LINE BUNDLES ON PROJECTIVE HOMOGENEOUS SPACES

BY

NIELS THOMAS HJORT LAURITZEN

Cand. scient., Københavns Universitet, 1989

M. S., University of Illinois, 1990

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INTRODUCTION

The topic of this thesis is projective homogeneous spaces in prime characteristic. Fix an algebraically closed field k of characteristic $p > 0$ and a semisimple algebraic group G. A variety X is called a G-space if it carries a morphism

$$G \times X \to X$$

such that the induced map on the functors of points $G(A) \times X(A) \to X(A)$ is an action of the group $G(A)$ on the set $X(A)$. A homogeneous space is a G-space X such that the induced action on k-points $G(k) \times X(k) \to X(k)$ is transitive. Standard examples of projective homogeneous G-spaces are the flag varieties G/P, where P is a parabolic subgroup of G. These are the projective homogeneous spaces with reduced stabilizer group schemes. The geometry of these spaces and their subvarieties has been studied extensively. Let me mention the Grassmannian $G(m, n)$, which is the variety of m-dimensional subspaces of an n-dimensional vector space as a prominent example. The study of Grassmannians and certain of their subvarieties, called Schubert varieties, dates back to the previous century. A central construction performed on flag varieties G/P and crucial for the representation theory of G is that global sections of homogeneous line bundles $\mathcal{L}$ on G/P are finite dimensional representations of G. Over fields of characteristic zero one gets all the simple G-representations in this way. In prime characteristic the simple G-modules are submodules of these spaces of global sections and one of the major open problems is to find a formula for the formal character of these. The formula is known to be a $\mathbb{Z}$-linear combination of Weyl character formulae and it has been conjectured by Lusztig [Ja, II.7.20 (3)] that the $\mathbb{Z}$-coefficients occurring are specializations of suitable Kazhdan-Lusztig polynomials for the affine Weyl group of G.

Kempf's vanishing theorem states that if $\mathcal{L}$ is an effective line bundle on a projective homogeneous space X under the assumption (A): the stabilizer group scheme of a point is reduced, then $\mathrm{H}^i(X, \mathcal{L}) = 0$ for $i > 0$. As a consequence, the character of the G-module $H^0(G/B, \mathcal{L})$, where B is a Borel subgroup of G, is given by Weyl's character formula. In his invited contribution [Ke] to the International Congress of Mathematicians, Helsinki 1978 Kempf writes:

> It would be interesting to know to what extent this theorem (Kempf's vanishing theorem) remains true when the assumption (A) is dropped

Certainly complete homogeneous spaces with non-reduced stabilizers abound. One might as examples take $G/B \ltimes P^{(n)}$, where $P^{(n)}$ denotes the Frobenius kernel of a parabolic subgroup containing the Borel subgroup B of G. In the last chapter of this thesis we give several examples in the shape of "twisted $\mathbb{P}^1$-fibrations" showing that Kempf's vanishing theorem breaks down for projective homogeneous spaces with non-reduced stabilizers.

Mehta and Ramanathan discovered in 1984 that flag varieties and their Schubert varieties have a remarkable property called Frobenius splitting. If X is a variety and $F : X \to X$ the Frobenius map, then X is called Frobenius split [MR] if the morphism

$$\mathcal{O}_X \to F_*\mathcal{O}_X$$

of $\mathcal{O}_X$-modules splits. The consequences of this innocent looking property are extraordinary. One gets the vanishing theorem for flag varieties and Schubert varieties (and as a byproduct Demazure's character formula) and a number of other nice properties [Ra].

Again one may ask: To what extent does the Frobenius splitting property generalizes to projective homogeneous spaces $G/\tilde{P}$, where $\tilde{P}$ is not reduced? One of the main results of this thesis is that projective homogeneous spaces with non-reduced stabilizers fail to be Frobenius split unless they are isomorphic to flag varieties (which happens rarely).

From a representation theoretic point of view, non-reduced parabolic subgroup schemes are interesting in that they give rise to new and interesting representations of G - mysterious creatures living inside the already known spaces of global sections of line bundles on flag varieties. To see how this happens, recall that the global sections of a homogeneous line bundle $\mathcal{L}(\lambda)$ for a dominant weight λ on G/B

$$\mathrm{H}^0(G/B, \mathcal{L}(\lambda)) = \{f : G \to k | f(x\,b) = \lambda(b)^{-1} f(x) \text{ for all } x \in G(A),\ b \in B(A) \text{ and all } k\text{-algebras } A\}$$

contains the irreducible G-module $L(\lambda)$ of highest weight λ. Now let $\tilde{B}$ be a subgroup scheme with reduced part B. If λ is a weight of $\tilde{B}$ (this imposes certain p-adic restrictions on the coefficients of the expansion of λ in terms of the fundamental dominant weights), then $H^0(G/\tilde{B}, \mathcal{L}(\lambda))$ is a submodule of $H^0(G/B, \mathcal{L}(\lambda))$, since $B \subseteq \tilde{B}$. This gives us the inclusions

$$L(\lambda) \hookrightarrow H^0(G/\tilde{B}, \mathcal{L}(\lambda)) \hookrightarrow H^0(G/B, \mathcal{L}(\lambda))$$

In view of the open problem about the formal character of $L(\lambda)$ it becomes interesting to find a character formula for the submodule $\mathrm{H}^0(G/\tilde{B}, \mathcal{L}(\lambda))$.

The main results of this work are[1]

(1) A complete description of the Frobenius splitting properties of projective homogeneous spaces

(2) A character formula for the Euler character of homogeneous line bundles on projective homogeneous spaces

(3) New counterexamples to Kempf's and Kodaira's vanishing theorem in prime characteristic.

Let me comment on these three items: The main result in 1) is the somewhat surprising fact, that a projective homogeneous space is Frobenius split only when it is isomorphic to a flag variety. This happens only when its stabilizer is the extension of a parabolic subgroup by a Frobenius kernel of G.

Using Nielsen's fixed point formula [Ni] and a computation of the action of a maximal torus on the tangent space of a projective homogeneous space at a point, we give a character formula for the Euler character of a homogeneous line bundle on a projective homogeneous

[1] The main results of this work have appeared or will appear in *Comptes Rendus* [La1], *Journal of Algebra* [La2] and the Conference Proceedings for the Steinberg Conference UCLA 1992 [HL].

space. This formula reduces to Weyl's character formula, when the projective homogeneous space is a flag variety.

Computer calculations with this character formula reveal the interesting fact that vanishing for ample line bundles on projective homogeneous spaces with non-reduced stabilizers breaks down. Ramanathan suggested that the natural generalization would be the Kodaira vanishing theorem. Using a modification of the character formula in the special case of "twisted $\mathbb{P}^1$-fibrations" and several hours of CPU-time on the dedicated research machine "orion" at the University of Illinois, a counter example to Kodaira's vanishing theorem was finally found in June 1992. The example is a 20-dimensional projective homogeneous space $G/\tilde{P}$ for G of type D_5 with an ample line bundle $\mathcal{L}$ such that $\mathcal{L} \otimes \omega_{G/\tilde{P}}$ has negative Euler characteristic. It answers negatively the following question raised by Raynaud in [Ray]:

> Soient X une variété propre et lisse sur k, $\mathcal{L}$ un faisceau ample sur X et ω le faisceau dualisant. Supposons que X et $\mathcal{L}$ relèvent en caractéristique zéro. Il resulte alors du théorème de Kodaira et des propriétés de spécialisation de la cohomologie que pour tout entier $i \geq 0$, on a:
>
> $$\chi^i(X, \mathcal{L} \otimes \omega) = \dim \mathrm{H}^i(X, \mathcal{L} \otimes \omega) - \dim \mathrm{H}^{i+1}(X, \mathcal{L} \otimes \omega) + \cdots \geq 0$$
>
> Ces propriétés restent-elles varies sans hypothèses de relèvement?

We begin with a chapter on preliminaries, where the basic facts on algebraic groups are recalled. We devote chapter 2 to the non-reduced parabolic subgroup schemes and the projective homogeneous spaces in the simplest case $G = SL_2(k)$. Chapter 3 covers the classification of parabolic subgroup schemes in characteristic $p > 3$ following [We1]. In chapter 4 we prove the rationality of projective homogeneous spaces along the lines of the proof in [HL]. Chapter 4 also contains results on the weights in the subrepresentations defined by non-reduced parabolic subgroup schemes along with a description of the canonical line bundle and the character formula alluded to above. Chapter 5 is devoted to Frobenius splitting and finally chapter 6 gives counterexamples to Kempf's and Kodaira's vanishing theorem in the shape of "twisted $\mathbb{P}^1$-fibrations".

CHAPTER I

PRELIMINARIES

In this chapter we review facts about algebraic groups and give definitions we shall use later on. In the rest of this work, k will denote an algebraically closed field of prime characteristic $p > 0$. All varieties are k-varieties. A knowledge of linear algebraic groups corresponding to for example Borel's book [Bo] is assumed. In what follows all algebraic groups are linear.

1.1 The Functor of Points.

Let X be a variety. The functor of points of X is the functor from k-algebras to sets given by

$$X(A) = \mathrm{Mor}_k(\mathrm{Spec}(A), X)$$

where Mor denotes the set of k-morphisms.

EXAMPLE 1. *Let* $X = \mathrm{Spec}(k[t_1, \ldots, t_n]/I)$ *be a closed subvariety of affine* n*-space. Suppose* I *is generated by the polynomials* $f_1, \ldots, f_r$*. It makes sense to talk about the solutions of the polynomial equations* $f_1, \ldots, f_r$ *over any* k*-algebra* C*. The set*

$$X(C) = \mathrm{Mor}_k(\mathrm{Spec}(C), X) \cong \mathrm{Hom}_{k-alg}(k[t_1, \ldots, t_n]/I, C)$$

is in 1-1 *correspondence with the solutions over* C*.*

The functor of points of an algebraic group $G = \mathrm{Spec}(A)$, where A is a Hopf algebra with comultiplication $\mu : A \to A \otimes A$, counit $\varepsilon : A \to k$ and coinverse $S : A \to A$, is group valued. Let $\varphi, \psi \in G(C) = \mathrm{Hom}_{k-alg}(A, C)$. Then the product $\varphi \cdot \psi$ of φ and ψ is given by the composite k-algebra homomorphism

$$A \xrightarrow{\mu} A \otimes A \xrightarrow{\varphi \otimes \psi} C \otimes C \longrightarrow C$$

where the last arrow is the multiplication map.

An ideal $I \subseteq A$ defines a subgroup scheme if and only if $\mu(I) \subseteq A \otimes I + I \otimes A$, $S(I) \subseteq I$ and $\varepsilon(I) = 0$. Such an ideal is called a Hopf ideal.

EXAMPLE 2. *Let* $\mathbb{G}_a = \operatorname{Spec} k[T]$ *be the affine line. In this case the Hopf algebra structure on* $k[T]$ *is given by* $\mu(T) = 1 \otimes T + T \otimes 1$, $\varepsilon(T) = 0$ *and* $S(T) = -T$. *The corresponding group functor maps a* k*-algebra* A *to its additive group* $(A, +)$. *Let* $I = (f(T))$ *be a Hopf ideal. Then* $f(0) = 0$, $f(T) = af(-T)$ *for some* $a \in k$ *and*

$$f(X+Y) = A(X,Y)f(X) + B(X,Y)f(Y)$$

for suitable $A, B \in k[X,Y]$.

Write $f(X+Y) - f(X) - f(Y) = A'(X,Y)f(X) + B'(X,Y)f(Y)$ *and compare* X*- and* Y*-degrees on both sides to conclude that* $f(X+Y) = f(X) + f(Y)$. *Now it follows easily that*

$$f(T) = a_0 T + a_1 T^p + \cdots + a_m T^{p^m}$$

for a suitable $m \in \mathbb{N}$ *and* $a_0, \ldots, a_m \in k$. *One consequence of this is that a connected subgroup scheme of* $\mathbb{G}_a$ *is*

$$\mathbb{G}_{a,m} = \operatorname{Spec} k[T]/(T^{p^m})$$

for suitable $m \in \mathbb{N}_0$ *or* $\mathbb{G}_a$ *itself.*

1.2 Semisimple Algebraic Groups.

Let G be a reduced and connected algebraic group with Lie algebra $\operatorname{Lie}(G)$ and let $R(G)$ denote the largest connected normal solvable subgroup of G. G is called semisimple if $R(G) = \{e\}$. In the following G will denote a semisimple group. Let T be a maximal torus in G with character group $X(T) = \operatorname{Hom}(T, \mathbb{G}_m)$. A finite dimensional rational representation V of G

$$\varphi : G \to GL(V)$$

decomposes into a direct sum of weight spaces $V_\lambda = \{v \in V | t.v = \lambda(t)v \text{ for all } t \in T\}$, where $\lambda \in X(T)$. Here $\chi \in X(T)$ is called a weight of V if $V_\chi \neq 0$.

Definition 1. *Let $R = \mathbb{Z}[X(T)]$ be the group ring of $X(T)$ with the canonical $\mathbb{Z}$-basis e^λ, where $\lambda \in X(T)$. The formal character of a finite dimensional rational representation $\varphi : G \to GL(V)$ is*

$$\operatorname{ch} V = \sum_{\chi \in X(T)} \dim V_\chi e^\chi \in R$$

where $V = \oplus_{\chi \in X(T)} V_\chi$.

The adjoint representation $G \to GL(\operatorname{Lie} G)$ decomposes into the T-eigenspaces

$$\operatorname{Lie} G = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in R} (\operatorname{Lie} G)_\alpha$$

Here $(\operatorname{Lie} G)_0 = \operatorname{Lie} T$ and the set R of non-zero weights of $\operatorname{Lie} G$ are called the roots of G with respect to T. For each $\alpha \in R$ we have the root homomorphism

$$x_\alpha : \mathbb{G}_a \to G$$

and

$$t x_\alpha(z) t^{-1} = x_\alpha(\alpha(t) z)$$

We denote the subgroup $x_\alpha(\mathbb{G}_{a,n})$ by $U_{\alpha,n}$, where $\mathbb{G}_{a,n}$ denotes the n-th Frobenius kernel $\operatorname{Spec} k[T]/(T^{p^n})$ of $\mathbb{G}_a$. As usual $x_\alpha(\mathbb{G}_a)$ will be denoted U_α.

For two roots α, β with $\alpha + \beta \neq 0$, we have the commutator formula (A any k-algebra, $a, b \in A$ and $c_{ij} \in k$)

(2) $$(x_\alpha(a), x_\beta(b)) = \prod_{\substack{i,j>0 \\ i\alpha + j\beta \in R}} x_{i\alpha+j\beta}(c_{ij} a^i b^j)$$

where the product is taken in any fixed order (c_{ij} depends on this order).

Let $Y(T) = \operatorname{Hom}(\mathbb{G}_m, T)$ denote the one-parameter subgroups of T. There is a perfect pairing $X(T) \times Y(T) \to \mathbb{Z}$ given by $(\phi, \lambda) \mapsto \langle \phi, \lambda \rangle$, where the composite homomorphism $\phi \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$ is $a \mapsto a^{\langle \phi, \lambda \rangle}$.

We let $W = N_G(T)/T$ denote the Weyl group of T. Recall that R is a root system in $\mathbb{R} \otimes X(T)$ of rank=rank $G = \dim T = l$. Now fix a Borel subgroup B containing T and let

S denote the simple roots in R corresponding to B. As a convention we choose *the roots of B to be the negative roots R^-.*

We let $\alpha^\vee \in Y(T)$ denote the coroot corresponding to $\alpha \in R$. As a convention we will denote $\langle \xi, \alpha^\vee \rangle$ by $\alpha^\vee(\xi)$, where $\xi \in X(T)$. The simple roots S form a basis of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes X(T)$ and $\{\alpha^\vee | \alpha \in S\}$ form a basis of $Y(T) \otimes \mathbb{Q} = (X(T) \otimes \mathbb{Q})^*$. There are $\omega_\alpha \in \mathbb{Q} \otimes X(T)$, where $\alpha \in S$ such that $\beta^\vee(\omega_\alpha) = \delta_{\alpha\beta}$ for $\alpha, \beta \in S$. These are called the fundamental dominant weights. G is called *simply connected* if $\omega_\alpha \in X(T)$ for $\alpha \in S$.

The Weyl group W is generated by the simple reflections $\{s_\alpha | \alpha \in S\}$. Let w_0 denote the longest element of W. The Bruhat decomposition states that G has a cell decomposition

$$G = \bigcup_{w \in W} BwB$$

Here Bw_0B is a dense open affine subset called the big cell.

Recall that a subgroup P is called parabolic if it is conjugate to a subgroup containing B. Let W_I denote the subgroup of W generated by the simple reflections corresponding to a subset $I \subseteq S$. There is a 1-1 correspondence between subsets $I \subseteq S$ and parabolic subgroups containing B given by

$$I \mapsto BW_IB = \bigcup_{w \in W_I} BwB$$

1.3 Homogeneous Vector Bundles.

A finite dimensional representation E of a subgroup scheme H of G gives rise [Ja, I.5.8] to a coherent homogeneous sheaf $\mathcal{L}(E)$ on the homogeneous space G/H given by

$$\mathcal{L}(E)(U) = \{f : \pi^{-1}(U) \to E | f(xh) = h^{-1} f(x) \\ \text{for all } x \in G(A),\ h \in H(A) \text{ and all } k\text{-algebras } A\}$$

where $\pi : G \to G/H$ is the projection. One can prove that $\mathcal{L}(E)$ is locally free using that it has constant rank $\dim E$. We call $\mathcal{L}(E)$ the homogeneous vector bundle on G/H induced by the H-representation E. The total space of the vector bundle associated with $\mathcal{L}(E)$ is $G \times^H E = G \times E/H$, where H acts on $G \times E$ via $(x, y)g = (xg, g^{-1}y)$. The fibration $G \times^H E \to G/H$ becomes G-equivariant, when G acts on $G \times^H E$ through left multiplication on the first factor.

1.4 The Frobenius Morphism.

Definition 1. *Let X be a variety. Define the variety X_n to have the same underlying topological space as X but with regular functions, which locally are p^n-th powers of regular functions on X. The morphism*

$$F_n : X \to X_n$$

which is the identity on point spaces and the inclusion $\mathcal{O}_{X_n} \hookrightarrow \mathcal{O}_X$ on the sheaf level is called the Frobenius morphism of order n. The variety X_n is called the Frobenius cover of order n of X. The image of a closed subvariety $Y \subseteq X$ under F_n will be denoted Y_n.

If G is an algebraic group, then $F_n : G \to G_n$ is a homomorphism of algebraic groups. Its kernel $G^{(n)}$ is called the Frobenius kernel of order n of G. Notice that $G^{(n)}$ is a characteristic subgroup of G and that the scheme theoretic inverse image $F_n^{-1}(H_n)$, where H is a subgroup of G, is the subgroup $G^{(n)}H$. We have an exact sequence

$$1 \to G^{(n)} \to G \to G_n \to 1$$

so that the regular functions on G_n are the regular functions on G, which are invariant under right translation by $G^{(n)}$.

Example 2. *If $G = GL_2(k)$ then the group homomorphism $F_n : G \to G_n$ is the Frobenius homomorphism on each matrix entry*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^{p^n} & b^{p^n} \\ c^{p^n} & d^{p^n} \end{pmatrix}$$

Remark 3. *If X is defined over $\mathbb{F}_p$, then X is isomorphic to X_n. Locally this amounts to the fact that the Frobenius homomorphism $x \mapsto x^{p^n}$ from an $\mathbb{F}_p$-algebra A to its subalgebra A_n generated by p^n-th powers is an isomorphism of $\mathbb{F}_p$-algebras. The base change of this morphism produces an isomorphism: $X \to X_n$ over the field k.*

By a classical result of Chevalley all semisimple algebraic groups G and their homogeneous spaces G/P, where P is a parabolic subgroup are defined over $\mathbb{F}_p$.

Definition 4. *Let* $\varphi : G \to GL(V)$ *be a finite dimensional rational representation of* G*. Then* φ *gives rise to a representation* $\varphi_n : G_n \to GL(V_n)$*. We call the composite*

$$G \xrightarrow{F_n} G_n \xrightarrow{\varphi_n} GL(V_n)$$

the n*-th Frobenius twist of the representation* V*. We denote this new representation by* $V^{[n]}$*.*

CHAPTER II

THE RANK 1 CASE

In this chapter we analyze the simplest case of one-dimensional projective homogeneous spaces for $G = SL_2(k)$. The classification of parabolic subgroup schemes for G is given by the parameter set $\mathbb{N} \cup \{\infty\}$.

2.1 Classification.

Let

$$T = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} | t \in k \smallsetminus \{0\}\}$$

$$U = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} | a \in k\}$$

$$U^- = \{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} | a \in k\}$$

Denote the upper triangular matrices by $B = TU$ and notice that $U^- B$ is the affine open set of $SL_2(k)$ given by

$$\Omega = \{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL_2(k) | x \neq 0\}$$

with functor of points $C \mapsto U^-(C)B(C) \subset G(C)$. Any closed subscheme of G with reduced part B must be a closed subscheme of Ω (an infinitesimal thickening of B lives only in between B and the open subset $\Omega \supset B$). In particular a subgroup scheme $\tilde{P}$ with reduced part B is contained in Ω. Now $U^-_{\tilde{P}} = U^- \cap \tilde{P}$ is a connected finite subgroup scheme of U^-. By Example I.1.2 a connected subgroup scheme of the affine line $\mathbb{G}_a$ is given by the Hopf ideal (T^{p^n}), $n \in \mathbb{N}_0 \cup \{\infty\}$, where we will use the convention $T^{p^\infty} = 0$. Therefore as a functor of points

$$U^-_{\tilde{P}}(C) = \{\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} | c^{p^n} = 0\}$$

In the shape of its functor of points $\tilde{P}$ is

$$\tilde{P}(C) = (\tilde{P} \cap U^-)(C)B(C) = U^-_{\tilde{P}}(C)B(C) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(C) | c^{p^n} = 0\}$$

Thus $\tilde{P}$ is the closed subgroup scheme given by the Hopf ideal (z^{p^n}). Notice also that $\tilde{P} = B \ltimes G^{(n)}$. We will denote the subgroup scheme $\tilde{P}$ of SL_2 by B^2_n, with the convention that $B^2_\infty = G$. We have proven that a subgroup scheme containing B has to be of the form B^2_n.

2.2 Characters.

The characters $X(B)$ of B are identified with $\mathbb{Z}$ in the natural way $\gamma_m \mapsto m$, where

$$\gamma_m : \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \mapsto t^m$$

A character m of B is character of B^2_n if and only if

$$a^m \, a_1^m = \gamma_m(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) = (a \, a_1 + b \, c_1)^m$$

for all k-algebras C and every $a, a_1, b, c_1 \in C$. Thus the characters of B^2_n are $p^n X(B) = p^n \mathbb{Z}$.

2.3 Line Bundles and Representations.

Recall that

$$X = G/B \cong \mathbb{P}^1$$

and fix a k-point e in $\mathbb{P}^1$ with stabilizer B. The line bundle induced by the character γ_m becomes the line bundle $\mathcal{O}(m)$ on $\mathbb{P}^1$. Let us analyze the structure of the variety X_n. We already know that X and X_n are isomorphic as varieties (since $\mathbb{P}^1$ is defined over $\mathbb{F}_p$). But X_n is certainly a G-homogeneous space in its own right through the group homomorphism $G \to G_n$. It is easy to see that the stabilizer of the k-point $e \in X_n(k)$ becomes $B \ltimes G^{(n)} = B^2_n$. We have proved the following miniature

PROPOSITION 1. *A projective homogeneous space X for SL_2 is G-isomorphic to a Frobenius cover $\mathbb{P}^1_n$ of the projective line $\mathbb{P}^1$.*

The geometry of this proposition is not very interesting - it basically says that every projective homogeneous spaces for SL_2 looks like $\mathbb{P}^1$. In order for the geometry to get interesting we need to pursue these matters in higher ranks. This is the topic of the later chapters. We can however get some information of representation theoretic interest. Recall that the global sections of $\mathcal{O}(m)$

$$\mathrm{H}^0(G/B, \mathcal{O}(m)) = \{f : G \to k | f(x\,b) = b^{-m} f(x),\, g \in G,\, b \in B\}$$

for $m \geq 0$ is a G-representation through left translation: If $g \in G$ and $\varphi \in H^0(G/B, \mathcal{O}(m))$, then $g.\varphi(x) = \varphi(g^{-1}x)$. In a souped down version this is just the ordinary representation of SL_2 on homogeneous polynomials $k[X, Y]_m$ of degree m:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} d\,X - b\,Y \\ -c\,X + a\,Y \end{pmatrix}$$

In the basis $e_0, \dots, e_r$, where $e_i = X^i Y^{m-i}$ we get the following

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} .e_i = t^{m-2i} e_i$$
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} .e_i = \sum_{j=0}^{i} \binom{i}{j} (-a)^{i-j} e_j$$

We identify G/B_n^2 with $\mathbb{P}_n^1$ and view $H^0(\mathbb{P}_n^1, \mathcal{O}_n(m))$ as a G-representation with basis $e_0^n, \dots, e_m^n$ via the Frobenius morphism $G \to G_n$. This time the action has a Frobenius twist:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} .e_i = t^{p^n(m-2i)} e_i$$
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} .e_i = \sum_{j=0}^{i} \binom{i}{j} (-a)^{p^n(i-j)} e_j$$

and by I.4.4, we get $H^0(\mathbb{P}_n^1, \mathcal{O}_n(m)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m))^{[n]}$

We summarize the above results in

Proposition 2. *Let $G = SL_2(k)$ and B_n^2 the parabolic subgroup scheme $B \ltimes G^{(n)}$, where B is the subgroup of upper triangular matrices. If $m = p^n r$ is a character on B,* *then the* G*-representation* $V_m^n = H^0(G/B_n^2, \mathcal{O}(m))$ *is the Frobenius twist* $H^0(G/B, \mathcal{O}(r))^{[n]}$. *If* $e_0, \ldots, e_r$ *is a basis of* V_m^n, *then the action of* B *is given by*

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} .e_i = t^{p^n(m-2i)} e_i$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} .e_i = \sum_{j=0}^{i} \binom{i}{j} (-a)^{p^n(i-j)} e_j$$

CHAPTER III

CLASSIFICATION

Let G be a semisimple algebraic group. Fix a Borel subgroup B of G and a maximal torus T contained in B. In this chapter we classify, following [We1], the parabolic subgroup schemes $\tilde{P}$ containing B and compute the character lattice $X(\tilde{P}) \subseteq X(B) = X(T)$.

3.1 The W-function of $\tilde{P}$.

First we introduce some notation, which will be used throughout this section. Let R be the root system of G with respect to T and let R^+ be the roots of the opposite Borel subgroup to B. We denote the simple roots of R^+ by S. Let $\tilde{P}$ denote a subgroup scheme containing B.

The reduced part $P = \tilde{P}_{\mathrm{red}}$ is a subgroup scheme of $\tilde{P}$ as the nil radical of a Hopf algebra over a perfect field is a Hopf ideal. In this way P is an ordinary parabolic subgroup. Since a parabolic subgroup is connected (a classical theorem of Chevalley says that $P = N_G(P)$) and $\tilde{P}_{\mathrm{red}} = P$, we see that $\tilde{P}$ is connected.

Let I be the unique subset of S, such that $P = BW_IB$ and let R_I denote the root system generated by I. Fix an ordering $\prec$ on the roots in $R^+ \setminus R_I$ such that $\alpha, \beta \prec \alpha + \beta$, whenever $\alpha, \beta \in R \setminus R_I$. Then the unipotent radical of P is $U_I \cong \prod_{\alpha \in R^+ \setminus R_I} U_{-\alpha}$, where the isomorphism is with respect to the ordering $\prec$. The opposite unipotent radical U_I^+ is $U_I^+ = \prod_{\alpha \in R^+ \setminus R_I} U_\alpha$. The image U_I^+P of U_I^+ and P under the product map is an open subset of G isomorphic to $U_I^+ \times P$ and its functor of points is given by $C \mapsto U_I^+(C)P(C) \subset G(C)$. The next lemma clarifies the structure of $\tilde{P}$ as a k-variety.

Lemma 1. *Let $\tilde{P}$ be a parabolic subgroup scheme. Then*

(1) *$\tilde{P}$, is a closed subscheme of U_I^+P.*

(2) *As a variety $\tilde{P}$ is isomorphic to*

$$\tilde{U} \times_k P \cong \tilde{U}\, P$$

where $\tilde{U} = U_I^+ \cap \tilde{P}$.

(3) *$\tilde{U}$ is a finite connected subgroup scheme of U_I^+*

(4) *$U_\alpha \cap \tilde{U} = U_{\alpha,n}$ for some $n \in \mathbb{N}$ and $\alpha \in R^+ \setminus R_I$*

Proof. If U is an open subset of an affine scheme $\mathrm{Spec}(A)$ containing the reduced closed subscheme $\mathrm{Spec}(A/I)$ and J is an ideal with $\sqrt{J} = I$ defining the infinitesimal neighborhood $\mathrm{Spec}(A/J)$ of $\mathrm{Spec}(A/I)$. Then $\mathrm{Spec}(A/J)$ is contained in U. (1) follows from this general fact.

Furthermore (1) implies that $\tilde{P}$ regarded as its functor of points is a subfunctor of the functor $U_I^+ P$. Clearly $\tilde{U}P$ is a subfunctor of $\tilde{P}$. Since $\tilde{P}$ is a right P-stable subfunctor of $U_I^+ P$, we find that $\tilde{P} = \tilde{U}P$. This proves (2). It follows from the isomorphism $\tilde{P} \cong \tilde{U} \times_k P$, that $\tilde{U}$ is a finite and connected subgroup scheme since $\tilde{P}$ is connected and $\dim \tilde{P} = \dim P$. This gives (3). Now (4) is a consequence of I.1.2. □

Proposition 2. *Let $\tilde{P}$ be a parabolic subgroup scheme and let $\{\alpha_1, \ldots, \alpha_l\} = R^+ \setminus R_I$. Denote $U_\alpha \cap \tilde{P}$ by U_{α,n_α}. Then*

$$U_{\alpha_1,n_{\alpha_1}} \times_k \cdots \times_k U_{\alpha_l,n_{\alpha_l}} \cong \prod_{i=1}^{l} U_{\alpha_i,n_{\alpha_i}} = \tilde{U}$$

Proof. The first isomorphism follows from the fact that the product map is an isomorphism. Now for the equality: By construction $U_{\alpha,n_\alpha} \subseteq \tilde{U}$, so the inclusion $\subseteq$ holds. We will prove the opposite inclusion by induction on the length s of the product decomposition

$$u = x_{\alpha_{i_1}}(a_{i_1}) \cdot \ldots \cdot x_{\alpha_{i_s}}(a_{i_s})$$

of the element $u \in \tilde{U}(A)$, where A is any k-algebra. For $s = 1$ we get that $u \in U_{\alpha_1,n_{\alpha_1}}$ and the result is clear. If $s > 1$, then for $t \in T(k)$ we get via the commutator formula that

$$\begin{aligned} u\,t\,u\,t^{-1} &= x_{\alpha_{i_1}}(a_{i_1}) \cdot \ldots \cdot x_{\alpha_{i_s}}(a_{i_s}) x_{\alpha_{i_1}}(\alpha_{i_1}(t)a_{i_1}) \cdot \ldots \cdot x_{\alpha_{i_s}}(\alpha_{i_s}(t)a_{i_s}) \\ &= x_{\alpha_{i_1}}((1+\alpha_{i_1}(t))a_{i_1}) x_{\alpha_{i_2}}((1+\alpha_{i_2}(t))a_{i_2}) x_{\alpha_{i_3}}(a'_{i_3}) \cdot \ldots \cdot x_{\alpha_{i_s}}(a'_{i_s}) \end{aligned}$$

where $a'_{i_3}, \ldots, a'_{i_s} \in A$. We have used the identity $x_\alpha(a)x_\beta(b) = x_\beta(b)x_\alpha(a)(x_\alpha(a), x_\beta(b))$ to bring the root subgroups in the desired order. Notice also that it is essential for the commutator computation that the order $\prec$ on the roots satisfies $\alpha, \beta \prec \alpha + \beta$ for $\alpha, \beta \in R^+ \setminus R_I$.

Finally α_{i_1} and α_{i_2} are distinct roots so their kernels are different, and we can choose $t_0 \in T(k)$, such that $\alpha_{i_2}(t_0) = -1 \neq \alpha_{i_1}(t_0)$. Inserting t_0 we get by induction, that $x_{\alpha_1}(c\, a_{i_1}) \in U_{\alpha_{i_1}, n_{\alpha_{i_1}}}$ for a non-zero constant $c \in k$. It follows that $x_{\alpha_{i_1}}(a_{i_1}) \in U_{\alpha_{i_1}, n_{\alpha_{i_1}}}$. Again by induction we get

$$u = x_{\alpha_{i_1}}(a_{i_1})(x_{\alpha_{i_1}}(a_{i_1})^{-1}u) \in \prod_{i=1}^{l} U_{\alpha_i, n_{\alpha_i}}$$

□

Definition 3. *A function $f : R^+ \to \mathbb{N}_0 \cup \{\infty\}$ is called a W-function if it satisfies*

$$f(\gamma) = \min\{f(\alpha) | \alpha \in \mathrm{Supp}(\gamma)\}$$

where $\mathrm{Supp}(\gamma) = \{\alpha \in S | \gamma^\vee(\omega_\alpha) \neq 0\}$.

Clearly a W-function is uniquely determined by its values on the simple roots $S \subseteq R^+$.

If $\tilde{P}$ is a parabolic subgroup scheme with reduced part P and $U_\alpha \cap \tilde{P} = U_{\alpha, n_\alpha}$ for $\alpha \in R^+ \setminus R_I$, then we define a function $f_{\tilde{P}} : R^+ \to \mathbb{N}_0 \cup \{\infty\}$ associated with $\tilde{P}$, by

$$f_{\tilde{P}}(\gamma) = \begin{cases} \infty, & \text{if } \gamma \text{ is a root of } P \\ n_\gamma, & \text{otherwise} \end{cases}$$

Notice that if $f_{\tilde{P}}$ is finite valued, then $\tilde{P}_{\mathrm{red}} = B$. Also if $f_{\tilde{P}}(R^+) \subseteq \{0, \infty\}$, then $\tilde{P}$ is reduced.

Lemma 4. *Let k be of characteristic $p > 3$. If $\tilde{P}$ is a parabolic subgroup scheme, then $f_{\tilde{P}}$ is a W-function.*

Proof. Let $\gamma \in R^+$ be a positive root with $\mathrm{ht}\, \gamma > 1$ and β a simple root, such that $\delta = \gamma - \beta \in R^+$. It suffices to prove that

$$f_{\tilde{P}}(\gamma) = \min(f_{\tilde{P}}(\gamma - \beta), f_{\tilde{P}}(\beta))$$

If $f_{\tilde{P}}(\gamma) = \infty$, then γ is a root of $\tilde{P}_{\text{red}}$ and so is β and $\gamma - \beta$. This proves that $f_{\tilde{P}}(\gamma - \beta) = f_{\tilde{P}}(\beta) = \infty$, so that the identity holds in this case. Assume that $f_{\tilde{P}}(\gamma) < \infty$. We will prove that $f_{\tilde{P}}(\gamma) \leq f_{\tilde{P}}(\beta)$. We can assume, that $f_{\tilde{P}}(\beta) < \infty$. Let A be any k-algebra and let $x_\gamma(a_\gamma) \in \tilde{U}(A)$ for $a_\gamma \in A$. By I.2.2 we get

$$(x_{-\delta}(1), x_\gamma(a_\gamma)) = \prod_{i,j>0} x_{-i\delta+j\gamma}(c_{ij}a_\gamma^j) \in \tilde{U}(A)$$

This implies by Proposition 2, that $x_{-\delta+\gamma}(c_{11}a_\gamma) = x_\beta(c_{11}a_\gamma) \in \tilde{U}(A)$. By [St, Lemma 15], c_{11} is the length of a root string and thereby ≤ 4. Since $p \geq 5$, we can conclude that $x_\beta(a_\gamma) \in \tilde{U}(A)$. This means, that for any k-algebra A and $a_\gamma \in A$ with $a_\gamma^{p^{f_{\tilde{P}}(\gamma)}} = 0$, we have $a_\gamma^{p^{f_{\tilde{P}}(\beta)}} = 0$. It follows that $f_{\tilde{P}}(\gamma) \leq f_{\tilde{P}}(\beta)$. Next we prove, that $f_{\tilde{P}}(\gamma) \leq f_{\tilde{P}}(\gamma - \beta)$. We can assume, that $f_{\tilde{P}}(\gamma - \beta) < \infty$. Now proceed in exactly the same way this time using I.2.2 on $(x_{-\beta}(1), x_\gamma(a_\gamma))$.

To complete the proof of the identity, we need to show, that

$$f_{\tilde{P}}(\gamma) \geq \min(f_{\tilde{P}}(\gamma - \beta), f_{\tilde{P}}(\beta))$$

Notice that the right hand side is always $< \infty$. Again let A be any k-algebra and let $a_\delta, a_\beta \in A$, such that $x_\delta(a_\delta), x_\beta(a_\beta) \in \tilde{P}(A)$. The commutator formula I.2.2 gives

$$(x_\delta(a_\delta), x_\beta(a_\beta)) = \prod_{i,j>0} x_{i\delta+j\gamma}(c_{ij}a_\delta^i a_\beta^j) \in \tilde{P}(A)$$

As before we get $x_{\delta+\beta}(c_{11}a_\delta a_\beta) = x_\gamma(c_{11}a_\delta a_\beta) \in \tilde{U}(A)$, so that $x_\gamma(a_\delta a_\beta) \in \tilde{U}(A)$. This gives that $(a_\delta a_\beta)^{p^{f_{\tilde{P}}(\gamma)}} = 0$. As above this implies that $f_{\tilde{P}}(\gamma) \geq \min(f_{\tilde{P}}(\gamma - \beta), f_{\tilde{P}}(\beta))$ $\square$

Remark 5. *If $\tilde{P}$ and $\tilde{Q}$ are parabolic subgroup schemes with f and g as W-functions respectively. Then*

(1) *$P \subseteq Q$ if and only if $f \leq g$*
(2) *$P \cap Q$ is a parabolic subgroup scheme with* $\min(f, g)$ *as W-function.*

Let $P(\alpha)$, for a simple root α, denote the maximal parabolic subgroup not containing the root subgroup U_α. Then $P(\alpha) \ltimes G^{(n)}$ is a parabolic subgroup scheme with a W-function, which is n at α and ∞ elsewhere.

Proposition 6. *Let* $S = \{\alpha_1, \dots, \alpha_l\}$ *and let* $f : R^+ \to \mathbb{N}_0 \cup \{\infty\}$ *be a W-function. With the convention* $G^{(\infty)} = G$

$$\tilde{P} = G^{(f(\alpha_1))} P(\alpha_1) \cap \dots \cap G^{(f(\alpha_l))} P(\alpha_l)$$

is a parabolic subgroup scheme with $f_{\tilde{P}} = f$.

5.2 The characters of $\tilde{P}$. In this section we compute the lattice of characters $X(\tilde{P}) = \mathrm{Hom}(\tilde{P}, \mathbb{G}_m)$ of a parabolic subgroup scheme $\tilde{P}$. We will assume that G is simply connected, so that the character lattice $X(T)$ is

$$\mathbb{Z}\omega_{\alpha_1} + \dots + \mathbb{Z}\omega_{\alpha_l}$$

where ω_{α_i} denotes the fundamental dominant weight corresponding to the simple root α_i. If $P_I = BW_IB$ is a parabolic subgroup given by $I = \{\alpha_{i_1}, \dots, a_{i_s}\}$, then

$$X(P_I) = \mathbb{Z}\omega_{\alpha_{i_1}} + \dots + \mathbb{Z}\omega_{\alpha_{i_s}}$$

The restriction map $X(P_I) \to X(B) = X(T)$ is injective and corresponds to the above inclusion of lattices. As a character is trivial on a unipotent subgroup it is easy to see, that the restriction $X(\tilde{P}) \to X(\tilde{P}_{\mathrm{red}})$ is injective for a parabolic subgroup scheme $\tilde{P}$. We will identify the characters $X(\tilde{P})$ of a parabolic subgroup scheme with its image in $X(B) = X(T)$ under the restriction map. Now the following

Lemma 1. *Let* $P(\alpha)$ *be the maximal parabolic subgroup not containing the root subgroup* U_α. *Then*

$$X(G^{(n)} P(\alpha)) = \mathbb{Z} p^n \omega_\alpha$$

Proof. When G is semisimple and simply connected, then a character on a Frobenius kernel $G^{(n)}$ has to be trivial [Ja II.3.15, Remarks 2]. From the first isomorphism theorem for groups we get

$$\begin{aligned} X(G^{(n)} P(\alpha)) &= X(G^{(n)} P(\alpha)/G^{(n)}) = X(P(\alpha)/G^{(n)} \cap P(\alpha)) \\ &= X(P(\alpha)/P(\alpha)^{(n)}) = X(P(\alpha)_n) = p^n X(P(\alpha)) \\ &= \mathbb{Z} p^n \omega_\alpha \end{aligned}$$

□

Let us recall the following from [Ja, p. 176]: For any $\alpha \in R$ there is a homomorphism

$$\varphi_\alpha : SL_2 \to G$$

such that

$$\varphi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_\alpha(a) \text{ and } \varphi_\alpha \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} = x_{-\alpha}(a)$$

and

$$\alpha^\vee(t) = \varphi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{G}_m$$

We are now ready to prove

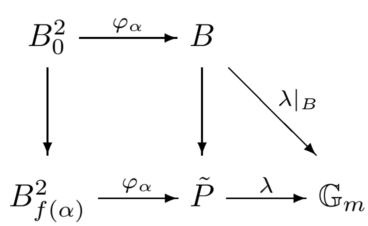
Proposition 2. *Let $\tilde{P}$ be a parabolic subgroup scheme with W-function f. With the p-adic convention $p^\infty = 0$, we have*

$$X(\tilde{P}) = \mathbb{Z}p^{f(\alpha_1)}\omega_{\alpha_1} + \cdots + \mathbb{Z}p^{f(\alpha_l)}\omega_{\alpha_l}$$

Proof. Since $\tilde{P} \subseteq G^{(f(\alpha))}P(\alpha)$ for a simple root α (with the convention $G^{(\infty)} = G$), we get by Lemma 1, that

$$\mathbb{Z}p^{f(\alpha_1)}\omega_{\alpha_1} + \cdots + \mathbb{Z}p^{f(\alpha_l)}\omega_{\alpha_l} \subseteq X(\tilde{P})$$

On the other hand if $\lambda \in X(\tilde{P})$ and α is a simple root with $f(\alpha) < \infty$, then $\varphi_\alpha(B^2_{f(\alpha)}) \subseteq \tilde{P}$ and we have the following diagram



Now $\lambda \circ \varphi_\alpha$ is a character of $B^2_{f(\alpha)}$ and thereby $p^{f(\alpha)} | \alpha^\vee(\lambda)$. Writing λ out as $\lambda = m_1\omega_{\alpha_1} + \cdots + m_l\omega_{\alpha_l}$, this shows that $p^{f(\alpha_i)} | m_i$. A moments consideration shows that we have actually proven, that

$$X(\tilde{P}) \subseteq \mathbb{Z}p^{f(\alpha_1)}\omega_{\alpha_1} + \cdots + \mathbb{Z}p^{f(\alpha_l)}\omega_{\alpha_l}$$

□

CHAPTER IV

THE SHEAF OF DIFFERENTIALS

In this chapter we first prove the rationality of projective homogeneous spaces. The proof we give runs along the same lines as the one in [HL]. We exhibit a T-stable neighborhood of the distinguished point equivariantly isomorphic to affine space with an appropriate T-action. The structure of this big cell can be used to prove a result on the T-weights in the representations occurring as global sections of line bundles on projective homogeneous spaces. This is the content of Proposition 2.1 and will be used later in a counterexample to Kempf's vanishing theorem. We also compute the homogenization of the sheaf of differentials and give a character formula for the formal Euler character of a homogeneous line bundle. In this chapter G will be simply connected and we will use the p-adic convention $p^\infty = 0$.

4.1 Rationality.

Proposition 1. *Let $\tilde{P} = \tilde{U}P$ be a parabolic subgroup scheme with W-function f and with reduced part $P = BW_IB$. Let x_0 denote the identity coset in the projective homogeneous space $X = G/\tilde{P}$ and let $R^+ \setminus R_I = \{\alpha_1, \ldots, \alpha_s\}$.*

There exists a T-stable affine neighborhood U of x_0, such that the algebra of functions $k[U]$ on U is generated by algebraically independent functions Y_{α_i}, $\alpha_i \in R^+ \setminus R_I$ each of weight $-p^{f(\alpha_i)}\alpha_i$ for the induced action of T on $k[U]$.

Proof. Let $\pi : G/P \to G/\tilde{P}$ denote the canonical morphism. If U denotes the opposite unipotent radical of P, then the image UP of U and P under the product morphism gives an open subset UP/P of G/P isomorphic to affine s-space. We will prove that the scheme theoretic image of UP/P under π is the desired T-neighborhood. We have the following chain of T-isomorphisms

$$\pi(UP/P) = UP/\tilde{P} = UP/\tilde{U}P \cong U/\tilde{U} \cong \operatorname{Spec} k[U]^{\tilde{U}}$$

where the last isomorphism follows from [Ja, I.5.5(6)] and $\tilde{P} = \tilde{U}P$.

We prove by induction on the number of elements M in $S \cap (R^+ \setminus R_I)$, that the algebra of invariants $k[U]^{\tilde{U}}$ under the finite group scheme $\tilde{U}$ is generated by algebraically independent functions of the desired weights.

If $M = 1$, then $\tilde{U}$ is $U^{(n)}$ for some n, as the W-function f is uniquely determined by its value on the simple root which is not in R_I. In this case the desired result follows from

$$k[U]^{\tilde{U}} = k[U]^{U^{(n)}} = k[U_n]$$

since the regular functions on U_n are p^n-th powers of the regular functions on U.

Now suppose $M > 1$ and let $R_0 = \{\alpha \in R | f(\alpha) = 0\}$ and $R_1 = \{\alpha \in R | f(\alpha) > 0\}$. Suppose that $R_0 \neq \emptyset$ and let $J = S \cap R_0$. Let U_0 be the opposite unipotent radical corresponding to the parabolic subgroup with roots R_1 and let U_1 be the smallest reduced unipotent subgroup containing $\tilde{U}$. Choose an isomorphism $U \cong U_0 \times U_1 \cong U_0U_1$. Under this isomorphism the functions $k[U_0]$ on U_0 are fixed under the action of $\tilde{U}$ on $k[U]$. For the ring of invariants we therefore have

$$k[U]^{\tilde{U}} \cong k[U_0] \otimes_k k[U_1]^{\tilde{U}}$$

We have algebraically independent functions in $k[U_0]$ of the desired weights. Now the result follows by induction since $|S \cap (R^+ \setminus R_{I \cup J})| < |S \cap (R^+ \setminus R_I)|$.

If $R_0 = \emptyset$, then $\tilde{U} = U^{(n)}\tilde{U}'$ for some n, where the W-function f' corresponding to $\tilde{U}'P$ is $f - n$. In this case we have

$$k[U]^{\tilde{U}} = k[U]^{U^{(n)}\tilde{U}'} \cong k[U_n]^{\tilde{U}'}$$

Working now with $U_n \subset G_n$, we have reduced to the case $R_0 \neq \emptyset$ and the proof is complete. □

4.2 Global Sections of Line Bundles.

Proposition 1. *Let $\tilde{P}$ be a parabolic subgroup scheme with associated W-function f and let $P = BW_IB$ be the reduced part of $\tilde{P}$. If $\lambda \in X(\tilde{P})$ is a character on the parabolic subgroup scheme $\tilde{P}$, then*

(1) $\mathrm{H}^0(G/\tilde{P}, \mathcal{L}(\lambda)) \neq 0$ *if and only if λ is dominant*
(2) $\mathcal{L}(\lambda)$ *is ample if and only if $\alpha^\vee(\lambda) > 0$ for every $\alpha \in S \setminus I$.*
(3) $\mathrm{H}^0(G/\tilde{P}, \mathcal{L}(\lambda))$ *is subrepresentation of* $\mathrm{H}^0(G/P, \mathcal{L}(\lambda))$ *and the T-weights occurring in* $\mathrm{H}^0(G/\tilde{P}, \mathcal{L}(\lambda))$ *are of the form*

$$\lambda - \sum_{\substack{\alpha \in R^+ \\ r_\alpha \geq 0}} r_\alpha p^{f(\alpha)} \alpha$$

Proof. Recall that

$$\mathrm{H}^0(G/H, \mathcal{L}(\lambda)) = \{f : G \to k \mid f(gh) = \lambda(h)^{-1} f(g), \text{for all } k\text{-algebras } A \text{ and } g \in G(A), h \in H(A)\}$$

for a subgroup scheme H and a character $\lambda \in X(H)$. We have here taken the view of the functor of points. The global sections $\mathrm{H}^0(G/H, \mathcal{L}(\lambda))$ is a G-representation through left translation. Recall also that for the parabolic subgroup P, we have $\mathrm{H}^0(G/P, \mathcal{L}(\lambda)) \neq 0$ if and only if λ is dominant. Let

$$\pi_{\tilde{P}} : G/P \to G/\tilde{P}$$

denote the morphism induced by the inclusion $P \subseteq \tilde{P}$. The pull back $\pi_{\tilde{P}}^* \mathcal{L}(\lambda)$ is the homogeneous line bundle $\mathcal{L}(\lambda|P)$ on G/P. From the injection

$$\mathcal{O}_{G/\tilde{P}} \hookrightarrow (\pi_{\tilde{P}})_* \mathcal{O}_{G/P}$$

of $\mathcal{O}_{G/\tilde{P}}$-modules and the projection formula, we get the injection

$$\mathrm{H}^0(G/\tilde{P}, \mathcal{L}(\lambda)) \hookrightarrow \mathrm{H}^0(G/P, \mathcal{L}(\lambda|P))$$

which gives the "if" part of (1). For the "only if" part, we need to construct a non-zero global section of $\mathcal{L}(\lambda)$ on $G/\tilde{P}$, where λ is dominant. This is done in the following way.

Let $P(\alpha)$ denote the maximal parabolic subgroup not containing the root subgroup U_α, where α is a simple root. Let μ be the fundamental dominant weight $\omega_\alpha \in X(P(\alpha))$. We can find a non-zero function $u_\alpha \in \mathrm{H}^0(G/P(\alpha), \mathcal{L}(\mu))$. Now for $n \in \mathbb{N}$, $u_\alpha^{p^n}$ is a regular function on G_n and thereby invariant under right translation by the Frobenius kernel $G^{(n)}$. It follows readily that

$$h_\alpha = u_\alpha^{p^n} \in \mathrm{H}^0(G/G^{(n)}P(\alpha), \mathcal{L}(p^n\omega_\alpha))$$

Let

$$\lambda = m_{\alpha_1}p^{f(\alpha_1)}\omega_{\alpha_1} + \cdots + m_{\alpha_l}p^{f(\alpha_l)}\omega_{\alpha_l}$$

where $\alpha_1, \ldots, \alpha_l$ denote the simple roots of R^+. Since λ is dominant, we have $m_{\alpha_i} \geq 0$. Now let

$$h = h_{\alpha_1}^{m_{\alpha_1}} \ldots h_{\alpha_l}^{m_{\alpha_l}}$$

where h_{α_i} is constructed according to m_{α_i} as above. We get

$$\begin{aligned} h(gp) &= h_{\alpha_1}(gp)^{m_{\alpha_1}} \ldots h_{\alpha_l}(gp)^{m_{\alpha_l}} \\ &= (p^{f(\alpha_1)}\omega_{\alpha_1})(p)^{-m_{\alpha_1}} h_{\alpha_1}(g) \ldots (p^{f(\alpha_l)}\omega_{\alpha_l})(p)^{-m_{\alpha_l}} h_{\alpha_l}(g) \\ &= \lambda(p)^{-1} f(g) \end{aligned}$$

since $\tilde{P} = G^{(f(\alpha_1))}P(\alpha_1) \cap \cdots \cap G^{(f(\alpha_l))}P(\alpha_l)$ by Proposition. We have constructed a non-zero function h, which is a section of $\mathcal{L}(\lambda)$ on $G/\tilde{P}$. This finishes the proof of (1).

Now $\pi_{\tilde{P}}$ is a finite and surjective morphism and it follows from [Ha2, I.4.4] that $\mathcal{L}(\lambda)$ is ample if and only if the pull back $\pi_{\tilde{P}}^*\mathcal{L}(\lambda)$ is ample. So (2) follows from the similar result [Ja, II.4.4] for G/P.

Let $h \in \mathrm{H}^0(G/\tilde{P}, \mathcal{L}(\lambda))$ be a highest weight function (of weight λ). The function h is uniquely determined by its restriction to the dense affine open subset U^oP, where U^o denotes the opposite unipotent radical of P. By Proposition 1.1

$$\mathcal{O}_{G/\tilde{P}}(U^oP/\tilde{P}) \cong k[Y_{\alpha_1}, \ldots, Y_{\alpha_s}]$$

where $\{\alpha_1, \ldots, \alpha_s\} = \{\alpha \in R^+ | f(\alpha) < \infty\}$ and Y_{α_i} is a T-eigenvector of weight $-p^{f(\alpha_i)}\alpha_i$. Now (3) follows from the T-equivariant inclusion

$$\mathrm{H}^0(G/\tilde{P}, \mathcal{L}(\lambda)) \hookrightarrow \mathrm{H}^0(U^oP/\tilde{P}, \mathcal{L}(\lambda)) = h\, k[Y_{\alpha_1}, \ldots, Y_{\alpha_s}]$$

□

4.3 Homogenization of the Sheaf of Differentials.

Let H be a subgroup scheme of G. To an H-representation V, we have the associated fibration $G \times^H V$ on G/H. A vector bundle with a G-action over G/H with G-equivariant projection has the form $G \times^H V$ for some H-representation V (take V as the fiber over the distinguished point $1\,H \in G/H$).

Our main result in this section is

Proposition 1. *Let $\tilde{P}$ be a parabolic subgroup scheme with W-function f and reduced part $P = BW_IB$. The sheaf of differentials Ω_X, where $X = G/\tilde{P}$ is homogeneous and the fiber of Ω_X at the distinguished point $\bar{e}$ is a $\tilde{P}$-representation with formal character*

$$\sum_{\alpha \in R^+ \smallsetminus R_I} e^{-p^{f(\alpha)}\alpha}$$

Proof. The sheaf of differentials is a homogeneous sheaf and we have an induced representation on the fibers

$$\frac{\Omega_{X,\bar{e}}}{\mathcal{M}_{\bar{e}}\Omega_{X,\bar{e}}} \cong \frac{\mathcal{M}_{\bar{e}}}{\mathcal{M}_{\bar{e}}^2}$$

where $\mathcal{M}_x$ denotes the maximal ideal of the stalk $\mathcal{O}_{X,x}$. Let U be the T-stable neighborhood of $\bar{e}$ given by Proposition 1.1. With the notation of this Proposition, we get the T-equivariant isomorphism

$$\frac{\mathcal{M}_{\bar{e}}}{\mathcal{M}_{\bar{e}}^2} \cong \frac{(Y_{\alpha_1}, \ldots, Y_{\alpha_s})}{(Y_{\alpha_1}, \ldots, Y_{\alpha_s})^2}$$

and the result follows, since the T-eigenvector Y_{α_i} has weight $-p^{f(\alpha_i)}\alpha_i$. □

Definition 2. *Let $\delta_f = \sum_{\alpha \in R^+} p^{f(\alpha)}\alpha$, where f is a W-function.*

Corollary 3. *Let $X = G/\tilde{P}$, where $\tilde{P}$ is a parabolic subgroup scheme with W-function f. Then the canonical line bundle ω_X is isomorphic to the induced line bundle $\mathcal{L}(-\delta_f)$.*

Proof. In general if V is a representation of a subgroup scheme H, then for the induced bundle $\mathcal{L}(V)$ on G/H

$$\wedge^n \mathcal{L}(V) \cong \mathcal{L}(\wedge^n V)$$

Now let V be the representation inducing the sheaf of differentials on $G/\tilde{P}$. By Proposition 1, we get that

$$\omega_X = \wedge^n \Omega_X = \mathcal{L}(\wedge^n V) = \mathcal{L}(-\delta_f)$$

□

4.4 A Character Formula.

THEOREM 1. *Let $\tilde{P}$ be a parabolic subgroup scheme with W-function f and reduced part $P = BW_IB$. For the homogeneous line bundle $\mathcal{L}(\xi)$ on $G/\tilde{P}$, where $\xi \in X(\tilde{P})$, the Euler character is given by*

$$\sum_i (-1)^i \operatorname{ch} \mathrm{H}^i(G/\tilde{P}, \mathcal{L}(\xi)) = \sum_{w \in W/W_I} \frac{e^{w(\xi)}}{\prod_{\alpha \in R^+ \setminus R_I} (1 - e^{-p^{f(\alpha)} w(\alpha)})}$$

PROOF. The character formula follows from Nielsen's fixed point formula [Ni][Iv, 9.4], which we now recall: Let X be a smooth projective variety with an action of a torus T. Suppose T acts with finitely many fixed points and let V be a T-homogeneous vector bundle on X. Then the following formula holds

$$\sum_i (-1)^i \operatorname{ch} \mathrm{H}^i(X, V) = \sum_{z \in X^T} \frac{\operatorname{ch} V_z}{\sum_i (-1)^i \operatorname{ch} \wedge^i T_z(X)^\vee}$$

where $T_z(X)$ denotes the tangent space at z.

The underlying space of $G/\tilde{P}$ is the same as the underlying space of G/P. This implies that the fixed points for the T-action on $G/\tilde{P}$ are parameterized by moving the identity coset x_0 around by representatives from W/W_I. Furthermore T acts through the character ξ on the fiber of $\mathcal{L}(\xi)$ at the identity coset and thus through the character $w(\xi)$ at the fiber of $\mathcal{L}(\xi)$ at the T-fixed point $w(x_0)$.

By Proposition 3.1 the formal character of the T-action on $T_{x_0}(G/\tilde{P})$ is

$$\operatorname{ch} T_{x_0}(G/\tilde{P})^\vee = \sum_{\alpha \in R^+ \setminus R_I} e^{-p^{f(\alpha)} \alpha}$$

Through this we get the following identity

$$\sum_i (-1)^i \operatorname{ch} \wedge^i T_{x_0}(G/\tilde{P})^\vee = \prod_{\alpha \in R^+ \setminus R_I} (1 - e^{-p^{f(\alpha)} \alpha})$$

Since the formal character of $T_{w(x_0)}(G/\tilde{P})$ is gotten by acting with w on the formal character of $T_{x_0}(G/\tilde{P})$, the character formula follows. □

Remark.

If $\tilde{P} = B$ meaning that the W-function f is identically zero, we get the formula

$$\sum_i (-1)^i \operatorname{ch} \mathrm{H}^i(G/B, \mathcal{L}(\xi)) = \sum_{w \in W} \frac{e^{w(\xi)}}{\prod_{\alpha \in R^+}(1 - e^{-w(\alpha)})}$$

This is the classical Weyl character formula in a mild disguise: Recall the definition of the operator J on the character ring $R = \mathbb{Z}[X(T)]$,

$$J(e^\lambda) = \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda)}$$

and the identity

$$\prod_{\alpha \in R^+} (1 - e^{-\alpha}) = e^{-\rho} J(e^\rho)$$

Now

$$\sum_{w \in W} \frac{e^{w(\xi)}}{\prod_{\alpha \in R^+}(1 - e^{-w(\alpha)})} = \sum_{w \in W} \frac{e^{w(\xi)} e^\rho}{w(J(e^\rho))} = \frac{J(e^{\xi+\rho})}{J(e^\rho)}$$

since $w(J(e^\rho)) = J(e^{w(\rho)}) = \operatorname{sgn}(w) J(e^\rho)$.

CHAPTER V

FROBENIUS SPLITTING

In this chapter we define the notion of Frobenius splitting of a variety. The main theorem is that projective homogeneous spaces with non-reduced stabilizer group schemes in general fail to be Frobenius split.

5.1 Vanishing and Frobenius Splitting.

Let X be a variety and let $F_1 : X \to X_1$ denote the Frobenius morphism. We call X Frobenius split if $\mathcal{O}_{X_1}$ is a direct summand in $\mathcal{O}_X$ as an $\mathcal{O}_{X_1}$-module. We can state this in a somewhat different way: Let $F : X \to X$ be the morphism, which is the identity on point spaces and where the morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to F_*\mathcal{O}_X$ is the p-th power map locally.

Definition 1 [MR]. *A variety X is called Frobenius split if there exists a morphism $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules such that the composite*

$$\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\sigma} \mathcal{O}_X$$

is the identity.

Examples 2.

(1) *The affine line $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ is Frobenius split. Take C to be the $k[t^p]$-submodule of $k[t]$ generated by t^i, where $p \nmid i$. Since $k[t] = C \oplus k[t^p]$, we get that $\mathbb{A}^1_k$ is Frobenius split.*

(2) *Let C be an elliptic curve. The morphism $F : C \to C$ induces a p-linear map $F^* : \mathrm{H}^1(C, \mathcal{O}_C) \to \mathrm{H}^1(C, \mathcal{O}_C)$. Since $\mathrm{H}^1(C, \mathcal{O}_C)$ is one-dimensional, F^* is either 0 or bijective. We call C supersingular if $F^* = 0$. If C is Frobenius split, then F^**

is injective. It follows that a supersingular elliptic curve is not Frobenius split. On the other hand, since $\mathcal{O}_C$ is the canonical line bundle of C, the injectivity of F^* can be seen to imply that C is Frobenius split.

Frobenius split varieties have beautiful vanishing properties. This is the content of the following

PROPOSITION 3. *Let X be a Frobenius split k-variety and $\mathcal{L}$ an ample line bundle on X. Then*

(1) $\mathrm{H}^i(X, \mathcal{L}) = 0$ *for* $i > 0$

(2) *Assume that X is a smooth projective variety with canonical line bundle ω_X. Then ("Kodaira's vanishing theorem")*

$$\mathrm{H}^i(X, \mathcal{L} \otimes \omega_X) = 0$$

for $i > 0$

PROOF. By tensoring $0 \to \mathcal{O}_X \to F_*\mathcal{O}_X$ and applying the additive functor $\mathrm{H}^i(X, -)$ for $i > 0$, we get

$$0 \to \mathrm{H}^i(X, \mathcal{L}) \to \mathrm{H}^i(X, \mathcal{L} \otimes F_*\mathcal{O}_X)$$

The projection formula gives, that $\mathcal{L} \otimes F_*\mathcal{O}_X = F_*F^*\mathcal{L}$. Since $F^*\mathcal{L} \cong \mathcal{L}^p$ and F is an affine morphism we end up with an injection

$$0 \to \mathrm{H}^i(X, \mathcal{L}) \to \mathrm{H}^i(X, F_*\mathcal{L}^p) = \mathrm{H}^i(X, \mathcal{L}^p)$$

By iterating this procedure we get for any $n \in \mathbb{N}$ an injection

$$0 \to \mathrm{H}^i(X, \mathcal{L}) \to \mathrm{H}^i(X, \mathcal{L}^{p^n})$$

But for $n \gg 0$ this last cohomology group is zero by Serre's vanishing theorem. Now (1) follows. Let $N = \dim X$. Using the same iteration as in (1) and Serre duality, we get for $n \in \mathbb{N}$ and $i > 0$

$$\mathrm{H}^i(X, \mathcal{L} \otimes \omega_X) \cong \mathrm{H}^{N-i}(X, \mathcal{L}^{-1})^\vee \hookrightarrow \mathrm{H}^{N-i}(X, \mathcal{L}^{-p^n})^\vee \cong \mathrm{H}^i(X, L^{p^n} \otimes \omega_X)$$

Again this last cohomology group is 0 for $n \gg 0$ by Serre's vanishing theorem and (2) follows. $\square$

We shall be interested in the Frobenius splitting properties of projective homogeneous space in what follows. The main theorem in [MR] is

THEOREM 4. *Let $X \subseteq G/P$ be Schubert variety, where P is a parabolic subgroup. Then X is Frobenius split and the restriction map*

$$H^0(G/P, \mathcal{L}) \to H^0(X, \mathcal{L})$$

is surjective, where $\mathcal{L}$ is an ample line bundle.

This theorem tells us that the flag varieties G/P, where P is a parabolic subgroup are Frobenius split. The results of this chapter indicates that Frobenius splitting in this setup has a lot to do with the fact that the stabilizer for the flag variety G/P is reduced. Once P is perturbed into a non-reduced subgroup scheme Frobenius splitting breaks down. We shall prove that a projective homogeneous space $G/\tilde{P}$ is Frobenius split if and only if the stabilizer $\tilde{P}$ is $\tilde{P}_{\text{red}} \ltimes G^{(n)}$.

5.2 Splitting and Duality for a Finite Morphism.

Call the variety Y (X, f)-split, where $f : X \to Y$ is a finite morphism, if morphism

$$\mathcal{O}_Y \to f_*\mathcal{O}_X$$

of $\mathcal{O}_Y$-modules splits. We shall mainly be interested in the special case of this situation, where $X = G/P$ and $G/\tilde{P}$ and f is the morphism induced by the inclusion $P \subseteq \tilde{P}$. In this section we outline the method given in [Ra, 1.16], which reformulates (X, f)-splitting in terms of sections in a specific line bundle.

Let $f : X \to Y$ be a finite morphism of varieties. A natural approach to the problem of splitting is to find conditions on global sections of $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)$ that guarantee, that they split the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$. We shall use duality for a finite morphism to approach this goal. Notice that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-algebra. Given a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, $f_*\mathcal{F}$ is a quasicoherent $\mathcal{O}_Y$-module, which is an $f_*\mathcal{O}_X$-module. Conversely given a quasicoherent $\mathcal{O}_Y$-module $\mathcal{G}$, which is an $f_*\mathcal{O}_X$-module, we get a quasicoherent $\mathcal{O}_X$-module $f^\natural\mathcal{G}$ by noting that $X \cong \mathbf{Spec} f_*\mathcal{O}_X$. In this way f_* induces an equivalence between the category of quasicoherent $\mathcal{O}_X$-modules and the category of quasicoherent $\mathcal{O}_Y$-modules, which are $f_*\mathcal{O}_X$-modules [Ha1, Exercise II.5.17].

Now if $\mathcal{G}$ is a quasicoherent $\mathcal{O}_Y$-module, then $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$ is a quasicoherent $\mathcal{O}_Y$ -module, which is an $f_*\mathcal{O}_X$-module in the natural way. Denote the quasicoherent $\mathcal{O}_X$-module $f^\natural\,\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$ by $f^!\mathcal{G}$. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$- module, then the natural map (duality for a finite morphism)

$$f_*\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G})$$

is an isomorphism.

We are primarily interested in the case where f is a finite morphism between smooth projective varieties of the same dimension. In this setup we have the following

Proposition 1. *Let $f : X \to Y$ be a finite morphism between smooth projective varieties of the same dimension and let K_X and K_Y denote the canonical line bundles on X and Y respectively. Then there is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \to f_*(K_X \otimes f^*K_Y^{-1})$$

Proof. Applying the duality for f to a vector bundle V and $f^!K_Y$ and then using Serre duality, it follows that $f^!K_Y$ is a dualizing sheaf for X and thereby isomorphic to K_X. The duality for f applied to $\mathcal{F} = f^*K_Y$ and $\mathcal{G} = K_Y$ gives

$$f_*\,\mathcal{H}om_{\mathcal{O}_X}(f^*K_Y, f^!K_Y) \cong \mathcal{H}om_{\mathcal{O}_Y}(f_*f^*K_Y, K_Y)$$

Since K_Y is locally free of rank 1, the right hand side is isomorphic to $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)$. Combining these two isomorphisms gives the result. $\square$

Let us apply this Proposition in the situation $X = G/P$, $Y = G/\tilde{P}$ and f the canonical morphism $\pi_{\tilde{P}}$.

Corollary 2. *Let $\tilde{P}$ be a parabolic subgroup scheme with reduced part P and W-function f. Let $K_{G/P} = \mathcal{L}(\omega)$ and $K_{G/\tilde{P}} = \mathcal{L}(\delta_f)$. We have the following isomorphism for the space of splittings*

$$\mathrm{Hom}_{\mathcal{O}_{G/\tilde{P}}}((\pi_{\tilde{P}})_*\mathcal{O}_{G/P}, \mathcal{O}_{G/\tilde{P}}) \cong \mathrm{H}^0(G/P, \mathcal{L}(\omega - \delta_f))$$

In the special case, where $X = Y = G/\tilde{P}$ and f is the Frobenius map, we get the following isomorphism for the space of splittings:

$$\mathrm{Hom}_{\mathcal{O}_{G/\tilde{P}}}(F_*\mathcal{O}_{G/\tilde{P}}, \mathcal{O}_{G/\tilde{P}}) \cong \mathrm{H}^0(G/\tilde{P}, \mathcal{L}(-(p-1)\delta_f))$$

If $-\delta_f$ is not dominant, then $G/\tilde{P}$ is not Frobenius split and not $(G/P, \pi_{\tilde{P}})$-split.

Proof. By Proposition 1 we get

$$\mathrm{Hom}_{\mathcal{O}_{G/\tilde{P}}}((\pi_{\tilde{P}})_*\mathcal{O}_{G/P}, \mathcal{O}_{G/\tilde{P}}) \cong \mathrm{H}^0(G/P, K_{G/P} \otimes \pi_{\tilde{P}}^* K_{G/\tilde{P}}^{-1}) = \mathrm{H}^0(G/P, \mathcal{L}(\omega - \delta_f))$$

Similarly in the special case we get

$$\mathrm{Hom}_{\mathcal{O}_{G/\tilde{P}}}(F_*\mathcal{O}_{G/\tilde{P}}, \mathcal{O}_{G/\tilde{P}}) \cong \mathrm{H}^0(G/\tilde{P}, K_{G/\tilde{P}} \otimes F^* K_{G/\tilde{P}}^{-1}) = \mathrm{H}^0(G/\tilde{P}, \mathcal{L}(-(p-1)\delta_f))$$

Now ω is known to be negative dominant, that is $\omega \in -X(T)_+$. In order for the respective spaces of splittings to be non-zero we must have that $-\delta_f \in X(T)_+$ by Proposition IV.2(1). $\square$

5.3 δ-Matrices for Root Systems.

The purpose of this section is to describe certain matrices associated with a given ordering of the simple roots in R^+. These matrices will be very helpful in computing the coordinates with respect to the fundamental dominant weights of the weight δ_f inducing the canonical line bundle on $G/\tilde{P}$, where $\tilde{P}$ has W-function f.

Let $\alpha_1, \dots, \alpha_l$ be the simple roots S in R^+. Write $\lambda = (\lambda_1, \dots, \lambda_l)$ to mean $\lambda_1\omega_{\alpha_1} + \cdots + \lambda_l\omega_{\alpha_l}$, where ω_{α_i} denotes the fundamental dominant weight associated with the simple root α_i. The root system generated by the subset $\{\alpha_{i_1}, \dots, \alpha_{i_s}\} \subseteq S$ will be denoted $[\alpha_{i_1}, \dots, \alpha_{i_s}]$.

Definition 1. *Let $\alpha_1, \ldots, \alpha_l$ be an ordering of $\Delta \subseteq R^+$. Let*

$$\begin{aligned}
\Gamma_1 &= [\alpha_1, \ldots, \alpha_l]^+ \setminus [\alpha_2, \ldots, \alpha_l]^+, & \theta_1 &= \sum_{\gamma \in \Gamma_1} \gamma \\
\Gamma_2 &= [\alpha_2, \ldots, \alpha_l]^+ \setminus [\alpha_3, \ldots, \alpha_l]^+, & \theta_2 &= \sum_{\gamma \in \Gamma_2} \gamma \\
&\vdots \\
\Gamma_l &= [\alpha_l]^+, & \theta_l &= \sum_{\gamma \in \Gamma_l} \gamma = \alpha_l
\end{aligned}$$

The δ-matrix of $\alpha_1, \ldots, \alpha_l$ is

$$M = (M_{ij}) = \begin{pmatrix} \alpha_1^\vee(\theta_1) & \ldots & \alpha_l^\vee(\theta_1) \\ \vdots & \alpha_j^\vee(\theta_i) & \vdots \\ \alpha_1^\vee(\theta_l) & \ldots & \alpha_l^\vee(\theta_l) \end{pmatrix}$$

Since R^+ is the disjoint union of $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ the sum of the rows in M is the row vector $(2, 2, \ldots, 2)$. Let $\tilde{P}$ be a parabolic subgroup scheme with W-function f. Order the simple roots Δ so that $f(\alpha_1) \leq \cdots \leq f(\alpha_l)$. By Corollary IV.3.3 the weight defining the inverse canonical line bundle is

$$p^{f(\alpha_1)}\theta_1 + \cdots + p^{f(\alpha_l)}\theta_l$$

The computation of the coordinates of this weight with respect to the fundamental dominant weights is greatly facilitated by the δ-matrix corresponding to the "increasing" order on the simple roots. The main observation is that these coordinates are available by multiplying the rows by $p^{f(\alpha_1)}, \ldots, p^{f(\alpha_l)}$ and then summing up the columns. We give some examples illustrating this

Example 2. *Suppose $R = A_5$ and that the simple roots $S = \{\alpha_1, \ldots, \alpha_5\}$ are given in the usual order in the Dynkin diagram* [Bou, PLANCHE I]. *Then the δ-matrix corresponding to this ordering is*

$$M = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ -4 & 5 & 0 & 0 & 0 \\ 0 & -3 & 4 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

On the other hand if S' is the ordering $\alpha_2, \alpha_1, \alpha_5, \alpha_3, \alpha_4$ then the δ-matrix is

$$M = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & 0 \\ -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

Notice that if we let $\tilde{B}_S$ be a parabolic subgroup scheme with W-function f such that $f(\alpha_1) \leq \cdots \leq f(\alpha_5)$, then the inverse canonical line bundle on $G/\tilde{B}_S$ is given by

$$(6p^{f(\alpha_1)} - 4p^{f(\alpha_2)}, 5p^{f(\alpha_2)} - 3p^{f(\alpha_3)}, 4p^{f(\alpha_3)} - 2p^{f(\alpha_4)}, 3p^{f(\alpha_4)} - p^{f(\alpha_5)}, 2p^{f(\alpha_5)})$$

On the other hand if $f(\alpha_2) \leq f(\alpha_1) \leq f(\alpha_5) \leq f(\alpha_3) \leq f(\alpha_4)$, then S' is ordered accordingly and the inverse canonical line bundle on $G/\tilde{B}_{S'}$ is given by

$$(2p^{f(\alpha_1)}, 6p^{f(\alpha_2)} - p^{f(\alpha_1)} - p^{f(\alpha_5)} - 2p^{f(\alpha_3)}, 3p^{f(\alpha_3)} - p^{f(\alpha_4)}, 2p^{f(\alpha_4)}, 4p^{f(\alpha_5)} - p^{f(\alpha_3)} - p^{f(\alpha_4)})$$

Proposition 3. *Let $\alpha_1, \ldots, \alpha_n$ be an ordering of the simple roots in R^+ and let M denote the corresponding δ-matrix. Then*

(1) *$M_{ij} \leq 0$ if $i > j$*

(2) *M is lower triangular*

(3) *If R is irreducible, then for $i > 1$ there exists j, such that $1 \leq j < i$ and $M_{ij} < 0$*

(4) *If R is irreducible, then $M_{11} \leq h$, where h denotes the Coxeter number of R.*

Proof. For $i \neq j$ we have that $\alpha_i^\vee(\alpha_j) < 0$. This gives (1). The column sum in M is 2 and $\alpha_i^\vee(\theta_i + \theta_{i+1} + \cdots + \theta_n) = 2$, since $[\alpha_i, \ldots, \alpha_n]$ is a root system. This implies (2).

If (3) does not hold, then there exists $i > 1$ such that $M_{ij} = 0$ for all $1 \leq j < i$. This splits up the root system in two orthogonal components contradicting the irreducibility.

We use the fact that $M_{11} = 2 - \sum_{\alpha \in [\alpha_2, \ldots, \alpha_n]^+} \alpha_1^\vee(\alpha)$ together with the classification of irreducible root systems to give case by case proof of (4). We will use the standard ordering of the simple roots as given in PLANCHES I-IX in [Bou]. This reference contains all the data about the positive roots of the irreducible root systems needed in the proof. As M_{11} only depends on the first simple root in the basis (and not the ordering of the remaining ones), we denote M_{11} by M_α, where α is the first simple root. Each of the cases proceeds by computing M_α for all possible choices of $\alpha \in S$. We let μ_α denote the number of times a specific simple root α occurs in the positive roots counted with multiplicity.

($\mathbf{A}_n$), $n \geq 1$: The Coxeter number of A_n is $n+1$. Deleting α_i breaks A_n up into A_{i-1} and A_{n-i}. Since $\mu_{\alpha_{i-1}} = i-1$ in this A_{i-1}, $\mu_{\alpha_{i+1}} = n-i$ and $\alpha_i^\vee(\alpha_{i-1}) = \alpha_i^\vee(\alpha_{i+1}) = -1$, we get $M_{\alpha_i} = 2 + (i-1) + (n-i) = n+1$.

($\mathbf{B}_n$), $n \geq 2$: The Coxeter number of B_n is $2n$. Deleting α_n leaves us with A_{n-1} and $M_{\alpha_n} = 2 + (n-1) = n+1 \leq 2n$ using that $\alpha_n^\vee(\alpha_{n-1}) = -1$. Deleting α_{n-1} leaves us with A_1 and A_{n-2} giving $M_{\alpha_{n-1}} = 2 + (n-2) + 2 = n+2 \leq 2n$ if $n \geq 2$, using $\alpha_{n-1}^\vee(\alpha_n) = -2$. Now suppose $i < n-1$ and $n > 2$. Deleting α_i splits B_n up into A_{i-1} and B_{n-i}. As $\mu_{\alpha_{i+1}} = 2(n-i) - 1$ in B_{n-i} [1, Planche II, (II)], we get $M_{\alpha_i} = 2 + (i-1) + 2(n-i) - 1 = 2n - i < 2n$.

($\mathbf{C}_n$), $n \geq 3$: The Coxeter number of C_n is $2n$. Deleting α_n leaves us with A_{n-1} and $M_{\alpha_n} = 2 + 2(n-1) = 2n$, using that $\alpha_n^\vee(\alpha_{n-1}) = -2$. Deleting α_{n-1} leaves us with an A_1 and an A_{n-2} giving $M_{\alpha_{n-1}} = 1 + (n-2) = n-1 < 2n$, using $\alpha_{n-1}^\vee(\alpha_n) = -1$. Now suppose $i < n-1$ and $n > 2$. Deleting α_i splits B_n up into A_{i-1} and C_{n-i}. As $\mu_{\alpha_{i+1}} = 2(n-i)$ in C_{n-i} [1, Planche III, (II)], we get $M_{\alpha_i} = 2 + (i-1) + 2(n-i) = 2n - i + 1 \leq 2n$.

($\mathbf{D}_n$), $n \geq 4$: The Coxeter number of D_n is $2(n-1)$. Deleting α_n leaves us with A_{n-1} and $\mu_{\alpha_{n-2}}$ in this A_{n-1} is [1, Planches I, (II)] $2(n-2)$, so that $M_{\alpha_n} = 2 + 2(n-2) = 2(n-1)$. $M_{\alpha_{n-1}} = 2(n-1)$ by symmetry. Deleting α_{n-2} splits D_n up into A_{n-3}, A_1 and A_1 giving $M_{\alpha_{n-2}} = n - 3 + 1 + 1 = n - 1 < 2(n-1)$. Deleting α_{n-3} gives us A_{n-4} and A_3. In this case $M_{\alpha_{n-3}} = 2 + (n-4) + 4 = n+2 \leq 2(n-1)$, as $n \geq 4$. Now suppose $n > 4$ and $i < n-3$. Then α_i breaks D_n up into A_{i-1} and D_{n-i}. Since $\mu_{\alpha_{i+1}} = 2(n-i) - 2$ in this D_{n-i} [1, Planche IV, II] we get $M_{\alpha_i} = 2 + i - 1 + 2(n-i) - 2 = 2n - i - 1 \leq 2(n-1)$

($\mathbf{E}_6$): The Coxeter number of E_6 is 12. Deleting α_1 gives us D_5 and $\mu_{\alpha_3} = 10$ in this D_5. So $M_{\alpha_1} = M_{\alpha_6} = 2 + 10 = 12$. Deleting α_2 we are left with A_5 and $\mu_{\alpha_4} = 9$ in this A_5, so that $M_{\alpha_2} = 2 + 9 = 11$. Similarly $M_{\alpha_3} = M_{\alpha_5} = 2 + 1 + 6 = 9$. Finally $M_{\alpha_4} = 2 + 2 + 1 + 2 = 7$.

($\mathbf{E}_7$): The Coxeter number of E_7 is 18. Deleting α_7 leaves us with E_6 and μ_{α_6} in this E_6 is 16 [1, Planche V, II] giving $M_{\alpha_7} = 2 + 16 = 18$. For the remaining roots we have $M_{\alpha_1} = 2 + 15 = 17$, $M_{\alpha_2} = 2 + 12 = 14$, $M_{\alpha_3} = 2 + 1 + 8 = 11$, $M_{\alpha_4} = 2 + 2 + 1 + 3 = 8$, $M_{\alpha_5} = 2 + 6 + 2 = 10$ and $M_{\alpha_6} = 2 + 10 + 1 = 13$.

($\mathbf{E}_8$): The Coxeter number of E_8 is 30. Deleting α_8 leaves us with E_7 and μ_{α_7} in this E_7 is 27 [1, Planche VI, II] giving $M_{\alpha_8} = 2 + 27 = 29$. For the remaining roots we have $M_{\alpha_1} = 2 + 21 = 23$, $M_{\alpha_2} = 2 + 15 = 17$, $M_{\alpha_3} = 2 + 1 + 10 = 13$, $M_{\alpha_4} = 2 + 2 + 1 + 4 = 9$, $M_{\alpha_5} = 2 + 6 + 3 = 11$ and $M_{\alpha_6} = 2 + 10 + 2 = 14$, $M_{\alpha_7} = 2 + 16 + 1 = 19$.

($\mathbf{F}_4$): The Coxeter number of F_4 is 12 and $M_{\alpha_1} = 2 + 6 = 8$, $M_{\alpha_2} = 2 + 1 + 2 \cdot 2 = 7$, $M_{\alpha_3} = 2 + 1 + 2 = 5$, $M_{\alpha 4} = 2 + 9 = 11$.

($\mathbf{G}_2$): The Coxeter number of G_2 is 6 and $M_{\alpha_1} = 2 + 1 = 3$, $M_{\alpha_2} = 2 + 3 \cdot 1 = 5$

□

5.4 The main theorem.

The following proposition is the key to our main theorem

Proposition 1. *Let f be a W-function on R^+, where R is an irreducible root system with Coxeter number h. Let*

$$\omega = \sum_{\gamma \in R^+} p^{f(\gamma)} \gamma$$

(1) *If $p > h$, then ω is dominant if and only if f is constant*

(2) *We can lower the bound in (1) to $p > 3$ if f is finite valued*

Proof. We give the proof of (2) first. Assume that f is non-constant. We will prove that ω is not dominant. Let the simple roots $\alpha_1, \ldots, \alpha_l$ be ordered such that $f(\alpha_1) \leq \cdots \leq f(\alpha_l)$. As f is a W-function we get

$$\omega = p^{f(\alpha_1)}\theta_1 + p^{f(\alpha_2)}\theta_2 + \cdots + p^{f(\alpha_l)}\theta_l$$

in the notation of Definition 3.1.

The assumption that f is not constant amounts to the existence of an index i, $1 \leq i < l$ so that $f(\alpha_i) < f(\alpha_{i+1})$. According to Proposition 3.3(3) we can find j, $1 \leq j < i+1$ such that $M_{i+1,j} = \alpha_j^\vee(\theta_{i+1}) < 0$. Now we have the following string of inequalities

$$\begin{aligned}
\alpha_j^\vee(\omega) &= p^{f(\alpha_j)}\alpha_j^\vee(\theta_j) + \cdots + p^{f(\alpha_i)}\alpha_j^\vee(\theta_i) + p^{f(\alpha_{i+1})}\alpha_j^\vee(\theta_{i+1}) + \cdots + p^{f(\alpha_l)}\alpha_j^\vee(\theta_l) \\
&\leq p^{f(\alpha_j)}\alpha_j^\vee(\theta_j + \cdots + \theta_i) + p^{f(\alpha_{i+1})}\alpha_j^\vee(\theta_{i+1} + \cdots + \theta_l) \\
&\leq p^{f(\alpha_j)}\alpha_j^\vee(\theta_j + \ldots \theta_i) + p^{f(\alpha_j)+1}(2 - \alpha_j^\vee(\theta_j + \cdots + \theta_i)) \\
&\leq 3p^{f(\alpha_j)} - p^{f(\alpha_j)+1} \\
&= p^{f(\alpha_j)}(3 - p) < 0
\end{aligned}$$

Since the coordinate of ω with respect to the fundamental dominant weight ω_{α_j} is negative, ω is not dominant.

The proof of (1) is similar. Assume that f is non-constant and let j be the index given in the proof of (2). Using Proposition 3.3(4) we get

$$\begin{aligned}
\alpha_j^\vee(\omega) &\leq \alpha_j^\vee(\theta_j)p^{f(\alpha_j)} + \alpha_j^\vee(\theta_{i+1})p^{f(\alpha_{i+1})} \\
&\leq hp^{f(\alpha_j)} - p^{f(\alpha_{i+1})} \\
&\leq p^{f(\alpha_j)}(h - p) < 0
\end{aligned}$$

and since the coordinate of ω with respect to ω_{α_j} is negative, ω is not dominant. $\square$

We can now state and prove

THEOREM 2. *Let G be an algebraic group of simple type. Let $\tilde{P}$ be a parabolic subgroup scheme with reduced part P and let f denote the W-function associated with P. Assume that $p > h$, where h denotes the Coxeter number of G. The following are equivalent*

(1) *$G/\tilde{P}$ is Frobenius split*
(2) *$G/\tilde{P}$ is $(G/P, \pi_{\tilde{P}})$-split*
(3) *$G/\tilde{P}$ is isomorphic to G/P as a variety*
(4) *f is constant*

If $\tilde{P}_{\text{red}} = B$ the W-function f is finite and the bound on p can be lowered to $p > 3$.

Proof. If f is constant say $= n$, then G/P is the Frobenius cover of G/B of order n and (1) and (2) are equivalent. As G/B is defined over $\mathbb{F}_p$, G/P and G/B are isomorphic. This proves that (4) implies (1), (2) and (3), since G/B is Frobenius split by Theorem 1.4.

To finish the proof of the theorem, it suffices to prove that if f is non-constant, then G/P is neither Frobenius split nor $(G/P, \pi_{\tilde{P}})$-split. By Corollary 2.2 it suffices to prove that ω is not dominant. This is exactly the content of Proposition 1(1), since the root system of G is irreducible. When the W-function is finite we use Proposition 1(2). $\square$

CHAPTER VI

TWISTED $\mathbb{P}^1$-FIBRATIONS

In this chapter we will study the special projective homogeneous spaces with stabilizers $\tilde{P}_{\alpha,n} = B \ltimes P_\alpha^{(n)}$, where P_α denotes the minimal parabolic subgroup corresponding to the simple root α. Notice that the W-function f of $\tilde{P}_{\alpha,n}$ is zero except at the simple root α, where it assumes the value n. We shall use the fibration

$$G/\tilde{P}_{\alpha,n} \to G/P_\alpha$$

which is a Frobenius twisted version of the usual $\mathbb{P}^1$-fibration $G/B \to G/P_\alpha$, to study the cohomology of line bundles on $G/\tilde{P}_{\alpha,n}$.

The results of this section were inspired by computer calculations with the character formula in IV.4. Unfortunately this character formula is quite difficult to compute with except in the rank 2 cases. In the special situation of this chapter the ordinary Lie theoretic algorithms like Freudenthal's algorithm for Weyl's character formula can be applied. From Theorem V.4.2 we know that if $n > 0$, then $G/\tilde{P}_{\alpha,n}$ is not Frobenius split and we may expect non-vanishing for ample line bundles and maybe even a "homogeneous counterexample" to Kodaira's vanishing theorem.

After the initial setup we give 3 examples of what might be termed unusual vanishing behavior using methods from the modular representation theory of algebraic groups. The first example is an ample line bundle $\mathcal{L}$ on a projective homogeneous space $X = G/\tilde{P}_{\alpha,1}$ for $G = SL_3(k)$ with $\mathrm{H}^1(X, \mathcal{L}) \neq 0$ in characteristics > 3. The second example is a counterexample to Kodaira's vanishing theorem with an ample line bundle $\mathcal{L}$ on a projective homogeneous space $X = G/\tilde{P}_{\alpha,1}$ for $G = SL_4(k)$ with $\mathrm{H}^1(X, \mathcal{L} \otimes \omega_X) \neq 0$ in all prime characteristics. This example relies heavily on Jantzen's sum formula from modular representation theory. Finally the third example is an ample line bundle $\mathcal{L}$ on a projective homogeneous space $X = G/\tilde{P}_{\alpha,1}$ for $G = SO_{10}(k)$, where $\mathcal{L} \otimes \omega_X$ has negative Euler characteristic. This example provides a negative answer to an old question of Raynaud.

6.1 Methods from Modular representation theory.

In this section we recall some basic facts along with three results from the modular representation theory of algebraic groups: Steinberg's tensor product theorem, the strong linkage principle and Jantzen's sum formula.

To each dominant weight $\lambda \in X(T)$ we have an indecomposable G-representation $\mathrm{H}^0(\lambda) := \mathrm{H}^0(G/B, \mathcal{L}(\lambda))$ [Ja, II.2.3], which is the dual of the Weyl module $V(\lambda)$ with highest weight λ. The Weyl module $V(\lambda) \cong \mathrm{H}^0(G/B, \mathcal{L}(-w_0\lambda))^*$ is by definition the extension to k of $U_{\mathbb{Z}}\, v^+$, where $U_{\mathbb{Z}}$ denotes Kostant's $\mathbb{Z}$-form in the enveloping algebra for the complex semisimple Lie algebra $\mathfrak{g}$ having the same root system as G and v^+ denotes a maximal weight vector of the irreducible representation of $\mathfrak{g}$ of highest weight λ.

THEOREM 1. *Let $\mathcal{L}(\lambda)$ be a homogeneous line bundle on the flag variety G/B. Then*

(1) *If λ is dominant, then $\mathrm{H}^i(G/B, \mathcal{L}(\lambda)) = 0$ for $i > 0$ (Kempf's vanishing theorem)*

(2) *The simple G-modules are classified by the weights in $X(T)_+$ via the $1-1$ correspondence*

$$\lambda \mapsto L(\lambda) = \mathrm{Soc}_G\, \mathrm{H}^0(\lambda)$$

(3) *The formal Euler character $\chi(\lambda)$ of $\mathcal{L}(\lambda)$ on G/B is given by Weyl's character formula*

$$\chi(\lambda) = \sum_i (-1)^i \,\mathrm{ch}\, \mathrm{H}^i(G/B, \mathcal{L}(\lambda)) = \frac{J(e^{\lambda+\rho})}{J(e^\rho)}$$

This expression is called the Weyl character of λ.

(4) *The Euler characteristic $\chi'(\lambda)$ of $\mathcal{L}(\lambda)$ on G/B is given by Weyl's dimension formula*

$$\chi'(\lambda) = \sum_i (-1)^i \dim \mathrm{H}^i(G/B, \mathcal{L}(\lambda)) = \frac{\prod_{\gamma \in \Phi^+} \gamma^\vee(\lambda+\rho)}{\prod_{\gamma \in \Phi^+} \gamma^\vee(\rho)}$$

PROOF. (1) follows from the fact that (Theorem V.1.4) G/B is Frobenius split - in this case Kempf's vanishing theorem is Kodaira's vanishing theorem for G/B. The proof of (2) is in [Ja, II.2.4]. (3) and (4) are consequences of the character formula in IV.4. $\square$

Theorem 2. *Let λ be a dominant weight with the p-adic expansion*

$$\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_n p^n$$

where $\lambda_i \in X_1(T) = \{\xi \in X(T)_+ | \alpha^\vee(\xi) < p$ for all simple roots $\alpha \in R^+\}$. Then

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \ldots \otimes L(\lambda_n)^{[n]}$$

Proof. [Ja, II.3.17] □

Definition 3. *The affine Weyl group W_p of G is the group generated by the reflections $s_{\beta,mp}$, where $s_{\beta,mp}(\nu) = s_\beta(\nu) + mp\beta$ and s_β denotes the reflection for the root β. We let $s_{\beta,mp} \cdot \nu = s_{\beta,mp}(\nu + \rho) - \rho = \nu - (\beta^\vee(\nu+\rho) - mp)\beta$ denote the translated action of W_p on $X(T)$.*

We shall need the following version of the strong linkage principle (it holds in general for all the cohomology groups)

Theorem 4. *Let $\chi \in X(T)_+$ and suppose $L(\lambda)$ is a composition factor of $\mathrm{H}^0(\chi)$. Then there exists $w \in W_p$, such that $\lambda = w \cdot \chi$.*

Proof. [A2, Theorem 1] □

Now for the sum formula

Theorem 5. *The Weyl module $V(\mu) = H^0(G/B, -w_0\mu)^*$ has a filtration*

$$V(\mu) = V(\mu)^0 \supset V(\mu)^1 \supset V(\mu)^2 \supset \ldots$$

with $V(\mu)/V(\mu)^1 \cong L(\mu)$, such that

$$\sum_{i>0} \operatorname{ch} V(\mu)^i = \sum_{\gamma \in \Phi^+} \sum_{0<mp<\gamma^\vee(\mu+\rho)} \nu_p(mp) \chi(s_{\gamma,mp} \cdot \mu)$$

where ν_p denotes the p-adic valuation on $\mathbb{Z}$.

Proof. [Ja, II.8.19] □

6.2 The fibration $G/\tilde{P}_{\alpha,n} \to G/P_\alpha$.

For the usual $\mathbb{P}^1$-fibration $G/B \to G/P_\alpha$ we have the following

Lemma 1. *Let f be the morphism $G/B \to G/P_\alpha$ induced by the inclusion $B \subset P_\alpha$. Then*

(1) *f is a trivial morphism in the sense that $R^i f_*(\mathcal{O}_{G/B}) = 0$ and $f_*\mathcal{O}_{G/B} = \mathcal{O}_{G/P_\alpha}$.*
(2) *If $\mathcal{L}(\lambda)$ is a homogeneous line bundle on G/B, then the push-down $f_*\mathcal{L}(\lambda)$ is a homogeneous line bundle on G/P_α induced by the P_α representation $\mathrm{H}^0(P_\alpha/B, \mathcal{L}(\lambda))$.*
(3) *If V is a vector bundle on G/B, then*

$$\mathrm{H}^i(G/B, f^*V) \cong \mathrm{H}^i(G/P_\alpha, V)$$

Proof. The fibers of f are isomorphic to $P_\alpha/B \cong \mathbb{P}^1$. So the statements in (1) follow from [Ha1, Corollary III.12.9]. Let $\pi : G \to G/B$ be the canonical projection. A section of $f_*\mathcal{L}(\lambda)$ over the open set $U \subseteq G/P_\alpha$ is a morphism $\varphi : \pi^{-1}(f^{-1}(U)) \to k$, such that $\varphi(xb) = b^{-\lambda}\varphi(x)$. Define g_U to be the morphism from $\pi^{-1}(f^{-1}(U))$ into $\mathrm{H}^0(P_\alpha/B, \mathcal{L}(\lambda))$, whose value on $x \in \pi^{-1}(f^{-1}(U))$ is $p \mapsto \varphi(xp)$. It is easy to check that g_U defines an isomorphism of sheaves from $f_*\mathcal{L}(\lambda)$ to $\mathcal{L}(\mathrm{H}^0(P_\alpha/B, \mathcal{L}(\lambda)))$. This proves (2).

The projection formula gives that $R^i f_*(\mathcal{O}_{G/B} \otimes f^*V) = R^i f_*(\mathcal{O}_{G/B}) \otimes V$. By (1) this vector bundle is zero for $i > 0$. This means that the Leray spectral sequence degenerates and we get

$$\mathrm{H}^i(G/B, f^*V) \cong \mathrm{H}^i(G/P_\alpha, f_*f^*V) \cong \mathrm{H}^i(G/P_\alpha, V)$$

for $i \geq 0$, where the last isomorphism follows from (1) and the projection formula. $\square$

Lemma 2. *Let $\lambda \in X(\tilde{P}_{\alpha,n})$ and suppose that $\alpha^\vee(\lambda) = r\,p^n \geq 0$. The push-down $(\pi_{\alpha,n})_*\mathcal{L}(\lambda)$ under the canonical morphism $\pi_{\alpha,n}$*

$$\begin{array}{ccc} G/B & \longrightarrow & G/\tilde{P}_{\alpha,n} \\ & \overset{\pi}{\searrow} & \downarrow{\scriptstyle \pi_{\alpha,n}} \\ & & G/P_\alpha \end{array}$$

of the line bundle $\mathcal{L}(\lambda)$ on $G/\tilde{P}_{\alpha,n}$ is induced by the representation $V^r_{\alpha,n} = H^0(P_\alpha/\tilde{P}_{\alpha,n}, \lambda)$ of P_α and for $i \geq 0$ we have

$$\mathrm{H}^i(G/\tilde{P}_{\alpha,n}, \mathcal{L}(\lambda)) \cong \mathrm{H}^i(G/B, \pi^*\mathcal{L}(V^r_{\alpha,n}))$$

For the P_α-representation $V^r_{\alpha,n}$ we have

(1) *$V^r_{\alpha,n}$ has a basis of T-eigenvectors $e_0^{\alpha,n},\dots,e_r^{\alpha,n}$, where $t.e_i^{\alpha,n}=(\lambda-i\,p^n\,\alpha)(t)e_i^{\alpha,n}$*

(2) $x_\alpha(z).e_i^{\alpha,n}=\sum_{j=0}^{i}\binom{i}{j}z^{p^n(i-j)}e_j^{\alpha,n}$

PROOF. The first statements are trivial modifications of the corresponding statements in Lemma 1 once it is noticed, that the fibers of $G/\tilde{P}_{\alpha,n}\to G/P_\alpha$ are isomorphic to the n-th Frobenius cover $\mathbb{P}^1_n$. (1) and (2) are consequences of Proposition II.3.2 after having reduced to the case $G=SL_2$ as in [Ja, Proposition II.5.2]. □

The above lemma "reduces" the computation of the cohomology of a line bundle on the space $G/\tilde{P}_{\alpha,n}$ to the computation of the cohomology of a vector bundle on the flag variety G/B. This leads to the following

PROPOSITION 3. *Let $\lambda\in X(\tilde{P}_{\alpha,n})$ be a dominant weight with $\alpha^\vee(\lambda)=r\,p^n$.*

(1) *The formal Euler character of $\mathcal{L}(\lambda)$ on $G/\tilde{P}_{\alpha,n}$ is given by*

$$\chi(\lambda)+\chi(\lambda-p^n\alpha)+\cdots+\chi(\lambda-r\,p^n\alpha)$$

where χ denotes the Weyl character. The Euler characteristic of $\mathcal{L}(\lambda)$ is given by

$$\chi'(\lambda)+\chi'(\lambda-p^n\alpha)+\cdots+\chi'(\lambda-r\,p^n\alpha)$$

where χ' denotes Weyl's dimension formula (Theorem 1.1(4)).

(2) *If furthermore $n=r=1$ then $H^i(G/\tilde{P}_{\alpha,1},\mathcal{L}(\lambda))=0$ for $i\geq 2$ and there is an exact sequence*

$$\begin{aligned}0\to &H^0(G/P_{\alpha,1},\mathcal{L}(\lambda))\to H^0(G/B,\mathcal{L}(\lambda))\to\\ &H^0(G/B,\mathcal{L}(\lambda-\alpha))\to H^1(G/P_{\alpha,1},\mathcal{L}(\lambda))\to 0\end{aligned}$$

PROOF. The vector bundle $\mathcal{L}(V^r_{\alpha,n})$ on G/B has a filtration with quotient line bundles induced by the characters in Lemma 2(1). Using Theorem 1.1 and the additivity of the Euler character and the Euler characteristic gives the statements in (1).

If $n=r=1$, $V^1_{\alpha,1}$ is the middle term in the short exact sequence of B-modules

$$0\to k_{\lambda-p\alpha}\to V^1_{\alpha,1}\to k_\lambda\to 0$$

This gives the long exact sequence on cohomology

$$0 \to H^0(G/B, \mathcal{L}(V^1_{\alpha,1})) \to H^0(G/B, \mathcal{L}(\lambda)) \to$$
$$H^1(G/B, \mathcal{L}(\lambda - p\alpha)) \to H^1(G/B, \mathcal{L}(V^1_{\alpha,1})) \to \cdots \to$$
$$0 \to H^i(G/B, \mathcal{L}(\lambda - p\alpha)) \to H^i(G/B, \mathcal{L}(V^1_{\alpha,1})) \to 0 \to \dots$$

by Kempf's vanishing theorem. Now since $\lambda - p\,\alpha = s_\alpha \cdot (\lambda - \alpha)$ and $0 \leq \alpha^\vee(\lambda - \alpha) < p$ the statement in (2) follows from the isomorphism [A1, 2.3.(ii)]

$$H^i(G/B, \mathcal{L}(\lambda - \alpha)) = H^{i+1}(G/B, \mathcal{L}(s_\alpha \cdot (\lambda - \alpha)))$$

□

We are now ready to give several examples related to the vanishing behavior of the cohomology of line bundles on $G/\tilde{P}_{\alpha,n}$.

6.3 Examples.

The following example shows that the vanishing theorem for ample line bundles (Kempf vanishing) fails for projective homogeneous spaces with non-reduced stabilizers:

Example 1. *Let $G = SL_3(k)$, where k is of characteristic $p > 3$. Fix a Borel subgroup B associated with roots $\{-\alpha, -\beta, -\alpha - \beta\}$, where $\alpha = 2\omega_\alpha - \omega_\beta$ and $\beta = -\omega_\alpha + 2\omega_\beta$. For the character $\lambda = \omega_\alpha + p\,\omega_\beta$ inducing the ample line bundle $\mathcal{L}(\lambda)$ on $G/\tilde{P}_{\beta,1}$ we have*

$$\dim H^1(G/\tilde{P}_{\beta,1}, \mathcal{L}(\lambda)) = \frac{(p-2)(p-3)}{2}$$

Proof. By Theorem 1.4 the highest weights of the possible composition factors of

$$H^0(G/B, \mathcal{L}(\lambda))$$

are

$$\{\omega_\alpha + p\,\omega_\beta, 2\omega_\alpha + (p-2)\omega_\beta, (p-4)\omega_\beta\}$$

On the other hand by Proposition IV.2.1(3) the dominant weights in $H^0(G/\tilde{P}_{\beta,1}, \mathcal{L}(\lambda))$ are

$$\{\omega_\alpha + p\,\omega_\beta, (p-1)\omega_\beta\}$$

It follows that $H^0(G/\tilde{P}_{\beta,1}, \mathcal{L}(\lambda))$ is the irreducible G-module $L(\lambda)$ of highest weight λ. By Steinberg's tensor product theorem (Theorem 1.2) $\dim H^0(G/\tilde{P}_{\beta,1}, \mathcal{L}(\lambda)) = \dim L(\lambda) = \dim L(\omega_\alpha) \dim L(\omega_\beta) = 9$.

From Proposition 3(2) we get

$$\begin{aligned}\dim H^1(G/\tilde{P}_{\beta,n}, \mathcal{L}(\lambda)) &= 9 - \dim H^0(G/B, \mathcal{L}(\lambda)) + \dim H^0(G/B, \mathcal{L}(\lambda - \beta)) \\ &= 9 - (p+1)(p+3) + \frac{3(p-1)(p+2)}{2} \\ &= \frac{(p-2)(p-3)}{2}\end{aligned}$$

□

Kodaira's vanishing theorem is true for the projective homogeneous space X in the above example. This follows from the fact that X embeds as a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ and is the zeros of a section in $\mathcal{O}(1) \times \mathcal{O}(p^n)$. Thereby X admits a flat lifting to $\mathbb{Z}$ and since $p > \dim X = 3$ it follows from [DI, 2.8], that Kodaira's vanishing theorem holds for X.

Confronted with an earlier version of the above example Ramanathan suggested, that the right generalization of Kempf's vanishing theorem to arbitrary projective homogeneous spaces could be a Kodaira type vanishing theorem. Unfortunately counterexamples exist already for $G = SL_4(k)$.

Example 2. *Let G be of type A_3 and $\alpha = \alpha_2$ the middle simple root in the Dynkin diagram*

Let α_1, α_3 denote the remaining simple roots in the Dynkin diagram. If ω_i denotes the fundamental dominant weight corresponding to the simple root α_i, then the canonical line bundle ω on $G/P_{\alpha,1}$ is given by $-2\omega_1 - 2\omega_2 - 2\omega_3 - (p-1)(-\omega_1 + 2\omega_2 - \omega_3) = (p-3)\omega_1 - 2p\omega_2 + (p-3)\omega_3$. Let $\mathcal{L}$ be the ample line bundle induced by the character $\lambda = \omega_1 + 3p\omega_2 + \omega_3$. Now $\mathcal{L} \otimes \omega$ is induced by $\mu = (p-2)\omega_1 + p\omega_2 + (p-2)\omega_3$ and we wish to prove that a G-homomorphism

$$\mathrm{H}^0(G/B, \mathcal{L}(\mu)) \to \mathrm{H}^0(G/B, \mathcal{L}(\mu - \alpha))$$

is not surjective. In view of proposition 3 (2) this gives that $\mathrm{H}^1(G/\tilde{P}_{\alpha,1}, \mathcal{L}(\mu)) \neq 0$.

Let s_1, s_2, s_3 *denote the simple reflections and* $\tilde{\alpha}$ *the long root of* A_3. *By Theorem 1.4 the highest weights of the possible composition factors of* $\mathrm{H}^0(G/B, \mathcal{L}(\mu))$ *are* $\mu, s_{2,p} \cdot \mu = (p-1)\omega_1 + (p-2)\omega_2 + (p-1)\omega_3$ *and* $(s_{\tilde{\alpha},2p}\, s_2) \cdot \mu = (p-2)\omega_2$. *Since* $\mu - \alpha = s_{2,p} \cdot \mu$, *the highest weights of the possible composition factors of* $\mathrm{H}^0(G/B, \mathcal{L}(\mu - \alpha))$ *are* $\mu - \alpha$ *and* $s_{\tilde{\alpha},2p} \cdot (\mu - \alpha) = (p-2)\omega_2$. *We will prove that the multiplicity* m *of* $L((p-2)\omega_2)$ *in* $H^0(G/B, \mathcal{L}(\mu))$ *is* 0 *and that the multiplicity* n *of* $L((p-2)\omega_2)$ *in* $H^0(G/B, \mathcal{L}(\mu - \alpha))$ *is* 1 *(for* $p > 2$*).*

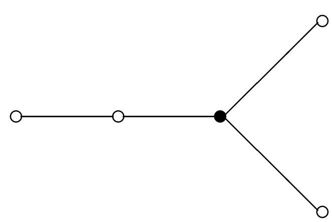
Using the sum formula in Theorem 1.5 it follows for $p > 2$ *that*

$$\mathrm{ch}\, L((p-1)\omega_1 + (p-2)\omega_2 + (p-1)\omega_3) = \chi((p-1)\omega_1 + (p-2)\omega_2 + (p-1)\omega_3) - \chi((p-2)\omega_2)$$

and that $n = 1$. *Evaluating the formula with* μ *yields* $\chi((p-1)\omega_1 + (p-2)\omega_2 + (p-1)\omega_3) - \chi((p-2)\omega_2)$. *Now a simple computation gives the final result* $m = 0$. *For* $p = 2$ *the same computations give* $m = 1$ *and* $n = 2$.

We finally give the “negative Euler characteristic” example.

Example 3. *Let* G *be of type* D_5 *and let* $\alpha = \alpha_3$ *be the marked simple root in the Dynkin diagram*



Let α_1 *be the leftmost and* α_2 *the second leftmost simple root in the Dynkin diagram. Denote the remaining simple roots in the Dynkin diagram by* α_4 *and* α_5. *If* ω_i *denotes the fundamental dominant weight corresponding to the simple root* α_i, *then the canonical line bundle* $\omega_{G/\tilde{P}_{\alpha,1}}$ *is induced by*

$$\begin{aligned}\delta &= -2\omega_1 - 2\omega_2 - 2\omega_3 - 2\omega_4 - 2\omega_5 - (p-1)(-\omega_2 + 2\omega_3 - \omega_4 - \omega_5) \\ &= -2\omega_1 + (p-3)\omega_2 - 2p\omega_3 + (p-3)\omega_4 + (p-3)\omega_5\end{aligned}$$

Now let $\mathcal{L}$ *be the ample line bundle induced by the character* $\lambda = 3\omega_1 + \omega_2 + 3p\omega_3 + \omega_4 + \omega_5$*. Then* $\mathcal{L} \otimes \omega_{G/\tilde{P}_{\alpha,1}}$ *is induced by* $\lambda + \delta = \omega_1 + (p-2)\omega_2 + p\omega_3 + (p-2)\omega_4 + (p-2)\omega_5$ *and by Proposition 2.3(1) the Euler characteristic of* $\mathcal{L} \otimes \omega_{G/\tilde{P}_{\alpha,1}}$ *is*

$$\chi'(\,\omega_1 + (p-2)\omega_2 + p\omega_3 + (p-2)\omega_4 + (p-2)\omega_5\,) -$$
$$\chi'(\,\omega_1 + (p-1)\omega_2 + (p-2)\omega_3 + (p-1)\omega_4 + (p-1)\omega_5\,)$$

By Theorem 1.1(4) we can evaluate the right hand side above by Weyl's dimension formula. This finally gives that that the Euler characteristic of $\mathcal{L} \otimes \omega_{G/\tilde{P}_{\alpha,1}}$ *is*

$$\frac{1}{87091200}[2^2 3p^5(p-1)(3p+1)^2(3p-1)^3\{2^7(p+1)^3(p-1)^2(2p-1)(5p-1)(5p+1)-$$
$$5(p+2)(2p+1)(2p-1)^3(4p-1)(4p+1)(5p-2)\}] =$$
$$-\frac{1}{7257600}[(2p-1)(p-1)p^5(3p-1)^3(3p+1)^2$$
$$\{320p^6 + 408p^5 + 6708p^4 - 1806p^3 - 3801p^2 + 48p + 148\}]$$

It follows easily that this expression is negative for $p \geq 2$*. So* $\mathcal{L}$ *furnishes an example of an ample line bundle on a smooth projective variety* X*, such that* $\mathcal{L} \otimes \omega_X$ *has negative Euler characteristic. This provides a negative answer to the question* [Ray, Remarques et questions 4] *raised by Raynaud.*

REFERENCES

[A1] Andersen, H. H., *Cohomology of line bundles on G/B*, Ann. Sci. Éc. Norm. Sup. (4) **12** (1979), 85–100.

[A2] ______, *The strong linkage principle*, J. reine angew. Math. **315** (1980), 53–59.

[Bo] Borel, A., *Linear algebraic groups* (GTM 126), Springer Verlag, New York, 1991.

[Bou] Bourbaki, N., *Groupes et algèbres de Lie, Chap. IV, V, VI*, 2ème édition, Masson, Paris, 1981.

[DI] Deligne, P., Illusie, L., *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270.

[HL] Haboush, W. and Lauritzen, N., *Varieties of unseparated flags*, Proc. of Conf. in honor of Robert Steinberg, UCLA, 1992 (to appear).

[Ha1] Hartshorne, R., *Algebraic Geometry* (GTM 52), Springer Verlag, New York, 1977.

[Ha2] ______, *Ample Subvarieties of Algebraic Varieties*, Lecture Notes in Mathematics **156** (1970), Springer Verlag.

[Iv] Iversen, B., *The Geometry of Algebraic Groups*, Adv. in Math. **20** (1976), 57–85.

[Ja] Jantzen, J. C., *Representations of Algebraic Groups*, Academic Press, Orlando, 1987.

[Ke] Kempf, G., *Algebraic Representations of Reductive Groups*, Proceedings of the International Congress of Mathematicians, Helsinki 1978 (1980), 575–577.

[La1] Lauritzen, N., *The Euler characteristic of a homogeneous line bundle*, C. R. Acad. Sci. Paris **315** (1992), 715–718.

[La2] ______, *Splitting Properties of Complete Homogeneous Spaces*, J. Algebra (to appear).

[MR] Mehta, V.B. and Ramanathan, A., *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. **122** (1985), 27–40.

[Ni] Nielsen, H. A., *Diagonalizably linearized coherent sheaves*, Bull. Soc. Math. France **102** (1974), 85–97.

[Ra] Ramanathan, A., *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Publ. Math. IHES **65** (1987), 61–90.

[Ray] Raynaud, M., *Contre-exemple au "Vanishing Theorem" en caractéristique* $p > 0$, C.P. Ramanujam - A tribute, Studies in Mathematics no. 8, Tata Institute of Fundamental Research, Springer Verlag, Bombay, 1978.

[St] Steinberg, R., *Lectures on Chevalley Groups*, Yale University, 1968.

[We1] Wenzel, C., *Classification of all Parabolic Subgroup Schemes of a Reductive Linear Algebraic Group over an Algebraically Closed Field*, Trans. Amer. Math. Soc. (to appear).

[We2] ______, *Rationality of* G/P *for a Non Reduced Parabolic Subgroup Scheme* P, Proc. Amer. Math. Soc. (to appear).