methods one could estimate cointegrating relationships on the model where the unit roots tested in the first stage are imposed. Thus, second stage methods incorporate in the estimation this type of information, allowing implementation of endogeneity corrections that leads to the desirable asymptotic properties. As a practical consequence, traditionally, the main difference between the two types of procedures is that first stage methods do not require knowledge of the integration orders involved, whereas second stage methods do (see, e.g., Phillips, 1991a).

In the context of fractional time series models, assuming knowledge of the integration orders involved is very unrealistic in general, even after pretesting. As will be seen below, we will present estimation and inference methods which share in many cases the optimal asymptotic properties of the second stage procedures without assuming knowledge of the integration orders.

4.1 Tests for fractional cointegration (rank)

As mentioned before, a fair number of tests have been proposed, including parametric, semiparametric, and nonparametric approaches. We will illustrate with some detail one method belonging to each category.

First, in a parametric setting, Breitung and Hassler (2002) generalized the univariate score test against fractional alternatives (see, e.g., Robinson, 1991, 1994b, Agiakloglou and Newbold, 1994, Tanaka, 1999) to a score-type test for fractional cointegration; see also the score test for cointegration in Nielsen (2004c) and the multivariate score tests in Nielsen (2004d, 2005a). Let Y_t be a p -dimensional vector of observables and suppose that, for $0 \leq r < p$, there exists a full rank orthonormal $p \times p$ matrix $R = (R_{p-r}, R_r)$, where R_{p-r}

and R_r have $p - r$ and r columns, respectively, such that

$$R'X_t = \begin{pmatrix} \Delta_+^{-d}I_{p-r} & 0 \\ 0 & \Delta_+^{-(d-b)}I_r \end{pmatrix} U_t \tag{55}$$

for some $d > 1/2$ and $b > 0$ and a p -dimensional process $U_t \in I(0)$. If $r = 0$ then $R = R_p$ and (55) becomes $R'X_t = \Delta_+^{-d}U_t$. Under (55), X_t is clearly cointegrated with rank r . We present Breitung and Hassler's (2002) proposal for the white noise case, where the p -dimensional vector $(\Delta_+^d X_t' R_{p-r}, \Delta_+^{d-b} X_t' R_r)'$ (or $\Delta_+^d X_t$ if $r = 0$) is a zero mean i.i.d. sequence with finite variance. Inspired by Johansen (1995), the test on cointegration rank is based on the eigenvalues derived as the solutions to

$$\left| \lambda \widehat{\Sigma} - S'_{10} S_{11}^{-1} S_{10} \right| = 0, \tag{56}$$

where $\widehat{\Sigma} = T^{-1} \sum_{t=1}^T \Delta_+^d X_t \Delta_+^d X_t'$, $S_{11} = T^{-1} \sum_{t=1}^T \sum_{j=1}^{t-1} j^{-1} \Delta_+^d X_{t-j} \sum_{k=1}^{t-1} k^{-1} \Delta_+^d X_{t-k}'$, and $S_{10} = T^{-1} \sum_{t=1}^T \sum_{j=1}^{t-1} j^{-1} \Delta_+^d X_{t-j} \Delta_+^d X_t'$. Here, the partial sum in Johansen's setting (where $d = 1$), $X_{t-1} = \sum_{j=1}^{t-1} \Delta_+ X_{t-j}$, is replaced by the weighted sum $\sum_{j=1}^{t-1} j^{-1} \Delta_+^d X_{t-j}$. The latter originates from evaluation of the derivative of $\Delta_+^d X_t$ with respect to d at $d = 0$ (e.g. Tanaka, 1999), noting that for $j \geq 1$, $\partial \pi_j(d) / \partial d|_{d=0} = j^{-1}$; see (4). Ordering the eigenvalues derived from (56) as $\widehat{\lambda}_1 \leq \widehat{\lambda}_2 \leq \dots \leq \widehat{\lambda}_p$, the statistic to test $H_0 : r = r_0$ proposed by Breitung and Hassler (2002) is

$$\Lambda_{r_0}(d) = \sum_{j=1}^{p-r_0} \widehat{\lambda}_j,$$

which is asymptotically χ^2 distributed with $(p - r_0)^2$ degrees of freedom under H_0 . Additionally, under the alternative $H_1 : r > r_0$, the authors claim that $\Lambda_{r_0}(d)$ diverges to ∞ at rate T . The test can be applied sequentially to estimate r by considering $r_0 = 0, 1, \dots$, and denoting by \widehat{r} the first non-rejected null value. Short-run dynamics and/or deterministic terms can be accommodated by replacing observables by appropriate residuals. Neatly, the results do not depend on the cointegrating gap b (although b presumably affects power), and in fact a unique cointegrating gap b is not even required. Importantly, however, the procedure relies on knowledge of d and this is a serious issue because replacing this parameter by an estimator affects the asymptotic null distribution. Equally importantly, there appears to be no way to verify whether the assumptions underlying the asymptotic distribution results are likely satisfied (i.e., to ensure that the lag structure is sufficiently rich); c.f. the discussion in Section 3.3.

Also in a parametric framework, in Subsection 4.4 below we will present a more detailed discussion of the fractionally cointegrated VAR model, which generalizes the cointegrated VAR model of Johansen (1995) to the fractional context. Among other things, this of course includes testing for the cointegrating rank.

In a semiparametric setting, Robinson and Yajima (2002) introduced a procedure to determine the cointegrating rank in a fractionally cointegrated system. Their model is quite general, allowing the components of the p -dimensional vector of observables X_t to have distinct integration orders, although belonging to the stationary and invertible region. Then, cointegration is inferred within blocks of variables sharing the same order. For this reason, without loss of generality, we exemplify their approach for the case where all observables have

the same integration order, d , with $|d| < 1/2$. We assume that the spectral density matrix of X_t behaves locally as in (53) but for a generic matrix G which is finite, non-negative definite, and with non-zero diagonal elements. Note that G is positive definite if and only if X_t is not cointegrated, see (54). The cointegrating rank is $r = p - \text{rank}(G)$ for $0 \leq r < p$, which is the basis of the procedure. Under regularity conditions, with an appropriate bandwidth m satisfying at least $m/T \rightarrow 0$ and a particular estimator \widehat{d} of d , a consistent estimator of G is

$$\widehat{G}(\widehat{d}) = \frac{1}{m} \sum_{j=1}^m \lambda_j^{2\widehat{d}} \text{Re}(I_X(\lambda_j)).$$

Assuming the $p-r$ nonzero eigenvalues of G are distinct, let λ_i (resp. $\widehat{\lambda}_i$) be the i 'th eigenvalue of G (resp. $\widehat{G}(\widehat{d})$), $i = 1, \dots, p$, ordered such that $\lambda_1 > \lambda_2 > \dots > \lambda_{p-r} > 0$, with $\lambda_{p-r+1} = \dots = \lambda_p$ if $r > 0$, and $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \dots \geq \widehat{\lambda}_p$. The main result in Robinson and Yajima (2002) is that, under regularity conditions, $m^{1/2}(\widehat{\lambda}_i - \lambda_i)$ are asymptotically independent for $i = 1, \dots, p$, converge in distribution to $N(0, \lambda_i^2)$ for $i = 1, \dots, p-r$, and are $o_p(1)$ for $i = p-r+1, \dots, p$ if $r > 0$. Defining, for a user chosen number $v(T) > 0$ which tends to zero as $T \rightarrow \infty$,

$$L(u) = v(T)(p-u) - \sum_{i=1}^{p-u} \widehat{\lambda}_i,$$

the cointegrating rank is estimated consistently by

$$\widehat{r} = \arg \min_{u=1, \dots, p-1} L(u).$$

A related procedure can be found in Chen and Hurvich (2003), see also Section 4.2, while Nielsen and Shimotsu (2007) extended Robinson and Yajima's (2002) approach to (asymptotically) stationary and nonstationary Type II fractionally integrated processes.

In a nonparametric setting, Nielsen (2010) proposed a variance ratio testing approach for fractional cointegration motivated by the following observation for an univariate series. Let $u_t \in I(0)$ and $x_t = \Delta_+^{-d} u_t$ with $d > 1/2$. For $d_1 > 0$, construct the partial sum $\tilde{x}_t = \Delta_+^{-d_1} x_t$. Under regularity conditions on u_t , it can be shown that, for $r \in [0, 1]$,

$$T^{1/2-d} x_{[rT]} \Rightarrow \sigma_u W_{d-1}(r) \text{ and } T^{1/2-d-d_1} \tilde{x}_{[rT]} \Rightarrow \sigma_u W_{d+d_1-1}(r),$$

see (23), where σ_u^2 is the long-run variance of u_t . Then, the (univariate) variance ratio statistic is

$$\rho(d_1) = T^{2d_1} \frac{\sum_{t=1}^T x_t^2}{\sum_{t=1}^T \tilde{x}_t^2} \rightarrow_d \frac{\int_0^1 W_{d-1}^2(r) dr}{\int_0^1 W_{d+d_1-1}^2(r) dr}, \quad (57)$$

where the limit follows by simple application of the continuous mapping theorem. For $d = d_1 = 1$, $\rho(1)$ was proposed by Breitung (2002) as the basis of a unit root test, but $\rho(1)$ is related to many well known statistics like the R/S, V/S, KPSS, and DW statistics; see also Giraitis, Kokoszka, Leipus, and Teyssiere (2003). Note that no parametric assumptions are made on u_t , so there is no risk of misspecification. Additionally, unlike classical nonparametric approaches, a bandwidth (or, alternatively, a lag length choice) is not needed. Finally, $\rho(d_1)$ does not depend on nuisance parameters because σ_u^2 is cancelled out by the ratio in (57).

Indeed, this cancellation is the main reason behind the design of the statistic because estimation of nuisance parameters, and in particular the long-run variance, is unnecessary. In principle, d_1 is a user-chosen parameter, but the choice is reflected in the asymptotic distribution of $\rho(d_1)$, and consequently it can be chosen so that the asymptotic local power is maximized (Nielsen, 2009, showed that $d_1 = 0.1$ appears to be a good choice). Deterministic components can be dealt with by using residuals from the corresponding regression specifications.

Nielsen (2010) exploited this idea to propose a cointegration rank test. Let X_t be a p -dimensional observable process generated by (55) with $d > 1/2$ and $d - b < 1/2$, and define $\tilde{X}_t = \Delta_+^{-d_1} X_t$ for $d_1 > 0$. Defining $A_T = \sum_{t=1}^T X_t X_t'$, $B_T = \sum_{t=1}^T \tilde{X}_t \tilde{X}_t'$, consider the ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_p$ of $R_T(d_1) = A_T B_T^{-1}$ as the solutions to

$$|\lambda B_T - A_T| = 0,$$

and the statistics

$$\Lambda_{p,r}(d_1) = T^{2d_1} \sum_{j=1}^{p-r} \lambda_j, \quad r = 0, \dots, p-1.$$

Nielsen (2010) showed that, for $r = 0, \dots, p-1$,

$$\Lambda_{p,r}(d_1) \rightarrow_d U_{p-r}(d, d_1)$$

for a random variable $U_{p-r}(d, d_1)$ which depends only on $p-r$, d , and d_1 . As in the univariate case, $\Lambda_{p,r}(d_1)$ does not depend on tuning parameters or on the short-run structure of the input process U_t , and the long-run covariance matrix does not need to be estimated. The statistic and its asymptotic null distribution also do not depend on the cointegration gap, b . The cointegrating rank r can be estimated by a sequential test procedure identical to that presented in relation to Breitung and Hassler's (2002) approach, but using $\Lambda_{p,r_0}(d_1)$ for $r_0 = 0, \dots, p-1$, instead. Nielsen's (2010) results ensure that the test is consistent, and, as in the univariate setting, X_t can be replaced by corresponding residuals where deterministic components have been removed. Unlike Breitung and Hassler's (2002) approach, the tests based on $\Lambda_{p,r_0}(d_1)$ do not require knowledge or estimation of d . In addition to providing an estimator of the cointegrating rank, Nielsen (2010) also proposed an estimator of the cointegrating space.

To conclude this subsection we comment briefly on some alternative approaches. Marinucci and Robinson (2001) proposed a Hausman-type procedure based on the comparison of two different estimators of the integration order of the observables, one of which is consistent under both the null of no cointegration and the alternative of cointegration (although relatively inefficient under the null), while the other is efficient under the null and inconsistent under the alternative. Robinson (2008a) formalized this idea for stationary and nonstationary settings. In a semiparametric setting and for nonstationary observables, Marmol and Velasco (2004) test the null of no cointegration (against the alternative of cointegration) by comparing OLS and generalized least squares (GLS) estimators of the projection or fundamental vector (i.e., the cointegrating vector, if cointegration exists). Again, their approach is based on a Hausman-type idea, but here the estimators have opposite consistency properties under the competing hypotheses. In a similar fashion, Hualde and Velasco (2008) introduced a procedure based on correctly orthogonalized residuals (obtained from estimating

the projection or fundamental vector) under the null of no cointegration which, unlike Marmol and Velasco (2004), enjoys a standard asymptotic null distribution. Their approach can be employed in parametric and semiparametric settings, covers stationary and nonstationary ranges, and deals effectively with observables with distinct integration orders. Finally, Hassler and Breitung (2006) introduced a modified score test for the null of no cointegration in a nonstationary time series. The test is applied to single equation regression residuals from a first-step regression with differenced variables and in a second step an endogeneity correction is implemented so the statistic obtains a standard asymptotic null distribution. However, like the procedure of Breitung and Hassler (2002), this procedure requires knowledge of the integration order of the observables, which is unrealistic in practice, and replacing this parameter by an estimator affects the asymptotic null distribution.

4.2 Semiparametric estimation of fractional cointegration

Within the topic of estimation of fractional cointegration, we present first several semiparametric approaches, where $f_u(\lambda)$ is considered an unknown nonparametric function. Initially, we will illustrate some estimators for the particular specification (50)–(51) with $r = 1$ known. The case where $r = 1$ is known has been routinely employed in both theoretical and empirical work, and an extension to allow $r > 1$ in this particular setup is non-trivial. On the other hand, an extension allowing the individual components of X_{2t} in (51) to have distinct integration orders is straightforward.

The most basic estimation approach of β in (50) is the OLS estimator, which is a first stage procedure. This estimator is

$$\hat{\beta}_{OLS} = \left(\sum_{t=1}^T X_{2t} X_{2t}' \right)^{-1} \sum_{t=1}^T X_{2t} X_{1t}. \tag{58}$$

For the particular $d = b = 1$ case, Phillips and Durlauf (1986) analyzed the asymptotic properties of $\hat{\beta}_{OLS}$ for general conditions on the error input series u_t . Specifically, they established its T -consistency and derived its limiting distribution, which in general is nonstandard and therefore unsuitable for inference. In fractional circumstances, the properties of the OLS estimate (58) can be very different from those in the traditional $d = b = 1$ situation. Robinson (1994c) showed the inconsistency of $\hat{\beta}_{OLS}$ in the model (50)–(51) when the observable X_t was a covariance stationary long-memory process, i.e. when $d < 1/2$. In this framework, the inconsistency of the OLS estimator is due to correlation between the stationary regressor and cointegrating error.

Robinson and Marinucci (2001, 2003) provided the asymptotic distribution of the OLS estimator (with or without intercept) for the case $d \geq 1/2$ and $d \geq b$ in a model very similar to (50)–(51), but where the processes belonged to a class closely related to, but wider than, the Type II fractionally integrated. They showed that the rate of convergence of the OLS estimator is $T^{\min(2d-1,b)}$, except for the case where $d > b$ and $2d - b = 1$, where the OLS is $T^b / \log T$ -consistent. In all cases, the OLS estimator has nonstandard limiting distributions which, as mentioned before, complicates statistical inference.

A first stage alternative to OLS is the narrow-band least squares (NBLS) estimator. For $l = 0, 1$ and an integer bandwidth m , with $l \leq m \leq T/2$, we can estimate β in (50) by

$$\tilde{\beta}_l(m) = \hat{F}_{X_2 X_2}^{-1}(l, m) \hat{F}_{X_2 X_1}(l, m), \tag{59}$$

where, given (perhaps identical) scalar or vector sequences a_t and b_t , $t = 1, \dots, T$,

$$\widehat{F}_{ab}(l, m) = 2 \operatorname{Re} \left\{ \frac{2\pi}{T} \sum_{j=l}^m I_{ab}(\lambda_j) \right\} - \frac{2\pi}{T} I_{ab}(\pi) \mathbb{I}(m = T/2)$$

is called the averaged (cross-)periodogram; see (26). Note that

$$\widehat{F}_{ab}(1, m) = \widehat{F}_{ab}(0, m) - \bar{a}\bar{b},$$

where \bar{a} and \bar{b} are the corresponding sample averages of a_t and b_t , such that omission of the zero frequency implies sample-mean correction. Under the bandwidth condition

$$\frac{1}{m} + \frac{m}{T} \rightarrow 0 \text{ as } T \rightarrow \infty, \tag{60}$$

the averaged (cross-)periodograms are based on a degenerating band of frequencies around 0, so that (59) only considers low-frequency components of the series. In this situation, $\widetilde{\beta}_l(m)$ is the NBLs estimator of β . In fact, since cointegration defines a long-run relationship, avoiding high-frequency components that could be both distortive and uninformative seems sensible. Note also that, from the orthogonality properties of the complex exponential, $\widetilde{\beta}_0(\lfloor T/2 \rfloor) = \widehat{\beta}_{OLS}$, whereas $\widetilde{\beta}_1(\lfloor T/2 \rfloor)$ is the OLS estimator in a model like (50) that includes an intercept.

The NBLs estimator was proposed by Robinson (1994c). It is related to the band-spectrum estimator proposed by Hannan (1963), developed later by Engle (1974) and analyzed by Phillips (1991b) in the context of standard $d = b = 1$ cointegration, with the fundamental difference that the band-spectrum estimator focuses on a nondegenerate band of frequencies, so (60) does not hold. Due to (60), NBLs resembles nonparametric spectral estimation, where now the focus is the parameter β instead of a spectral density at a given fixed frequency.

The NBLs estimator is particularly interesting in the case of so-called “stationary cointegration”, i.e. with $d \in (0, 1/2)$ and $d - b \in [0, 1/2)$ in the model (50)–(51), where, as mentioned before, OLS is inconsistent. Somewhat surprisingly, Robinson (1994c) proved consistency of the NBLs estimator in this case. Intuitively, consistency of NBLs is obtained because focusing on a slowly degenerating band of low frequencies reduces the bias due to the endogeneity of X_{2t} . Robinson and Marinucci (2003) gave a rate of convergence (which they conjectured as sharp) for the NBLs estimator under stationary cointegration. Christensen and Nielsen (2006) improved on this result by providing a better rate than that of Robinson and Marinucci (2003) and showing that the NBLs estimator has an asymptotic normal distribution. This was under the additional conditions that the collective memory satisfies $2d - b < 1/2$ and that the coherency between the weakly dependent processes u_{1t} and u_{2t} is zero at frequency zero. The latter amounts to a type of long-run exogeneity that can be a strong condition in some contexts.

For the nonstationary case, Robinson and Marinucci (2001, 2003) also exploited the bias reduction achieved by focusing on a degenerating band of frequencies around zero, and showed that in case $2d - b < 1$ or $2d - b = 1$ with $d > b$, the rates of convergence previously given for the OLS can in fact be improved upon. For NBLs, these rates are $T^b m^{2d-b-1}$ if $2d - b < 1$, $T^b / \log m$ if $2d - b = 1$ with $d > b$, and T^b otherwise, noting (60). Like OLS, NBLs in general has nonstandard limiting distributions. For $d = b = 1$, the convergence rates of

$\tilde{\beta}_l(m)$ with (60) and $\tilde{\beta}_l(\lfloor T/2 \rfloor)$ are identical, but $\tilde{\beta}_1(m)$ eliminates the “second-order bias” present in the asymptotic distribution of $\tilde{\beta}_1(\lfloor T/2 \rfloor)$. The superiority of NBLs over OLS does not appear when comparing $\tilde{\beta}_0(m)$ and $\tilde{\beta}_0(\lfloor T/2 \rfloor)$, however, for the standard $d = b = 1$ case.

Focusing also on a narrow-band approach, Nielsen (2005b) proposed a semiparametric version of the weighted least squares estimator of Robinson and Hidalgo (1997). In the context of stationary cointegration in the model (50)–(51), Nielsen’s (2005b) estimator is

$$\hat{\beta}_{\delta,m} = \left(\sum_{j=1}^m \lambda_j^{2\delta} I_{X_2 X_2}(\lambda_j) \right)^{-1} \sum_{j=1}^m \lambda_j^{2\delta} I_{X_2 X_1}(\lambda_j), \quad (61)$$

where δ is a user-chosen number. Clearly, if $\delta = 0$ (and $m < T/2$ satisfies (60)) then $\hat{\beta}_{0,m}$ is the NBLs estimator of β . However, the appeal of (61) is mainly for the infeasible choice $\delta = d - b$, in which case $\hat{\beta}_{d-b,m}$ was denoted the narrow-band GLS estimator. The estimator $\hat{\beta}_{d-b,m}$ also maximizes the local Whittle likelihood (see (31)) constructed for (50). Nielsen (2005b) showed that, under regularity conditions that include the zero coherence condition (as in Christensen and Nielsen, 2006), $\hat{\beta}_{\delta,m}$ has a Gaussian limiting distribution. In particular, for the $p = 2$ case, he showed that

$$\sqrt{m} \lambda_m^{-b} (\hat{\beta}_{d-b,m} - \beta) \rightarrow_d N \left(0, \frac{f_u^{(1,1)}(0)}{f_u^{(2,2)}(0)} \left(\frac{1}{2} - b \right) \right), \quad (62)$$

where $f_u^{(i,j)}(\lambda)$ is the (i, j) element of $f_u(\lambda)$. The result (62) contrasts with that of Christensen and Nielsen (2006), where

$$\sqrt{m} \lambda_m^{-b} (\hat{\beta}_{0,m} - \beta) \rightarrow_d N \left(0, \frac{f_u^{(1,1)}(0)}{f_u^{(2,2)}(0)} \frac{(1/2 - d)^2}{(1/2 - (2d - b))} \right)$$

with the additional condition that $2d - b < 1/2$ (which is not required for (62)). The ratio between the respective asymptotic variances of $\hat{\beta}_{0,m}$ and $\hat{\beta}_{d-b,m}$ is

$$(1/2 - d)^2 / ((1/2 - d)^2 - (d - b)^2) \geq 1, \quad (63)$$

with equality holding if and only if $d - b = 0$, so that (63) confirms the anticipated higher asymptotic efficiency of $\hat{\beta}_{d-b,m}$ over $\hat{\beta}_{0,m}$. As mentioned before, $\hat{\beta}_{d-b,m}$ is infeasible. However, Nielsen (2005b) showed that, for a log T -consistent estimator $\widehat{d - b}$, the feasible estimator $\widehat{\beta}_{\widehat{d-b},m}$ has the same asymptotic distribution as $\hat{\beta}_{d-b,m}$.

Still in the context of stationary cointegration in the model (50)–(51), Nielsen (2007) proposed a local Whittle estimator $\hat{\theta}$ of $\theta = (d, b, \beta)'$. The main result, derived again under the local orthogonality condition that $f_u^{(1,2)}(0) = 0$, is that the joint asymptotic distribution of the appropriately normalized and centered $\hat{\theta}$ is Gaussian with the following particularities (mainly due to the zero coherence condition). First, the distribution of the estimators of the integration orders, which are \sqrt{m} -consistent, is identical to that if β were a known parameter (this is especially relevant for the estimator of $d - b$). Second, the result for the estimator of β is the same as (62). Third, the estimators of the integration orders are asymptotically independent of the estimator of the cointegrating vector. Velasco (2003b)

and Shimotsu (2012) obtained similar results in a nonstationary bivariate setting using, respectively, tapering in a two-step approach (based on consistent initial estimators) and the exact local Whittle approach.

Despite their Gaussian limit in some cases, all previous estimators are first stage methods, because they do not implement the type of correction needed to avoid the endogeneity created by regressor X_{2t} and which is removed in some cases by the zero coherence condition. We next present three second stage procedures.

Robinson (2008b) proposed a bivariate version of the model (50)–(51) with stationary cointegration, where an unknown phase parameter $\gamma \in (-\pi, \pi]$ between $\Delta^{-(d-b)}u_{1t}$ and $\Delta^{-d}u_{2t}$ at frequency zero is allowed. In the leading ARFIMA case $\gamma = b\pi/2$. Robinson (2008b) studied the properties of a local Whittle estimator, obtaining corresponding results to Nielsen (2007) but without the $f_u^{(1,2)}(0) = 0$ condition. In fact, Robinson (2008b) assumed that $f_u^{(1,2)}(0) \neq 0$, such that γ is identifiable, but his theory can be easily extended to cover the case with known γ and $f_u^{(1,2)}(0) = 0$. As in Nielsen (2007), the estimators have a Gaussian joint asymptotic distribution and the same convergence rates, including now \sqrt{m} -consistency for the estimator of the phase parameter. However, unlike Nielsen’s (2007) results, the estimators are asymptotically dependent due to the $f_u^{(1,2)}(0) \neq 0$ condition.

Considering the more general Type II fractionally integrated model (47) with (46) and $d_1 \geq 0$, Hualde and Robinson (2010) proposed identification conditions on Υ and on the integration orders such that (47) represents a cointegrated process with multiple cointegrating subspaces. Their framework is very general and notationally involved, so, for simplicity, we illustrate it for a particular example with $p = 4$. Suppose there is a cointegrating space $S^{(1)} \subset \mathbb{R}^4$ of dimension $r_1 = 2$; that is, there exists a full rank 4×2 matrix $\beta(1)$ such that $\beta(1)'X_t \in I(d_2)$ with $d_2 < d_4$ (in the sense of Definition 1). Suppose also that there is another cointegrating subspace $S^{(2)}$ of dimension $r_2 = 1$, i.e. that there exists a 4×1 vector $\beta(2)$, which is a linear combination of the columns of $\beta(1)$, such that $\beta(2)'X_t \in I(d_1)$ with $d_1 < d_2$. Here, $S^{(1)}$ is plane in \mathbb{R}^4 , and all vectors belonging to that plane are cointegrating vectors. Similarly, $S^{(2)}$ is a line in the plane $S^{(1)}$ that leads to a greater reduction in the integration orders of the observables, that is from d_4 to d_1 (instead of the reduction from d_4 to d_2 achieved by the rest of vectors in the plane $S^{(1)}$ not belonging to $S^{(2)}$). Note that the order d_2 is unique in the sense that any other full rank 4×2 matrix whose columns are cointegrating vectors leads to the same order reduction (from d_4 to d_2), while d_1 is unique in the sense that any vector $\phi \in S^{(2)}$ implies $\phi'X_t \in I(d_1)$.

As explained in Hualde and Robinson (2010), data-based procedures can be applied to determine the number of cointegrating subspaces (two in this example), and also their dimensions r_1 and r_2 . Moreover, this cointegration structure leads, for a particular ordering of the observables in X_t (which can be also inferred from data), to restrictions in Υ and the integration orders given by

$$\Upsilon = \begin{pmatrix} 1 & -\beta_{12} & -\beta_{13} & -\beta_{14} \\ 0 & 1 & -\beta_{23} & -\beta_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{64}$$

and

$$d_1 < d_2 < d_3 = d_4, \tag{65}$$

where the β 's in (64) are unrestricted parameters. We say that model (47) with (64) and (65) is written in generalized triangular form. In a general and much more complex setting, considering unknown orders of integration and unknown $f_u(\lambda)$ treated as a nonparametric function, Hualde and Robinson (2010) proposed two frequency domain GLS (FDGLS) type of estimators for the unrestricted parameters. One relies on the inverse spectral weighting $f_u^{-1}(\lambda_j)$, while the other is unweighted, but involved the inverse of $f_u(0)$. The motivation for the second estimator is, apart from its simplicity, that, under regularity conditions, $f_u^{-1}(\lambda)$ and $f_u^{-1}(0)$ behave similarly for λ close to zero. Therefore, employing a narrow-band approach in FDGLS estimation, an estimate of $f_u^{-1}(\lambda)$ can be replaced by an estimate of $f_u^{-1}(0)$ without affecting asymptotic properties. Note that using estimates of $f_u^{-1}(\lambda)$ or $f_u^{-1}(0)$ corrects the endogeneity problem caused by correlation between regressors and cointegrating errors, so both estimators are second stage. Also, they have the advantage of a closed form representation, although they depend on estimates of the nuisance parameters ($f_u(\lambda)$ and orders of integration), so the choice of several bandwidth parameters is involved.

In the previous example, the properties of the estimators depend crucially on whether $d_4 - d_2 > 1/2$ (so, also, $d_4 - d_1 > 1/2$), which is described as “strong” fractional cointegration, or $d_4 - d_1 < 1/2$ (so, also, $d_4 - d_2 < 1/2$), which is the so-called “weak” fractional cointegration. In the former case, estimators have a mixed Gaussian asymptotic distribution and the following convergence rates: estimators of β_{23} and β_{24} are $T^{d_4 - d_2}$ -consistent, whereas those of β_{12} , β_{13} , β_{14} have in general the rate $T^{d_2 - d_1}$, but the estimator of β_{13} has the faster rate $T^{d_4 - d_1}$ if $\beta_{23} = 0$, whereas that of β_{14} has the faster rate $T^{d_4 - d_1}$ if $\beta_{24} = 0$. In the “weak” situation, the limiting distribution of the estimators is Gaussian with the following convergence rates, where m is a bandwidth parameter such that $m \rightarrow \infty$ as $T \rightarrow \infty$: estimators of β_{23} and β_{24} are $\sqrt{m}\lambda_m^{d_4 - d_2}$ -consistent, whereas those of β_{12} , β_{13} , β_{14} have in general rate $\sqrt{m}\lambda_m^{d_2 - d_1}$, but the estimator of β_{13} has the faster rate $\sqrt{m}\lambda_m^{d_4 - d_1}$ if $\beta_{23} = 0$, whereas that of β_{14} has the faster rate $\sqrt{m}\lambda_m^{d_4 - d_1}$ if $\beta_{24} = 0$. The case with mixed “strong” and “weak” situations is also studied, where both mixed Gaussian and Gaussian limits co-exist but are mutually independent. Interestingly, in all cases these asymptotic distribution results lead to a χ^2 -limit for Wald statistics based on the proposed estimators.

In a bivariate situation, Hualde and Iacone (2019) derived the asymptotic theory of related semiparametric estimators when the bandwidth m is kept fixed in the limit. They showed that this alternative limit theory provides a more accurate approximation to the sampling distribution of the classical Wald statistics, which are typically oversized when confronted with the standard limit theory.

A different type of endogeneity correction is proposed by Nielsen and Frederiksen (2011) based on a fully-modified approach to NBLs estimation. In particular, they derive an expression for the asymptotic bias of the NBLs estimator, which, in terms of model (47) depends on d_1, \dots, d_p and $f_u(0)$. This asymptotic bias is consistently estimated, and based on this an appropriate correction of the NBLs estimator is implemented, thus obtaining the fully-modified NBLs estimator, which, again, is a second stage procedure. Their results apply to the “weak” fractional cointegration case, where there is a unique nontrivial cointegrating relation, and their estimator enjoys a Gaussian limit with identical convergence rates to those in Hualde and Robinson (2010).

Finally, a different but related modeling approach of fractional cointegration can be found in Chen and Hurvich (2003). This is based on a common components representation and

their focus is estimating the cointegrating space using a semiparametric narrow-band approach. Assuming a known cointegrating rank r , this space is estimated by the eigenvectors corresponding to the r smallest eigenvalues of the averaged tapered periodogram matrix of the observables, which are assumed to have identical memory. Chen and Hurvich (2006) extended this setting so that observables, which again are assumed to have a common memory, are modeled as a linear combination of unobservable components with potentially different memories. This prompts consideration of the issue of different cointegrating subspaces, and Chen and Hurvich's (2006) approach leads to estimates of the whole cointegration structure; that is, the cointegrating space and possible subspaces.

4.3 Parametric estimation of fractional cointegration

We present some parametric approaches where $f_u(\lambda)$ is considered a known parametric function up to a vector of unknown parameters. Early research (e.g., Cheung and Lai, 1993, or Baillie and Bollerslev, 1994a,b) focused on a three-step approach assuming a model like (50)–(51): first, d is estimated or assumed to take a particular value, e.g., $d = 1$ (perhaps as a result of the outcome of a unit root test); then β is estimated using a first stage procedure like OLS; and finally $d - b$ is estimated from the corresponding residuals of the potentially cointegrated relation. Testing for (fractional) cointegration, for example, then amounts to finding statistical evidence in favor of $b > 0$. These steps were combined by Dueker and Startz (1998) who proposed joint maximum likelihood estimation of d , b , and β in a Type I version of (50)–(51) where $p = 2$, $r = 1$ (so X_{1t} and X_{2t} are scalars) and $d < 1/2$, assuming also that u_t is a vector ARMA process. Their proposal relied on previous unpublished work by Sowell (1986), but no asymptotic results were derived. Jeganathan (1999) proposed a MLE of β in a Type I version of (50)–(51), focusing on the case where u_t is white noise with known distribution and assuming known d and b . He showed that this estimator has a mixed Gaussian asymptotic distribution and provided some discussion about a feasible alternative in which d and b are estimated.

Robinson and Hualde (2003) proposed two parametric GLS-type of estimators of β in model (50)–(51) with $p = 2$, $r = 1$, and $d \geq b > 1/2$, which captures the “strong” fractionally cointegrated case. The motivation behind their estimators can be illustrated by focusing first on the traditional $d = b = 1$ case. In that case, Phillips (1991a) justified that the second-order simultaneity bias due to correlation between the error terms in (50) and (51) can be eliminated and the optimal properties of a second stage estimator, including a mixed Gaussian limiting distribution, can be recovered. This is achieved by pseudo maximum likelihood (PML) estimation of β in the model (50)–(51), which is equivalent to OLS in the augmented regression model

$$X_{1t} = \beta X_{2t} + \rho \Delta X_{2t} + u_{1.2t}, \tag{66}$$

where $\rho = \omega_{12}/\omega_{22}$, $\omega_{12} = \text{Cov}(u_{1t}, u_{2t})$, $\omega_{22} = \text{Var}(u_{2t})$, and $u_{1.2t} = u_{1t} - \rho u_{2t}$, noting that by construction $\text{Cov}(u_{1.2t}, u_{2t}) = 0$.

This idea was extended to the fractional setting by Robinson and Hualde (2003), where, focusing again on the white noise case for illustration, the equivalent to (66) is

$$\Delta^{d-b} X_{1t} = \beta \Delta^{d-b} X_{2t} + \rho \Delta^d X_{2t} + u_{1.2t}, \tag{67}$$

so it could be expected that OLS estimation in (67) would lead to optimality properties (see also Nielsen, 2004c, for an identical approach). Robinson and Hualde (2003) confirmed this

result and showed that the feasible version of this estimator where d and b are replaced by corresponding estimates (with convergence rates T^ϕ for $\phi > \max\{0, 1 - b\}$), has an identical limiting distribution to the infeasible OLS estimator in (67). Furthermore, their results apply for a general $I(0)$ error u_t with parametric spectral density. Based on this parametric structure, they proposed time domain (based on an autoregressive transformation) and frequency domain estimators which approximate GLS (and in fact PML). Both estimators have the same mixed Gaussian asymptotic distribution, leading to Wald test statistics with χ^2 null limit distributions for known or unknown integration orders.

Corresponding results for the “weak” cointegration case where $b < 1/2$ were provided by Hualde and Robinson (2007). This case is more difficult to handle because of the smaller cointegration gap, b , which could potentially be very small so the strength of the cointegrating relation is weaker compared to the “strong” case. Hualde and Robinson’s (2007) estimator, again for the scalar $p = 2$, $r = 1$ case, is designed for the VAR(k) situation, where in model (50)–(51), the error term is

$$u_t = \sum_{j=1}^k B_j u_{t-j} + \varepsilon_t, \tag{68}$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ is a zero mean white noise process and all zeros of $\det\{I_2 - \sum_{j=1}^k B_j z^j\}$ lie outside the unit circle. Defining $\zeta = (1, 0)'$ and letting B_{ij} be the i 'th row of B_j , the orthogonalization equivalent to that in (67) is, for $t > k$,

$$\begin{aligned} \Delta^{d-b} X_{1t} &= \beta \Delta^{d-b} X_{2t} + \rho \Delta^d X_{2t} + \sum_{j=1}^k (B_{1j} - \rho B_{2j}) (\Delta^{d-b} X_{1,t-j}, \Delta^d X_{2,t-j})' \\ &\quad - \beta \sum_{j=1}^k (B_{1j} - \rho B_{2j}) \zeta \Delta^{d-b} X_{2,t-j} + \varepsilon_{1.2t}, \end{aligned} \tag{69}$$

where $\varepsilon_{1.2t} = \varepsilon_{1t} - \rho \varepsilon_{2t}$ and $\rho = E(\varepsilon_{1t} \varepsilon_{2t}) / E(\varepsilon_{2t}^2)$. It is shown that the OLS estimator of β in (69) is \sqrt{T} -consistent and asymptotically normal, but is infeasible because it depends on d and b . Moreover, a feasible estimator of β with the same properties as the infeasible estimator is unachievable in the “weak” cointegration case because estimators of the integration orders are at most \sqrt{T} -consistent. Instead, Hualde and Robinson (2007) proposed convenient estimators of the nuisance parameters d and b based on simple univariate optimizations. The feasible estimator of β based on these estimators is inefficient relative to the PML estimator, but has an optimal rate of convergence, is asymptotically normal, and is computationally much simpler than the PML estimator.

4.4 Fractionally cointegrated VAR model

The fractionally cointegrated VAR (FCVAR) model can be seen as a generalization of the CVAR model (Johansen, 1995) to fractional time series, and also as a generalization of the fractional AR model (Johansen and Nielsen, 2010) discussed in Section 3.4 to multivariate time series. For a p -dimensional time series X_t , the FCVAR model can be derived from the usual CVAR model by the same steps as in (41)–(43) to arrive at

$$\Delta^d X_t = \alpha \beta' L_b \Delta^{d-b} X_t + \sum_{i=1}^k \Gamma_i L_b^i \Delta^d X_t + \varepsilon_t, \tag{70}$$

where ε_t is assumed i.i.d. with mean zero and variance Ω . An alternative derivation starts with (50)–(51), though using γ instead of β in (50) because β has a standard meaning in (70). From there it follows (Johansen, 2008) that $\Delta^d X_t = \alpha\beta' L_b \Delta^{d-b} X_t + v_t$ with $v_t = \alpha(\beta'\alpha)^{-1}u_{1t} + \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}u_{2t}$ and the parameters are $\beta' = (I_r, \gamma')$ and $\alpha'_\perp = (0_{p-r,r}, I_{p-r})$. Here, for a matrix A , the matrix A_\perp is such that $A'A_\perp = A'_\perp A = 0$.

The FCVAR model (70) was proposed by Granger (1986) and Johansen (2008), although Granger had an additional lag in the error-correction term, $L_b \Delta^{d-b} X_{t-1}$, which would appear to be a typo. The FCVAR model (70) was analyzed by Johansen and Nielsen (2012a, 2018, 2019) who list (at least) five desirable properties for a fractional cointegration model that are all achieved by the FCVAR model: (i) the parameters should have interesting interpretations, (ii) there exists testable criteria on the parameters for when the solution is cointegrated (and for the cointegration rank), (iii) estimation is relatively simple, (iv) asymptotic theory is straightforward to apply, and (v) generic computer programs can be made to estimate the model and test the most interesting hypotheses.

Model (70) includes the Johansen (1995) CVAR model as the special case $d = b = 1$. Some of the parameters are well-known from the CVAR model and they have the usual interpretations in the FCVAR model. The most important of these are the long-run parameters α and β , which are $p \times r$ matrices with $0 \leq r \leq p$ and r being the cointegration, or cofractional, rank. The columns of β constitute the r cointegration vectors such that $\beta' X_t$ are the long-run equilibrium relations. The parameters in α are the adjustment or loading coefficients which represent the speed of adjustment towards equilibrium for each of the variables. The short-run dynamics of the variables are governed by the parameters $\Gamma_1, \dots, \Gamma_k$ in the autoregressive augmentation.

The FCVAR model has two additional parameters compared with the CVAR model, namely the fractional order of the observables, d , and the cointegration gap, b . These parameters are estimated jointly with the remaining parameters. This model thus has the same main structure as the standard CVAR model in that it allows for modeling of both cointegration and adjustment towards equilibrium, but is more general since it accommodates fractional integration and fractional cointegration.

The representation theory (solution) for the model (70) was derived by Johansen (2008) and Johansen and Nielsen (2012a), who showed that

$$X_t = C\Delta_+^{-d}\varepsilon_t + \Delta_+^{-(d-b)}Y_t + \mu_t,$$

where $C = \beta_\perp(\alpha'_\perp(I_p - \sum_{i=1}^k \Gamma_i)\beta_\perp)^{-1}\alpha'_\perp$ such that $\beta'C = 0$, $Y_t \in I(0)$, and μ_t is a function of initial values X_{-n} , $n \geq 0$. Thus, it is clear that $X_t \in I(d)$ while $\beta'X_t \in I(d-b)$. Furthermore, the rank of $\alpha\beta'$ ($= \Pi$) is the cointegration rank, which is testable. The initial values term, μ_t , is most often assumed to be zero (and known) in the literature, but Johansen and Nielsen (2012a, 2016) analyze more general conditions under which μ_t induces a bias in finite samples that can be alleviated by either conditioning the analysis on the first part of the sample or by including a so-called “level parameter”. The latter is related to the notions of restricted and unrestricted constant terms (Johansen and Nielsen, 2012a, 2016, and Dolatabadi, Nielsen, and Xu, 2015).

The parameters $(d, b, \alpha, \beta, \Gamma_1, \dots, \Gamma_k, \Omega)$ of model (70), along with possible deterministic terms or level parameters, can be estimated jointly by maximum likelihood. For fixed (d, b) ,

maximum likelihood is reduced rank regression, which leaves a numerical optimization problem over the two parameters (d, b) .

The asymptotic distribution theory for the MLEs is justified by Johansen and Nielsen (2012a, 2018, 2019). They show that, when $b < 1/2$, all estimators are jointly asymptotically Gaussian. On the other hand, when $b > 1/2$, $\hat{\beta}$ is asymptotically mixed normal while the remaining estimators are jointly asymptotically Gaussian and independent of $\hat{\beta}$. The important consequence, of course, is that likelihood ratio tests of hypotheses on any of the parameters are χ^2 -distributed, regardless of the value of b . Furthermore, the likelihood ratio test for cointegration rank is asymptotically χ^2 -distributed when $b < 1/2$, but is a functional of fractional Brownian motion when $b > 1/2$. This dependence on the real-valued nuisance parameter b complicates tabulation, but a plug-in approach can be used to compute p -values in practice (MacKinnon and Nielsen, 2014).

Thus, estimation is quite simple and asymptotic theory is straightforward to apply, and computer programs are available in Matlab and R. For details on practical implementation and computation, see Nielsen and Popiel (2018) and Morin, Nielsen, and Popiel (2021). We note that the asymptotic analysis in Johansen and Nielsen (2012a) assumed that d and b are such that $0 \leq d - b < 1/2$, i.e. that the long-run equilibrium errors have non-negative memory and are (asymptotically) stationary. The former condition is restrictive because it implies that the usual CVAR model with $d = b = 1$ lies on the boundary of the parameter space, which complicates testing this restriction. The latter condition is likely restrictive in practice, where the possibility of nonstationary cointegration with $d - b \geq 1/2$ may be desired. However, these restrictions were removed in Johansen and Nielsen (2018, 2019), while relaxing at the same time a strong moment condition in Johansen and Nielsen (2012a).

5 Applications of fractional integration and cointegration

Many empirical researchers have applied the previous methodological contributions to describe and illustrate features of real data, especially since the early 1990s. Some of the earlier works are nicely summarized in Baillie (1996) or Henry and Zaffaroni (2003), but, nevertheless, we first provide a brief overview of some early results and subsequently focus on some applications that we find particularly interesting.

In one of the earliest empirical investigations involving long memory behavior in economics or finance, Diebold and Rudebusch (1989) analyzed aggregate output and found evidence of fractional integration through estimation of ARFIMA models. In particular, their estimates of d typically range from 0.5 to 0.90, depending on the particular series, and with quarterly series being apparently more persistent than annual ones. This indicates that macroeconomic growth shocks appear to be less persistent than under the classical $I(1)$ prescription, although the results need to be taken with caution due to relatively wide confidence intervals. In a similar fashion, Sowell (1992) used an ARFIMA process to model the first differences of US quarterly real GNP. Sowell (1992) estimated an ARFIMA(3, d , 2) model (selected by AIC) by maximum likelihood and obtained $\hat{d} = -0.59$ (d being the integration order of the first differences of the quarterly real GNP). Neither of the hypotheses $H_0 : d = 0$ (GNP is difference stationary) or $H_0 : d = -1$ (GNP is trend stationary) were rejected, so the data appear to be consistent with both rival models.

Extending the analysis to other macroeconomic variables, Crato and Rothman (1994a) applied ARFIMA models to an extended version of the data set used by Nelson and Plosser

(1982) and found evidence supporting the difference stationary model for many macroeconomic time series. In a similar vein, Gil-Alana and Robinson (1997) applied Robinson's (1994b) parametric frequency domain Lagrange multiplier test to the extended Nelson and Plosser dataset and concluded that prices and money stock are the most nonstationary series, while unemployment rate and industrial production are the closest to stationarity.

Focusing on inflation data, Hassler and Wolters (1995) examined the question of whether inflation has a unit root, which is an important issue in monetary policy. Using ARFIMA modeling and semiparametric estimation methods they concluded that, for all considered countries (United States, United Kingdom, France, Germany, Italy), estimates are significantly different from both 0 and 1, with estimated integration orders typically around the stationary/nonstationary boundary of 0.5. A similar result was obtained by Baillie, Chung and Tieslau (1996) who used an approximate maximum likelihood estimator in an ARFIMA-GARCH setting. Using monthly post-World War II data for ten countries, they found clear evidence of long memory with mean reverting behaviour (so $0 < d < 1$) except for Japan, for which the hypothesis that $d = 0$ could not be rejected.

There are numerous different applications to exchange rates and the analysis of the purchasing power parity (PPP). This is a key equilibrium condition in international economics where nominal exchange rates and prices adjust, so real exchange rates revert to a parity value. Thus, this condition represents a cointegrating relation and, in the fractional context, it was first analyzed by Diebold, Husted, and Rush (1991). They constructed a long data set for 16 real exchange rates covering approximately a century of the gold standard period and estimated the corresponding integration orders by maximum likelihood. In contrast to the assumed $I(1)$ condition for nominal exchange rates, their memory estimates for real exchange rates were in all cases significantly less than 1, and in some other cases also significantly different from zero. In their view, these results justify that the PPP holds in the long run. Similar analyses can be found in Cheung and Lai (1993), Baillie and Bollerslev (1994a), Crato and Rothman (1994b), Nielsen (2004c), and Gil-Alana and Hualde (2009).

Long memory in interest rates have been intensively studied (e.g., Shea, 1991, Lai, 1997, Tsay, 2000, Meade and Maier, 2003, Nielsen, 2004d, 2005a), and there are also some studies in the fractional cointegration literature. Actually, the methodological contribution of Dueker and Startz (1998) mentioned in Section 4.3 was illustrated by analyzing the potential cointegration between US and Canadian bond rates. Their results implied that the observables were nonstationary but mean reverting processes, whereas the cointegrating error was stationary long memory, supporting the possibility of a "weak" cointegration relation. The term structure of interest rates is particularly interesting in the context of (fractional) cointegration because the unbiasedness hypothesis in this model implies a single common stochastic trend. Chen and Hurvich (2003) analyzed fractional cointegration among daily US interest rates with eight different maturities ranging from 3 months to 30 years as an illustration of their semiparametric inference procedure. They found a clear evidence of cointegration with an estimated cointegrating rank of $\hat{r} = 6$ or $\hat{r} = 7$, and with some of the cointegrating relationships being stronger than others. Nielsen (2010) addressed a similar phenomenon by analyzing four daily US Treasury Bill interest rates with maturities ranging from 3 months to 2 years. His results indicated that each series was $I(1)$ and found evidence of three cointegrating relations with a single common stochastic trend. Nielsen (2010) also applied the parametric procedures of Johansen (1995) and Breitung and Hassler (2002) and

conjectured why they both fail to detect the correct rank.

Another important strand of empirical research has focused on analyzing the volatility of financial time series. Baillie, Bollerslev, and Mikkelsen (1996) introduced the fractionally integrated generalized autoregressive conditionally heteroskedastic (FIGARCH) model. Here, the main novelty with respect to the popular GARCH model is that the conditional variance is fractionally integrated. This model was further extended by Bollerslev and Mikkelsen (1996) to include the exponential (asymmetric) feature. Both these papers included empirical studies of stock market volatility and found strong evidence of fractional integration. Andersen, Bollerslev, Diebold, and Ebens (2001) studied “realized” daily equity return volatilities and correlations from high-frequency intraday transaction prices on some individual stocks. They showed that the behaviour of these statistics is coherent with long memory and also found comovements on volatilities and correlations across assets. Similarly, Andersen, Bollerslev, Diebold, and Labys (2001) analyzed daily exchange rate volatility and correlation showing long-memory dynamics in their behavior.

Beltratti and Morana (2006) analyzed the linkages between stock market and macroeconomic volatility. They found that, after accounting for some common structural breaks, the series appeared to be long memory and displayed fractional cointegration. Specifically, they justified three cointegrating relationships among stock market, money growth, inflation, the Federal funds rate, and output growth volatilities.

Following this general overview of some relevant empirical works, we next present in more details applications of fractional integration and cointegration that we find particularly interesting and relevant.

5.1 The Deaton paradox

This paradox refers to an apparently contradictory result related to the permanent income hypothesis (PIH) under rational expectations; in particular the variability of consumption changes versus that of income innovations. Following Diebold and Rudebusch (1991b) and letting C_t and Y_t be consumption and income, respectively, the PIH implies that

$$\Delta C_t = r \sum_{i=0}^{\infty} \beta^i (E_t Y_{t+i} - E_{t-1} Y_{t+i}), \tag{71}$$

where r is the real interest rate, $\beta = r/(1+r)$ is a discount factor, and E_t is the conditional expectation formed at time t . Thus, (71) implies that the behavior of consumption depends on the specification of the income process. It is typically assumed that Y_t is $I(1)$, so letting

$$\Delta Y_t = \gamma + a(L)\epsilon_t, \tag{72}$$

where ϵ_t is a zero mean white noise and $a(z) = 1 + a_1z + a_2z^2 + \dots$ has all roots outside the unit circle, it can be shown that

$$\Delta C_t = c_{\infty}^{\beta} \epsilon_t,$$

where the multiplier is $c_{\infty}^{\beta} = 1 + \sum_{i=1}^{\infty} \beta^i a_i$. For many ARIMA specifications in (72) and realistic assumptions about r , c_{∞}^{β} is substantially above 1.

Noting that

$$\text{std}(\Delta C_t) = c_{\infty}^{\beta} \text{std}(\epsilon_t),$$

the ARIMA specification in (72) implies that the variability of consumption changes should be greater than the variability of income innovations when $c_{\infty}^{\beta} > 1$. However, in real data the opposite is observed, i.e. consumption is too smooth. This contradictory result has been named the “Deaton paradox.”

Diebold and Rudebusch (1991b) proposed to model Y_t as an ARFIMA(p, d, q) process instead of (72). This alternative model is important in the context of the Deaton paradox because the multiplier c_{∞}^{β} depends critically on the impact of past innovations to ΔY_t , which is driven by the chosen specification for Y_t . Specifically, Diebold and Rudebusch (1991b) show how the observed excess smoothness of consumption is theoretically supported by values of d less than one. Unfortunately, estimates of d are typically close to one, with confidence intervals including one, so it cannot be assessed whether their results are a departure from the predictions of the PIH. Either way, a key message of the paper is that the introduction of a more flexible specification (ARFIMA vs. ARIMA) shows that departures from the classical PIH, in this case excess smoothness of consumption, could be due to misspecification of the time series properties of the data.

5.2 Long memory inflation uncertainty and the term structure of interest rates

Backus and Zin (1993) examined the apparent contradiction between the predictions of theory and observed data in the relationship of short-term interest rates and long-term yields (or, equivalently, long forward rates). Specifically, if the short rate process is a stationary ARMA(p, q) process, then theory implies that the expected long-term yield converges to a constant while its variance tends to zero exponentially. This result contrasts with observed data, where, typically, the mean yield curve flattens out for longer maturities and the variance of the long yields declines with maturity although far from exponentially. The former is coherent with theory, but the latter is not. An alternative $I(1)$ specification of the short rate implies that expected long-term yield is linear in maturity while the variance is constant, which is also at odds with observed data.

A possible explanation of this contradictory behavior was proposed by Backus and Zin (1993) who suggested modeling the short rate by a stationary fractionally integrated noise. They showed that this alternative model implies that the variability of yields decreases with maturity, but at a much slower rate than for the stationary ARMA process. In particular, for values of d around 0.3, the theoretically implied shape of the yield curve, both in terms of expected yield and variance of the yield, is clearly superior to the alternative ARMA and $I(1)$ specifications. Unfortunately, empirical support for a stationary fractionally integrated short rate process is weak. Backus and Zin (1993) reported stronger evidence of fractional behavior for inflation and money growth, which is relevant because they conjecture that the behavior of short rates is driven to some extent by those variables.

5.3 Fractional beta-convergence

As noted by Michelacci and Zaffaroni (2000), there are three stylized facts in modern empirical macroeconomics which seem to be inconsistent: unit root in output per capita, output per capita of different economies converging to their long-run steady state value at a uniform exponential rate of 2% per year (beta-convergence), and, finally, that the steady state output could well be represented by a smooth linear trend. The first stylized fact implies that output is not mean reverting. This does not necessarily contradict beta-convergence because current

and steady state outputs may be cointegrated. However, if the smooth trend representation for steady state output is correct and current output has a unit root, testing beta-convergence would be equivalent to testing trend stationarity in output which is contradictory.

However, Michelacci and Zaffaroni (2000) proposed an alternative model for current output which reconciles these findings. Their solution is to model the de-trended current per capita output as a fractionally integrated process with memory d such that $1/2 < d < 1$. This is a nonstationary but mean reverting process, which is consistent with beta-convergence (to a smooth trend steady state), and which could easily lead to non-rejections of the unit root hypothesis. In addition, the authors found conditions under which long memory in output arises (by aggregation of heterogeneous units as in Section 2.4) and provided an extensive empirical analysis with evidence in favor of nonstationary and mean reverting behavior of per capita output and also of unconditional convergence across OECD countries with similar rates of convergence.

5.4 The forward premium anomaly

This is a puzzle in the international finance literature which refers to the inability of forward exchange rates to forecast future spot rates. Denote by s_t the log-spot exchange rate at time t and $f_{t,k}$ the log-forward rate with k being the length of the forward contract. According to the forward rate unbiasedness hypothesis, the forward rate is an unbiased forecaster of the future spot rate. That is, $E_t s_{t+k} = f_{t,k}$, where E_t is the conditional expectation formed at time t . This hypothesis implies market efficiency, rational expectations, and risk neutrality. Tests of unbiasedness have commonly been implemented in a regression of the spot return, $s_{t+k} - s_t$, on the forward premium, $f_{t,k} - s_t$, i.e.,

$$s_{t+k} - s_t = \alpha + \beta(f_{t,k} - s_t) + \varepsilon_{t+k}, \quad (73)$$

where the null hypothesis of unbiasedness would be $H_0 : \alpha = 0, \beta = 1$ or sometimes just $H_0 : \beta = 1$ to allow for a constant risk premium. Surprisingly, estimates are typically significantly smaller than one and even negative in some cases (see Ballie and Bollerslev, 2000, for references).

Ballie and Bollerslev (2000) described this anomaly as a statistical problem related to the intrinsic dependence structures of the regressor and dependent variable in (73). There is strong evidence in favour of s_t and $f_{t,k}$ both being $I(1)$, implying that $s_{t+k} - s_t$ is $I(0)$, whereas $f_{t,k}$ and s_t appear to be cointegrated such that the forward premium $f_{t,k} - s_t$ is $I(d)$ with most likely $1/2 < d < 1$. This implies an imbalance in the regression (73) caused by the different integration orders of the regressor and regressand. Maynard and Phillips (2001) provided formal theoretical justification of the consequences of this imbalance when the forward premium is nonstationary but mean reverting. Specifically, the OLS estimator in (73) converges in probability to zero at rate T^{1-2d} , so that the appropriately normalized estimator tends to a non-standard random variable whose sign depends on the one-sided long-run covariance, say λ , between the spot return and the forward premium. A non-standard behaviour is also reported for the t -statistic, which, even if the OLS estimator of β converges to zero, diverges to $+\infty$ if $\lambda > 0$ or to $-\infty$ if $\lambda < 0$, at rate T^{1-d} . To overcome this problem, Maynard, Smallwood, and Wohar (2013) provided a two-step procedure to rebalance the regression by fractionally differencing the regressor with an appropriate integration order obtained in the first step, and use their procedure to test the forward rate unbiasedness

hypothesis.

5.5 The implied-realized volatility relation

A similar unbiasedness hypothesis as in the previous subsection can be found in other contexts. For example, under market efficiency and rationality, option prices should reflect all available information about expected future return volatility of the underlying asset. Early work (e.g., Christensen and Prabhala, 1998, and the references therein) considered regression (in logs) of realized volatility of the underlying asset, $\sigma_{RV,t}$, on the volatility implied by option prices, $\sigma_{IV,t}$, i.e.,

$$\log \sigma_{RV,t} = \alpha + \beta \log \sigma_{IV,t} + \varepsilon_t, \quad (74)$$

where the unbiased hypothesis of interest is $H_0 : \beta = 1$ allowing for a constant risk premium. The regression (74) was estimated by OLS with estimated values of β being significantly less than one.

Bandi and Perron (2006) and Christensen and Nielsen (2006) analyzed this problem from a stationary fractional cointegration perspective, noting that log-volatilities appear to be stationary long memory processes. In this case, as discussed in Section 4.2, OLS is inconsistent and this could explain the results obtained by OLS estimation in earlier work. Christensen and Nielsen (2006) estimated the integration orders of $\log \sigma_{RV,t}$ and $\log \sigma_{IV,t}$ using the local Whittle procedure, and for different bandwidth choices obtained values in the range 0.35–0.48. They then estimated (74) by NBLs obtaining estimated β 's in the range 0.84–0.89 (again, for different bandwidth choices) and insignificantly different from $\beta = 1$. Finally, estimates of the integration order of ε_t , obtained by local Whittle applied to the NBLs residuals, were obtained in the range 0.09–0.11 and thus supported the existence of fractional cointegration with weakly dependent cointegrating errors. These results are consistent with those in Bandi and Perron (2006) who, in independent work, applied a narrow-band approach with subsampling and provided further support to the $\beta = 1$ hypothesis. Nielsen (2007) revisited this issue using instead a local Whittle quasi-maximum likelihood estimation approach, where the integration orders of the regressors and error and the cointegration parameter in (74) are jointly estimated. This alternative methodology allows for testing interesting joint hypotheses like $H_0 : d_\varepsilon = 0, \beta = 1$, where d_ε is the integration order of ε_t . This particular hypothesis is rejected for some bandwidth choices, though not for others, but in any case the existence of stationary fractional cointegration is supported by Nielsen's (2007) analysis.

5.6 Political science

Several political science questions have been addressed by fractional integration and cointegration techniques. The appeal of this methodology is that, in many circumstances, data in political studies are formed through aggregation of heterogeneous dynamic behaviour at individual level, which, as explained in Section 2.4, is one of the sources of long memory. Using the aggregation argument and a model of voter behavior, Box-Steffensmeier and Smith (1996), Byers, Davidson, and Peel (1997), and Dolado, Gonzalo, and Mayoral (2002) showed that aggregate opinion poll data may be best modeled using fractional time series models, and empirical estimates of integration orders around 0.8 were found.

The relationship between measures of political support and economic indicators, a phenomenon known as economic voting, has been investigated by Box-Steffensmeier and Tom-

linson (2000) and Davidson (2003) in the context of fractional integration and cointegration. Their results are inconclusive about the existence of fractional cointegration between the two types of variables. Based on a fractionally integrated vector error correction model, Davidson, Byers, and Peel (2006) analyzed the relationship between the approval of the performances of prime ministers and governments in the UK, providing some evidence supporting the existence of fractional cointegration. A similar analysis was conducted by Jones, Nielsen, and Popiel (2014) using the FCVAR model (Section 4.4) to study the potential cointegration linkages between economic performance and political support both in Canadian terms and also relative to US performance. Interestingly, they found that support to the Progressive Conservative and Liberal parties depend substantially on the economic situation: periods of high interest rates and low unemployment benefit the conservatives, while the opposite economic conditions lead to higher liberal support. Additionally, their results indicate that US economic performance does not appear to have an effect on Canadian political support.

In a similar fashion the working paper version of MacKinnon and Nielsen (2014) analyzed the possibility of fractional cointegration between the support for the Conservative and Labour parties in UK using monthly Gallup opinion poll data from 1951 to 2000. They applied the FCVAR methodology and provided evidence in favour of the nonstationary mean reverting behaviour of both observable series, which is in line with previous studies cited above, but found no evidence that the two series are cointegrated. Furthermore, they paid particular attention to the important issue of initial values. In a nonstationary setting, due to the truncation inherent in Definition 4 of fractional models, it is necessary to condition on some observed (but not modeled) initial values. The conditioning argument was proposed and rigorously justified by Johansen and Nielsen (2010, 2012a, 2016), while Hualde and Robinson (2010) provided an heuristic solution based on omitting observations at the beginning of the sample. In addition to theoretical arguments, Johansen and Nielsen (2016) gave a detailed analysis of the polling data and the consequences of alternative specifications of the initial conditions.

A forecasting analysis based on polling data of political support in the UK for 2010–2015 was provided by Nielsen and Shibaev (2018). They compared the FCVAR model with relevant competing models, and their results showed evidence of superior performance of the former model with the relative forecast improvement being higher at longer forecast horizons, which is expected.

5.7 Risk-return relationship in asset pricing

An important research topic in the finance literature is the risk-return trade-off. Simple CAPM and factor models suggest a strong relation between the conditional mean and variance of returns. This relation has been extensively studied in the empirical finance literature, but the evidence is mixed and in particular there does not appear to be any agreement in the literature on the sign of the risk-return relation. Christensen and Nielsen (2007) noted that, even though the stationary long memory properties of volatility have been extensively documented in the literature, previous empirical and theoretical work on the risk-return relationship has omitted this important feature. Moreover, standard specifications that incorporate volatility in returns in a linear fashion would imply long memory in returns, which is not empirically warranted.

Consequently, Christensen and Nielsen (2007) specify and estimate new vector ARFIMA

models for the joint dynamics of stock returns and volatility that allow for long memory in volatility without imposing this property on returns. They also show how asset pricing theory implies testable cross-equation restrictions on the system, and these are not rejected in their preferred specifications. The latter include a strong financial leverage effect, a positive risk-return trade-off, long memory in volatility, but a small and short lived effect of volatility shocks on stock prices.

5.8 The purchasing power parity revisited

We have already mentioned several studies of the PPP in a fractional context. The PPP involves the relationship between the domestic log-price index (p_t), foreign log-price index (p_t^*), and log-exchange rate (e_t). One approach would be to analyze the complete cointegration structure of the vector $z_t = (p_t, p_t^*, e_t)'$ and check whether the condition $p_t = p_t^* - e_t$, which is the absolute version of the PPP, is supported by the data. This approach is further complicated by the possibility that the individual integration orders of the observables in z_t might be distinct.

Hualde and Robinson (2010) used their semiparametric methodology (see Section 4.2) and the step-wise procedure described in Hualde (2008) on US/UK data and concluded that the cointegrating rank is $r_1 = 2$ with a further cointegrating subspace of dimension $r_2 = 1$. That is, there exists a two-dimensional cointegration space spanned by the two linearly independent cointegrating vectors $\gamma_1 = (\beta_{21}, 1, 0)'$ with $\beta_{21} \neq 0$ and $\gamma_2 = (0, 0, 1)'$. The first vector reflects that p_t and p_t^* have the same integration order and are cointegrated; the second simply reflects that e_t has a smaller integration order than p_t and p_t^* , so there exists a trivial cointegrating relation. Additionally, the existence of a cointegrating subspace means that the order-reducing linear combination of p_t and p_t^* cointegrates with e_t to get a linear combination involving the three variables with a further reduced order. Hualde and Robinson (2010) thus described the cointegrating structure as

$$\beta_{11}p_t + \beta_{12}p_t^* + e_t = \alpha_1 + \Delta_+^{-d_1}u_{1t}, \tag{75}$$

$$\beta_{21}p_t + p_t^* = \alpha_2 + \Delta_+^{-d_2}u_{2t}, \tag{76}$$

$$p_t = \alpha_3 + \Delta_+^{-d_3}u_{3t}, \tag{77}$$

where $d_1 < d_2 < d_3$; see also (47)–(48). They obtained estimates of d_1 , d_2 , and d_3 of about 1.5–1.8, around 1.3, and around 1, respectively. Based on their estimates of β_{11} , β_{12} , and β_{21} , they were unable to reject the hypothesis $\beta_{11} = 1, \beta_{12} = -1$, which represents the absolute version of the PPP, but their results do not clarify the question of whether the deviations from equilibrium in (75) are mean reverting or not.

5.9 Empirical applications of the FCVAR model

Given the enormous impact of Johansen’s (1995) CVAR methodology as a tool to analyze the standard $I(1)/I(0)$ cointegration setting (as well as $I(2)$ situations), the development of extensions to cover fractional time series is a very relevant research area. We have addressed the main theoretical achievements, and in particular the FCVAR model, in Section 4.4. There are now a large number of empirical applications of this methodology facilitated by freely available software packages in both Matlab (Nielsen and Popiel, 2018) and R (Morin, Nielsen, and Popiel, 2021). In addition to the applications involving political science and opinion

polls mentioned previously, we will now mention a small selection of empirical applications of the FCVAR model in a variety of areas.

In an early application, Osterrieder and Schotman (2011) analyzed the joint behaviour of real-estate returns and the rent-to-price ratio. Using a long time series of 355 years of real-estate returns and rent-to-price ratios they estimated an FCVAR model and compared the predictive performance of this model to that of a triangular specification. Their results support the superiority of the FCVAR model. Bollerslev, Osterrieder, Sizova, and Tauchen (2013) estimated an FCVAR model using high-frequency data of futures contracts for the S&P500 and the corresponding VIX volatility index. Specifically, they proposed a trivariate model to explain the joint evolution of two log-volatility measures and the returns. Their main conclusion is that both volatilities appear to be $I(d)$ (with estimated d about 0.4) and cointegrated with an $I(0)$ cointegration error, while the returns appear to be $I(0)$ as expected. Interestingly, the cointegrating relation between the volatilities is related to the variance risk premium (which is linked to aggregate economic uncertainty), and the joint modeling provides nontrivial return predictability. In another application to volatilities, Rossi and De Magistris (2013) estimated a bivariate FCVAR model for two volatility measures, the futures and spot log-daily ranges, $\log \sigma_{t,F}$ and $\log \sigma_{t,S}$, respectively. The model is estimated assuming the cointegrating relation $\log \sigma_{t,F} + \beta \log \sigma_{t,S}$. Here, the value $\beta = -1$ represents the non-arbitrage condition and the authors provide evidence supporting it. Moreover, while the observables appear to be stationary long memory processes (with estimated memory close to but below 0.5), the cointegration error appears to be $I(0)$. Furthermore, the out-of-sample forecast superiority of the proposed FCVAR model over other model specifications is illustrated. Related to the volatility applications and the risk-return applications mentioned earlier, Chen, Chiang, and Härdle (2018) analyze down-side risk, volatility spill-overs, and comovement of stock returns in an FCVAR model. Their results indicate, for example, that the downside risk for each G7 market is cointegrated with that of the world market.

Finally, Dolatabadi, Nielsen, and Xu (2015, 2016) analyze equilibrium relations between spot and futures prices of several commodities and implications for price discovery. Specifically, for each commodity they estimated a bivariate FCVAR model and in each case found evidence of cointegration. They compared their results with the standard non-fractional CVAR and showed that the cointegration errors appear to be stationary but fractionally integrated, so that the fractional model is more appropriate than the standard CVAR model. In a similar fashion, Dolatabadi, Narayan, Nielsen, and Xu (2018) analyzed the relationship between spot and futures prices for many commodities and demonstrated the superiority of the FCVAR model against the CVAR model in terms of in-sample fit and out-of-sample forecasting, and further illustrated this by means of “economic significance.”

6 Concluding remarks and further readings

We have presented a general overview of the literature on fractional integration and cointegration, both from theoretical and empirical perspectives. Our aim has been to provide a good introduction to most of the relevant issues in these research areas, but, undoubtedly, some important topics have been omitted or not covered in detail. To conclude our review we briefly outline a few such topics.

A prominent example relates to stationary ARCH-type models that have been introduced to capture long memory features of asset return volatility. Some of the most relevant ideas

regarding this general topic are covered in the survey of Giraitis, Leipus, and Surgailis (2009). The most well-known of these models appears to be the integrated FIGARCH process introduced by Baillie, Bollerslev, and Mikkelsen (1996) briefly mentioned in Section 5 in the context of empirical applications, but this is only one particular ARCH-type model with long memory. A general model that may exhibit power law decay in the autocovariances of squared returns is the ARCH(∞) model introduced by Robinson (1991), which is a nonparametric generalization of the GARCH(p, q) model. Other processes which display long memory in squares are the Linear ARCH of Giraitis, Robinson, and Surgailis (2000, 2004), the bilinear model of Giraitis and Surgailis (2002), the HYGARCH model of Davidson (2004), and the long memory stochastic volatility model introduced by Breidt, Crato, and de Lima (1998) and Harvey (1998); see also Robinson and Zaffaroni (1998) and Robinson (2001).

Another important class of models that we have not covered so far are those where the singularity of the spectral density of the process does not occur at the zero frequency as in (9), but instead at a seasonal or cyclical frequency. This type of process is relevant for the treatment of seasonally unadjusted time series when the dependence among seasonal (or cyclical) observations display the typical slow decay of standard long memory processes. Properties of these models and relevant estimation methods have been analyzed by, among others, Gray, Zhang, and Woodward (1989), Robinson (1994b), Giraitis and Leipus (1995), Arteché and Robinson (1999, 2000), Arteché (2002, 2020), Nielsen (2004a), and Arteché and Velasco (2005). Within this literature, a particularly interesting inferential problem appears when, in addition to the exponent of the spectral singularity (the memory parameter), the location of the pole is also unknown and needs to be estimated from the data. The location of the pole is known for seasonal processes, but in cases where the interest is to measure the length of a cycle (e.g., when modeling macroeconomic observables), it is realistic to assume that the location of the pole is unknown. In the latter case, inference is substantially more complicated. Estimation of both the location of the pole and the related memory parameter is analyzed by Giraitis, Hidalgo, and Robinson (2001), Hidalgo and Soulier (2004), and Hidalgo (2005).

Related to fractional cointegration and focusing on a bivariate case, Hualde (2006) considered a situation where the integration orders of the two observables are different, but their corresponding balanced versions are cointegrated in the usual sense. The balanced versions are obtained by fractionally differencing one series with the appropriate parameter, termed the imbalance parameter, such that both series have identical integration orders. Hualde (2006) termed this concept unbalanced cointegration, and it is a particular case of polynomial (fractional) cointegration. In the context of the FCVAR model, Johansen (2008) and Franchi (2010) gave conditions under which polynomial cointegration can arise. Interestingly, in this situation both the cointegration parameter and the imbalance parameter drive the long-run linkages between the observables. Hualde (2014) proposed a model of unbalanced cointegration and a semiparametric estimator of the cointegration and imbalance parameters, and described their limiting properties. Johansen and Nielsen (2021) proposed a generalization of the FCVAR model in which each observable has its own memory parameter. They showed that the regression-based unbalanced cointegration model analyzed by Hualde (2006, 2014) arises as a special case of their model, and further discussed the concept of unbalanced cointegration, model properties, as well as maximum likelihood estimators and their asymptotic properties.

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