# A local Gaussian bootstrap method for realized volatility and realized beta*

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**Abstract**

This paper introduces a local Gaussian bootstrap method useful for the estimation of the asymptotic distribution of high-frequency data-based statistics such as functions of realized multivariate volatility measures as well as their asymptotic variances. The new approach consists of dividing the original data into non-overlapping blocks of $M$ consecutive returns sampled at frequency $h$ (where $h^{-1}$ denotes the sample size) and then generating the bootstrap observations at each frequency within a block by drawing them randomly from a mean zero Gaussian distribution with a variance given by the realized variance computed over the corresponding block.

Our main contributions are as follows. First, we show that the local Gaussian bootstrap is first-order consistent when used to estimate the distributions of realized volatility and realized betas under assumptions on the log-price process which follows a continuous Brownian semimartingale process. Second, we show that the local Gaussian bootstrap matches accurately the first four cumulants of realized volatility up to $o(h)$, implying that this method provides third-order refinements. This is in contrast with the wild bootstrap of Gonçalves and Meddahi (2009), which is only second-order correct. Third, we show that the local Gaussian bootstrap is able to provide second-order refinements for the realized beta, which is also an improvement of the existing bootstrap results in Dovonon, Gonçalves and Meddahi (2013) (where the pairs bootstrap was shown not to be second-order correct under general stochastic volatility). In addition, we highlight the connection between the local Gaussian bootstrap and the local Gaussianity approximation of continuous semimartingales established by Mykland and Zhang (2009) and show the suitability of this bootstrap method to deal with the new class of estimators introduced in that paper. Lastly, we provide Monte Carlo simulations and use empirical data to compare the finite sample accuracy of our new bootstrap confidence intervals for integrated volatility with the existing results.

**JEL Classification**: C15, C22, C58

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# 1 Introduction

Realized measures of volatility have become extremely popular in the last decade as higher and higher frequency returns are available. Despite the fact that these statistics are measured over large samples, their finite sample distributions are not necessarily well approximated by their asymptotic mixed-Gaussian distributions. This is especially true for realized statistics that are not robust to market microstructure noise since in this case researchers usually face a trade-off between using large sample sizes and incurring in market microstructure biases. This has spurred an interest in developing alternative approximations based on the bootstrap. In particular, Gonçalves and Meddahi (2009) proposed bootstrap methods for realized volatility whereas Dovonon, Gonçalves and Meddahi (2013) have studied the application of the bootstrap in the context of realized regressions.

In this paper, we propose and analyze a new bootstrap method for realized measures which we term the local Gaussian bootstrap. In a univariate setting, the new bootstrap approach consists of dividing the original data into non-overlapping blocks of $M$ consecutive observations and then generating the bootstrap observations at each frequency within a block by drawing a random draw from a normal distribution with mean zero and variance given by the realized variance over the corresponding block. By construction, conditional on the observed data, the local Gaussian bootstrap's returns are homoscedastic within blocks. In practice, the volatility of asset returns is highly persistent, implying that it is at least locally nearly constant. Thus, we may expect the local Gaussian bootstrap to provide a good approximation of the asymptotic distribution of high-frequency data-based statistics. We focus on two realized measures: realized variance and realized regression coefficients. The latter can be viewed as a smooth function of the realized covariance matrix and, in this case, we extend the univariate idea and generate bootstrap observations for the vector of intraday returns entering the regression model. Specifically, we generate bootstrap observations on the vector of variables of interest by drawing a random vector from a multivariate normal distribution with mean zero and covariance matrix given by the realized covariance matrix computed over the corresponding block.

We prove the first-order asymptotic validity of the local Gaussian bootstrap method for realized volatility and realized regression coefficients under general assumptions on the log-price process, allowing the latter to follow a continuous Brownian semimartingale process with stochastic volatility of general nonparametric form, which accommodates drift and leverage effects. Next, we consider a model where we impose more restrictive assumptions on the data generating process that rules out these effects. For this simplified model, we show that the local Gaussian bootstrap yields higher-order asymptotic refinements.

Our findings for realized volatility are as follows: when $M$ is fixed, the local Gaussian bootstrap is asymptotically correct but it does not offer any asymptotic refinements. More specifically, the first four bootstrap cumulants of the $t$-statistic based on realized volatility and studentized with a variance estimator that is based on a block size of $M$ do not match the cumulants of the original $t$-statistic to higher order (although they are consistent). Note that when $M = 1$, the new bootstrap method coincides with the wild bootstrap of Gonçalves and Meddahi (2009) based on a $N(0,1)$ external random variable. As Gonçalves and Meddahi (2009) show, this is not an optimal choice, which is in line with our results. Therefore, our result generalizes that of Gonçalves and Meddahi (2009) to the case of a fixed $M > 1$. For any finite values of $M \geq 1$, the rate of convergence of the local Gaussian bootstrap and the normal approximation is the same. In such a situation, to see whether we can still improve upon the existing first-order asymptotic theory (and /or the wild bootstrap method of Gonçalves and Meddahi (2009)) by relying on our new local Gaussian bootstrap method, we follow Shao and Tu (1995) (cf. Section 3.3) and use the asymptotic relative bootstrap error as a criterion of comparison. We show that under this criterion, for any fixed $M \geq 3$ the local Gaussian bootstrap is better than the normal approximation. Whereas for any fixed $M \geq 2$, the comparison between the local Gaussian bootstrap and the wild bootstrap method of Gonçalves and Meddahi (2009) favors the local Gaussian bootstrap.

When the block length $M \to \infty$, as $h \to 0$ such that $Mh \to 0$, (where $h^{-1}$ denotes the sample size), we show that the local Gaussian bootstrap is able to provide an asymptotic refinement. In particular, we show that the first and third bootstrap cumulants of the $t$-statistic converge to the corresponding cumulants at the rate $o\left(h^{1/2}\right)$, which implies that the local Gaussian bootstrap offers a second-order refinement. In this case, the local Gaussian bootstrap is an alternative to the optimal two-point distribution wild bootstrap proposed by Gonçalves and Meddahi (2009). More interestingly, our results also highlight that the local Gaussian bootstrap is able to match the second and fourth order cumulants up to $o(h)$, small enough error to yield a third-order asymptotic refinement. This is in contrast to the optimal wild bootstrap methods of Gonçalves and Meddahi (2009), which cannot deliver third-order asymptotic refinements. We also show that the local Gaussian bootstrap variance is a consistent estimator of the (conditional) asymptotic variance of realized volatility. Then, we provide a proof of the first-order asymptotic validity of the local Gaussian bootstrap method for unstudentized (percentile) intervals. This is in contrast to the best existing bootstrap methods of Gonçalves and Meddahi (2009), which cannot consistently estimate the asymptotic variance of realized volatility.

For the realized regression estimator, the local Gaussian bootstrap matches the cumulants of the $t$-statistics through order $o\left(h^{1/2}\right)$ when $M \to \infty$. Thus, this method can provide second-order refinements. This contrasts with the pairs bootstrap studied by Dovonon, Gonçalves and Meddahi (2013), which is only first-order correct. In addition, we show that the local Gaussian bootstrap method can be used to estimate the variance of smooth function of realized covariance matrix, in particular the variance of realized beta. This is also an improvement of the existing bootstrap results in Dovonon, Gonçalves and Meddahi (2013), where the pairs bootstrap cannot consistently estimate the asymptotic variance of smooth function of realized covariance matrix. Our result provides an alternative to the recent wild blocks of blocks bootstrap method studied by Hounyo (2017) (see also Hounyo, Gonçalves and Meddahi (2017)), which is also a valid approach when estimating the asymptotic variance of smooth function of realized covariance matrix. Note that, although the results in Hounyo (2017) are derived under general assumption including the presence of market microstructure noise, jumps and non-synchronous observations, there are no results about a possible higher-order correctness of their bootstrap method.

The structure of our local Gaussian bootstrap method can be related to the ideas in Mykland and Zhang (2009). As these authors explain, one useful way of thinking about inference in the context of realized measures is to consider that returns have constant variance and are conditionally Gaussian over blocks of consecutive $M$ observations. Roughly speaking, a high-frequency return of a given asset is equal in law to the product of its volatility (the spot volatility) multiplied by a normal standard distribution. Mykland and Zhang (2009) show that this local Gaussianity approximation of continuous semimartingales observed at high-frequency is useful in deriving the asymptotic theory for the estimators used in this literature by providing an analytical tool to find the asymptotic behaviour without calculations being too cumbersome. They also introduced a class of block-based estimators. When observation times are equidistant, the block-based procedures of Mykland and Zhang (2009) have the advantage of yielding more efficient estimators by increasing the size of the block (see also the related work of Renault, Sarisoy and Werker (2017)). We analyze the connection between the local Gaussian bootstrap and the local Gaussianity approximation of continuous semimartingales observed at high-frequency established by Mykland and Zhang (2009) and show the suitability of our bootstrap method to deal with the new class of estimators introduced in that paper. In particular, we show how and to what extent the local Gaussianity approximation of continuous semimartingales observed at high-frequency can be explored to generate a valid bootstrap method for Mykland and Zhang's (2009) realized beta-type estimator. Our Monte Carlo simulations suggest that the new bootstrap method we propose improves upon the first-order asymptotic theory in finite samples and outperforms the existing bootstrap methods.

The rest of this paper is organized as follows. In the next section, we first introduce the setup, our assumptions and describe the local Gaussian bootstrap. In Sections 3 and 4 we establish the consistency of this method for realized volatility and realized beta, respectively. Section 5 contains the higher-order asymptotic properties of the bootstrap cumulants. Section 6 provides a study of the local Gaussian bootstrap in the context of Mykland and Zhang's (2009) local Gaussianity framework. Section 7 contains simulations. One empirical application is presented in Section 8. We conclude in Section 9 by describing how the local Gaussian bootstrap can be extended to a setting which allows for irregularly spaced data, non-synchronicity, market microstructure noise and/or jumps in the log-price process. Appendix A contains the tables with simulation results whereas Appendix B contains the proofs of the main results. Last, online supplementary material provides technical lemmas and the remaining proofs. Further details are provided at Cambridge Journals Online.[1]

[1]Readers may refer to the supplementary material for this article as Hounyo (2018), available at Cambridge Journals Online (journals.cambridge.org/ect).

## 2 Framework and the local Gaussian bootstrap

The statistics of interest in this paper can be written as smooth functions of the realized multivariate volatility matrix. In this section, we describe the theoretical framework for multivariate high-frequency returns, as well as propose a new resampling method - the local Gaussian bootstrap. Sections 3 and 4 will consider in detail the theoretical properties of this method for the special cases of realized volatility and realized beta, respectively.

We follow Mykland and Zhang (2009) and assume that the log-price process $X_t = \left(X_t^{(1)}, \ldots, X_t^{(d)}\right)'$ of a $d$-dimensional vector of assets is defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. We model $X$ as a Brownian semimartingale process,

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s,\ t \geq 0, \tag{1}$$

where $\mu = (\mu_t)_{t\geq 0}$ is a $d$-dimensional predictable locally bounded drift vector, $W = (W_t)_{t\geq 0}$ is $d$-dimensional Brownian motion and $\sigma = (\sigma_t)_{t\geq 0}$ is an adapted càdlàg $d \times d$ locally bounded process such that $\Sigma_t \equiv \sigma_t \sigma_t'$ is the spot covariance matrix of $X$ at time $t$.

We follow Barndorff-Nielsen et al. (2006) and assume that the matrix process $\sigma_t$ is invertible and satisfies the following assumption

$$\sigma_t = \sigma_0 + \int_0^t a_s ds + \int_0^t b_s dW_s + \int_0^t v_s dZ_s, \tag{2}$$

where $a$, $b$, and $v$ are all locally bounded, càdlàg and adapted processes, with $a$ also being predictable, and $Z$ is a vector Brownian motion independent of $W$.

The representation in (1) and (2) is rather general as it allows for leverage and drift effects.[2] Assumption 2 of Mykland and Zhang (2009) or equation (1) of Mykland and Zhang (2011) also impose a Brownian semimartingale structure on the instantaneous covolatility matrix $\sigma$. Equation (2) rules out jumps in volatility, but this can be relaxed (see Assumption H1 of Barndorff-Nielsen et al. (2006) for a weaker assumption on $\sigma$).

The parameters of interest in this paper are functions of the elements of the integrated covariance matrix of $X$, i.e., the process

$$\Gamma_t \equiv \int_0^t \Sigma_s ds,\ \ t \in [0,1].$$

Without loss of generality, we let $t = 1$, define $\Gamma = \Gamma_1 = \int_0^1 \Sigma_s ds$ as the integrated covariance of $X$ over the period $[0,1]$, and we let $\Gamma_{kl}$ denote the $(k,l)$-th element of $\Gamma$. Suppose we observe $X$ over a fixed time interval $[0,1]$ at regular time points $t_i = ih$, for $i = 0, \ldots, 1/h$, from which we compute $1/h$ intraday returns at frequency $h$,

$$y_i \equiv X_{ih} - X_{(i-1)h} = \int_{(i-1)h}^{ih} \mu_t dt + \int_{(i-1)h}^{ih} \sigma_t dW_t, \quad i = 1, \ldots, \frac{1}{h}, \tag{3}$$

[2]However, for the higher-order asymptotic properties of the bootstrap cumulants in Section 5 we do require the absence of leverage and drift effects.

where we will let $y_{k,i}$ to denote the $i$-th intraday return on asset $k$, $k = 1, \ldots, d$.

Next, we introduce a new resampling method called the *local Gaussian bootstrap*. To generate the local Gaussian bootstrap vector of returns, we only need an estimator of the spot covariance matrix and a normal external random vector. Divide the fixed time interval $[0, 1]$ into blocks, each of which containing $M$ high-frequency returns. For each block $j = 1, \ldots, \frac{1}{Mh}$, let $\hat{C}_{(j)}$ be the realized covolatility computed over the block $j$ containing returns $y_{1+(j-1)M}, \ldots, y_{jM}$. To be precise, $\hat{C}_{(j)}$ is such that $\hat{C}_{(j)}\hat{C}'_{(j)} = \frac{1}{Mh}\sum_{i=1}^{M} y_{i+(j-1)M} y'_{i+(j-1)M}$. Inside each block $j$ of size $M$ ($j = 1, \ldots, \frac{1}{Mh}$), we generate the $M$ vector of returns as follows,

$$y^*_{i+(j-1)M} = \hat{C}_{(j)} \cdot \left(\sqrt{h}\eta^*_{i+(j-1)M}\right), \text{ for } i = 1, \ldots, M, \tag{4}$$

where $\eta^*_{i+(j-1)M} \sim i.i.d.N(0, I_d)$ across $(i, j)$, and $I_d$ is a $d \times d$ identity matrix. The local Gaussian bootstrap data-generating process (DGP) is motivated by the toy model $X_t = \int_0^t \sigma_s dW_s$, where $\sigma$ is independent to $W$ and the volatility process $\sigma_t$ is piecewise constant. Specifically, the volatility process $\sigma_t$ is invertible and such that for $j = 1, \ldots, \frac{1}{Mh}$,

$$\sigma_t = \sigma_{(j-1)Mh}, \text{ for } t \in ((j-1)\,Mh, jMh]. \tag{5}$$

Under these assumptions,

$$y_{i+(j-1)Mh} = C_{(j)} \cdot \left(\sqrt{h}\eta_{i+(j-1)M}\right), \text{ for } 1 \leq i \leq M,$$

where $\eta_{i+(j-1)M} \sim i.i.d.N(0, I_d)$ and $C_{(j)} = \sigma_{(j-1)Mh}$. Although (4) is motivated by this very simple model, as we show in this paper, the local Gaussian bootstrap method remains valid more generally. In particular, its validity extends to the general stochastic volatility models described by (1) and (2). In Section 6, we will discuss the connection between the local Gaussian bootstrap and the local Gaussianity framework recently analyzed by Mykland and Zhang (2009).

In this paper, and as usual in the bootstrap literature, $P^*$ ($E^*$ and $Var^*$) denotes the probability measure (expected value and variance) induced by the bootstrap resampling, conditional on a realization of the original time series. In the remainder of the paper, we will need the following notation; $\upsilon$ denotes a given probability measure (it could be the true probability distribution $P$ or an approximate probability measure $Q_{h,M}$). For a sequence of bootstrap statistics $Z^*_{h,M}$, we write $Z^*_{h,M} = o_{P^*}(1)$ in probability under $\upsilon$, or $Z^*_{h,M} \to^{P^*} 0$, as $h \to 0$, in probability under $\upsilon$, if for any $\varepsilon > 0$, $\delta > 0$, $\lim_{h\to 0} \upsilon \left[P^*\left(\left|Z^*_{h,M}\right| > \delta\right) > \varepsilon\right] = 0$. Similarly, we write $Z^*_{h,M} = O_{P^*}(1)$ as $h \to 0$, in probability under $\upsilon$ if for all $\varepsilon > 0$ there exists a $M_\varepsilon < \infty$ such that $\lim_{h\to 0} \upsilon \left[P^*\left(\left|Z^*_{h,M}\right| > M_\varepsilon\right) > \varepsilon\right] = 0$. Finally, we write $Z^*_{h,M} \to^{d_*} Z$ as $h \to 0$, in probability under $\upsilon$, if conditional on the sample, $Z^*_{h,M}$ converges weakly to $Z$ under $P^*$, for all samples contained in a set with probability converging to one under $\upsilon$.

# 3 Results for realized volatility

In this section, we introduce the statistics and CLT of interest as well as analyze the first-order asymptotic properties of the bootstrap. In particular, we review some known (but nontrivial) asymptotic results that are relevant for introducing the local Gaussian bootstrap method for realized volatility. Next, we study the theoretical properties of this bootstrap method for realized volatility. Throughout this section we let $d = 1$.

## 3.1 Existing asymptotic theory

To describe the asymptotic properties of realized volatility, we need to introduce some notation. In the following we assume that $1/Mh$ is an integer. For any $q > 0$, define the realized $q$-th order power variation as

$$R_q \equiv Mh \sum_{j=1}^{1/Mh} \left( \frac{RV_{j,M}}{Mh} \right)^{q/2}. \tag{6}$$

where $RV_{j,M} = \sum_{i=1}^{M} y_{i+(j-1)M}^2$ is the realized volatility over the period $[(j-1)\,Mh, jMh]$ for $j = 1, \ldots, \frac{1}{Mh}$. Note that when $q = 2$, $R_2 = RV \equiv \sum_{i=1}^{1/h} y_i^2$ (i.e., the realized volatility), and $R_2$ is not a function of $M$. Similarly, for any $q > 0$, define the integrated power variation by

$$\overline{\sigma^q} \equiv \int_0^1 \sigma_u^q du.$$

Finally, define $c_{M,q} \equiv E\left( \left( \frac{\chi_M^2}{M} \right)^{q/2} \right)$ where $\chi_M^2$ has the standard $\chi^2$ distribution with $M$ degrees of freedom, and note that $c_{M,q}$ can be written as

$$c_{M,q} = \left( \frac{2}{M} \right)^{q/2} \frac{\Gamma\left(\frac{q+M}{2}\right)}{\Gamma\left(\frac{M}{2}\right)}, \tag{7}$$

where $\Gamma$ is the Gamma function.

For any bounded $M$, under (1) and (2) Mykland and Zhang (2009) (cf. equation (58) in conjunction with Remark 8) show that as $h \to 0$

$$\sqrt{h^{-1}} \left( \frac{1}{c_{M,q}} R_q - \overline{\sigma^q} \right) \to^{st} N(0,1) \sqrt{\frac{M\left(c_{M,2q} - c_{M,q}^2\right)}{c_{M,q}^2} \overline{\sigma^{2q}}}, \tag{8}$$

under $P$, where $\to^{st}$ denotes stable convergence. An implication of (8) is that for any bounded $M$, under (1) and (2) as $h \to 0$

$$\frac{1}{c_{M,q}} R_q \xrightarrow{P} \overline{\sigma^q}. \tag{9}$$

When $M$ goes to infinity, a number of recent papers extends these asymptotic results for block-based statistics in several directions, see e.g., Jacod and Rosenbaum (2013) (cf. equations (3.3), (3.4), (3.8) and (3.11)) and Li, Todorov and Tauchen (2017) (cf. Theorems 2 and 3). See also Jacod and

Protter (2012) (cf. Theorem 9.4.1) and Mykland and Zhang (2011) for similar results. In particular, when $M \to \infty$ as $h \to 0$ such that $Mh \to 0$, a sufficient assumption such that $R_q \xrightarrow{P} \overline{\sigma^q}$, is (1) and (2). Hence, the consistency result in (9) still holds when $M$ goes to infinity by noting that for any $q > 0$, $c_{M,q} \to 1$ as $M \to \infty$. As a result, under (1) and (2), if $M$ is fixed or $M \to \infty$ as $h \to 0$ such that $Mh \to 0$, then as $h \to 0$,

$$\hat{V}_{\bar{\sigma^2},h,M} \equiv 2 \cdot \frac{1}{c_{M,4}} R_4 \xrightarrow{P} V_{\bar{\sigma^2}} \equiv 2\overline{\sigma^4}. \tag{10}$$

By combining the asymptotic results in (8) for $q = 2$ and the consistency result in (10), we obtain the feasible CLT result for realized volatility. Specifically, under (1) and (2), if $M$ is fixed or $M \to \infty$ as $h \to 0$ such that $Mh \to 0$, then as $h \to 0$, we get

$$\frac{\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)}{\sqrt{\hat{V}_{\bar{\sigma^2},h,M}}} \xrightarrow{d} N(0,1). \tag{11}$$

To better understand how the CLT result in (11) holds for both $M$ fixed and $M \to \infty$, first note that with $q = 2$, given (7), we have for any $M > 0$, $c_{M,2} = 1$, $c_{M,4} = \frac{M+2}{M}$, hence $\frac{M\left(c_{M,4} - c_{M,2}^2\right)}{c_{M,2}^2} = 2$. Then, let

$$T_{\bar{\sigma^2}} = \frac{\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)}{\sqrt{V_{\bar{\sigma^2}}}}, \tag{12}$$

and notice that $T_{\bar{\sigma^2}}$ *is not a function of* $M$. In particular, the CLT result for $T_{\bar{\sigma^2}}$ does not depend either on the condition $M$ fixed or the condition $M \to \infty$. A sufficient assumption to obtain $T_{\bar{\sigma^2}} \xrightarrow{d} N(0,1)$ as $h \to 0$ is (1) and (2) (see e.g., Barndorff-Nielsen et al. (2006) (cf. Theorem 2.3), see also Jacod and Protter (2012) (cf. equations (5.6.3) and (5.6.4)) for similar results). Finally, to complete the picture, we write

$$\frac{\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)}{\sqrt{\hat{V}_{\bar{\sigma^2},h,M}}} = T_{\bar{\sigma^2}} \sqrt{\frac{V_{\bar{\sigma^2}}}{\hat{V}_{\bar{\sigma^2},h,M}}} = \underbrace{T_{\bar{\sigma^2}}}_{\xrightarrow{d} N(0,1)} \underbrace{\sqrt{\frac{\overline{\sigma^4}}{\frac{1}{c_{M,4}} R_4}}}_{\xrightarrow{P} 1}.$$

We can use the feasible asymptotic distribution result in (11) to build confidence intervals for integrated volatility. In particular, a two-sided feasible $100(1-\alpha)\%$ level interval for $\overline{\sigma^2}$ is given by:

$$IC_{Feas,1-\alpha} = \left(R_2 - z_{1-\alpha/2} n^{-1/2} \sqrt{\hat{V}_{\bar{\sigma^2},h,M}}, R_2 + z_{1-\alpha/2} n^{-1/2} \sqrt{\hat{V}_{\bar{\sigma^2},h,M}}\right), \tag{13}$$

where $z_{1-\alpha/2}$ is such that $\Phi\left(z_{1-\alpha/2}\right) = 1 - \alpha/2$, and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. For instance, $z_{0.975} = 1.96$ when $\alpha = 0.05$. When the block size $M$ is fixed such that $M = 1$, the result in (11) is indeed well-known in the literature. This is equivalent to the CLT result for realized volatility derived by Barndorff-Nielsen and Shephard (2002). In particular, $\hat{V}_{\bar{\sigma^2},h,1} = \frac{2}{3} h^{-1} \sum_{i=1}^{1/h} y_i^4$. Notice that, when $M > 1$, the realized volatility $RV$ using the blocking approach is the same realized volatility studied by Barndorff-Nielsen and Shephard (2002), but the $t$-statistic is different because $\hat{V}_{\bar{\sigma^2},h,M}$ changes with $M$. Under regular sampling schemes, one advantage of the block-based estimator (given by (6)) is to improve efficiency by increasing the size of the block, see e.g., Mykland and Zhang (2009) and the recent work of Renault, Sarisoy and Werker (2017).

### 3.2 Bootstrap consistency

In this section, we show that the proposed local Gaussian bootstrap, described in Section 2, is consistent when applied to realized volatility. Specifically, given (4) with $d = 1$, for $j = 1, \ldots, 1/Mh$, we let

$$y_{i+(j-1)M}^{*} = \sqrt{\frac{RV_{j,M}}{M}} \eta_{i+(j-1)M}^{*},\ i = 1, \ldots, M, \tag{14}$$

where $\eta_{i+(j-1)M}^{*} \sim$ i.i.d.$N(0,1)$ across $(i,j)$. Note that this bootstrap method is related to the wild bootstrap approach proposed by Gonçalves and Meddahi (2009). In particular, when $M = 1$ and $d = 1$, $\frac{RV_{j,M}}{M} = y_j^2$ this amounts to the wild bootstrap based on a standard normal external random variable. As the results of Gonçalves and Meddahi (2009) show, this choice of $\eta^*$ is not optimal for inference on $\overline{\sigma^2}$ because it does not replicate the higher order cumulants of $RV$ up to $o\left(h^{1/2}\right)$. Instead, they propose a two-point distribution that matches the first three cumulants up to $o\left(h^{1/2}\right)$ and therefore provides a second-order refinement. We show that by replacing $y_j^2$ with the local average $\frac{RV_{j,M}}{M}$, yields an asymptotically valid bootstrap method for $RV$ and can match the first four cumulants of $RV$ through order $o\left(h\right)$ even when volatility is stochastic.

We define the bootstrap realized volatility estimator as follows

$$R_2^* = \sum_{i=1}^{1/h} y_i^{*2} = \sum_{j=1}^{1/Mh} RV_{j,M}^*,$$

where $RV_{j,M}^* = \sum_{i=1}^{M} y_{i+(j-1)M}^{*2}$. Letting

$$\frac{1}{M}\sum_{i=1}^{M} \eta_{i+(j-1)M}^{*2} \equiv \frac{\chi_{j,M}^2}{M},$$

it follows that $RV_{j,M}^* = \frac{\chi_{j,M}^2}{M} RV_{j,M}$. Hence, we can easily show that

$$E^*\left(R_2^*\right) = \sum_{j=1}^{1/Mh} E^*\left(\frac{\chi_{j,M}^2}{M}\right) RV_{j,M} = c_{M,2} R_2 = R_2.$$

Similarly, we have

$$\begin{aligned} V_{\bar{\sigma^2},h,M}^* &\equiv Var^*\left(h^{-1/2} R_2^*\right) = h^{-1} Var^*\left[\sum_{j=1}^{1/Mh}\left(\frac{\chi_{j,M}^2}{M}\right) RV_{j,M}\right] \\ &= M^2 h\left[\sum_{j=1}^{1/Mh}\left(\frac{RV_{j,M}}{Mh}\right)^2 Var^*\left(\frac{\chi_{j,M}^2}{M}\right)\right] = \underbrace{M\left(c_{M,4} - c_{M,2}^2\right)}_{=2} R_4 = \frac{M+2}{M}\hat{V}_{\bar{\sigma^2},h,M}. \end{aligned}$$

Our next result studies the convergence of $V_{\bar{\sigma^2},h,M}^*$ when $M$ is either fixed or $M \to \infty$ as $h \to 0$. We also provide a theoretical justification for using the local Gaussian bootstrap to consistently estimate the distribution of $\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)$.

**Theorem 3.1.** *Suppose (1), (2) and (4) hold. We have that*

**(a)** *For any fixed $M > 0$, as $h \to 0$*

$$V^*_{\bar{\sigma^2},h,M} \xrightarrow{P} \frac{M+2}{M} V_{\bar{\sigma^2}}.$$

*Whereas, when $M \to \infty$ as $h \to 0$ such that $Mh \to 0$, as $h \to 0$*

$$V^*_{\bar{\sigma^2},h,M} \xrightarrow{P} V_{\bar{\sigma^2}}.$$

**(b)** *If $M$ is fixed or $M \to \infty$ as $h \to 0$ such that $M = o\left(h^{-1/2}\right)$, then as $h \to 0$,*

$$\sup_{x \in \Re} \left| P^* \left( \sqrt{\frac{M}{M+2}} \sqrt{h^{-1}} \left(R_2^* - R_2\right) \leq x \right) - P\left( \sqrt{h^{-1}} \left(R_2 - \overline{\sigma^2}\right) \leq x \right) \right| \to 0,$$

*in probability under $P$.*

Part (a) of Theorem 3.1 shows that the local Gaussian bootstrap variance estimator $V^*_{\bar{\sigma^2},h,M}$, without adjustment, is not consistent for $V_{\bar{\sigma^2}}$ when the block size $M$ is finite. In particular, for fixed $M$ (small) the local Gaussian bootstrap variance converges in probability to $\frac{M+2}{M} V_{\bar{\sigma^2}} \neq V_{\bar{\sigma^2}}$. The consistency of $V^*_{\bar{\sigma^2},h,M}$ towards $V_{\bar{\sigma^2}}$ holds only when $M \to \infty$. This explains the presence of the scale factor $\sqrt{\frac{M}{M+2}}$ in $\sqrt{\frac{M}{M+2}} \sqrt{h^{-1}} \left(R_2^* - R_2\right)$, in order to restore consistency of the local Gaussian bootstrap variance even when $M$ is finite. Part (b) of Theorem 3.1 justifies using the local Gaussian bootstrap to estimate the entire distribution of $\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)$, thus to construct bootstrap unstudentized (percentile) intervals for integrated volatility. Bootstrap percentile intervals are easier to implement since they have the advantage of not requiring an explicit estimator of the (conditional) asymptotic variance of $R_2$. Specifically, a $100\,(1-\alpha)\,\%$ symmetric bootstrap percentile interval for integrated volatility based on the local Gaussian bootstrap is given by

$$IC^*_{perc,1-\alpha} = \left(R_2 - n^{-1/2} p^*_{1-\alpha}, R_2 + n^{-1/2} p^*_{1-\alpha}\right), \tag{15}$$

where $p^*_{1-\alpha}$ is the $1-\alpha$ quantile of the bootstrap distribution of $\left| \sqrt{\frac{M}{M+2}} \sqrt{h^{-1}} \left(R_2^* - R_2\right) \right|$.

In the univariate context considered in this section, the estimator of $V_{\bar{\sigma^2}}$ is rather simple[3] (it is given by a (scaled) version of $R_4$), but this is not necessarily the case for other applications. For instance, for realized regression coefficients defined by the Mykland and Zhang's (2009) blocking approach the bootstrap percentile method could be useful in that context. In general, when the variance estimator of the statistic of interest is hard to compute, bootstrap percentile intervals are a very attractive method in these cases. Although, the bootstrap percentile intervals are simple to compute, they may not be

[3]Note that even in the simple univariate context, estimators of integrated quarticity $\overline{\sigma^4}$ (and thus of $V_{\bar{\sigma^2}}$) can have very poor finite sample properties see e.g., Andersen, Dobrev and Schaumburg (2014).

necessarily very accurate unless the sample size $n$ is large (see e.g., the discussion in Shao and Tu (1995) (cf. Section 4.1.2) and Gonçalves, Hounyo and Meddahi (2014) (cf. p. 682)). Next, we propose the following consistent estimator of $V^*_{\bar{\sigma^2},h,M}$ that allows us to form bootstrap studentized (percentile-$t$) intervals:

$$\hat{V}^{*\text{MZ}}_{\bar{\sigma^2},h,M} = \frac{M\left(c_{M,4} - c^2_{M,2}\right)}{c_{M,4}} R^*_4 = 2\frac{M}{M+2} R^*_4,$$

where for any $q > 0$, we let $R^*_q \equiv Mh \sum_{j=1}^{1/Mh} \left(\frac{RV^*_{j,M}}{Mh}\right)^{q/2}$. In the following theorem, we provide a theoretical justification for using the bootstrap distribution of $\left(\hat{V}^{*\text{MZ}}_{\bar{\sigma^2},h,M}\right)^{-1/2} \sqrt{h^{-1}}\left(R^*_2 - R_2\right)$ to estimate the entire distribution of $\left(\hat{V}_{\bar{\sigma^2},h,M}\right)^{-1/2} \sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)$.

**Theorem 3.2.** *Suppose (1), (2) and (4) hold. If $M$ is fixed or $M \to \infty$ as $h \to 0$ such that $M = o\left(h^{-1/2}\right)$, then as $h \to 0$,*

$$\sup_{x\in\Re}\left|P^*\left(\left(\hat{V}^{*MZ}_{\bar{\sigma^2},h,M}\right)^{-1/2}\sqrt{h^{-1}}\left(R^*_2 - R_2\right) \le x\right) - P\left(\left(\hat{V}_{\bar{\sigma^2},h,M}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right) \le x\right)\right| \to 0,$$

*in probability under $P$.*

Theorem 3.2 also justifies constructing bootstrap percentile-$t$ intervals. In particular, a $100\,(1-\alpha)\,\%$ symmetric bootstrap percentile-$t$ interval for integrated volatility is given by

$$IC^*_{perc\text{-}t,1-\alpha} = \left(R_2 - q^*_{1-\alpha} n^{-1/2}\sqrt{\hat{V}_{\bar{\sigma^2},h,M}},\, R_2 + q^*_{1-\alpha} n^{-1/2}\sqrt{\hat{V}_{\bar{\sigma^2},h,M}}\right), \tag{16}$$

where $q^*_{1-\alpha}$ is the $(1-\alpha)$-quantile of the bootstrap distribution of $\left|\left(\hat{V}^{*\text{MZ}}_{\bar{\sigma^2},h,M}\right)^{-1/2}\sqrt{h^{-1}}\left(R^*_2 - R_2\right)\right|$. Once again, it is worth emphasizing that these results hold under general stochastic volatility models described by (1) and (2), which allow for the presence of drifts and leverage effects under $P$. As the proof of Theorem 3.2 shows, the asymptotic validity of the local Gaussian bootstrap depends on the availability of a CLT result of $R_2$ and a law of large numbers for $R_q$, which hold directly under the conditions of Theorem 3.2.

To see the gain from the new local Gaussian bootstrap procedure, one should compare these results with those of Gonçalves and Meddahi (2009). The results in Theorem 3.1 are in contrast to the (i.i.d. and wild) bootstrap methods studied by Gonçalves and Meddahi (2009), which are only valid for percentile-$t$ intervals i.e., when the statistic is normalized by its (bootstrap) standard deviation. In the following, we denote Gonçalves and Meddahi's (2009) bootstrap intraday $h$-period returns as $y^*_{i,\text{GM}}$. For the i.i.d. bootstrap, $y^*_{i,\text{GM}}$ is obtained by resampling i.i.d. from $\{y_i : i = 1, \ldots, 1/h\}$. Whereas for the wild bootstrap, $y^*_{i,\text{GM}} = y_i \nu^*_i$, where $\nu^*_i$ are i.i.d. with moments given by $\mu^*_q \equiv E^*\left|\nu^*\right|^q$. We can show that the i.i.d. bootstrap variance estimator for the asymptotic variance of the realized volatility studied by Gonçalves and Meddahi (2009) is given by

$$Var^*\left(h^{-1/2}\sum_{i=1}^{1/h} y_{i,\text{GM}}^{*2}\right) = h^{-1}\sum_{i=1}^{1/h} y_i^4 - \left(\sum_{i=1}^{1/h} y_i^2\right)^2 \xrightarrow{P} 3\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2,$$

(see e.g., part (a1) of Lemma S.5 in the online supplement of Gonçalves and Meddahi (2009)) which is equal to $V_{\bar{\sigma^2}}$ (i.e. the asymptotic conditional variance of the realized volatility) only when the volatility is constant. Similarly, we can show that the wild bootstrap variance estimator of the asymptotic variance of the realized volatility is given by

$$Var^*\left(h^{-1/2}\sum_{i=1}^{1/h} y_{i,\text{GM}}^{*2}\right) = \left(\mu_4^* - \mu_2^{*2}\right)\left(h^{-1}\sum_{i=1}^{1/h} y_i^4\right) \xrightarrow{P} \underbrace{\left(\mu_4^* - \mu_2^{*2}\right)\cdot 3\overline{\sigma^4}}_{\equiv V_{\text{GM}}}, \tag{17}$$

(see e.g., the remark before equation (6) in Gonçalves and Meddahi (2009)). Next, using the best existing choice of $\nu^*$ (i.e. the optimal two-point distribution) suggested in Proposition 4.5 of Gonçalves and Meddahi (2009) yield:

$$\mu_q^* = \left(\frac{1}{5}\sqrt{31+\sqrt{186}}\right)^q \cdot p + \left(-\frac{1}{5}\sqrt{31-\sqrt{186}}\right)^q \cdot (1-p)$$

where $p = \frac{1}{2} - \frac{3}{\sqrt{186}}$. Thus, we have

$$\mu_2^* \approx 0.991, \text{ and } \mu_4^* \approx 1.217.$$

As a result, in view of (17), $V_{\text{GM}}$ becomes

$$V_{\text{GM}} \approx 0.679 \cdot \overline{\sigma^4} \neq V_{\bar{\sigma^2}}.$$

This means that we would neither use the i.i.d. bootstrap nor the best existing bootstrap method of realized volatility (i.e. the optimal two-point wild bootstrap of Gonçalves and Meddahi (2009)) to compute standard errors of statistics based on functionals of $R_2$. However, note that although the bootstrap methods in Gonçalves and Meddahi (2009) do not consistently estimate $V_{\bar{\sigma^2}}$, their bootstrap methods are still asymptotically valid for bootstrap percentile-$t$ intervals. It is also possible to consistently estimate the asymptotic variance of $R_2$, by using the wild bootstrap of Gonçalves and Meddahi (2009) with certain external random variables. Specifically, given (17), we can show that a necessary condition for the consistency of the wild bootstrap variance in Gonçalves and Meddahi (2009) is to choose the external random variable $\nu^*$ such that $\mu_4^* - \mu_2^{*2} = \frac{2}{3}$. Unfortunately, it is easy to see that the latter does not deliver an asymptotic refinement.

Although, the local Gaussian bootstrap method can be used to consistently estimate $V_{\bar{\sigma^2}}$ when $M \to \infty$, we would like to note that for the local Gaussian bootstrap, as for all blocking methods, there is an inherent trade-off at stake by increasing the block size $M$. To gain further insight, let us consider a simplified model $dX_t = \sigma dW_t$ (i.e. the constant volatility model without drift), where $y_i \overset{d}{=} \sigma h^{1/2}\nu_i$, with $\nu_i \sim$ i.i.d. $N(0,1)$, and '$\overset{d}{=}$' denotes equivalence in distribution. It is easy to see that for this model $V_{\bar{\sigma}^2} = 2\sigma^4$, $V^*_{\bar{\sigma}^2,h,M} = 2\sigma^4 \frac{h}{M} \sum_{j=1}^{1/Mh} \left(\chi^2_{M,j}\right)^2$ where $\chi^2_{M,j}$ is i.i.d.$\sim \chi^2_M$. A straightforward calculation shows that

$$Bias\left(V^*_{\bar{\sigma}^2,h,M}\right) = \frac{4\sigma^2}{M} = O\left(\frac{1}{M}\right) \text{ and } Var\left(V^*_{\bar{\sigma}^2,h,M}\right) = 4Mh\frac{(M+2)\left(M^2+9M+24\right)}{M^3}\sigma^4 = O\left(Mh\right),$$

see Lemma C1.3 in the online supplement to this paper (Hounyo (2018)) for further details. Then the bias of $V^*_{\bar{\sigma}^2,h,M}$ decreases with increasing $M$, whereas the variance increases with increasing $M$. Hence, in order to pick up any biases due to the blocking in our local Gaussian bootstrap procedure, one should choose $M$ very large. Of course, this comes with a cost: large variance of $V^*_{\bar{\sigma}^2,h,M}$. It follows that the mean squared error of $V^*_{\bar{\sigma}^2,h,M}$ is of order $O\left(1/M^2\right) + O\left(Mh\right)$. Hence, to minimize it, one should pick $M$ proportional to $h^{-1/3}$, in which case $MSE\left(V^*_{\bar{\sigma}^2,h,M}\right) = O\left(h^{-2/3}\right)$. However, the optimal block size for mean squared error may be suboptimal for the purpose of distribution estimation; see Hall, Horowitz and Jing (1995). As a result, we will not pursue this approach here and leave it for further research.

To summarize, Theorems 3.1 and 3.2 provide a theoretical justification for using the local Gaussian bootstrap approach as an alternative to the existing bootstrap methods proposed by Gonçalves and Meddahi (2009). Compared with the latter, there is no gain to using the local Gaussian bootstrap resampling method when $M = 1$, since it boils down to the wild bootstrap of Gonçalves and Meddahi (2009) based on a $N(0,1)$ external random variable. However, our result in Theorem 3.1 (even for $M = 1$) complements results in Theorem 3.1 of Gonçalves and Meddahi (2009).

It is worth stressing that Theorems 3.1 and 3.2 do not answer the following question: Does the local Gaussian bootstrap help to gradually improve upon the performance of the bootstrap method of Gonçalves and Meddahi (2009) by moving for instance from $M = 1$ to $M = 2$ or from $M = 2$ to $M = 3$, etc.? The usual way of comparing the bootstrap to asymptotic theory-based inference or more generally comparing two different asymptotically valid approximations of the distribution of a given statistic is by means of Edgeworth expansions. In Section 5.1, using a simpler model that rules out drift and leverage effects, and based on Edgeworth expansions, we will show that the answer to this question is yes.

When $M$ is fixed, the rate of convergence of both approximations (local Gaussian bootstrap and the asymptotic theory-based) is the same. In such a situation, we follow Shao and Tu (1995) and rely on the asymptotic relative bootstrap error to quantify the magnitude of the bootstrap error relative to the error of the normal approximation. We show that the asymptotic relative bootstrap error is a strictly decreasing function of $M$, implying that one can improve upon the performance of the bootstrap method of Gonçalves and Meddahi (2009) (see also Figure 1) by progressively increasing $M$. When $M > 1$, with $M$ large, in addition to consistently estimate the (conditional) asymptotic variance of realized volatility, we will see shortly (see Section 5.1) that one can also reduce the local Gaussian bootstrap error in estimating $P\left(\left|\left(\hat{V}_{\bar{\sigma}^2,h,1}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)\right| \le x\right)$ up to $o_P\left(h\right)$. More specifically, if the block length $M \to \infty$, as $h \to 0$ such that $Mh \to 0$, then the local Gaussian bootstrap is an improvement of the best existing bootstrap methods of Gonçalves and Meddahi (2009). However, note that in contrast to Gonçalves and Meddahi (2009), our new bootstrap method requires the choice of an additional tuning parameter, specifically the block size $M$.

# 4 Results for realized regression

Our aim in this section is to study the first-order asymptotic properties of the local Gaussian bootstrap method in multivariate setting. In particular, we investigate the validity of our approach for the estimator of the slope coefficient in a linear-time regression model for the continuous martingale parts of two semimartingales observed over the fixed time interval $[0,1]$ studied in Barndorff-Nielsen and Shephard (2004a). To simplify the discussion, in the following, we consider the bivariate case where $d = 2$ and look at results for assets $k$ and $l$, whose $i$th high-frequency returns are written as $y_{ki}$ and $y_{li}$, respectively.

Let $\hat{C}_{(j)} \equiv \begin{pmatrix} \hat{C}_{kk(j)} & 0 \\ \hat{C}_{lk(j)} & \hat{C}_{ll(j)} \end{pmatrix} = \begin{pmatrix} \sqrt{\hat{\Gamma}_{kk(j)}} & 0 \\ \frac{\hat{\Gamma}_{kl(j)}}{\sqrt{\hat{\Gamma}_{kk(j)}}} & \sqrt{\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}^2_{kl(j)}}{\hat{\Gamma}_{kk(j)}}} \end{pmatrix}$ be the Cholesky decomposition of $\frac{1}{Mh}\sum_{i=1}^{M} y_{i+(j-1)M} y'_{i+(j-1)M}$. Given (4) with $d = 2$, for $j = 1, \ldots, 1/Mh$, and $i = 1, \ldots, M$,

$$\begin{pmatrix} y^*_{k,i+(j-1)M} \\ y^*_{l,i+(j-1)M} \end{pmatrix} = \sqrt{h} \begin{pmatrix} \sqrt{\hat{\Gamma}_{kk(j)}}\eta^*_{k,i+(j-1)M} \\ \frac{\hat{\Gamma}_{kl(j)}}{\sqrt{\hat{\Gamma}_{kk(j)}}}\eta^*_{k,i+(j-1)M} + \sqrt{\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}^2_{kl(j)}}{\hat{\Gamma}_{kk(j)}}}\eta^*_{l,i+(j-1)M} \end{pmatrix}, \tag{18}$$

where

$$\hat{\Gamma}_{kk(j)} = \frac{1}{Mh}\sum_{i=1}^{M} y^2_{k,i+(j-1)M},\ \hat{\Gamma}_{ll(j)} = \frac{1}{Mh}\sum_{i=1}^{M} y^2_{l,i+(j-1)M},\ \hat{\Gamma}_{kl(j)} = \frac{1}{Mh}\sum_{i=1}^{M} y_{k,i+(j-1)M} y_{l,i+(j-1)M},$$

and $\begin{pmatrix} \eta^*_{k,i+(j-1)M} \\ \eta^*_{l,i+(j-1)M} \end{pmatrix} \sim i.i.d.N(0, I_2)$, with $I_2$ a $2 \times 2$ identity matrix.

## 4.1 Barndorff-Nielsen and Shephard's (2004a) type estimator

Under (1) and given our assumptions on $\mu$ and $\sigma$ (Barndorff-Nielsen and Shephard (2004a) (cf. equations (25) and (26)) in conjunction with, e.g., Jacod and Protter (2012)), we have that

$$S_h = \sqrt{h^{-1}}\left( vech\left(\sum_{i=1}^{1/h} y_i y_i'\right) - vech\left(\Gamma\right)\right) = \sqrt{h^{-1}} \begin{pmatrix} \sum_{i=1}^{1/h} y^2_{k,i} - \int_0^1 \Sigma_{kk,s} ds \\ \sum_{i=1}^{1/h} y_{k,i} y_{l,i} - \int_0^1 \Sigma_{kl,s} ds \\ \sum_{i=1}^{1/h} y^2_{l,i} - \int_0^1 \Sigma_{ll,s} ds \end{pmatrix} \xrightarrow{st} N\left(0, \Pi\right), \tag{19}$$

and

$$\Pi=\begin{pmatrix} 2\int_0^1 \Sigma_{kk,s}^2 ds & 2\int_0^1 \Sigma_{kk,s}\Sigma_{kl,s} ds & 2\int_0^1 \Sigma_{kl,s}^2 ds \\ 2\int_0^1 \Sigma_{kk,s}\Sigma_{kl,s} ds & \int_0^1 \Sigma_{kk,s}\Sigma_{ll,s} ds+\int_0^1 \Sigma_{kl,s}^2 ds & 2\int_0^1 \Sigma_{ll,s}\Sigma_{kl,s} ds \\ 2\int_0^1 \Sigma_{kl,s}^2 ds & 2\int_0^1 \Sigma_{ll,s}\Sigma_{kl,s} ds & 2\int_0^1 \Sigma_{ll,s}^2 ds \end{pmatrix}. \tag{20}$$

Barndorff-Nielsen and Shephard (2004a) propose the following consistent estimator of $\Pi$ :

$$\hat{\Pi}_h=h^{-1}\sum_{i=1}^{1/h} x_i x_i' - \frac{1}{2}h^{-1}\sum_{i=1}^{1/h-1}\left(x_i x_{i+1}'+x_{i+1}x_i'\right),$$

where

$$x_i=\begin{pmatrix} y_{k,i}^2 & y_{k,i}y_{l,i} & y_{l,i}^2 \end{pmatrix}'.$$

Thus

$$T_h=\hat{\Pi}_h^{-1/2}S_h \xrightarrow{st} N\left(0, I_{\frac{d(d+1)}{2}}\right), \tag{21}$$

where here $\frac{d(d+1)}{2}=3$, since we let $d=2$. We consider a general class of nonlinear transformations that satisfy the following assumption. Throughout we let $\nabla f$, ($d\times 1$ vector-valued function) denote the gradient of $f$.

**Assumption F:** The function $f:\mathbb{R}^{\frac{d(d+1)}{2}}\to\mathbb{R}$ is continuously differentiable with $\nabla f\left(vech\left(\Gamma\right)\right)$ is non-zero for any sample path of $\Gamma$.

The corresponding statistics are defined as

$$S_{f,h}=\sqrt{h^{-1}}\left(f\left(vech\left(\sum_{i=1}^{1/h} y_i y_i'\right)\right)-f\left(vech\left(\Gamma\right)\right)\right) \text{ and } T_{f,h}=\left(\hat{\Pi}_{f,h}\right)^{-1/2} S_{f,h},$$

where

$$\hat{\Pi}_{f,h}\equiv\nabla' f\left(vech\left(\sum_{i=1}^{1/h} y_i y_i'\right)\right)\hat{\Pi}_h \nabla f\left(vech\left(\sum_{i=1}^{1/h} y_i y_i'\right)\right)\xrightarrow{P}\Pi_f. \tag{22}$$

In particular, we consider the question of estimating the integrated beta, i.e., the parameter

$$\beta_{lk}=\frac{\int_0^1 \Sigma_{kl,s} ds}{\int_0^1 \Sigma_{kk,s} ds}. \tag{23}$$

Given (21) and by using the delta method, the asymptotic distribution of the realized regression coefficient

$$\hat{\beta}_{lk}=\left(\sum_{i=1}^{1/h} y_{k,i}^2\right)^{-1}\sum_{i=1}^{1/h} y_{k,i}y_{l,i}, \tag{24}$$

obtained from regressing $y_{l,i}$ on $y_{k,i}$ is

$$T_{\beta,h}=\frac{S_{\beta,h}}{\sqrt{\hat{V}_\beta}}\xrightarrow{st} N\left(0,1\right), \tag{25}$$

where

$$S_{\beta,h} = \sqrt{h^{-1}}\left(\hat{\beta}_{lk} - \beta_{lk}\right), \quad \hat{V}_{\beta} = \left(\sum_{i=1}^{1/h} y_{k,i}^2\right)^{-2} h^{-1}\hat{g}_{\beta_{lk}} \text{ such that}$$

$$\hat{g}_{\beta_{lk}} = \sum_{i=1}^{1/h} x_{\beta i}^2 - \sum_{i=1}^{1/h-1} x_{\beta i}x_{\beta,i+1}, \text{ and } x_{\beta i} = y_{l,i}y_{k,i} - \hat{\beta}_{lk}y_{k,i}^2.$$

The local Gaussian bootstrap version of $S_{f,h}$ is given by

$$S_{f,h,M}^* = \sqrt{h^{-1}}\left(f\left(vech\left(\sum_{i=1}^{1/h} y_i^* y_i^{*\prime}\right)\right) - f\left(vech\left(\sum_{i=1}^{1/h} y_i y_i^{\prime}\right)\right)\right).$$

For the raw statistic

$$S_{f,h,M}^* = S_{h,M}^* = \sqrt{h^{-1}}\sum_{i=1}^{1/h}\left(x_i^* - x_i\right),$$

where

$$x_i^* = \begin{pmatrix} y_{k,i}^{*2} & y_{k,i}^* y_{l,i}^* & y_{l,i}^{*2} \end{pmatrix}^{\prime}.$$

It is elementary (though somewhat tedious) to show that

$$E^*\left(S_{h,M}^*\right) = 0,$$

and

$$\Pi_{h,M}^* \equiv Var^*\left(S_{h,M}^*\right) = Mh\sum_{j=1}^{1/Mh}\begin{pmatrix} 2\hat{\Gamma}_{kk(j)}^2 & 2\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & 2\hat{\Gamma}_{kl(j)}^2 \\ 2\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} + \hat{\Gamma}_{kl(j)}^2 & 2\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ 2\hat{\Gamma}_{kl(j)}^2 & 2\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} & 2\hat{\Gamma}_{ll(j)}^2 \end{pmatrix} \tag{26}$$

(see Lemma C2.6 in Hounyo (2018)). Similarly, we denote the bootstrap analogue of $T_{f,h}$, $\hat{\beta}_{lk}$, $S_{\beta,h}$, and $T_{\beta,h}$ by $T_{f,h,M}^*$, $\hat{\beta}_{lk}^*$, $S_{\beta,h,M}^*$, and $T_{\beta,h,M}^*$, respectively. In particular, we define

$$T_{f,h,M}^* \equiv \left(\hat{\Pi}_{f,h,M}^*\right)^{-1/2} S_{f,h.M}^*,$$

where

$$\hat{\Pi}_{f,h,M}^* \equiv \nabla^{\prime} f\left(vech\left(\sum_{i=1}^{1/h} y_i^* y_i^{*\prime}\right)\right)\hat{\Pi}_{h,M}^*\nabla f\left(vech\left(\sum_{i=1}^{1/h} y_i^* y_i^{*\prime}\right)\right),$$

with

$$\begin{aligned} \hat{\Pi}_{h,M}^* &= Mh\sum_{j=1}^{1/Mh}\begin{pmatrix} 2\hat{\Gamma}_{kk(j)}^{*2} & 2\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^* & 2\hat{\Gamma}_{kl(j)}^{*2} \\ 2\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^* & \hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{ll(j)}^* + \hat{\Gamma}_{kl(j)}^{*2} & 2\hat{\Gamma}_{kl(j)}^*\hat{\Gamma}_{ll(j)}^* \\ 2\hat{\Gamma}_{kl(j)}^{*2} & 2\hat{\Gamma}_{kl(j)}^*\hat{\Gamma}_{ll(j)}^* & 2\hat{\Gamma}_{ll(j)}^{*2} \end{pmatrix} \\ &\quad + \frac{1}{M}\cdot Mh\sum_{j=1}^{1/Mh}\begin{pmatrix} 4\hat{\Gamma}_{kk(j)}^2 & 4\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & 2\left(\hat{\Gamma}_{kl(j)}^2 + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}\right) \\ 4\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} + 3\hat{\Gamma}_{kl(j)}^2 & 4\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ 2\left(\hat{\Gamma}_{kl(j)}^2 + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}\right) & 4\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} & \frac{4}{M}\hat{\Gamma}_{ll(j)}^2 \end{pmatrix}, \end{aligned}$$

such that

$$\hat{\Gamma}^*_{kk(j)} = \frac{1}{Mh}\sum_{i=1}^{M} y^{*2}_{k,i+(j-1)M},\ \hat{\Gamma}^*_{ll(j)} = \frac{1}{Mh}\sum_{i=1}^{M} y^{*2}_{l,i+(j-1)M},\ \hat{\Gamma}^*_{kl(j)} = \frac{1}{Mh}\sum_{i=1}^{M} y^{*}_{k,i+(j-1)M} y^{*}_{l,i+(j-1)M}.$$

We also let

$$\hat{\beta}^*_{lk} = \left(\sum_{i=1}^{1/h} y^{*2}_{k,i}\right)^{-1} \sum_{i=1}^{1/h} y^*_{k,i} y^*_{l,i},$$

$$S^*_{\beta,h,M} = \sqrt{h^{-1}}\left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk}\right),\ \text{and } T^*_{\beta,h,M} = \frac{S^*_{\beta,h,M}}{\sqrt{\hat{V}^*_{\beta,h,M}}}\quad \text{such that } \hat{V}^*_{\beta,h,M} = \left(\sum_{i=1}^{1/h} y^{*2}_{k,i}\right)^{-2} h^{-1}\hat{g}^*_{\beta_{lk}},$$

with

$$\hat{g}^*_{\beta_{lk}} = \sum_{i=1}^{1/h} x^{*2}_{\beta i} - \sum_{i=1}^{1/h-1} x^*_{\beta i} x^*_{\beta,i+1},\ \text{and } x^*_{\beta i} = y^*_{l,i} y^*_{k,i} - \hat{\beta}^*_{lk} y^{*2}_{k,i}.$$

**Theorem 4.1.** *Suppose (1), (2) and (4) hold. Under Assumption F and the probability distribution $P$, if $M \to \infty$ as $h \to 0$ such that $M = o\left(h^{-1/2}\right)$, then as $h \to 0$, the following hold*

**(a)**

$$\Pi^*_{f,h,M} \equiv \nabla' f\left(vech\left(\sum_{i=1}^{1/h} y_i y_i'\right)\right)\Pi^*_{h,M}\nabla f\left(vech\left(\sum_{i=1}^{1/h} y_i y_i'\right)\right) \xrightarrow{P} \Pi_f,$$

*where* $\Pi_f = p\lim_{h\to 0} Var\left(S_{f,h}\right)$, *in particular*

$$Var^*\left(S^*_{\beta,h,M}\right) \xrightarrow{P} V_\beta,$$

*where* $V_\beta = p\lim_{h\to 0} Var\left(S_{\beta,h}\right) \equiv \left(\int_0^1 \Sigma_{kk,s} ds\right)^{-2} B$, *with*

$$B = \int_0^1 \left(\Sigma^2_{kl,s} + \Sigma_{ll,s}\Sigma_{kk,s} - 4\beta_{lk}\Sigma_{kl,s}\Sigma_{kk,s} + 2\beta^2_{lk}\Sigma^2_{kk,s}\right) ds.$$

**(b)**

$$\sup_{x\in\mathbb{R}}\left|P^*\left(S^*_{f,h,M} \le x\right) - P\left(S_{f,h} \le x\right)\right| \xrightarrow{P} 0,$$

*in particular*

$$\sup_{x\in\mathbb{R}}\left|P^*\left(S^*_{\beta,h,M} \le x\right) - P\left(S_{\beta,h} \le x\right)\right| \xrightarrow{P} 0.$$

**(c)**

$$\sup_{x\in\mathbb{R}}\left|P^*\left(T^*_{f,h,M} \le x\right) - P\left(T_{f,h} \le x\right)\right| \xrightarrow{P} 0,$$

*in particular*

$$\sup_{x\in\mathbb{R}}\left|P^*\left(T^*_{\beta,h,M} \le x\right) - P\left(T_{\beta,h} \le x\right)\right| \xrightarrow{P} 0.$$

Theorem 4.1 justifies using the local Gaussian bootstrap to estimate the distribution (and functionals of it such as the variance) of smooth function of realized covariance matrix $\sum_{i=1}^{1/h} y_i y_i'$, in particular the realized beta. This contrasts with results in Dovonon et al. (2013) (cf. Theorem 5.1) where the traditional pairs bootstrap is not able to mimic the score heterogeneity.

# 5 Higher-order properties

In this section, we investigate the asymptotic higher order properties of the bootstrap cumulants. Section 5.1 considers the case of the realized volatility whereas Section 5.2 considers the realized beta estimator as studied by Barndorff-Nielsen and Shephard (2004a). The ability of the bootstrap to accurately match the cumulants of the statistic of interest is a first step to showing that the bootstrap offers an asymptotic refinement.

The results in this section are derived under the assumption of zero drift and no leverage (i.e. $W$ is assumed independent of $\sigma$). As in Dovonon et al. (2013), a nonzero drift changes the expressions of the cumulants derived here. For instance, for the realized volatility, the effect of the drift on $T_h$ is $O_P\left(\sqrt{h}\right)$. While this effect is asymptotically negligible at first-order, it is not at higher orders. See also the recent work of Hounyo and Veliyev (2016) (cf. equation (3.8)) where an additional term shows up in the Edgeworth expansions for the realized volatility estimator when the drift $\mu_t$ is non zero. The no leverage assumption is mathematically convenient as it allows us to condition on the path of volatility when computing the cumulants of our statistics. Allowing for leverage is a difficult but promising extension of the results derived here.

We introduce some notation. For any statistics $T_h$ and $T_h^*$, we write $\kappa_j\left(T_h\right)$ to denote the $j^{th}$ order cumulant of $T_h$ and $\kappa_j^*\left(T_h^*\right)$ to denote the corresponding bootstrap cumulant. For $j = 1$ and 3, $\kappa_j$ denotes the coefficient of the terms of order $O\left(\sqrt{h}\right)$ of the asymptotic expansion of $\kappa_j\left(T_h\right)$, whereas for $j = 2$ and 4, $\kappa_j$ denotes the coefficients of the terms of order $O\left(h\right)$. The bootstrap coefficients $\kappa_{j,h,M}^*$ are defined similarly.

## 5.1 Higher order cumulants of realized volatility

Here, we focus on the $t$-statistic $T_{\bar{\sigma^2},h}$ and the bootstrap $t$-statistic $T^*_{\bar{\sigma^2},h,M}$ defined by

$$T_{\bar{\sigma^2},h} = \frac{\sqrt{h^{-1}}\left(R_2 - \mu_2 \overline{\sigma^2}\right)}{\sqrt{\hat{V}_{\bar{\sigma^2},h}}}, \tag{27}$$

where $\hat{V}_{\bar{\sigma^2},h} = \frac{\left(\mu_4 - \mu_2^2\right)}{\mu_4} h^{-1} \sum_{i=1}^{1/h} y_i^4$, and

$$T^*_{\bar{\sigma^2},h,M} = \frac{\sqrt{h^{-1}}\left(R_2^* - \mu_2 R_2\right)}{\sqrt{\hat{V}^*_{\bar{\sigma^2},h,M}}}, \tag{28}$$

where

$$\hat{V}^*_{\bar{\sigma^2},h,M} = \frac{\left(\mu_4 - \mu_2^2\right)}{\mu_4} h^{-1} \sum_{i=1}^{h^{-1}} y_i^{*4} = \frac{\left(\mu_4 - \mu_2^2\right)}{\mu_4} Mh \sum_{i=1}^{1/Mh} \left(\frac{RV_{j,M}}{Mh}\right)^2 \frac{1}{M} \sum_{i=1}^{M} \left(\eta^{*4}_{i+(j-1)M} - \mu_4\right),$$

such that $\eta^*_{i+(j-1)M} \sim i.i.d.N(0,1)$, $\mu_q = E\left|\eta^*\right|^q$ for $q > 0$. Let $\sigma_{q,p} \equiv \frac{\overline{\sigma^q}}{\left(\overline{\sigma^p}\right)^{q/p}}$, for any $q, p > 0$, and $R_{q,p} \equiv \frac{R_q}{(R_p)^{q/p}}$. We make the following assumption.

**Assumption H.** The log price process follows (1) with $\mu_t = 0$ and $\sigma_t$ is independent of $W_t$, where the volatility $\sigma$ is a càdlàg process, bounded away from zero, and satisfies the following regularity condition:

$$\lim_{h \to 0} h^{1/2} \sum_{i=1}^{1/h} \left|\sigma^r_{\eta_i} - \sigma^r_{\xi_i}\right| = 0,$$

for some $r > 0$ and for any $\eta_i$ and $\xi_i$ such that $0 \le \xi_1 \le \eta_1 \le h \le \xi_2 \le \eta_2 \le 2h \le \ldots \le \xi_{1/h} \le \eta_{1/h} \le 1$.

Assumption H is stronger than required to prove the CLT for $R_2$ and the first-order validity of the local Gaussian bootstrap, but it is a convenient assumption to derive the cumulants expansions of $T_{\bar{\sigma^2},h}$ and $T^*_{\bar{\sigma^2},h,M}$. We note that Assumption H was already used in Barndorff-Nielsen and Shephard (2003, 2004b) and Gonçalves and Meddahi (2009). Specifically, under Assumption H, Barndorff-Nielsen and Shephard (2004b) show that for any $q > 0$, $\overline{\sigma^q_h} - \overline{\sigma^q} = o(\sqrt{h})$, where $\overline{\sigma^q_h} = h^{1-q/2} \sum_{s=1}^{1/h} \left(\int_{(s-1)h}^{sh} \sigma_u^2 du\right)^{q/2}$, a result on which we subsequently rely on to establish the cumulants expansion of $T_{\bar{\sigma^2},h}$.

The following result gives the expressions of $\kappa_j$ and $\kappa^*_{j,h,M}$ for $j = 1, 2, 3$ and 4. We need to introduce some notation. Let

$$\begin{aligned}
A_1 &= \frac{\mu_6 - \mu_2\mu_4}{\mu_4\left(\mu_4 - \mu_2^2\right)^{1/2}} = 2\sqrt{2}, \\
B_1 &= \frac{\left(\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3\right)}{\left(\mu_4 - \mu_2^2\right)^{3/2}} = 2\sqrt{2}, \\
A_2 &= \frac{\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + 2\mu_2^2\mu_4}{\mu_4\left(\mu_4 - \mu_2^2\right)} = 12, \\
B_2 &= \frac{\mu_8 - 4\mu_2\mu_6 + 12\mu_2^2\mu_4 - 6\mu_2^4 - 3\mu_4^2}{\left(\mu_4 - \mu_2^2\right)^2} = 12, \\
C_1 &= \frac{\mu_8 - \mu_4^2}{\mu_4^2} = \frac{32}{3}.
\end{aligned}$$

**Theorem 5.1.** *Suppose (1) and (2) hold with $\mu = 0$ and $W$ independent of $\sigma$. Furthermore assume (4). Under Assumption H, conditionally on $\sigma$ and under $P$, it follows that, as $h \to 0$*

**(a)**

$$\begin{aligned}
\kappa_1\left(T_{\bar{\sigma}^2,h}\right) &= \sqrt{h}\kappa_1 + o(h), \ \textit{with } \kappa_1 = -\frac{A_1}{2}\sigma_{6,4}, \\
\kappa_2\left(T_{\bar{\sigma}^2,h}\right) &= 1 + h\kappa_2 + o(h), \ \textit{with } \kappa_2 = (C_1 - A_2)\sigma_{8,4} + \frac{7}{4}A_1^2\sigma_{6,4}^2, \\
\kappa_3\left(T_{\bar{\sigma}^2,h}\right) &= \sqrt{h}\kappa_3 + o(h), \ \textit{with } \kappa_3 = (B_1 - 3A_1)\sigma_{6,4}, \\
\kappa_4\left(T_{\bar{\sigma}^2,h}\right) &= h\kappa_4 + o(h), \ \textit{with } \kappa_4 = (B_2 + 3C_1 - 6A_2)\sigma_{8,4} + \left(18A_1^2 - 6A_1B_1\right)\sigma_{6,4}^2.
\end{aligned}$$

**(b)**

$$\begin{aligned}
\kappa_1^*\left(T^*_{\bar{\sigma}^2,h,M}\right) &= \sqrt{h}\kappa^*_{1,h,M} + o_P(h), \ \textit{with } \kappa^*_{1,h,M} = -\frac{A_1}{2}R_{6,4}, \\
\kappa_2^*\left(T^*_{\bar{\sigma}^2,h,M}\right) &= 1 + h\kappa^*_{2,h,M} + o_P(h), \ \textit{with } \kappa^*_{2,h,M} = (C_1 - A_2)R_{8,4} + \frac{7}{4}A_1^2R_{6,4}^2, \\
\kappa_3^*\left(T^*_{\bar{\sigma}^2,h,M}\right) &= \sqrt{h}\kappa^*_{3,h,M} + o_P(h), \ \textit{with } \kappa^*_{3,h,M} = (B_1 - 3A_1)R_{6,4}, \\
\kappa_4^*\left(T^*_{\bar{\sigma}^2,h,M}\right) &= h\kappa^*_{4,h,M} + o_P(h), \ \textit{with } \kappa^*_{4,h,M} = (B_2 + 3C_1 - 6A_2)R_{8,4} + \left(18A_1^2 - 6A_1B_1\right)R_{6,4}^2.
\end{aligned}$$

**(c)** *For* $j = 1, 2, 3$ *and* 4, $\underset{h\to 0}{p\lim}\kappa^*_{j,h,M} - \kappa_j$ *is nonzero if* $M$ *is finite and it is zero if* $M \to \infty$ *as* $h \to 0$ *such that* $M = o\left(h^{-1/2}\right)$.

Theorem 5.1 states our main findings for realized volatility. Part (a) of Theorem 5.1 is well known in the literature (see e.g., Gonçalves and Meddahi (2009) (cf. Theorem A.1) and Hounyo and Veliyev (2016)). These results are only given here for completeness. The remaining results in parts (b) and (c) are new. Part (b) gives the corresponding results for the local Gaussian bootstrap. Part (c) shows that the cumulants of $T_{\bar{\sigma}^2,h}$ and $T^*_{\bar{\sigma}^2,h,M}$ do not agree up to $o\left(h^{1/2}\right)$ when the block size $M$ is fixed (although they are consistent), implying that the bootstrap does not provide a higher-order asymptotic refinement for finite values of $M$. The main reason why the local Gaussian bootstrap is not able to match cumulants up to $o\left(h^{1/2}\right)$ when $M$ is finite is that $\underset{h\to 0}{p\lim}R_{q,p} - \sigma_{q,p}$ does not always equal to zero. See Remark 1 for further details. Nevertheless, when $M \to \infty$ as $h \to 0$ such that $Mh \to 0$, $\underset{h\to 0}{p\lim}R_{q,p} - \sigma_{q,p} = 0$, then the bootstrap matches the first and third order cumulants through order $O\left(h^{1/2}\right)$, which implies that it provides a second-order refinement, i.e. the bootstrap distribution $P^*\left(T^*_{\bar{\sigma}^2,h,M} \leq x\right)$ consistently estimates $P\left(T_{\bar{\sigma}^2,h} \leq x\right)$ with an error that vanishes as $o\left(h^{1/2}\right)$ (assuming the corresponding Edgeworth expansions exist).[4] This is in contrast with the first-order asymptotic Gaussian distribution whose error converges as $O\left(h^{-1/2}\right)$. Note that Gonçalves and Meddahi (2009) also proposed a choice of the external random variable for their wild bootstrap

[4]Recently, Hounyo and Veliyev (2016) rigorously justify the Edgeworth expansions for realized volatility derived by Gonçalves and Meddahi (2009). As a consequence the cumulants expansions of $T_{\bar{\sigma}^2,h}$ exist under our assumptions. Our focus in parts (b) and (c) of Theorem 5.1 is on using formal expansions to explain the superior finite sample properties of the new local Gaussian bootstrap theoretically (see e.g., Mammen (1993), Davidson and Flachaire (2001) and Gonçalves and Meddahi (2009) for a similar approach). This approach does not seek to determine the conditions under which the relevant expansions are valid.

method which delivers second-order refinements. Our results for the bootstrap method based on the local Gaussianity are new. We will compare the two methods in the simulation section.

Theorem 5.1 also shows that the new bootstrap method we propose is able to match the second and fourth order cumulants of $T_{\bar{\sigma^2},h}$ through order $O(h)$ when $M \to \infty$ as $h \to 0$. These results imply that the bootstrap distribution of $\left|T^*_{\bar{\sigma^2},h,M}\right|$ consistently estimate the distribution of $\left|T_{\bar{\sigma^2},h}\right|$ through order $o(h)$, in which case the bootstrap offers a third-order asymptotic refinement (this again assumes that the corresponding Edgeworth expansions of $T^*_{\bar{\sigma^2},h,M}$ exist, something we have not attempted to prove in this paper). If this is the case, then the local Gaussian bootstrap will deliver symmetric percentile-$t$ intervals for integrated volatility with coverage probabilities that converge to zero at the rate $o(h)$. In contrast, the coverage probability implied by the asymptotic theory-based intervals converge to the desired nominal level at the rate $O(h)$.

In the following remark, we rely on the formal second-order Edgeworth expansions of $P\left(T_{\bar{\sigma^2},h} \leq x\right)$ and $P^*\left(T^*_{\bar{\sigma^2},h,M} \leq x\right)$ to gain further insight into the property of the local Gaussian bootstrap for finite values of $M$.

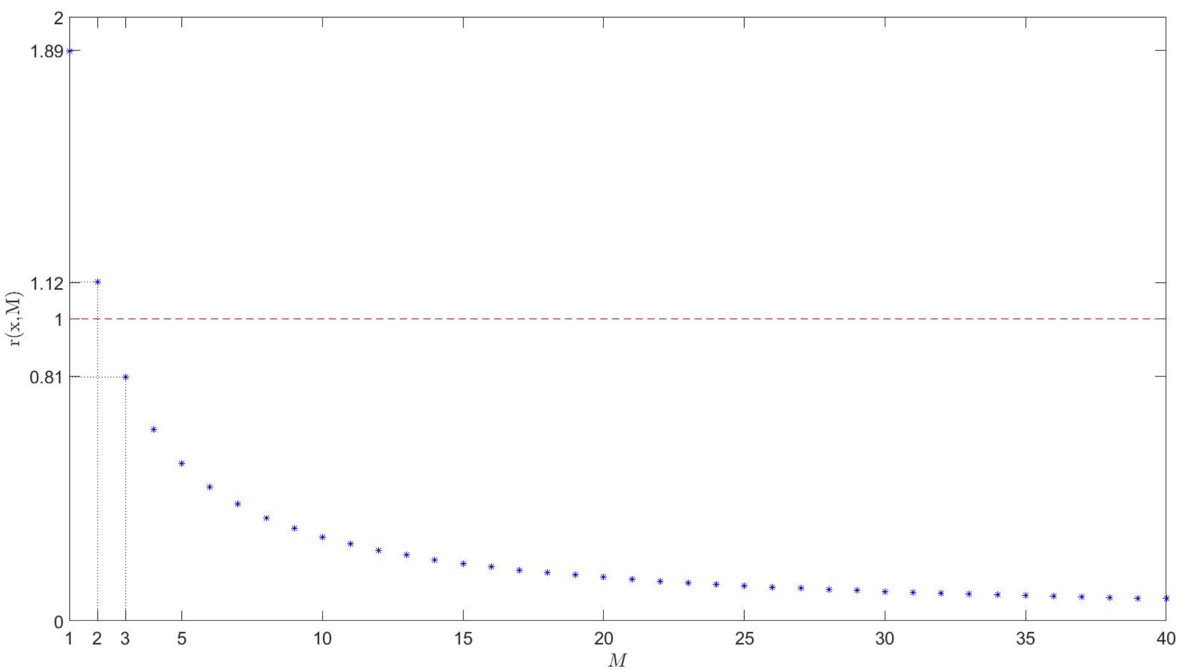


Figure 1: The asymptotic relative bootstrap error for the realized volatility estimator.

**Remark 1.** *Under the same conditions as in Theorem 5.1, we have*

$$P^*\left(T^*_{\bar{\sigma^2},h,M} \leq x\right) - P\left(T_{\bar{\sigma^2},h} \leq x\right) = \sqrt{h}\left[p\lim_{h\to 0} q^*_{h,M}(x) - q(x)\right]\phi(x) + o_P\left(\sqrt{h}\right),$$

*where* $q(x) = -\left(\kappa_1 + \frac{1}{6}\kappa_3\left(x^2-1\right)\right)$, $q^*_{h,M}(x) = -\left(\kappa^*_{1,h,M} + \frac{1}{6}\kappa^*_{3,h,M}\left(x^2-1\right)\right)$ *and* $\phi(\cdot)$ *is the density* *(with respect to the Lebesgue measure) of the standard normal distribution on $\mathbb{R}$. Parts (a) and (b) of Theorem 5.1 imply that, when $M$ is finite*

$$p\lim_{h\to 0} q_{h,M}^{*}(x) - q(x) = \left(\frac{c_{M,6}}{(c_{M,4})^{3/2}} - 1\right) q(x) = \left(\frac{M+4}{\sqrt{M(M+2)}} - 1\right) q(x),$$

*showing that the leading term of the bootstrap error is $\sqrt{h}\left(\frac{M+4}{\sqrt{M(M+2)}} - 1\right) q(x)\phi(x)$. Instead, the error of the normal approximation is equal to $\sqrt{h}q(x)\phi(x) = O\left(h^{1/2}\right)$. Since both errors are of the same order of magnitude the bootstrap does not offer a second-order refinement when $M$ is finite. However, in such a situation we can use for instance the asymptotic relative bootstrap error as a criterion of comparison (see e.g., Shao and Tu (1995) (cf. Section 3.3)). For any $x \in \mathbb{R}$, and $M \geq 1$ we approximate the asymptotic relative local Gaussian bootstrap error by*

$$r(x,M) = \left|\frac{p\lim_{h\to 0} q_{h,M}^{*}(x) - q(x)}{q(x)}\right| = \left|\frac{M+4}{\sqrt{M(M+2)}} - 1\right|.$$

*As a result, under the same conditions as in Theorem 5.1, $r(x,M)$ is a function that depends only on $M$, but not on $x$ and $\sigma$. It turns out that for any fixed $M \geq 3$, $\sup_{x\in\mathbb{R}} |r(x,M)| \leq 1$. Thus, $r(z_\alpha, M) \leq 1$ for values of $z_\alpha \in \mathbb{R}$, where $z_\alpha$ is the $\alpha$ quantile of a standard normal distribution (for instance, when $\alpha = 0.05$, $z_\alpha = 1.645$), showing that the bootstrap critical values are at least as accurate as those of the standard normal approximation. In addition, for any $x \in \mathbb{R}$, and $M \geq 1$, $r(x,M)$ is a strictly decreasing function of $M$, implying that increasing the block size $M$ decreases the asymptotic relative bootstrap error uniformly over $x$ (see Figure 1). This proves that the local Gaussian bootstrap improves upon the performance of the bootstrap method of Gonçalves and Meddahi (2009) (which amounts to $M = 1$).*

**Remark 2.** *We would like to highlight that results in Theorem 5.1 are stated for the t-statistics $T_{\bar{\sigma^2},h}$ and $T_{\bar{\sigma^2},h,M}^{*}$ (see (27) and (28)) but not for the t-statistic defined in (11) and its bootstrap analogue. However, under the same conditions as in Theorem 5.1 but strengthened by piecewise constant assumption on the volatility process $\sigma_t$, it is easy to see that analogue results as in Theorem 5.1 hold true for $\left(\hat{V}_{\bar{\sigma^2},h,M}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)$ and $\left(\hat{V}_{\bar{\sigma^2},h,M}^{*MZ}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2^{*} - R_2\right)$ under $P$. More specifically, if the volatility process $\sigma_t$ is invertible and satisfies (5) for any sample path of $\sigma$, then we can deduce that, when $M \to \infty$ as $h \to 0$, we also have*

$$P^{*}\left(\left(\hat{V}_{\bar{\sigma^2},h,M}^{*MZ}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2^{*} - R_2\right) \leq x\right) - P\left(\left(\hat{V}_{\bar{\sigma^2},h,M}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right) \leq x\right) = o_P\left(h^{1/2}\right),$$

$$P^{*}\left(\left|\left(\hat{V}_{\bar{\sigma^2},h,M}^{*MZ}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2^{*} - R_2\right)\right| \leq x\right) - P\left(\left|\left(\hat{V}_{\bar{\sigma^2},h,M}\right)^{-1/2}\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)\right| \leq x\right) = o_P(h),$$

*under $P$. The proof follows exactly the same line as the proof of Theorem 5.1. For reasons of space we leave the details for the reader. When $M \to \infty$ condition (5) amounts almost to say that the volatility is constant ($\sigma_t = \sigma$).*

The potential for the local Gaussian bootstrap intervals to yield third-order asymptotic refinements is particularly interesting because Gonçalves and Meddahi (2009) show that their wild bootstrap method is not able to deliver such refinements. Thus, our method is an improvement not only of the Gaussian asymptotic distribution but also of the best existing bootstrap methods for realized volatility in the context of no microstructure effects (where prices are observed without any error).

## 5.2 Higher order cumulants of realized beta

In this section, we provide the first and third order cumulants of realized beta estimator given in (24). These cumulants enter the Edgeworth expansions of the one-sided distribution functions of $T_{\beta,h}$ and $T^*_{\beta,h,M}$, i.e., $P^*\left(T^*_{\beta,h,M} \leq x\right)$ and $P\left(T_{\beta,h} \leq x\right)$, respectively. To describe the Edgeworth expansions, we need to introduce additional notation. Let

$$
\begin{aligned}
\tilde{A}_0 &= \int_0^1 \left(\Sigma_{kk,s}\Sigma_{kl,s} + \beta_{lk}\Sigma^2_{kk,s}\right) ds, \\
\tilde{A}_1 &= \int_0^1 \left(2\Sigma^3_{kl,s} + 6\Sigma_{kk,s}\Sigma_{kl,s}\Sigma_{ll,s} - 18\beta_{lk}\Sigma_{kk,s}\Sigma^2_{kl,s} - 6\beta_{lk}\Sigma^2_{kk,s}\Sigma_{ll,s} + 24\beta^2_{lk}\Sigma^2_{kk,s}\Sigma_{kl,s} - 8\beta^3_{lk}\Sigma^3_{kk,s}\right) ds, \\
\tilde{B} &= \int_0^1 \left(\Sigma^2_{kl,s} + \Sigma_{kk,s}\Sigma_{ll,s} - 4\beta_{lk}\Sigma_{kk,s}\Sigma_{kl,s} + 2\beta^2_{lk}\Sigma^2_{kk,s}\right) ds, \\
\tilde{H}_1 &= \frac{4\tilde{A}_0}{\Gamma_{kk}\sqrt{\tilde{B}}} \text{ and } \tilde{H}_2 = \frac{\tilde{A}_1}{\tilde{B}^{3/2}}.
\end{aligned}
$$

Similarly we let

$$
\begin{aligned}
\tilde{A}^*_{0,h,M} &= Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} - \hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}, \\
\tilde{A}^*_{1,h,M} &= Mh\sum_{j=1}^{1/Mh}\left(\begin{array}{c} 2\hat{\Gamma}^3_{kl(j)} + 6\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} - 18\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)} \\ -6\hat{\beta}_{lk}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)} + 24\hat{\beta}^2_{lk}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)} - 8\hat{\beta}^3_{lk}\hat{\Gamma}^3_{kk(j)} \end{array}\right), \\
\tilde{B}^*_{h,M} &= Mh\sum_{j=1}^{1/Mh}\left(\hat{\Gamma}^2_{kl(j)} + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - 4\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} + 2\hat{\beta}^2_{lk}\hat{\Gamma}^2_{kk(j)}\right), \\
\tilde{R}^*_{1,h,M} &= -\frac{1}{\tilde{B}^{*3/2}_{h,M}}\left[\begin{array}{c} 3Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)} + 5Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ -19\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)} - 5\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)} \\ +24\hat{\beta}^2_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)} - 8\hat{\beta}^3_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)} \end{array}\right], \\
\tilde{H}^*_{1,h,M} &= \frac{4\tilde{A}^*_{0,h,M}}{\hat{\Gamma}_{kk}\sqrt{\tilde{B}^*_{h,M}}} \text{ and } \tilde{H}_{2,h,M} = \frac{\tilde{A}^*_{1,h,M}}{\tilde{B}^{*3/2}_{h,M}}, \text{ where } \hat{\Gamma}_{kk} = \sum_{i=1}^{1/h} y^2_{k,i}\ .
\end{aligned}
$$

**Theorem 5.2.** *Suppose (1) and (2) hold with $\mu = 0$ and $W$ independent of $\sigma$. Furthermore assume (4). Under Assumption H, conditionally on $\sigma$ and under $P$, it follows that, as $h \to 0$*

**(a)**

$$\begin{aligned}\kappa_1\left(T_{\beta,h}\right) &= \sqrt{h}\kappa_1 + o\left(\sqrt{h}\right),\ \textit{with } \kappa_1 = \frac{1}{2}\left(\tilde{H}_1 - \tilde{H}_2\right),\\ \kappa_3\left(T_{\beta,h}\right) &= \sqrt{h}\kappa_3 + o\left(\sqrt{h}\right),\ \textit{with } \kappa_3 = 3\tilde{H}_1 - 2\tilde{H}_2,\end{aligned}$$

**(b)**

$$\begin{aligned}\kappa_1^*\left(T^*_{\beta,h,M}\right) &= \sqrt{h}\kappa^*_{1,h,M} + o_P\left(\sqrt{h}\right),\ \textit{with } \kappa^*_{1,h,M} = \frac{1}{2}\left(\tilde{H}^*_{1,h,M} - \tilde{H}^*_{2,h,M}\right) + \frac{\tilde{R}^*_{1,h,M}}{M},\\ \kappa_3^*\left(T^*_{\beta,h,M}\right) &= \sqrt{h}\kappa^*_{3,h,M} + o_P\left(\sqrt{h}\right),\ \textit{with } \kappa^*_{3,h,M} = 3\tilde{H}^*_{1,h,M} - 2\tilde{H}^*_{2,h,M} + o_P\left(\frac{1}{M}\right),\end{aligned}$$

**(c)** *For $j = 1$ and 3, $\underset{h\to 0}{p\lim}\kappa^*_{j,h,M} - \kappa_j$ is nonzero if $M$ is finite and it is zero if $M \to \infty$ as $h \to 0$ such that $M = o\left(h^{-1/2}\right)$.*

Theorem 5.2 shows that the cumulants of $T_{\beta,h,M}$ and $T^*_{\beta,h,M}$ agree through order $O\left(\sqrt{h}\right)$, which implies that the error of the bootstrap approximation $P^*\left(T^*_{\beta,h,M} \leq x\right)$ to the distribution of $T_{\beta,h,M}$ is of order $o\left(\sqrt{h}\right)$. Since the normal approximation has an error of the order $O\left(\sqrt{h}\right)$, this implies that the local Gaussian bootstrap is second-order correct. This result is an improvement over the bootstrap results in Dovonon, Gonçalves and Meddahi (2013), who showed that the pairs bootstrap is not second-order correct in the general case of stochastic volatility. Thus, for the realized beta, the local Gaussian bootstrap is able to replicate the first and third cumulants of $\hat{\beta}_{lk}$ through order $o\left(\sqrt{h}\right)$ when $M \to \infty$. We conjecture that similar analysis at higher-order (possibly third-order asymptotic refinement) holds for the realized beta estimator, but a full exploration of this is left for future research.

# 6 Mykland and Zhang's (2009) local Gaussianity framework revisited

In this section, we discuss the connection between the local Gaussian bootstrap and the local Gaussianity approximation of continuous semimartingales observed at high-frequency established by Mykland and Zhang (2009) and show the suitability of our bootstrap method to deal with the new class of estimators introduced in that paper.

## 6.1 Statistical risk neutral measure

Suppose (1) and (2). As equation (3) shows, the intraday returns $y_i$ depend on the drift $\mu$, unfortunately when carrying out inference for observations in a fixed time interval the process $\mu_t$ cannot be consistently estimated. For most purposes it is only a nuisance parameter. To deal with this, Mykland and Zhang (2009) propose to work with a new probability measure $Q$ which is measure-theoretic equivalent to $P$ and under which there is no drift (a statistical risk neutral measure). Specifically, we have that under $Q$, (cf. equations (7), (8), and (9) in Mykland and Zhang (2009))

$$dX_t = \sigma_t dW_t^Q,\ X_0 = x_0,$$

where $W_t^Q$ is a $Q$-Brownian motion such that

$$log\frac{dQ}{dP} = -\int_0^1 \sigma_t^{-1}\mu_t dW_t - \frac{1}{2}\int_{0t}^1 \mu_t' \left(\sigma_t\sigma_t'\right)^{-1}\mu_t dt$$

with

$$dW_t^Q = dW_t + \sigma_t^{-1}\mu_t dt.$$

Note that the process $X_t$ does not change from $P$ to $Q$, but its distribution changes. Mykland and Zhang (2009) pursue the analysis further and use the measure change idea but defined on the discretized observations $X_{ih}$ *only*, for which the volatility is constant on each of the $\frac{1}{Mh}$ non overlapping blocks of size $M$. Then $M$ is the number of high-frequency returns within a block so $M \leq \frac{1}{h}$. Specifically, consider the approximate measure termed $Q_{h,M}$, satisfying $X_0 = x_0$ and for each $j = 1, \ldots, \frac{1}{Mh}$, and for $i = 1, \ldots, M$,

$$y_{i+(j-1)Mh} = \sigma_{(j-1)Mh}\left(W_{(i-1)h+(j-1)Mh}^{Q_{h,M}} - W_{ih+(j-1)Mh}^{Q_{h,M}}\right), \tag{29}$$

where $\sigma_{(j-1)Mh}$ is the value of $\sigma_t$ at the beginning of the $j$-th block, the random variables $W_{(i-1)h+(j-1)Mh}^{Q_{h,M}} - W_{ih+(j-1)Mh}^{Q_{h,M}}$ are independent across $(i,j)$ for fixed $h$, and $W_{(i-1)h+(j-1)Mh}^{Q_{h,M}} - W_{ih+(j-1)Mh}^{Q_{h,M}}$ is conditionally normal with mean zero and variance $hI_d$, where $I_d$ is a $d \times d$ identity matrix. Notice that the process $W_t^{Q_{h,M}}$ is characterized on the sampling times *only*. It may not exist in continuous time or for other $t$ than the sampling times. To formally define $Q_{h,M}$, we go through two definitions (cf. Definitions 3 and 5 in Mykland and Zhang (2009)). In particular, we first specify the time discrete process subject to measure change.

**Definition 1.** *Let*

$$\begin{aligned} U_{ih}^{(1)} &= X_{ih}, \\ U_{ih}^{(2)} &= \left(\sigma_{ih}, \frac{d}{dt}\left[\sigma, W\right]_{ih}, \frac{d}{dt}\left[\sigma,\sigma\right]_{ih}\right), \\ U_{ih} &= \left(U_{ih}^{(1)}, U_{ih}^{(2)}\right), \end{aligned}$$

*for* $i = 0, \ldots, 1/h$. *Here, the quantity* $\frac{d}{dt}\left[\sigma, W\right]_t$ *is a three-dimensional* $(d \times d \times d)$ *object (tensor) consisting of elements* $\frac{d}{dt}\left[\sigma^{(s_1,s_2)}, W^{(s_3)}\right]_t$, $(s_1 = 1, \ldots, d, s_2 = 1, \ldots, d, s_3 = 1, \ldots, d)$. *Similarly,* $\frac{d}{dt}\left[\sigma,\sigma\right]_t$ *is a four-dimensional tensor with elements of the form* $\frac{d}{dt}\left[\sigma^{(s_1,s_2)}, \sigma^{(s_3,s_4)}\right]_t$.

Note that $\frac{d}{dt}\left[\sigma, W\right]_t$ and $\frac{d}{dt}\left[\sigma,\sigma\right]_t$ are derivatives of quadratic variations, but they are only observed at the time $t = ih$.

**Definition 2.** *Define the approximate measure* $Q_{h,M}$ *recursively as follows:*

**(i)** $U_0$ *has the same distribution under* $Q_{h,M}$ *as under* $Q$.

**(ii)** *For* $i \geq 0$, *the conditional* $Q_{h,M}$*-distribution of* $U_{(i+1)h}^{(1)}$ *given* $U_0, \ldots, U_{ih}$ *is given by (29), where* $W_{i(h-1)}^{Q_{h,M}} - W_{ih}^{Q_{h,M}}$ *is conditionally normal with mean zero and variance* $hI_d$, *and*

**(iii)** *For $i \geq 0$, the conditional $Q_{h,M}$-distribution of $U_{ih}^{(2)}$ given $U_0, \ldots, U_{ih}, U_{(i+1)h}^{(1)}$ is the same as under $Q$.*

It follows then that under $Q_{h,M}$, in each block $j = 1, \ldots, \frac{1}{Mh}$, we have,

$$y_{i+(j-1)Mh} = C_{(j)} \cdot \left(\sqrt{h}\eta_{i+(j-1)M}\right), \text{ for } 1 \leq i \leq M, \tag{30}$$

where $\eta_{i+(j-1)M} \sim i.i.d.N(0, I_d)$ and $C_{(j)} = \sigma_{(j-1)Mh}$.

The true distribution is $P$, but we may prefer to work with $Q_{h,M}$ since their calculations are much simpler (see e.g., Section 6.2 where we derive results under $Q_{h,M}$). Afterwards we have to adjust results back to $P$ using the likelihood ratio (Radon-Nikodym derivative) $dQ_{h,M}/dP$. We emphasize that the paper of Mykland and Zhang (2009) consider a much more general framework when defining the approximate measure $Q_{h,M}$, allowing observations times to be irregularly spaced (but nonrandom).

**Remark 3.** *As pointed out in Mykland and Zhang's (2009) Theorem 3, under (1) and (2) the measure $P$ and its approximation $Q_{h,M}$ are contiguous on the observables. This is to say that for any sequence $\mathcal{A}_{h,M}$ of sets, $P(\mathcal{A}_{h,M}) \to 0$ if and only if $Q_{h,M}(\mathcal{A}_{h,M}) \to 0$. In particular, if an estimator is consistent under $Q_{h,M}$, it is also consistent under $P$. Rates of convergence (typically $h^{-1/2}$) are also preserved, but the asymptotic distribution may change. More specifically, when adjusting from $Q_{h,M}$ to $P$, the asymptotic variance of the estimator is unchanged (due to the preservation of quadratic variation under limit operations), while the asymptotic bias may change. It appears that a given sequence $Z_{h,M}$ of martingales will have exactly the same asymptotic distribution under $Q_{h,M}$ and $P$, when the $Q_{h,M}$ martingale part of the log likelihood ratio $log(dP/dQ_{h,M})$ has zero asymptotic covariation with $Z_{h,M}$. In this case, we do not need to adjust the distributional result from $Q_{h,M}$ to $P$. An important example where this is true is Mykland and Zhang's (2009) realized beta-type estimator, which we will study in detail in Section 6.2.*

**Remark 4.** *In the particular case where the window length $M$ increases with the sample size $h^{-1}$ such that $M = o(h^{-1/2})$, there is also no contiguity adjustment (see Remark 2 of Mykland and Zhang (2011)). However, it is important to highlight that all results using contiguity arguments in Mykland and Zhang (2009) apply only to the case with a bounded $M$. For the case $M \to \infty$, Mykland and Zhang (2011) use a different representation than (30). It should be noted that throughout this paper (in particular, in Section 6.2 where we derive results under $Q_{h,M}$), we will focus* only *on the representation given in (30) (see Definition 2). As a consequence, we would not assume that the approximate measure $Q_{h,M}$ is contiguous to the measure $P$ when $M$ grows to infinity.*

It is easy to see that the idea of the local Gaussian bootstrap method introduced by (4) is closely related to the structure of intraday returns $y_i$ under the approximate measure $Q_{h,M}$ with representation given by (30). In particular, the local Gaussian bootstrap replaces $C_{(j)}$ by its estimate $\hat{C}_{(j)}$ and use the local Gaussianity inherent in the semimartingale continuous time model driving $X_t$ to generate the bootstrap observations. This relationship with Mykland and Zhang (2009), suggests that our local Gaussian bootstrap approach may be applied very generally, in particular, for the new class of statistics introduced in that paper. In Section 6.2., we establish the first-order asymptotic validity of the local Gaussian bootstrap for the realized beta estimator in the context of Mykland and Zhang's (2009) blocking approach. The full exploration of other block-based statistics introduced in Mykland and Zhang (2009) is beyond the scope of this paper.

It is informative to stress that the existence of a possible contiguity relation between $Q_{h,M}$ and $P$ is not a necessary condition to the validity of the local Gaussian bootstrap approach. As we have already shown in Sections 3 and 4, the local Gaussian bootstrap method remains asymptotically valid when $M \to \infty$, i.e., a setting where we do not know whether the approximate measure $Q_{h,M}$ (with representation given by (30)) is contiguous to $P$. In such situations, for the proof of the validity of the bootstrap the key aspect is that we have to proceed directly under the true probability measure $P$, without using any contiguity arguments.

The following result is crucial in obtaining our bootstrap results only in the context where those results are derived under the approximate measure $Q_{h,M}$ (which are the case of Theorem 6.2 and Theorem 6.3).

**Theorem 6.1.** *Let $Z^*_{h,M}$ be a sequence of bootstrap statistics. Given the probability measure $P$ and its approximation $Q_{h,M}$, we have that*

*$Z^*_{h,M} \to^{P^*} 0$, as $h \to 0$, in probability under $P$, if and only if $Z^*_{h,M} \to^{P^*} 0$, as $h \to 0$, in probability under $Q_{h,M}$.*

Theorem 6.1 provides a theoretical justification to derive bootstrap consistency results under the approximate measure $Q_{h,M}$ as well as under $P$. This may simplify the bootstrap inference. We will subsequently rely on this theorem to establish the bootstrap consistency results, when necessary (see e.g., Theorem 6.2 and Theorem 6.3).

### 6.2 New variance estimator of Mykland and Zhang's (2009) realized beta-type estimator and bootstrap consistency

We here study the realized beta in the context of Mykland and Zhang's (2009) blocking approach. Specifically, we complement the literature and provide a feasible asymptotic CLT result and a theoretical proof of the first-order validity of the local Gaussian bootstrap method for Mykland and Zhang's (2009) realized beta-type estimator. To obtain a feasible CLT result for the latter, we propose a consistent estimator of its asymptotic (conditional) variance.

In this section, to establish our results, we consider a setting much simpler than the general assumptions on the DGP in Section 4. More specifically, we use the approach of Dovonon et al. (2013) (cf. Section 5.1) and suppose that $\sigma$ is independent of $W$, which in particular excludes the leverage effect. Allowing dependence between $\sigma$ and $W$ in this section would complicate substantially our analysis. When $\sigma$ is independent of $W$, it is possible for us to condition on the path of $\sigma$ and then use the independence property of increments to compute the moments of cross products of high-frequency returns. In the presence of correlation between $\sigma$ and $W$, this approach breaks down, since we only have a martingale difference sequence instead of the independence property. Providing a consistent variance estimator of Mykland and Zhang's (2009) realized beta-type estimator in the presence of the leverage effect is an interesting question which we leave for future research. However, it is worth emphasizing that contrary to Dovonon et al. (2013), we do not need here to assume away the presence of drift, i.e., $\mu_t = 0$ for all $t$. The main reason is because we can directly proceed under the approximate measure $Q_{h,M}$ (we will see shortly how). In Dovonon et al. (2013), letting $\mu_t = 0$ is mathematically convenient to have mean zero high-frequency returns when computing the moments of cross products of high-frequency returns. The mean zero high-frequency returns property is obtained directly under the statistical risk neutral measure $Q_{h,M}$ without assuming that $\mu_t = 0$.

Before introducing our results for Mykland and Zhang's (2009) realized beta-type estimator, we further discuss the results in Dovonon et al. (2013). Under the assumption that $\mu_t = 0$ and $\sigma$ is independent of $W$, we can simply write a high-frequency return of a given asset as the product of its volatility (the spot volatility) multiplied by a normal standard distribution under the true probability measure $P$. Specifically, Dovonon et al. (2013) (cf. Section 5.1) assume that $dX_t = \sigma_t dW_t$, where $\sigma$ is independent of $W$. Then, following Dovonon et al. (2013) (cf. equation (5)), conditionally on $\sigma$, for a given frequency of the observations, we can write

$$y_{li} = \beta_{lki} y_{ki} + u_i, \tag{31}$$

where independently across $i = 1, \ldots, 1/h$, $u_i | y_{ki} \sim N(0, V_i)$, with $V_i \equiv \Gamma_{li} - \frac{\Gamma_{lki}^2}{\Gamma_{ki}}$, and $\beta_{lki} \equiv \frac{\Gamma_{lki}}{\Gamma_{ki}}$, with $\Gamma_{lki} = \int_{(i-1)h}^{ih} \Sigma_{lk}(u)\, du$. Notice that (31) only makes sense in discrete time.[5]

As Dovonon et al. (2013) argue, the conditional mean parameters of realized regression models are heterogeneous under stochastic volatility. This heterogeneity justifies why the pairs bootstrap method that they studied is not second-order accurate.

Here we consider the general stochastic volatility model described by (1), but we rule out the leverage effects. Given (30) (without assuming that $\mu_t = 0$) high-frequency returns have similar representation but under $Q_{h,M}$. It follows that under $Q_{h,M}$, $y_{k,i+(j-1)M} = C_{kk(j)}\sqrt{h}\eta_{k,i+(j-1)M}$ and $y_{l,i+(j-1)M} = C_{lk(j)}\sqrt{h}\eta_{k,i+(j-1)M} + C_{ll(j)}\sqrt{h}\eta_{l,i+(j-1)M}$, for $i = 1, \ldots, M$ and $j = 1, \ldots, \frac{1}{Mh}$ where

$$C_{(j)} \equiv \begin{pmatrix} C_{kk(j)} & 0 \\ C_{lk(j)} & C_{ll(j)} \end{pmatrix}, \; \eta_{i+(j-1)M} \equiv \begin{pmatrix} \eta_{k,i+(j-1)M} \\ \eta_{l,i+(j-1)M} \end{pmatrix} \sim i.i.d.N(0, I_2),$$

$I_2$ is a $2\times2$ identity matrix, $C_{(j)}$ is the Cholesky decomposition of $\Sigma_{(j)} \equiv C_{(j)}C'_{(j)}$ with $C_{(j)} = \sigma_{(j-1)Mh}$, i.e., the value of $\sigma_t$ at the beginning of the $j$-th block. Under the approximate measure $Q_{h,M}$ for the observables in the $j$th block ($j = 1, \ldots, \frac{1}{Mh}$), the regression (31) becomes

$$y_{l,i+(j-1)M} = \beta_{lk(j)} y_{k,i+(j-1)M} + u_{i+(j-1)M}, \tag{32}$$

[5] The underlying data generating process is in continuous time while one observes a discrete time sample of the process.

where $u_{i+(j-1)M}|y_{k,i+(j-1)M} \sim$i.i.d.$N\left(0, V_{(j)}\right)$, for $i = 1, \ldots, M$, with $V_{(j)} = hC^2_{ll(j)}$, and $\beta_{lk(j)} \equiv \frac{C_{lk(j)}}{C_{kk(j)}}$. Let us denote by $\check{\beta}_{lk(j)}$ the ordinary least squares (OLS) estimator of $\beta_{lk(j)}$. To estimate the parameter given by $\beta_{lk} = \int_0^1 \beta_{lk,s} ds$, Mykland and Zhang (2009) proposed to use $\check{\beta}_{lk}$ defined as follows,

$$\check{\beta}_{lk} = Mh \sum_{j=1}^{1/Mh} \check{\beta}_{lk(j)} = Mh \sum_{j=1}^{1/Mh} \left(\sum_{i=1}^{M} y^2_{k,i+(j-1)M}\right)^{-1} \left(\sum_{i=1}^{M} y_{k,i+(j-1)M} y_{l,i+(j-1)M}\right). \tag{33}$$

Note that the realized beta estimator discussed in Section 4.1 (see equation (23) and (24)) and also studied among others by Dovonon et al. (2013) is a different statistic than the term given by (33). Here, the realized beta estimator $\check{\beta}_{lk}$ is not directly a least squares estimator, but is the result of the average of $\check{\beta}_{lk(j)}$, the OLS estimators for each block. Since under $Q_{h,M}$, the volatility matrix is constant in each block $j$, implying consequently that the score is not heterogeneous and has mean zero. This simplifies the asymptotic inference on $\beta_{lk(j)}$, and on $\beta_{lk}$. It is worth emphasizing that contrary to what we have observed in the case of the realized volatility estimator in Section 3.1[6], here when $M = 1$, the realized beta estimator using the blocking approach becomes

$$\check{\beta}_{lk} = h \sum_{i=1}^{1/h} \frac{y_{l,i}}{y_{k,i}},$$

which is a different statistic than the statistic studied by Barndorff-Nielsen and Shephard (2004a). But when $M = h^{-1}$, both estimators are equivalent. However, as Mykland and Zhang (2011) pointed out, when $M \to \infty$ with the sample size $h^{-1}$, the local approximation is good only when $M = O(h^{-1/2})$. It follows then that we are not comfortable to contrast Mykland and Zhang (2009) block-based "realized beta" estimator asymptotic results with those of Barndorff-Nielsen and Shephard (2004a) when $M = h^{-1}$. The first reason is that in this case (i.e. $M = h^{-1}$) the local approximation is not accurate enough. The second and main reason is that when $M \to \infty$, to the best of our knowledge we do not know whether the approximate measure $Q_{h,M}$ with representation given by (30) is contiguous to $P$.

Mykland and Zhang (2009) provide a CLT result for $\beta_{lk}$. In particular, we have under $P$ and $Q_{h,M}$, as the number of intraday observations increases to infinity (i.e. if $h \to 0$), by using Section 4.2 of Mykland and Zhang (2009), as $h \to 0$, for any $\delta > 0$ such that $M > 2\,(1+\delta)$ with $M = O(1)$,

$$\frac{\sqrt{h^{-1}}\left(\check{\beta}_{lk} - \beta_{lk}\right)}{\sqrt{V_{\check{\beta}}}} \xrightarrow{d} N(0,1), \tag{34}$$

where

$$V_{\check{\beta}} = \frac{M}{M-2} \int_0^1 \left(\frac{\Sigma_{ll,s}}{\Sigma_{kk,s}} - \beta^2_{lk,s}\right) ds.$$

In practice, this result is infeasible since the asymptotic variance $V_{\check{\beta}}$ depends on unobserved quantities. Mykland and Zhang (2009) did not provide any consistent estimator of $V_{\check{\beta}}$. One of our contributions is to propose a consistent estimator of $V_{\check{\beta}}$. To this end, we exploit the special structure of the regression

[6]Where the realized volatility estimator $R_2$ using Mykland and Zhang's (2009) blocking approach with $M = 1$, is the same realized volatility studied by Barndorff-Nielsen and Shephard (2002).

model. To find the asymptotic variance of realized regression estimator $\check{\beta}_{lk}$, we can write

$$\sqrt{h^{-1}}\left(\check{\beta}_{lk}-\beta_{lk}\right)=M\sqrt{h}\sum_{j=1}^{1/Mh}\left(\check{\beta}_{lk(j)}-\beta_{lk(j)}\right).$$

Since $\check{\beta}_{lk(j)}$ are independent across $j$, it follows that

$$V_{\check{\beta},h,M}\equiv Var\left(\sqrt{h^{-1}}\left(\check{\beta}_{lk}-\beta_{lk}\right)\right)=M^2h\sum_{j=1}^{1/Mh}Var\left(\check{\beta}_{lk(j)}-\beta_{lk(j)}\right). \tag{35}$$

To compute (35), note that from standard regression theory, we have that under $Q_{h,M}$,

$$Var\left(\check{\beta}_{lk(j)}-\beta_{lk(j)}\right)=E\left(\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\right)V_{(j)},$$

which implies that

$$V_{\check{\beta},h,M}=M^2h\sum_{j=1}^{1/Mh}E\left(\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\right)V_{(j)}. \tag{36}$$

Note that we can contrast $V_{\check{\beta}}$ with equation (72) of Mykland and Zhang (2009). In fact, we can write under $Q_{h,M}$, $\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\overset{d}{=}hC_{kk(j)}^2\sum_{i=1}^{M}v_{i+(j-1)M}^2\overset{d}{=}hC_{kk(j)}^2\chi_{j,M}^2$, where $v_{i+(j-1)M}\sim i.i.d.N\left(0,1\right)$ and $\chi_{j,M}^2$ follow the standard $\chi^2$ distribution with $M$ degrees of freedom. Then for any integer $M>2$ and conditionally on the volatility path, by using the expectation of the inverse of a Chi square distribution we have,

$$E\left(\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\right)=E\left(\frac{1}{\chi_{j,M}^2}\right)h^{-1}C_{kk(j)}^{-2}=\frac{1}{M-2}h^{-1}C_{kk(j)}^{-2}. \tag{37}$$

It follows then that

$$V_{\check{\beta},h,M}=\frac{M}{M-2}Mh\sum_{j=1}^{1/Mh}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2.$$

By using the structure of (36), a natural consistent estimator of $V_{\check{\beta},h,M}$ is

$$\hat{V}_{\check{\beta},h,M}\equiv M^2h\sum_{j=1}^{1/Mh}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\frac{1}{M-1}\sum_{i=1}^{M}\hat{u}_{i+(j-1)M}^2\right), \tag{38}$$

where $\hat{u}_{i+(j-1)M}=y_{l,i+(j-1)M}-\check{\beta}_{lk(j)}y_{k,i+(j-1)M}$ (see Lemma C3.13 and Lemma C3.14 in Hounyo (2018)). Together with the CLT result (34), we have under $P$ and $Q_{h,M}$ the feasible result

$$T_{\check{\beta},h,M}\equiv\frac{\sqrt{h^{-1}}(\check{\beta}_{lk}-\beta_{lk})}{\sqrt{\hat{V}_{\check{\beta},h,M}}}\to^d N\left(0,1\right).$$

Next we show that the local Gaussian bootstrap method is consistent when applied to $\check{\beta}_{lk}$ i.e., the realized beta estimator given by (33). Recall (18), and let $\check{\beta}_{lk(j)}^*$ denote the OLS bootstrap estimator from the regression of $y_{l,i+(j-1)M}^*$ on $y_{k,i+(j-1)M}^*$ inside the block $j$. The bootstrap realized beta estimator is

$$\check{\beta}_{lk}^{*} = Mh \sum_{j=1}^{1/Mh} \check{\beta}_{lk(j)}^{*}.$$

It is easy to check that $\check{\beta}_{lk}^{*} - \check{\beta}_{lk} = o_{P^*}(1)$, in probability under $Q_{h,M}$ and $P$, where

$$\check{\beta}_{lk} = Mh \sum_{j=1}^{1/Mh} E^{*} \left( \left( \sum_{i=1}^{M} y_{k,i+(j-1)M}^{2*} \right)^{-1} \left( \sum_{i=1}^{M} y_{k,i+(j-1)M}^{*} y_{l,i+(j-1)M}^{*} \right) \right).$$

The bootstrap analogue of the regression error $u_{i+(j-1)M}$ in model (32) is thus $u_{i+(j-1)M}^{*} = y_{l,i+(j-1)M}^{*} - \check{\beta}_{lk(j)} y_{k,i+(j-1)M}^{*}$, whereas the bootstrap OLS residuals are defined as $\hat{u}_{i+(j-1)M}^{*} = y_{l,i+(j-1)M}^{*} - \check{\beta}_{lk(j)}^{*} y_{k,i+(j-1)M}^{*}$. Thus, conditionally on the observed vector of returns $y_{i+(j-1)M}$, it follows that $u_{i+(j-1)M}^{*} | y_{k,i+(j-1)M}^{*} \sim i.i.d.N\left(0, \hat{V}_{(j)}\right)$, for $i = 1, \ldots, M$, where

$$\hat{V}_{(j)} \equiv h\hat{C}_{ll(j)}^{2}.$$

We can show that for fixed $M$,

$$Var^{*}\left(\sqrt{h^{-1}}(\check{\beta}_{lk}^{*} - \check{\beta}_{lk})\right) = \frac{M-1}{M-2}\hat{V}_{\check{\beta},h,M}.$$

Hence, for any fixed $M$, we would not use the local Gaussian bootstrap variance estimator to estimate $V_{\check{\beta}}$. However, since we know the scaled factor $\frac{M-1}{M-2}$, this does not create a problem if we adjust the bootstrap statistic accordingly. Our next theorem summarizes these results.

**Theorem 6.2.** *Consider DGP (1), (2) and suppose (4) holds. Let $M > 2(1+\delta)$ for some $\delta > 0$ such that $M$ is bounded, conditionally on $\sigma$, as $h \to 0$, the following hold under $Q_{h,M}$ and $P$*

**(a)**

$$\begin{aligned} V_{\check{\beta},h,M}^{*} &\equiv Var^{*}\left(\sqrt{h^{-1}}(\check{\beta}_{lk}^{*} - \check{\beta}_{lk})\right) \\ &\xrightarrow{P} \frac{M-1}{M-2} V_{\check{\beta}}, \end{aligned}$$

**(b)**

$$\sup_{x \in \mathbb{R}} \left| P^{*}\left( \sqrt{\frac{M-2}{M-1}} \sqrt{h^{-1}} \left(\check{\beta}_{lk}^{*} - \check{\beta}_{lk}\right) \leq x \right) - P\left(\sqrt{h^{-1}}\left(\check{\beta}_{lk} - \beta_{lk}\right)\right) \right| \xrightarrow{P} 0.$$

Part (a) of Theorem 6.2 shows that the bootstrap variance estimator is not consistent for $V_{\check{\beta}}$ when the block size $M$ is finite. Note that for large $M$, but bounded $V_{\check{\beta},h,M}^{*} \to V_{\check{\beta}}$, (approximately) and $\left(\frac{M-1}{M-2}\right)^{-1} V_{\check{\beta},h,M}^{*} \to V_{\check{\beta}}$, for any $M > 2(1+\delta)$ for some $\delta > 0$. Results in part (b) imply that the bootstrap realized beta estimator has a first-order asymptotic normal distribution with mean zero and covariance matrix $V_{\check{\beta}}$. This is in line with the existing results in the cross section regression context, where the wild bootstrap and the pairs bootstrap variance estimator of the least squares estimator are robust to heteroskedasticity in the error term.

Bootstrap percentile intervals do not promise asymptotic refinements. Next, we propose a consistent bootstrap variance estimator that allows us to form bootstrap percentile-$t$ intervals. More specifically, we can show that the following bootstrap variance estimator consistently estimates $V^*_{\check{\beta},h,M}$:

$$\hat{V}^*_{\check{\beta},h,M} \equiv M^2 h \sum_{j=1}^{1/Mh} \left( \sum_{i=1}^{M} y^{*2}_{k,i+(j-1)M} \right)^{-1} \left( \frac{1}{M-1} \sum_{i=1}^{M} \hat{u}^{*2}_{i+(j-1)M} \right). \tag{39}$$

Our proposal is to use this estimator to construct the bootstrap $t$-statistic, associated with the bootstrap realized regression coefficient $\check{\beta}^*_{lk}$. Let

$$T^*_{\check{\beta},h,M} \equiv \frac{\sqrt{h^{-1}} \left( \check{\beta}^*_{lk} - \check{\beta}_{lk} \right)}{\sqrt{\hat{V}^*_{\check{\beta},h,M}}}, \tag{40}$$

be the bootstrap analogue of $T_{\check{\beta},h,M}$.

**Theorem 6.3.** *Consider DGP (1), (2) and suppose (4) holds. Let $M > 4\,(2+\delta)$ for some $\delta > 0$ such that $M$ is bounded, conditionally on $\sigma$, as $h \to 0$, the following hold*

$$T^*_{\check{\beta},h,M} \to^{d^*} N(0,1)\,, \text{ in probability, under } Q_{h,M} \text{ and } P.$$

Note that when the block size $M$ is finite the bootstrap is also first-order asymptotically valid when applied to the $t$-statistic $T^*_{\check{\beta},h,M}$ (defined in (40) without any scaled factor), as our Theorem 6.3 proves. This first-order asymptotic validity occurs despite the fact that $V^*_{\check{\beta},h,M}$ does not consistently estimate $V_{\check{\beta}}$ when $M$ is fixed. The key aspect is that we studentize the bootstrap OLS estimator with $\hat{V}^*_{\check{\beta},h,M}$ (defined in (39)), a consistent estimator of $V^*_{\check{\beta},h,M}$, implying that the asymptotic variance of the bootstrap $t$-statistic is one.

# 7 Monte Carlo results

In this section we assess by Monte Carlo simulation the accuracy of the feasible asymptotic theory approach of Mykland and Zhang (2009). We find that this approach leads to important coverage probability distortions when returns are not sampled too frequently. We also compare the finite sample performance of the new local Gaussian bootstrap method with the existing bootstrap method for realized volatility proposed by Gonçalves and Meddahi (2009).

For integrated volatility, we consider two data generating processes in our simulations. First, following Zhang, Mykland and Aït-Sahalia (2005), we use the one-factor stochastic volatility (SV1F) model of Heston (1993) as our data-generating process, i.e.

$$dX_t = (\mu - \nu_t/2)\, dt + \sigma_t dB_t,$$

and

$$d\nu_t = \kappa\, (\alpha - \nu_t)\, dt + \gamma\, (\nu_t)^{1/2}\, dW_t,$$

where $\nu_t = \sigma_t^2$, $B$ and $W$ are two Brownian motions, and we assume $Corr(B, W) = \rho$. The parameter values are all annualized. In particular, we let $\mu = 0.05/252$, $\kappa = 5/252$, $\alpha = 0.04/252$, $\gamma = 0.05/252$, $\rho = -0.5$.

We also consider the two-factor stochastic volatility (SV2F) model analyzed by Barndorff-Nielsen et al. (2008) and Gonçalves and Meddahi (2009), see also the recent work of Hounyo and Varneskov (2017), where[7]

$$\begin{aligned} dX_t &= \mu dt + \sigma_t dB_t, \\ \sigma_t &= s\text{-}exp\left(\beta_0, +\beta_1 \tau_{1t} + \beta_2 \tau_{2t}\right), \\ d\tau_{1t} &= \alpha_1 \tau_{1t} dt + dB_{1t}, \\ d\tau_{2t} &= \alpha_2 \tau_{2t} dt + \left(1 + \phi \tau_{2t}\right) dB_{2t}, \\ corr\left(dW_t, dB_{1t}\right) &= \varphi_1, \; corr\left(dW_t, dB_{2t}\right) = \varphi_2. \end{aligned}$$

We follow Huang and Tauchen (2005) and set $\mu = 0.03$, $\beta_0 = -1.2$, $\beta_1 = 0.04$, $\beta_2 = 1.5$, $\alpha_1 = -0.00137$, $\alpha_2 = -1.386$, $\phi = 0.25$, $\varphi_1 = \varphi_2 = -0.3$. We initialize the two factors at the start of each interval by drawing the persistent factor from its unconditional distribution, $\tau_{10} \sim N\left(0, \frac{-1}{2\alpha_1}\right)$ and by starting the strongly mean-reverting factor at zero.

For integrated beta, the design of our Monte Carlo study is roughly identical to that used by Barndorff-Nielsen and Shephard (2004a), and Dovonon Gonçalves and Meddahi (2013) with a minor difference. In particular, we add a constant drift component to the design of Barndorff-Nielsen and Shephard (2004a). Here we briefly describe the Monte Carlo design we use. We assume that $dX(t) = \sigma(t) dW(t) + \mu d(t)$, with $\sigma(t) \sigma'(t) = \Sigma(t)$, where

$$\Sigma(t) = \begin{pmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{pmatrix} = \begin{pmatrix} \sigma_1^2(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_2^2(t) \end{pmatrix}, \text{ and } \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

and $\sigma_{12}(t) = \sigma_1(t) \sigma_2(t) \rho(t)$. As Barndorff-Nielsen and Shephard (2004a), we let $\sigma_1^2(t) = \sigma_1^{2(1)}(t) + \sigma_1^{2(2)}(t)$, where for $s = 1, 2$, $d\sigma_1^{2(s)}(t) = -\lambda_s(\sigma_1^{2(s)}(t) - \xi_s)dt + \omega_s \sigma_1^{(s)}(t)\sqrt{\lambda_s} db_s(t)$, where $b_i$ is the $i$-th component of a vector of standard Brownian motions, independent from $W$. We let $\lambda_1 = 0.0429$, $\xi_1 = 0.110$, $\omega_1 = 1.346$, $\lambda_2 = 3.74$, $\xi_2 = 0.398$, and $\omega_2 = 1.346$. Our model for $\sigma_2^2(t)$ is the GARCH(1,1) diffusion studied by Andersen and Bollerslev (1998): $d\sigma_2^2(t) = -0.035(\sigma_2^2(t) - 0.636)dt + 0.236\sigma_2^2(t)db_3(t)$. We follow Barndorff-Nielsen and Shephard (2004a), and let $\rho(t) = (e^{2x(t)} - 1)/(e^{2x(t)} + 1)$, where $x$ follows the GARCH diffusion: $dx(t) = -0.03(x(t) - 0.64)dt + 0.118x(t)db_4(t)$. Finally, we let $\mu_1 = \mu_2 = 0.03$.

We simulate data for the unit interval $[0, 1]$. The observed log-price process $X$ is generated using an Euler scheme. We then construct the $h$-horizon returns $y_i \equiv X_{ih} - X_{(i-1)h}$ based on samples of size $1/h$.

[7]The function s-exp is the usual exponential function with a linear growth function splined in at high values of its argument: $s\text{-}exp(x) = exp(x)$ if $x \leq x_0$ and $s\text{-}exp(x) = \frac{exp(x_0)}{\sqrt{x_0 - x_0^2 + x^2}}$ if $x > x_o$, with $x_0 = log(1.5)$.

Tables 1 and 2 give the actual rates of 95% confidence intervals of integrated volatility and integrated beta, computed over 10,000 replications. Results are presented for six different samples sizes: $1/h$ 1152, 576, 288, 96, 48, and 12, corresponding to "1.25-minute", "2.5-minute", "5-minute", "15-minute", "half-hour" and "2-hour" returns. In Table 1, for each sample size we have computed the coverage rate by increasing the block size, whereas in Table 2 we summarize results by selecting the optimal block size. We also report results for confidence intervals based on a logarithmic version of the statistic $T^{\mathrm{MZ}}_{\sigma^2,h,M}$ and its bootstrap version.

In our simulations, bootstrap intervals use 999 bootstrap replications for each of the 10,000 Monte Carlo replications. We consider the studentized (percentile-$t$) symmetric bootstrap confidence interval method computed at the 95% level.

As for all blocking methods, to implement our bootstrap methods, we need to choose the block size $M$. We follow Politis and Romano (1999) and Hounyo, Gonçalves and Meddahi (2017) and use the Minimum Volatility Method. Although the block length $M$ that results for this approach may be suboptimal for the purpose of distribution estimation (in the context of realized volatility) as we do in our application, it is readily accessible to practitioners, typically performs well, and allows meaningful comparisons of the local Gaussian bootstrap and asymptotic normal theory-based methods. Here we describe the algorithm we employ for a two-sided confidence interval.

**Algorithm: Choice of the block size $M$ by minimizing confidence interval volatility**

**(i)** For $M = M_{small}$ to $M = M_{big}$ compute a bootstrap interval for the parameter of interest (integrated volatility or integrated beta) at the desired confidence level, this resulting in endpoints $IC_{M,low}$ and $IC_{M,up}$.

**(ii)** For each $M$ compute the volatility index $VI_M$ as the standard deviation of the interval endpoints in a neighborhood of $M$. More specifically, for a smaller integer $l$, let $VI_M$ equal to the standard deviation of the endpoints $\{IC_{M-l,low}, \ldots, IC_{M+l,low}\}$ plus the standard deviation of the endpoints $\{IC_{M-l,up}, \ldots, IC_{M+l,up}\}$, i.e.

$$VI_M \equiv \sqrt{\frac{1}{2l+1}\sum_{i=-l}^{l}\left(IC_{M+i,low} - \bar{IC}_{low}\right)^2} + \sqrt{\frac{1}{2l+1}\sum_{i=-l}^{l}\left(IC_{M+i,up} - \bar{IC}_{up}\right)^2},$$

where $\bar{IC}_{low} = \frac{1}{2l+1}\sum_{i=-l}^{l} IC_{M+i,low}$ and $\bar{IC}_{up} = \frac{1}{2l+1}\sum_{i=-l}^{l} IC_{M+i,up}$.

**(iii)** Pick the value $M^*$ corresponding to the smallest volatility index and report $\{IC_{M^*,low}, IC_{M^*,up}\}$ as the final confidence interval.

One might ask what is a selection of reasonable $M_{small}$ and $M_{big}$? In our experience, for a sample size $1/h = 1152$, the choices $M_{small} = 1$ and $M_{big} = 12$ usually suffice, for the samples sizes : $1/h =$ 1152, 576, 288, 96, and 48, we have used $M_{small} = 1$ and $M_{big} = 12$. For results in Table 2, we used $l = 2$ in our simulations. Some initial simulations (not recorded here) show that the actual coverage rate of the confidence intervals using the bootstrap is not sensitive to reasonable choice of $l$, in particular, for $l = 1, 2, 3$.

Starting with integrated volatility, the Monte Carlo results in Tables 1 and 2 show that for both models (SV1F and SV2F), the asymptotic intervals tend to undercover. The degree of undercoverage is especially large, when sampling is not too frequent. It is also larger for the raw statistics than for the log-based statistics. The SV2F model exhibits overall larger coverage distortions than the SV1F model, for all sample sizes. When $M = 1$, the local Gaussian bootstrap method is equivalent to the wild bootstrap of Gonçalves and Meddahi (2009) that uses the normal distribution as the external random variable. One can see that the bootstrap replicates their simulations results. In particular, the Gaussian bootstrap intervals tend to overcover across all models. The actual coverage probabilities of the confidence intervals using the Gaussian bootstrap are typically monotonically decreasing in $M$, and does not tend to decrease very fast in $M$ for larger values of sample size.

A comparison of the local Gaussian bootstrap with the best existing bootstrap methods for realized volatility[8] shows that, for smaller samples sizes, the confidence intervals based on the Gaussian bootstrap are conservative, yielding coverage rates larger than 95% for the SV1F model. The confidence intervals tend to be closer to the desired nominal level for the SV2F than the best bootstrap proposed by Gonçalves and Meddahi (2009). For instance, for the SV1F model, the Gaussian bootstrap covers 96.51% of the time when $h^{-1} = 12$ whereas the best bootstrap of Gonçalves and Meddahi (2009) does only 87.42%. These rates decrease to 93.21% and 80.42% for the SV2F model, respectively.

We also consider intervals based on the i.i.d. bootstrap studied by Gonçalves and Meddahi (2009). Despite the fact that the i.i.d. bootstrap does not theoretically provide an asymptotic refinement for two-sided symmetric confidence intervals, it performs well.

While none of the intervals discussed here (bootstrap or asymptotic theory-based) allow for $M = h^{-1}$, we have also studied this setup which is nevertheless an obvious interest in practice. For the SV1F model, results are not very sensitive to the choice of the block size, whereas for the SV2F model coverage rates for intervals using a very large value of block size ($M = h^{-1}$) are systematically much lower than 95% even for the largest sample sizes. When $M = h^{-1}$, the realized volatility $R_2$ using the blocking approach is the same realized volatility studied by Barndorff-Nielsen and Shephard (2002), but the estimator of integrated quarticity using the blocking approach is $\frac{h^{-1}+2}{h^{-1}} R_2^2$. This means that asymptotically we replace $\int_0^1 \sigma_t^4 dt$ by $\left(\int_0^1 \sigma_t^2 dt\right)^2$, which is only valid under constant volatility. By Cauchy-Schwarz inequality, we have $\left(\int_0^1 \sigma_t^2 dt\right)^2 \leq \int_0^1 \sigma_t^4 dt$, it follows then that we underestimated the asymptotic variance of the realized volatility estimator. This explains the poor performance of the theory based on the blocking approach when the block size is too large.

For the realized beta, we see that intervals based on the feasible asymptotic procedure using Myk-

[8] The wild bootstrap based on Proposition 4.5 of Gonçalves and Meddahi (2009).

land and Zhang's (2009) blocking approach and the bootstrap tend to be similar for larger sample sizes whereas, at the smaller sample sizes, intervals based on the asymptotic normal distribution are quite severely distorted. For instance, the coverage rate for the feasible asymptotic theory of Mykland and Zhang (2009) when $h^{-1} = 12$ (cf. $h^{-1} = 48$) is only equal to 88.49% (92.86%), whereas it is equal to 95.17% (94.84%), for the local Gaussian bootstrap (the corresponding symmetric interval based on the pairs bootstrap of Dovonon Gonçalves and Meddahi (2013) yields a coverage rate of 93.59% (93.96%), better than Mykland and Zhang (2009) but worse than the local Gaussian bootstrap-based interval). Our Monte Carlo results also confirm that for a good approximation, a very large block size is not recommended.

Overall, all methods behave similarly for larger sample sizes, in particular the coverage rate tends to be closer to the desired nominal level. The local Gaussian bootstrap performance is quite remarkable and outperforms the existing methods, especially for smaller samples sizes ($h^{-1} = 12$ and 48).

# 8 Empirical results

As a brief illustration, in this section we implement the local Gaussian bootstrap method with real high-frequency financial intraday data, and compare it to the existing feasible asymptotic procedure of Mykland and Zhang (2009). The data consists of transaction log prices of General Electric (GE) shares carried out on the New York Stock Exchange (NYSE) in August 2011. Before analyzing the data we have cleaned the data. For each day, we consider data from the regular exchange opening hours from time stamped between 9:30 a.m. till 4 p.m. Our procedure for cleaning data is exactly identical to that used by Barndorff-Nielsen et al. (2008). We detail in Appendix A the cleaning we carried out on the data.

We implemented the realized volatility estimator of Mykland and Zhang (2009) on returns recorded every $S$ transactions, where $S$ is selected each day so that there are 96 observations a day. This means that on average these returns are recorded roughly every 15 minutes. Table 3 in the Appendix provides the number of transactions per day, and the sample size used. Typically each interval corresponds to about 131 transactions.

This choice is motivated by the empirical study of Hansen and Lunde (2006), who investigate 30 stocks of the Dow Jones Industrial Average, in particular they have presented detailed work for the GE shares. They suggest to use 10 to 15 minutes horizon for liquid assets to avoid the market microstructure noise effect. Hence the main assumptions underlying the validity of the Mykland and Zhang (2009) block-based method and our new bootstrap method are roughly satisfied and we feel comfortable to implement them on this data. To implement the realized volatility estimator, we need to choose the block size $M$. We use the Minimum Volatility Method described in Section 7 to choose $M$.

We consider bootstrap percentile-$t$ intervals, computed at the 95% level. The results are displayed in

Figure 2 in the appendix in terms of daily 95% confidence intervals (CIs) for integrated volatility. Two types of intervals are presented: our proposed new local Gaussian bootstrap method and the feasible asymptotic theory using the Mykland and Zhang (2009) blocking approach. The realized volatility estimate $R_2$ is in the center of both confidence intervals by construction. A comparison of the local Gaussian bootstrap intervals with the intervals based on the feasible asymptotic theory using Mykland and Zhang (2009) block-based approach suggests that both types of intervals tend to be similar. The width of these intervals varies through time. However, there are instances where the bootstrap intervals are wider than the asymptotic theory-based interval. These days often correspond to days with large estimated volatility. We consider whether this is due to jumps and have implemented the jumps test using the blocked bipower variation of Mykland, Shephard and Sheppard (2012). We have found no evidence of jumps at 5% significance level for these two days. The figures also show a lot of variability in the daily estimate of integrated volatility.

# 9 Concluding remarks

In this paper, we propose the local Gaussian bootstrap as a method of inference for statistics that are smooth functions of the realized multivariate volatility matrix. We show the first-order asymptotic validity of this new bootstrap method for realized volatility and realized regression coefficients. We use Monte Carlo simulations and derive higher order expansions for cumulants to compare the accuracy of the bootstrap and the normal approximations at estimating confidence intervals for integrated volatility and integrated beta. Based on these expansions, we show that the local Gaussian bootstrap provides second-order refinements for the realized beta, whereas it provides a third-order asymptotic refinement for realized volatility. This is an improvement of the existing bootstrap results. Our new bootstrap method also generalizes the wild bootstrap of Gonçalves and Meddahi (2009). Monte Carlo simulations suggest that the local Gaussian bootstrap improves upon the first-order asymptotic theory in finite samples and outperforms the existing bootstrap methods for realized volatility and realized betas.

An important assumption we make throughout this paper is that asset prices are observed at regular time intervals and without any error (so that markets are frictionless). If prices are nonequidistant (irregularly spaced (but nonrandom)) and non noisy, results in Mykland and Zhang (2009) show that a contiguity argument holds and the same variance estimators of the realized covariation measures remain consistent. Consequently, the same $t$-statistics as those considered here can be used for inference purposes. In this case, we can rely on the same local Gaussian bootstrap $t$-statistics to approximate their distributions. Recently, Renault, Sarisoy and Werker (2017) show that irregular sampling invalidates the advantage of the block-based estimator (given by (6)) to achieve efficiency asymptotically by increasing the size of the block. We conjecture that in such a situation, the local Gaussian bootstrap method may lose its higher-order refinement property. When the observation times are possibly random but predictable, we expect that a natural way to preserve the higher-order correctness property of the local Gaussian bootstrap would consist in replacing in equation (14) (the bootstrap DGP), the spot realized volatility estimator by another estimator of the spot volatility (appropriately scaled) which can get arbitrarily close to the nonparametric bound, similar to that of Renault, Sarisoy and Werker (2017) (see also the related work of Kristensen (2010)).

The case of noisy data is much more challenging because in this case the realized covariation measures studied in this paper are not consistent estimators of their integrated volatility measures. In the context of inference with noisy data, the noise robust measures based on a pre-averaging approach (a different kind of blocking method) of Podolskij and Vetter (2009) and Jacod et al. (2009) exploit the locally constant approximation of stochastic volatility. In particular, in these papers, the latent semimartingale is itself given a locally constant approximation. Therefore, we conjecture that contiguity results can be found under common types of noise. However, for the bootstrap, it is worth emphasising that we have to distinguish two cases: The non-overlapping pre-averaging based estimators as in Podolskij and Vetter (2009), and the overlapping pre-averaging estimators as in Jacod et al. (2009). For the latter, the local Gaussian bootstrap would not be valid, since the pre-averaged returns are very strongly dependent because they rely on many common high-frequency returns. A different bootstrap method such as the wild blocks of blocks bootstrap studied by Hounyo, Gonçalves and Meddahi (2017) (see also Hounyo (2017)) will be appropriate. But we conjecture that, when the pre-averaging method is applied on non-overlapping intervals (as in Podolskij and Vetter (2009) and Gonçalves, Hounyo and Meddahi (2014)), the local Gaussian bootstrap method remains valid. However, it is important to highlight that in this case, we should apply the local Gaussian bootstrap method to the pre-averaged returns, instead of the raw returns. Since in the presence of noise, the non-overlapping pre-averaged returns remain asymptotically Gaussian and conditionally independent, which is not the case for noisy high-frequency raw returns.

Another promising extension would be to allow for jumps in the log-price process. Following the initial draft of this paper, Dovonon et al. (2014) generalize the local Gaussian bootstrap method and use it for jump tests. In particular, instead of the local Gaussian bootstrap observations within a given block will have a normal distribution with variance equal to the realized variance over the corresponding block. They propose to generate the bootstrap observations with normal distribution, but now with a variance given by a local jumps-robust realized measure of volatility, for instance the local block multipower variation estimator, similar to that of Mykland, Shephard and Sheppard (2012).

We also supposed in the multivariate framework that prices on different assets are observed synchronously. If prices are non-synchronous, different robust measures (e.g., Hayashi and Yoshida (2005) covariance estimator) are required. Another important extension is to prove the validity of the Edgeworth expansions derived here and to provide a theoretical optimal choice of the block size $M$ for confidence interval construction. The extension of the local Gaussian bootstrap to these alternative estimators and test statistics is left for future research.

# Appendix A

This appendix is organized as follows. First, we detail the cleaning we carried out on the data. Second, we report simulation results. Finally we report empirical results.

### Data Cleaning

In line with Barndorff-Nielsen et al. (2009) we perform the following data cleaning steps:

**(i)** Delete entries outside the 9:30pm and 4pm time window.

**(ii)** Delete entries with a quote or transaction price equal to be zero.

**(iii)** Delete all entries with negative prices or quotes.

**(iv)** Delete all entries with negative spreads.

**(v)** Delete entries whenever the price is outside the interval $[bid - 2 * spread \; ; \, ask + 2 * spread]$.

**(vi)** Delete all entries with the spread greater than or equal to 50 times the median spread of that day.

**(vii)** Delete all entries with the price greater than or equal to 5 times the median mid-quote of that day.

**(viii)** Delete all entries with the mid-quote greater than or equal to 10 times the mean absolute deviation from the local median mid-quote.

**(ix)** Delete all entries with the price greater than or equal to 10 times the mean absolute deviation from the local median mid-quote.

We report in Table 1 below, the actual coverage rates for the feasible asymptotic theory approach and for our bootstrap methods. In Table 2 we summarize results using the optimal block size by minimizing confidence interval volatility. Table 3 provides some statistics of GE shares in August 2011.

**Table 1. Coverage rates of nominal 95% CI for integrated volatility and integrated beta**

| | Integrated volatility | | | | | | | | | Integrated beta | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | SV1F | | | | SV2F | | | | | | |
| | Raw | | Log | | Raw | | Log | | | Raw | |
| $M$ | CLT | Boot | CLT | Boot | CLT | Boot | CLT | Boot | $M$ | CLT | Boot |
| | | | | | $1/h = 12$ | | | | | | |
| 1 | 85.44 | 98.49 | 90.08 | 97.86 | 80.38 | 96.62 | 86.17 | 96.24 | 2 | 83.58 | 95.67 |
| 2 | 85.56 | 97.31 | 90.31 | 96.80 | 80.43 | 94.70 | 86.27 | 94.73 | 3 | 87.57 | 94.98 |
| 3 | 85.71 | 96.46 | 90.84 | 96.08 | 80.34 | 93.77 | 85.89 | 93.70 | 4 | 89.15 | 94.81 |
| 4 | 85.88 | 96.20 | 90.97 | 95.93 | 80.34 | 92.88 | 85.52 | 92.89 | 6 | 90.65 | 94.47 |
| 12 | 86.11 | 94.84 | 91.27 | 94.87 | 77.66 | 88.89 | 81.65 | 86.97 | 12 | 90.46 | 93.62 |
| | | | | | $1/h = 48$ | | | | | | |
| 1 | 92.04 | 98.55 | 93.51 | 97.71 | 88.28 | 97.09 | 90.93 | 96.67 | 3 | 92.36 | 95.65 |
| 2 | 92.10 | 97.28 | 93.59 | 96.50 | 88.13 | 95.63 | 91.08 | 95.48 | 4 | 92.69 | 95.29 |
| 4 | 92.20 | 96.40 | 93.80 | 95.80 | 88.16 | 94.55 | 91.10 | 94.53 | 8 | 92.91 | 94.71 |
| 8 | 92.33 | 95.60 | 93.88 | 95.18 | 87.89 | 93.32 | 90.33 | 93.20 | 12 | 92.62 | 93.75 |
| 48 | 92.74 | 95.06 | 94.22 | 95.04 | 81.83 | 86.63 | 82.92 | 84.57 | 48 | 91.61 | 92.45 |
| | | | | | $1/h = 96$ | | | | | | |
| 1 | 93.35 | 97.94 | 94.09 | 97.10 | 90.20 | 97.06 | 92.10 | 96.66 | 3 | 92.60 | 95.51 |
| 2 | 93.43 | 96.78 | 93.99 | 96.06 | 90.37 | 95.84 | 92.24 | 95.67 | 4 | 93.12 | 95.01 |
| 4 | 93.47 | 95.78 | 94.03 | 95.61 | 90.46 | 94.70 | 92.09 | 94.83 | 8 | 93.78 | 94.81 |
| 8 | 93.50 | 95.26 | 94.09 | 95.32 | 90.07 | 93.81 | 91.75 | 94.01 | 12 | 93.79 | 94.55 |
| 96 | 93.42 | 94.80 | 94.35 | 94.87 | 81.93 | 84.61 | 82.79 | 83.60 | 96 | 91.97 | 92.36 |
| | | | | | $1/h = 288$ | | | | | | |
| 1 | 94.57 | 97.09 | 94.61 | 96.25 | 93.39 | 97.44 | 93.96 | 96.76 | 3 | 93.81 | 95.72 |
| 2 | 94.56 | 96.00 | 94.61 | 95.67 | 93.51 | 96.35 | 93.95 | 95.95 | 4 | 94.73 | 95.61 |
| 4 | 94.62 | 95.48 | 94.67 | 95.36 | 93.50 | 95.57 | 93.98 | 95.28 | 8 | 94.92 | 95.45 |
| 8 | 94.55 | 95.26 | 94.81 | 95.19 | 93.43 | 95.06 | 93.82 | 94.75 | 12 | 94.65 | 94.99 |
| 288 | 94.46 | 94.78 | 94.84 | 94.99 | 82.43 | 83.86 | 83.34 | 83.53 | 288 | 90.08 | 90.33 |
| | | | | | $1/h = 576$ | | | | | | |
| 1 | 94.53 | 96.12 | 94.75 | 95.84 | 94.19 | 96.96 | 94.49 | 96.52 | 3 | 93.93 | 95.58 |
| 2 | 94.57 | 95.53 | 94.68 | 95.41 | 94.17 | 96.23 | 94.52 | 95.78 | 4 | 94.49 | 95.36 |
| 4 | 94.74 | 95.15 | 94.70 | 95.16 | 94.32 | 95.59 | 94.56 | 95.45 | 8 | 94.51 | 94.88 |
| 8 | 94.67 | 95.08 | 94.72 | 94.96 | 94.22 | 95.38 | 94.46 | 95.16 | 12 | 94.48 | 94.86 |
| 576 | 94.58 | 94.85 | 94.76 | 94.92 | 82.01 | 82.37 | 82.05 | 82.32 | 576 | 87.09 | 87.09 |
| | | | | | $1/h = 1152$ | | | | | | |
| 1 | 95.06 | 96.06 | 95.16 | 95.70 | 94.51 | 96.52 | 94.47 | 95.95 | 3 | 94.75 | 95.91 |
| 2 | 95.13 | 95.68 | 95.20 | 95.65 | 94.53 | 95.79 | 94.47 | 95.42 | 4 | 94.90 | 95.45 |
| 4 | 95.05 | 95.49 | 95.20 | 95.31 | 94.42 | 95.21 | 94.50 | 95.11 | 8 | 94.83 | 95.13 |
| 8 | 95.15 | 95.47 | 95.18 | 95.20 | 94.39 | 95.03 | 94.47 | 94.85 | 12 | 94.95 | 94.85 |
| 1152 | 94.86 | 94.97 | 94.83 | 94.91 | 82.60 | 82.73 | 82.85 | 82.89 | 1152 | 81.69 | 81.60 |

Notes: CLT-intervals based on the Normal; Boot-intervals based on our proposed new local Gaussian bootstrap (percentile-$t$ intervals); $M$ is the block size used to compute confidence intervals. 10,000 Monte Carlo trials with 999 bootstrap replications each.

**Table 2. Coverage rates of nominal 95% intervals for integrated volatility and integrated beta using the optimal block size**

| | Integrated volatility | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | SV1F | | | | | | | | | SV2F | | | | | | | | |
| | | Raw | | | | Log | | | | | Raw | | | | Log | | | |
| $n$ | $M^*$ | CLT | iidB | WB | Boot | CLT | iidB | WB | Boot | $M^*$ | CLT | iidB | WB | Boot | CLT | iidB | WB | Boot |
| 12 | 3.75 | 86.44 | 93.66 | 87.42 | 96.51 | 90.79 | 96.11 | 88.35 | 96.07 | 3.84 | 80.42 | 90.82 | 79.41 | 93.21 | 86.70 | 93.43 | 80.41 | 93.35 |
| 48 | 5.37 | 90.89 | 94.62 | 93.98 | 95.76 | 92.31 | 95.45 | 94.71 | 95.35 | 5.75 | 85.40 | 92.77 | 89.98 | 93.63 | 87.59 | 94.15 | 90.82 | 93.80 |
| 96 | 5.86 | 93.01 | 94.67 | 94.38 | 95.29 | 93.68 | 95.48 | 94.85 | 95.11 | 5.81 | 88.94 | 94.05 | 92.01 | 94.28 | 90.48 | 94.28 | 93.11 | 94.21 |
| 288 | 5.66 | 94.49 | 94.70 | 94.98 | 95.05 | 94.60 | 94.86 | 94.81 | 94.96 | 5.94 | 93.32 | 94.85 | 94.18 | 95.02 | 93.62 | 94.99 | 94.36 | 94.90 |
| 576 | 5.82 | 94.50 | 94.54 | 94.51 | 94.95 | 94.58 | 94.65 | 94.66 | 94.98 | 6.06 | 94.08 | 94.89 | 94.47 | 95.19 | 94.38 | 94.90 | 94.81 | 95.03 |
| 1152 | 6.01 | 95.05 | 94.87 | 95.13 | 95.04 | 95.15 | 94.85 | 95.14 | 95.02 | 6.25 | 94.41 | 94.83 | 94.56 | 94.92 | 94.38 | 94.92 | 94.76 | 94.83 |

| | Integrated beta | | | |
|---|---|---|---|---|
| $n$ | $M^*$ | CLT | PairsB | Boot |
| 12 | 3.62 | 88.49 | 93.60 | 95.16 |
| 48 | 4.96 | 92.87 | 93.96 | 94.85 |
| 96 | 5.60 | 93.70 | 94.98 | 94.62 |
| 288 | 5.76 | 94.55 | 94.75 | 94.98 |
| 576 | 5.79 | 94.25 | 94.72 | 94.64 |
| 1152 | 5.94 | 94.75 | 94.68 | 94.76 |

Notes: CLT-intervals based on the Normal; iidB-intervals based on the i.i.d. bootstrap of Gonçalves and Meddahi (2009); WB-wild bootstrap based on Proposition 4.5 of Gonçalves and Meddahi (2009); Boot-intervals based on our proposed new local Gaussian bootstrap (percentile-$t$ intervals); PairsB-intervals based on the pairs bootstrap of Dovonon Gonçalves and Meddahi (2013); $M^*$ is the optimal block size selected by using the Minimum Volatility method. 10,000 Monte Carlo trials with 999 bootstrap replications each.

**Table 3. Summary statistics**

| Days | Trans | $n$ | $S$ |
|---|---|---|---|
| 1 Aug | 11303 | 96 | 118 |
| 2 Aug | 13873 | 96 | 145 |
| 3 Aug | 13205 | 96 | 138 |
| 4 Aug | 16443 | 96 | 172 |
| 5 Aug | 16212 | 96 | 169 |
| 8 Aug | 18107 | 96 | 189 |
| 9 Aug | 18184 | 96 | 190 |
| 10 Aug | 15826 | 96 | 165 |
| 11 Aug | 15148 | 96 | 158 |
| 12 Aug | 12432 | 96 | 130 |
| 15 Aug | 12042 | 96 | 126 |
| 16 Aug | 10128 | 96 | 106 |
| 17 Aug | 9104 | 96 | 95 |
| 18 Aug | 15102 | 96 | 158 |
| 19 Aug | 11468 | 96 | 120 |
| 22 Aug | 10236 | 96 | 107 |
| 23 Aug | 11518 | 96 | 120 |
| 24 Aug | 10429 | 96 | 109 |
| 25 Aug | 9794 | 96 | 102 |
| 26 Aug | 9007 | 96 | 94 |
| 29 Aug | 10721 | 96 | 112 |
| 30 Aug | 9131 | 96 | 96 |
| 31 Aug | 10724 | 96 | 112 |

"Trans" denotes the number of transactions, $n$ the sample size used to compute the realized volatility, and sampling of every $S$'th transaction price, so the period over which returns are calculated is roughly 15 minutes.

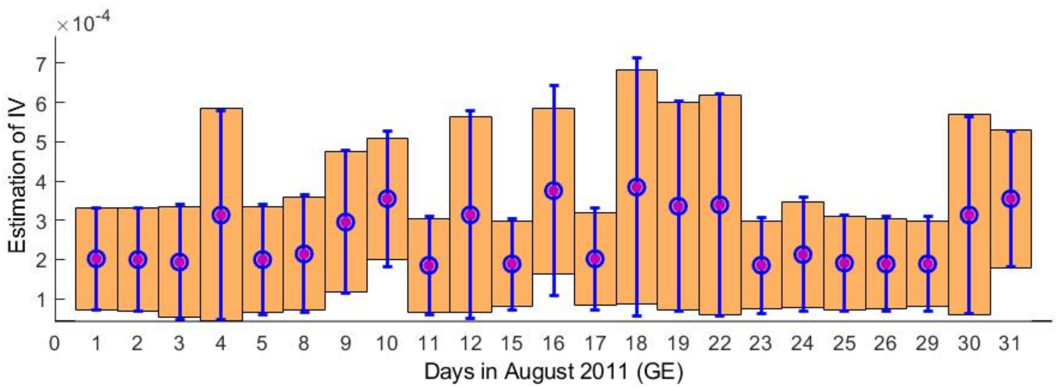


Figure 2: 95% Confidence Intervals (CI's) for the daily $\overline{\sigma^2}$, for each regular exchange opening days in August 2011, calculated using the asymptotic theory of Mykland and Zhang (CI's with bars), and the new wild bootstrap method (CI's with lines). The realized volatility estimator is the middle of all CI's by construction. Days on the $x$-axis.

# Appendix B: Proofs

**Proof of Theorem 3.1 part (a).** Given the definitions of $\hat{V}_{\bar{\sigma^2},h,M} = \frac{2}{c_{M,4}} R_4$, $c_{M,4} = \frac{M+2}{M}$ and $V^*_{\bar{\sigma^2},h,M} = 2R_4$, we can deduce that $V^*_{\bar{\sigma^2},h,M} = \frac{M+2}{M} \hat{V}_{\bar{\sigma^2},h,M}$. The consistency results follow directly by noting that if $M$ is fixed or $M \to \infty$ as $h \to 0$ such that $Mh \to 0$, then as $h \to 0$, we have

$$\hat{V}_{\bar{\sigma^2},h,M} - V_{\bar{\sigma^2}} = o_P(1),$$

and when $M \to \infty$, $\frac{M+2}{M} \to 1$.

**Proof of Theorem 3.1 part (b).** We show that, if $M$ is fixed or $M \to \infty$ as $h \to 0$ such that $M^2 h \to 0$, then as $h \to 0$,

$$\sup_{x \in \Re} \left| P^* \left( \tilde{T}^{*\mathrm{MZ}}_{\bar{\sigma^2},h,M} \leq x \right) - P \left( \tilde{T}^{\mathrm{MZ}}_{\bar{\sigma^2},h,M} \leq x \right) \right| \to 0,$$

in probability under $P$, where $\tilde{T}^{*\mathrm{MZ}}_{\bar{\sigma^2},h,M} = \sqrt{\frac{M}{M+2}} \sqrt{h^{-1}} \left( R_2^* - R_2 \right)$ and $\tilde{T}^{\mathrm{MZ}}_{\bar{\sigma^2},h,M} = \sqrt{h^{-1}} \left( R_2 - \overline{\sigma^2} \right)$. To this end, let $\tilde{T}^{*\mathrm{MZ}}_{\bar{\sigma^2},h,M} = \sum_{j=1}^{1/Mh} z_j^*$, where

$$z_j^* = \sqrt{\frac{M}{M+2}} \sqrt{h^{-1}} \left( RV^*_{j,M} - E^* \left( RV^*_{j,M} \right) \right).$$

Note that $E^* \left( \sum_{j=1}^{1/Mh} z_j^* \right) = 0$, and $Var^* \left( \sum_{j=1}^{1/Mh} z_j^* \right) = \hat{V}_{\bar{\sigma^2},h,M} \xrightarrow{P} V_{\bar{\sigma^2}}$. Moreover, since $z_1^*, \ldots, z_{1/Mh}^*$ are conditionally independent, by the Berry-Esseen bound, for some small $\delta > 0$ and for some constant $C > 0$ (which changes from line to line),

$$\sup_{x \in \Re} \left| P^* \left( \tilde{T}^{*\mathrm{MZ}}_{\bar{\sigma^2},h,M} \leq x \right) - \Phi \left( x / \sqrt{V_{\bar{\sigma^2}}} \right) \right| \leq C \sum_{j=1}^{1/Mh} E^* \left| z_j^* \right|^{2+\delta},$$

which converges to zero in probability under $P$ for any $M \geq 1$ such that $M \approx C h^{-\alpha}$ with $\alpha \in [0, 1/2)$, as $h \to 0$. Indeed, we have that

$$\begin{aligned}
\sum_{j=1}^{1/Mh} E^* \left| z_j^* \right|^{2+\delta} &= \sum_{j=1}^{1/Mh} E^* \left| \sqrt{\frac{M}{M+2}} \sqrt{h^{-1}} \left( RV^*_{j,M} - E^* \left( RV^*_{j,M} \right) \right) \right|^{2+\delta} \\
&\leq 2 \left( \frac{M}{M+2} \right)^{\frac{2+\delta}{2}} h^{\frac{-(2+\delta)}{2}} \sum_{j=1}^{1/Mh} E^* \left| RV^*_{j,M} \right|^{2+\delta} \\
&= 2 \left( \frac{M}{M+2} \right)^{\frac{2+\delta}{2}} h^{\frac{-(2+\delta)}{2}} E^* \left| \frac{\sum_{i=1}^{M} \eta^{*2}_{(j-1)M+i}}{M} \right|^{2+\delta} \sum_{j=1}^{1/Mh} \left| RV_{j,M} \right|^{2+\delta},
\end{aligned}$$

where the inequality follows from the $C_r$ and the Jensen inequalities. Then, given the definitions of

$c_{M,2(2+\delta)}$ and $R_{2(2+\delta)}$, we can write

$$\begin{aligned}\sum_{j=1}^{1/Mh} E^*\left|z_j^*\right|^{2+\delta} &\leq 2\left(\frac{M}{M+2}\right)^{\frac{2+\delta}{2}} c_{M,2(2+\delta)} M^{1+\delta} h^{\frac{\delta}{2}} R_{2(2+\delta)} \\ &\leq C\left(\frac{M}{M+2}\right)^{\frac{2+\delta}{2}} c_{M,2(2+\delta)}^2 h^{\frac{\delta}{2}-\alpha(1+\delta)} \frac{1}{c_{M,2(2+\delta)}} R_{2(2+\delta)}.\end{aligned}$$

Note that for any $\delta > 0$, $M \geq 1$ such that $M \approx Ch^{-\alpha}$ with $\alpha \in [0, 1/2)$, as $h \to 0$, $\left(\frac{M}{M+2}\right)^{\frac{2+\delta}{2}} = O(1)$, $\frac{\delta}{2} - \alpha(1+\delta) > 0$, $\frac{1}{c_{M,2(2+\delta)}} R_{2(2+\delta)} \overset{P}{\to} \overline{\sigma^{2(2+\delta)}} = O_P(1)$, and $c_{M,2(2+\delta)} \to 1$. Thus, results follow, in particular, we have

$$\begin{aligned}\sum_{j=1}^{1/Mh} E^*\left|z_j^*\right|^{2+\delta} &= O_P\left(h^{\frac{\delta}{2}-\alpha(1+\delta)} c_{M,2(2+\delta)}^2\right) \\ &= o_P(1).\end{aligned}$$

To be precise, for a bounded $M$ (i.e., $\alpha = 0$), the consistency result $\frac{1}{c_{M,2(2+\delta)}} R_{2(2+\delta)} \overset{P}{\to} \overline{\sigma^{2(2+\delta)}} = O_P(1)$ follows from Mykland and Zhang (2009). Whereas when $M \to \infty$, in particular for $\alpha \in (0, 1/2)$, the consistency result still holds since as $M \to \infty$, $c_{M,2(2+\delta)} \to 1$ and using e.g., Jacod and Rosenbaum (2013) (cf. equations (3.8) and (3.11)) or the recent work of Li, Todorov and Tauchen (2017) (cf. Theorems 2 and 3), we have $R_{2(2+\delta)} \overset{P}{\to} \overline{\sigma^{2(2+\delta)}}$. See also Theorem 9.4.1 of Jacod and Protter (2012) for similar result.

**Proof of Theorem 3.2.** Given that $T_{\bar{\sigma}^2,h,M}^{\text{MZ}} \equiv \frac{\sqrt{h^{-1}}\left(R_2 - \overline{\sigma^2}\right)}{\sqrt{\hat{V}_{\bar{\sigma}^2,h,M}}} \overset{d}{\to} N(0,1)$, it suffices that $T_{\bar{\sigma}^2,h,M}^{*\text{MZ}} \equiv \frac{\sqrt{h^{-1}}\left(R_2^* - R_2\right)}{\sqrt{\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}}} \overset{d^*}{\to} N(0,1)$ in probability under $P$. Let

$$H_{\bar{\sigma}^2,h,M}^* = \frac{\tilde{T}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}}{\sqrt{\frac{M}{M+2} V_{\bar{\sigma}^2,h,M}^{*\text{MZ}}}} = \frac{\sqrt{h^{-1}}\left(R_2^* - R_2\right)}{\sqrt{V_{\bar{\sigma}^2,h,M}^{*\text{MZ}}}},$$

and note that

$$T_{\bar{\sigma}^2,h,M}^{*\text{MZ}} = H_{\bar{\sigma}^2,h,M}^* \sqrt{\frac{V_{\bar{\sigma}^2,h,M}^*}{\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}}} = H_{\bar{\sigma}^2,h,M}^* \sqrt{\frac{M+2}{M} \frac{\hat{V}_{\bar{\sigma}^2,h,M}}{\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}}},$$

where $V_{\bar{\sigma}^2,h,M}^*$, $\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}$ and $\hat{V}_{\bar{\sigma}^2,h,M}$ are defined in the main text whereas $\tilde{T}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}$ is defined in the proof of Theorem 3.1. Theorem 3.1 proved that $H_{\bar{\sigma}^2,h,M}^* \overset{d^*}{\to} N(0,1)$ in probability under $P$. Thus, it suffices to show that $\frac{\hat{V}_{\bar{\sigma}^2,h,M}}{\frac{M}{M+2}\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}} \overset{P^*}{\to} 1$ in probability under $P$. In particular, we show that (1) $Bias^*\left(\frac{M}{M+2}\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}\right) = 0$, and (2) $Var^*\left(\frac{M}{M+2}\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}}\right) \to 0$, in probability under $P$. Notice that by definition, we have

$$\hat{V}_{\bar{\sigma}^2,h,M}^{*\text{MZ}} = 2\frac{M}{M+2} R_4^* = \frac{2}{c_{M,4}}\left[Mh \sum_{j=1}^{1/Mh}\left(\frac{\chi_{j,M}^2}{M}\right)^2 \left(\frac{RV_{j,M}}{Mh}\right)^2\right] \text{ and } \frac{M}{M+2} V_{\bar{\sigma}^2,h,M}^* = \hat{V}_{\bar{\sigma}^2,h,M}.$$

It follows that

$$\begin{aligned} Bias^{*}\left(\frac{M}{M+2}\hat{V}^{*\mathrm{MZ}}_{\bar{\sigma}^2,h,M}\right) &= \frac{2}{c_{M,4}^{2}}E^{*}\left[Mh\sum_{j=1}^{1/Mh}\left(\frac{\chi^2_{j,M}}{M}\right)^{2}\left(\frac{RV_{j,M}}{Mh}\right)^{2}\right]-\hat{V}_{\bar{\sigma}^2,h,M} \\ &= \frac{2}{c_{M,4}}\left(Mh\sum_{j=1}^{1/Mh}\left(\frac{RV_{j,M}}{Mh}\right)^{2}\right)-\hat{V}_{\bar{\sigma}^2,h,M}=\hat{V}_{\bar{\sigma}^2,h,M}-\hat{V}_{\bar{\sigma}^2,h,M}=0. \end{aligned}$$

Similarly, we have

$$Var^{*}\left(\frac{M}{M+2}\hat{V}^{*\mathrm{MZ}}_{\bar{\sigma}^2,h,M}\right)=\left(\frac{M}{M+2}\right)^{2}Var^{*}\left(\hat{V}^{*}_{\bar{\sigma}^2,h,M}\right), \tag{41}$$

where

$$\begin{aligned} Var^{*}\left(\hat{V}^{*}_{\bar{\sigma}^2,h,M}\right) &= E^{*}\left(\hat{V}^{*}_{\bar{\sigma}^2,h,M}-V^{*}_{\bar{\sigma}^2,h,M}\right)^{2}-\left(E^{*}\left(\hat{V}^{*}_{\bar{\sigma}^2,h,M}-V^{*}_{\bar{\sigma}^2,h,M}\right)\right)^{2} \\ &= M^{2}\left(\frac{c_{M,4}-c_{M,2}^{2}}{c_{M,4}}\right)^{2}(Mh)^{-2}E^{*}\left(\sum_{j=1}^{1/Mh}\left(RV^{2*}_{j,M}-c_{M,4}RV^{2}_{j,M}\right)\right)^{2} \\ &= M^{2}\left(\frac{c_{M,4}-c_{M,2}^{2}}{c_{M,4}}\right)^{2}(Mh)^{-2}\sum_{j=1}^{1/Mh}RV^{4}_{j,M}E^{*}\left(\left(\frac{\chi^2_{j,M}}{M}\right)^{2}-c_{M,4}\right)^{2}, \end{aligned}$$

then given (41), the definitions of $c_{M,2}, c_{M,4}$, $c_{M,8}$ and $R_8$, we can write

$$\begin{aligned} Var^{*}\left(\frac{M}{M+2}\hat{V}^{*\mathrm{MZ}}_{\bar{\sigma}^2,h,M}\right) &= \left(\frac{M}{M+2}\right)^{2}M^{2}\left(\frac{c_{M,4}-c_{M,2}^{2}}{c_{M,4}}\right)^{2}(Mh)^{-2}\left(c_{M,8}-c_{M,4}^{2}\right)\sum_{j=1}^{1/Mh}RV^{4}_{j,M} \\ &= \left(\frac{M}{M+2}\right)^{2}M^{2}\left(\frac{c_{M,4}-c_{M,2}^{2}}{c_{M,4}}\right)^{2}(Mh)\left(c_{M,8}-c_{M,4}^{2}\right)R_8 \\ &= \left(\frac{M}{M+2}\right)^{2}h\left(\frac{2M}{M+2}\right)^{2}\frac{(M+2)(M+4)(M+6)-M(M+2)^{2}}{M^{2}}R_8 \\ &= O_P(Mh)\to 0 \end{aligned}$$

in probability under $P$, as long as $Mh\to 0$ as $h\to 0$, which completes the proof of Theorem 3.2.

**Proof of Theorem 5.1** Part (a) follows under our assumptions by using Theorem A.1 of Gonçalves and Meddahi (2009). The proof of part (b) uses the same technique as in the proof of Theorem A.3 in Gonçalves and Meddahi (2009). In particular, the first four cumulants of $T^{*}_{\bar{\sigma}^2,h,M}$ are given by (e.g., Hall (1992) (cf. p.42)):

$$\begin{aligned} \kappa_1^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right) &= E^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right), \\ \kappa_2^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right) &= E^{*}\left(T^{*2}_{\bar{\sigma}^2,h,M}\right)-\left(E^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right)\right)^{2}, \\ \kappa_3^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right) &= E^{*}\left(T^{*3}_{\bar{\sigma}^2,h,M}\right)-3E^{*}\left(T^{*2}_{\bar{\sigma}^2,h,M}\right)E\left(T^{*}_{\bar{\sigma}^2,h,M}\right)+2\left(E^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right)\right)^{3}, \\ \kappa_4^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right) &= E^{*}\left(T^{*4}_{\bar{\sigma}^2,h,M}\right)-4E^{*}\left(T^{*3}_{\bar{\sigma}^2,h,M}\right)E^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right)-3\left(E^{*}\left(T^{*2}_{\bar{\sigma}^2,h,M}\right)\right)^{2} \\ &\quad +12E^{*}\left(T^{*2}_{\bar{\sigma}^2,h,M}\right)\left(E^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right)\right)^{2}-6\left(E^{*}\left(T^{*}_{\bar{\sigma}^2,h,M}\right)\right)^{4}. \end{aligned}$$

Our goal is to identify the terms of order up to $O_P(h)$ in the asymptotic expansions of these four cumulants. We will first provide asymptotic expansions through order $O(h)$ for the first four moments of $T^*_{\bar{\sigma}^2,h,M}$ by using a Taylor expansion. For a fixed value $k$, a second-order Taylor expansion of $f(x)=(1+x)^{-k/2}$ around 0 yields $f(x)=1-\frac{k}{2}x+\frac{k}{4}(\frac{k}{2}+1)x^2+O(x^3)$. We have that for any fixed integer $k$,

$$\begin{aligned}
T^{*k}_{\bar{\sigma}^2,h,M} &= S^{*k}_{\bar{\sigma}^2,h,M}\left(1+\sqrt{h}U^*_{\bar{\sigma}^2,h,M}\right)^{-k/2}+O_{P^*}(h^{3/2}),\\
&= S^{*k}_{\bar{\sigma}^2,h,M}-\frac{k}{2}\sqrt{h}S^{*k}_{\bar{\sigma}^2,h,M}U^*_{\bar{\sigma}^2,h,M}+\frac{k}{4}(\frac{k}{2}+1)hS^*_{\bar{\sigma}^2,h,M}U^{*2}_{\bar{\sigma}^2,h,M}+O_{P^*}(h^{3/2})\\
&\equiv \hat{T}^{*k}_{\bar{\sigma}^2,h,M}+O_{P^*}(h^{3/2}).
\end{aligned}$$

For $k=1,\cdots,4$, the moments of $T^{*k}_{\bar{\sigma}^2,h,M}$ up to order $O(h^{3/2})$ are given by[9]

$$\begin{aligned}
E^*\left(T^*_{\bar{\sigma}^2,h,M}\right) &= 0-\frac{\sqrt{h}}{2}E^*\left(S^*_{\bar{\sigma}^2,h,M}U^*_{\bar{\sigma}^2,h,M}\right)+\frac{3}{8}hE^*\left(S^*_{\bar{\sigma}^2,h,M}U^{*2}_{\bar{\sigma}^2,h,M}\right)\\
E^*\left(T^{*2}_{\bar{\sigma}^2,h,M}\right) &= 1-\sqrt{h}E^*\left(S^{*2}_{\bar{\sigma}^2,h,M}U^*_{\bar{\sigma}^2,h,M}\right)+hE^*\left(S^{*2}_{\bar{\sigma}^2,h,M}U^{*2}_{\bar{\sigma}^2,h,M}\right)\\
E^*\left(T^{*3}_{\bar{\sigma}^2,h,M}\right) &= E^*\left(S^{*3}_{\bar{\sigma}^2,h,M}\right)-\sqrt{h}\frac{3}{2}E^*\left(S^{*3}_{\bar{\sigma}^2,h,M}U^*_{\bar{\sigma}^2,h,M}\right)+\frac{15}{8}hE^*\left(S^{*3}_{\bar{\sigma}^2,h,M}U^{*2}_{\bar{\sigma}^2,h,M}\right)\\
E^*\left(T^{*4}_{\bar{\sigma}^2,h,M}\right) &= E^*\left(S^{*4}_{\bar{\sigma}^2,h,M}\right)-2\sqrt{h}E^*\left(S^{*4}_{\bar{\sigma}^2,h,M}U^*_{\bar{\sigma}^2,h,M}\right)+3hE^*\left(S^{*4}_{\bar{\sigma}^2,h,M}U^{*2}_{\bar{\sigma}^2,h,M}\right).
\end{aligned}$$

where we used $E^*\left(S^*_{\bar{\sigma}^2,h,M}\right)=0$, and $E^*\left(S^{*2}_{\bar{\sigma}^2,h,M}\right)=1$. By Lemma C1.2 in Hounyo (2018), we have that

$$\begin{aligned}
E^*\left(T^*_{\bar{\sigma}^2,h,M}\right) &= \sqrt{h}\left(-\frac{A_1}{2}R_{6,4}\right)+O_P(h^{3/2}),\\
E^*\left(T^{*2}_{\bar{\sigma}^2,h,M}\right) &= 1+\sqrt{h}\left(\left(C_1-A_2\right)R_{8,4}+C_2R^2_{6,4}\right)+O_P(h^2)\\
E^*\left(T^{*3}_{\bar{\sigma}^2,h,M}\right) &= \sqrt{h}\left(B_1-\frac{3}{2}A_3\right)R_{6,4}+O_P(h^{3/2})\\
E^*\left(T^{*4}_{\bar{\sigma}^2,h,M}\right) &= 3+h\left(\left(B_2-2D_1+3E_1\right)R_{8,4}+\left(3E_2-2D_2\right)R^2_{6,4}\right)+O_P(h^2).
\end{aligned}$$

Thus $\kappa^*_1\left(T^*_{\bar{\sigma}^2,h,M}\right)=-\sqrt{h}\frac{A_1}{2}R_{6,4}$, this proves the first result. The remaining results follow similarly.

For part (c), results follows by noting that $\frac{1}{c_{M,q}}R_q\to\overline{\sigma^q}$ in probability under $P$. For fixed $M$, use results in Section 4.1 of Myklang and Zhang (2009), whereas when $M\to\infty$ use Jacod and Rosenbaum (2013) (cf. equations (3.8) and (3.11)).

**Proof of Theorem 4.1.** We prove results for $f(z)=z$; the delta method implies the result for nonlinear $f$.

Part (a). Result follows from Theorem 3 of Li, Todorov and Tauchen (2017) by noting the following: First the elements of $\Pi^*_{h,M}$ are all of the form of $Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{k'l'(j)}$. Second, since $X$ is continuous in our framework, the truncation in equation (3.1) of Li, Todorov and Tauchen (2017) is useless: one may use a threshold $v_n=\infty$ which reduce to our definition of $\hat{\Gamma}_{kl(j)}$.

[9] To be strictly rigorous, we should be taking expected value of $\hat{T}^{*k}_{\bar{\sigma}^2,h,M}$ rather than of $T^{*k}_{\bar{\sigma}^2,h,M}$. See e.g. Hall (1992) (cf. p. 72) or Gonçalves and Meddahi (2009) (cf. p. 302) for similar approach.

Part (b). Given the definition of $S_{h,M}^*$, we have

$$S_{h,M}^* = \sqrt{h^{-1}} \sum_{j=1}^{1/Mh} \sum_{i=1}^{M} \left( x_{i+(j-1)M}^* - x_{i+(j-1)M} \right) = \sqrt{h^{-1}} \sum_{j=1}^{1/Mh} \left[ \sum_{i=1}^{M} x_{i+(j-1)M}^* - E^* \left( \sum_{i=1}^{M} x_{i+(j-1)M}^* \right) \right].$$

The proof follows from showing that for any $\lambda \in \mathbb{R}^{\frac{d(d+1)}{2}}$ such that $\lambda'\lambda = 1$, $\sup_{x\in\mathbb{R}} |P^*(\sum_{j=1}^{1/Mh} \tilde{x}_j^* \leq x) - \Phi(x/(\lambda'\Pi\lambda))| \xrightarrow{P} 0$, where $\tilde{x}_j^* = \sqrt{h^{-1}}\lambda' \left[ \sum_{i=1}^{M} x_{i+(j-1)M}^* - E^* \left( \sum_{i=1}^{M} x_{i+(j-1)M}^* \right) \right]$. Note that, $E^* \left( \sum_{j=1}^{1/Mh} \tilde{x}_j^* \right) = 0$ and $Var^* \left( \sum_{j=1}^{1/Mh} \tilde{x}_j^* \right) = \lambda \Pi_{h,M}^* \lambda \xrightarrow{P} \lambda'\Pi\lambda$ by part (a). Thus, by Katz's (1963) Berry-Essen Bound, for some small $\varepsilon > 0$ and some constant $C > 0$ which changes from line to line,

$$\sup_{x\in\mathbb{R}} \left| P^* \left( \sum_{j=1}^{1/Mh} \tilde{x}_j^* \leq x \right) - \Phi(x/(\lambda'\Pi\lambda)) \right| \leq C \sum_{j=1}^{1/Mh} E^*|\tilde{x}_j^*|^{2+\epsilon}.$$

Next, we show that $\sum_{j=1}^{1/Mh} E^*|\tilde{x}_j^*|^{2+\varepsilon} = o_p(1)$. We have that

$$\begin{aligned}
\sum_{j=1}^{1/Mh} E^*|\tilde{x}_j^*|^{2+\varepsilon} &= \sum_{j=1}^{1/Mh} E^* \left| \sqrt{h^{-1}}\lambda' \left[ \sum_{i=1}^{M} x_{i+(j-1)M}^* - E^* \left( \sum_{i=1}^{M} x_{i+(j-1)M}^* \right) \right] \right|^{2+\varepsilon} \\
&\leq 2^{2+\epsilon} h^{\frac{-(2+\varepsilon)}{2}} \sum_{j=1}^{1/Mh} E^* \left| \lambda' \sum_{i=1}^{M} x_{i+(j-1)M}^* \right|^{2+\varepsilon} \\
&\leq 2^{2+\epsilon} h^{\frac{-(2+\varepsilon)}{2}} \sum_{j=1}^{1/Mh} E^* \left| \sum_{i=1}^{M} x_{i+(j-1)M}^* \right|^{2+\varepsilon} \\
&\leq 2^{2+\epsilon} C h^{\frac{-(2+\varepsilon)}{2}} M^{1+\varepsilon} \sum_{j=1}^{1/Mh} \sum_{i=1}^{M} E^* \left| x_{i+(j-1)M}^* \right|^{2+\varepsilon},
\end{aligned}$$

where the first inequality follows from the $C_r$ and the Jensen inequalities; the second inequality uses the Cauchy-Schwarz inequality and the fact that $\lambda'\lambda = 1$; and the third inequality uses the $C_r$ inequality.

We let $|z|^2 = (z'z)$ for any vector $z$. Then, we have

$$\begin{aligned}
\sum_{j=1}^{1/Mh} E^* |\tilde{x}_j^*|^{2+\varepsilon} &\leq Ch^{\frac{-(2+\varepsilon)}{2}} M^{1+\varepsilon} \sum_{j=1}^{1/Mh} \sum_{i=1}^{M} E^* \left| \sum_{k=1}^{d} \sum_{l=1}^{d} \left( y_{k,i+(j-1)M}^* y_{l,i+(j-1)M}^* \right)^2 \right|^{1+\varepsilon/2} \\
&\leq Ch^{\frac{-(2+\varepsilon)}{2}} M^{1+\varepsilon} \sum_{j=1}^{1/Mh} \sum_{i=1}^{M} E^* \left| y_{k,i+(j-1)M}^* y_{l,i+(j-1)M}^* \right|^{2+\varepsilon} \\
&\leq Ch^{\frac{2+\varepsilon}{2}} \underbrace{E^* \left| \eta_{k,1}^* \right|^{2(2+\varepsilon)}}_{=O(1)} M^{2+\varepsilon} \sum_{j=1}^{1/Mh} \left| \hat{\Gamma}_{kl(j)} \right|^{2+\varepsilon} \\
&\quad + Ch^{\frac{2+\varepsilon}{2}} \underbrace{E^* \left| \eta_{k,1}^* \eta_{l,1}^* \right|^{2+\varepsilon}}_{=O(1)} M^{2+\varepsilon} \sum_{j=1}^{1/Mh} \left| \sqrt{\hat{\Gamma}_{kk(j)} \hat{\Gamma}_{ll(j)} - \hat{\Gamma}_{kl(j)}^2} \right|^{2+\varepsilon} \\
&\leq C \underbrace{h^{\frac{\varepsilon}{2}} M^{1+\varepsilon}}_{=o(1)} \underbrace{Mh \sum_{j=1}^{1/Mh} \left[ \left| \hat{\Gamma}_{kl(j)} \right|^{2+\varepsilon} + \left| \hat{\Gamma}_{kk(j)} \hat{\Gamma}_{ll(j)} \right|^{\frac{2+\varepsilon}{2}} \right]}_{=O_P(1)} = o_P(1).
\end{aligned}$$

where the second and third inequalities follow from the $C_r$ inequality, the third inequality uses in addition the definition of $y_{k,i+(j-1)M}^* y_{l,i+(j-1)M}^*$. Finally, the last inequality and the consistency follow from Minkowski inequality, Lemma C2.4 in Hounyo (2018) and the fact that for any $M$ such that $M \approx Ch^{-\alpha}$ with $\alpha \in \left(0, \frac{\varepsilon}{2(1+\varepsilon)}\right)$, $h^{\frac{\varepsilon}{2}} M^{1+\varepsilon} \to 0$, as $h \to 0$.

Part (c). Given that $T_h \overset{d}{\to} N\left(0, I_{\frac{d(d+1)}{2}}\right)$, it suffices that $T_{h,M}^* \overset{d^*}{\to} N\left(0, I_{\frac{d(d+1)}{2}}\right)$ in probability under $P$. Next, note that from Lemma C2.5 in Hounyo (2018)

$$\hat{\Pi}_{h,M}^* - \Pi_{h,M}^* \overset{P^*}{\longrightarrow} 0,$$

in probability under $P$. In addition, both $\hat{\Pi}_{h,M}^*$ and $\Pi_{h,M}^*$ are non singular in large samples with probability approaching one, as $h \to 0$. Then, using results in parts (a) and (b) of Theorem 4.1, it follows that

$$T_{h,M}^* = \underbrace{\left(\hat{\Pi}_{h,M}^*\right)^{-1/2} \left(\Pi_{h,M}^*\right)^{1/2}}_{\overset{P^*}{\to} I_{\frac{d(d+1)}{2}}} \underbrace{\left(\Pi_{h,M}^*\right)^{-1/2} S_{h,M}^*}_{\overset{d^*}{\to} N\left(0, I_{\frac{d(d+1)}{2}}\right)} \overset{d^*}{\to} N\left(0, I_{\frac{d(d+1)}{2}}\right),$$

in probability-$P$.

**Proof of Theorem 5.2.** Part (a) follows under our assumed conditions by using Theorem 5.2 of Dovonon et al. (2013). The proof of part (b) use the same technique as in the proof of Theorem A.3 in Gonçalves and Meddahi (2009). Given equation (C2.1) and Lemma C2.10 in Hounyo (2018), we may decompose

$$T_{\beta,h,M}^* = S_{\beta,h,M}^* \left(1 + \sqrt{h} \left(\check{U}_{\beta,h,M}^* + U_{4,\beta,h,M}^* + O_{P^*}\left(\sqrt{h}\right)\right)\right)^{-1/2}.$$

Hence, for any fixed integer $k$, we have that

$$T_{\beta,h,M}^{*k} = S_{\beta,h,M}^{*k} \left(1 - \sqrt{h} \frac{k}{2} \left(\check{U}_{\beta,h,M}^* + U_{4,\beta,h,M}^*\right)\right) + O_{P^*}(h) \equiv \tilde{T}_{\beta,h,M}^* + O_{P^*}(h).$$

For $k = 1, 2, 3$, the moments of $\tilde{T}^*_{\beta,h,M}$ are given by

$$E^*\left(\tilde{T}^{*k}_{\beta,h,M}\right) = E^*\left(S^{*k}_{\beta,h,M}\right) - \sqrt{h}\frac{k}{2}E^*\left(S^{*k}_{\beta,h,M}\check{U}^*_{\beta,h,M}\right) - \sqrt{h}\frac{k}{2}E^*\left(S^{*k}_{\beta,h,M}U^*_{4,\beta,h,M}\right).$$

The first and third cumulants of $\tilde{T}^*_{\beta,h,M}$ are given by

$$\begin{aligned}
\kappa_1^*\left(\tilde{T}^*_{\beta,h,M}\right) &= E^*\left(\tilde{T}^*_{\beta,h,M}\right)\\
\kappa_3^*\left(\tilde{T}^*_{\beta,h,M}\right) &= E^*\left(\tilde{T}^{*3}_{\beta,h,M}\right) - 3E^*\left(\tilde{T}^{*2}_{\beta,h,M}\right)E^*\left(\tilde{T}^*_{\beta,h,M}\right) + 2\left[E^*\left(\tilde{T}^*_{\beta,h,M}\right)\right]^3.
\end{aligned}$$

By Lemma C2.9 in Hounyo (2018), we deduce

$$\begin{aligned}
E^*\left(\tilde{T}^*_{\beta,h,M}\right) &= -\frac{\sqrt{h}}{\tilde{B}^{*3/2}_{h,M}}\left[Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)} + 3Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}\right]\\
&-\frac{\sqrt{h}}{\tilde{B}^{*3/2}_{h,M}}\left[-3\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)} - \hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)}\right]\\
&-\frac{\sqrt{h}}{\tilde{B}^{*3/2}_{h,M}}\left[-6\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)} - 2\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)} + 8\hat{\beta}^2_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)}\right]\\
&-\frac{\sqrt{h}}{\tilde{B}^{*3/2}_{h,M}}\left[4\hat{\beta}^2_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)} - 4\hat{\beta}^3_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)}\right] + \sqrt{h}\frac{2\tilde{A}^*_{0,h,M}}{\sqrt{\tilde{B}^*_{h,M}}\sum_{i=1}^{1/h}y^2_{k,i}}\\
&-\frac{\sqrt{h}}{\tilde{B}^{*3/2}_{h,M}M}\left[3Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)} + 5Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} - 19\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)}\right]\\
&-\frac{\sqrt{h}}{\tilde{B}^{*3/2}_{h,M}M}\left[-5\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)} + 24\hat{\beta}^2_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)} - 8\hat{\beta}^3_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)}\right].
\end{aligned}$$

The result follows since we can write,

$$E^*\left(\tilde{T}^*_{\beta,h,M}\right) = \sqrt{h}\left(\frac{2\tilde{A}^*_{0,h,M}}{\sqrt{\tilde{B}^*_{h,M}}\sum_{i=1}^{1/h}y^2_{k,i}} - \frac{\tilde{A}^*_{1,h,M}}{2\tilde{B}^{*3/2}_{h,M}} + \frac{\tilde{R}^*_{1,h,M}}{M}\right),$$

where $\tilde{A}^*_{0,h,M}$, $\tilde{A}^*_{1,h,M}$, $\tilde{B}^*_{h,M}$ and $\tilde{R}^*_{1,h,M}$ are defined in the main text. The remaining results follow similarly. For part (c), apply Theorem 3 of Li, Todorov and Tauchen (2017).

**Proof of Theorem 6.1** For any $\varepsilon > 0$, $\delta > 0$, letting $\mathcal{A}_{h,M} \equiv \left\{P^*\left(\left|Z^*_{h,M}\right| > \delta\right) > \varepsilon\right\}$, we have that $Z^*_{h,M} \to^{P^*} 0$, as $h \to 0$, in probability under $P$, if for any $\varepsilon > 0$, $\delta > 0$, $\lim_{h\to 0} P\left(\mathcal{A}_{h,M}\right) = 0$. This is equivalent to $\lim_{h\to 0} Q_{h,M}\left(\mathcal{A}_{h,M}\right) = 0$, since for any sequence $\mathcal{B}_{h,M}$ of sets, $P(\mathcal{B}_{h,M}) \to 0$ if and only if $Q_{h,M}(\mathcal{B}_{h,M}) \to 0$ (i.e., $P$ and $Q_{h,M}$ are contiguous). It follows then that $Z^*_{h,M} \to^{P^*} 0$, as $h \to 0$, in probability under $Q_{h,M}$. The inverse follows similarly.

**Proof of Theorem 6.2** For part (a), the proof follows the same steps as the proof of $V_{\check{\beta},h,M}$ which we explain in the main text, in particular, given the definition of $\hat{\beta}_{lk}^*$, we have that

$$\begin{aligned} V_{\check{\beta},h,M}^* &= Var^*\left(\sqrt{h^{-1}}(\check{\beta}_{lk}^* - \check{\beta}_{lk})\right) \\ &= M^2 h \sum_{j=1}^{1/Mh} Var^*\left(\check{\beta}_{lk(j)}^* - \check{\beta}_{lk(j)}\right) \\ &= M^2 h \sum_{j=1}^{1/Mh} E^*\left(\left(\sum_{i=1}^{M} y_{k,i+(j-1)M}^{2*}\right)^{-1}\right)\hat{V}_{(j)} \\ &= \frac{M^2 h}{M-2}\sum_{j=1}^{1/Mh}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2 = \frac{M-1}{M-2}\hat{V}_{\check{\beta},h,M}, \end{aligned}$$

then the result follows, given Lemma C3.13 in Hounyo (2018) or part (a4) of Lemma C3.14 in Hounyo (2018).

For part (b), we have $\sqrt{\frac{M-2}{M-1}}\sqrt{h^{-1}}\left(\check{\beta}_{lk}^* - \check{\beta}_{lk}\right) = \sum_{j=1}^{1/Mh} z_{j,\check{\beta}}^*$, where

$$z_{j,\check{\beta}}^* = \sqrt{\frac{M-2}{M-1}} M\sqrt{h}\left(\sum_{i=1}^{M} y_{k,i+(j-1)M}^{*2}\right)^{-1}\left(\sum_{i=1}^{M} y_{k,i+(j-1)M}^* u_{i+(j-1)M}^*\right).$$

Note that $E^*\left(z_{j,\check{\beta}}^*\right) = 0$, and that

$$Var^*\left(\sum_{j=1}^{1/Mh} z_j^*\right) = \frac{M-2}{M-1}V_{\check{\beta},h,M}^* = \hat{V}_{\check{\beta},h,M} \xrightarrow{P} V_{\check{\beta}},$$

by part (a) moreover, since $z_{1,\check{\beta}}^*, \ldots, z_{1/Mh,\check{\beta}}^*$ are conditionally independent, by the Berry-Esseen bound, for some small $\delta > 0$ and for some constant $C > 0$,

$$\sup_{x\in\Re}\left|P^*\left(\sqrt{\frac{M-2}{M-1}}\sqrt{h^{-1}}(\check{\beta}_{lk}^* - \check{\beta}_{lk}) \le x\right) - \Phi\left(\frac{x}{V_{\check{\beta}}}\right)\right| \le C\sum_{j=1}^{1/Mh} E^*\left|z_{j,\check{\beta}}^*\right|^{2+\delta},$$

Next, we show that $\sum_{j=1}^{1/Mh} E^*\left|z_{j,\check{\beta}}^*\right|^{2+\delta} = o_p(1)$. We have that

$$\begin{aligned} \sum_{j=1}^{1/Mh} E^*\left|z_{j,\check{\beta}}^*\right|^{2+\delta} &= \left(\frac{M-2}{M-1}\right)^{\frac{2+\delta}{2}}\left(M\sqrt{h}\right)^{2+\delta}\sum_{j=1}^{1/Mh} E^*\left(\left(\sum_{i=1}^{M} y_{k,i+(j-1)M}^{*2}\right)^{-(2+\delta)}\left|\sum_{i=1}^{M} y_{k,i+(j-1)M}^* u_{i+(j-1)M}^*\right|^{2+\delta}\right) \\ &\equiv \left(\frac{M-2}{M-1}\right)^{\frac{2+\delta}{2}}\left(M\sqrt{h}\right)^{2+\delta}\sum_{j=1}^{1/Mh} E^*\left(A_j^* B_j^*\right), \end{aligned}$$

it follows then by using Cauchy-Schwarz inequality that

$$\begin{aligned} E^*\left(A_j^* B_j^*\right) &\leq \sqrt{E^*\left(\sum_{i=1}^{M} y_{k,i+(j-1)M}^{*2}\right)^{-2(2+\delta)}} \sqrt{E^*\left(\left|\sum_{i=1}^{M} y_{k,i+(j-1)M}^{*} u_{i+(j-1)M}^{*}\right|^{2(2+\delta)}\right)} \\ &\leq \mu_{2(2+\delta)} b_{M,4(2+\delta)}^{\frac{1}{2}} M^{1+\frac{\delta}{2}} \hat{\Gamma}_{k(j)}^{-\frac{2+\delta}{2}} \hat{V}_{(j)}^{\frac{2+\delta}{2}} \\ &= \mu_{2(2+\delta)} b_{M,4(2+\delta)}^{\frac{1}{2}} \left(\frac{\hat{\Gamma}_{l(j)}}{\hat{\Gamma}_{k(j)}} - \left(\frac{\hat{\Gamma}_{lk(j)}}{\hat{\Gamma}_{k(j)}}\right)^2\right)^{\frac{2+\delta}{2}}, \end{aligned}$$

where the second inequality used parts (a1) and (a2) of Lemma C3.15 in Hounyo (2018) and $\mu_{2(2+\delta)} = E\left|v\right|^{2(2+\delta)}$ with $v \sim N(0,1)$. Finally, given the definition of $R_{\beta,2+\delta}$, we can write

$$\begin{aligned} \sum_{j=1}^{1/Mh} E^*\left|z_{j,\breve{\beta}}^*\right|^{2+\delta} &\leq \left(\frac{M-2}{M-1}\right)^{\frac{2+\delta}{2}} \mu_{2(2+\delta)} b_{M,4(2+\delta)}^{\frac{1}{2}} M^{2+\delta} h^{1+\frac{\delta}{2}} \sum_{j=1}^{1/Mh} \left(\frac{\widehat{C}_{ll(j)}}{\widehat{C}_{kk(j)}}\right)^{2+\delta} \\ &= \left(\frac{M-2}{M-1}\right)^{\frac{2+\delta}{2}} \mu_{2(2+\delta)} b_{M,4(2+\delta)}^{\frac{1}{2}} \left(\frac{M-1}{M}\right)^{\frac{2+\delta}{2}} b_{M,2+\delta} c_{M-1,2+\delta} M^{1+\delta} h^{\frac{\delta}{2}} R_{\beta,2+\delta} \\ &= O_P\left(h^{\frac{\delta}{2}} b_{M,4(2+\delta)}^{\frac{1}{2}} \left(\frac{M-2}{M}\right)^{\frac{2+\delta}{2}} b_{M,2+\delta} c_{M-1,2+\delta}\right) \\ &= o_P(1). \end{aligned}$$

Since for any $\delta > 0$, $\mu_{2(2+\delta)} = E\left|v\right|^{2(2+\delta)} \leq \Delta < \infty$ where $v \sim N(0,1)$, moreover as $h \to 0$, $c_{M-1,2+\delta} = O(1)$, $b_{M,4(2+\delta)} = O(1)$, $b_{M,2+\delta} = O(1)$ and by using Lemma C3.14 in Hounyo (2018) we have $R_{\beta,2+\delta} = O_P(1)$.

**Proof of Theorem 6.3** Let

$$H_{\breve{\beta},h,M}^* = \frac{\sqrt{\frac{M-2}{M-1}}\sqrt{h^{-1}}(\breve{\beta}_{lk}^* - \breve{\beta}_{lk})}{\sqrt{\hat{V}_{\breve{\beta},h,M}}} = \frac{\sqrt{h^{-1}}(\breve{\beta}_{lk}^* - \breve{\beta}_{lk})}{\sqrt{V_{\breve{\beta},h,M}^*}},$$

and note that

$$T_{\breve{\beta},h,M}^* = H_{\breve{\beta},h,M}^* \sqrt{\frac{V_{\breve{\beta},h,M}^*}{\hat{V}_{\breve{\beta},h,M}^*}},$$

where $\hat{V}_{\breve{\beta},h,M}^*$ is defined in the main text. Theorem 6.2 proved that $H_{\breve{\beta},h,M}^* \xrightarrow{d^*} N(0,1)$ in probability. Thus, it suffices to show that $\hat{V}_{\breve{\beta},h,M}^* - V_{\breve{\beta},h,M}^* \xrightarrow{P^*} 0$ in probability under $Q_{h,M}$ and $P$. In particular, we show that (1) $Bias^*\left(\hat{V}_{\breve{\beta},h,M}^*\right) = 0$, and (2) $Var^*\left(\hat{V}_{\breve{\beta},h,M}^*\right) \xrightarrow{P} 0$. The result follows directly by using the bootstrap analogue of parts (a1), (a2) and (a3) of Lemma C3.13 in Hounyo (2018).

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