# The Real Computational Complexity of Minmax Value and Equilibrium Refinements in Multi-player Games*

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**Abstract**

We show that for several solution concepts for finite $n$-player games, where $n \geq 3$, the task of simply verifying its conditions is computationally equivalent to the decision problem of the existential theory of the reals. This holds for trembling hand perfect equilibrium, proper equilibrium, and CURB sets in strategic form games and for (the strategy part of) sequential equilibrium, trembling hand perfect equilibrium, and quasi-perfect equilibrium in extensive form games of perfect recall. For obtaining these results we first show that the decision problem for the minmax value in $n$-player games, where $n \geq 3$, is also equivalent to the decision problem for the existential theory of the reals.

Our results thus improve previous results of NP-*hardness* as well as Sqrt-Sum-*hardness* of the decision problems to *completeness* for $\exists\mathbb{R}$, the complexity class corresponding to the decision problem of the existential theory of the reals. As a byproduct we also obtain a simpler proof of a result by Schaefer and Štefankovič giving $\exists\mathbb{R}$-completeness for the problem of deciding existence of a probability constrained Nash equilibrium.

## 1 Introduction

From a computational point of view, finite games with three or more players present unique challenges compared to finite 2-player games. This is already indicated by the example due to Nash of a 3-player game with no rational Nash equilibrium [31], but even more strikingly so by constructions of Bubelis [6] and Datta [13]. More precisely, Bubelis constructs a 3-player game with a unique Nash equilibrium giving as equilibrium payoff an arbitrary algebraic number to some player, and Datta shows that any real algebraic variety is isomorphic to the set of fully mixed Nash equilibria in a 3-player game.

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The problem of *computing* a Nash equilibrium in finite strategic form games was characterized in seminal work by Daskalakis, Goldberg, and Papadimitriou [12] and Chen and Deng [10] as PPAD-complete for 2-player games and by Etessami and Yannakakis [16] as FIXP-complete for $n$-player games, when $n \geq 3$.

While a Nash equilibrium is guaranteed to exist, one might be interested in Nash equilibria satisfying certain properties, e.g. having at least a certain social welfare. The corresponding decision problems were characterized as being NP-complete for 2-player games by Gilboa and Zemel [20] and by Conitzer and Sandholm [11]. For the analogous problems in games with three or more players a precise characterization was only obtained very recently. Schaefer and Štefankovič [35] obtained the first such result giving $\exists\mathbb{R}$-completeness for deciding the existence of a probability constrained Nash equilibrium. Subsequent work by Garg et al. [18] and by Bilò and Mavronicolas [3] extended this to $\exists\mathbb{R}$-completeness for all the analogous decision problems studied for two players. Thus these problems are not only to be considered computationally intractable, as implied by the corresponding results for 2-player games, but are in fact computationally equivalent to each other. Prior to obtaining $\exists\mathbb{R}$-completeness this was an open problem.

In this paper we are interested in several standard *refinements* of Nash equilibrium, all guaranteed to exist. Since any such refinement is in particular a Nash equilibrium it follows that computing these is PPAD-hard for 2-player games and FIXP-hard for $n$-player games, when $n \geq 3$. For 2-player games in strategic form, computing a (symbolic) proper equilibrium was shown to be in PPAD by Sørensen [38]. For 2-player games in extensive form with perfect recall, computing a quasi-perfect equilibrium was shown to be in PPAD by Miltersen and Sørensen [29], and computing a trembling hand perfect equilibrium was shown to be in PPAD by Farina and Gatti [17]. Together these results completely settle the complexity of computing the Nash equilibrium refinements considered in this paper for the case of 2-player games. For $n$-player games in strategic form, computing an approximation to a trembling hand perfect equilibrium was shown to be in $\text{FIXP}_a$ by Etessami et al. [15]. Recently Hansen and Lund [24] showed that computing an approximation to a proper equilibrium is likewise in $\text{FIXP}_a$. For $n$-player games in extensive form with perfect recall, Etessami [14] proved membership in $\text{FIXP}_a$ for approximating a quasi-perfect equilibrium and for approximating trembling hand perfect equilibrium. Thus these results settle the complexity of approximating the Nash equilibrium refinements considered in this paper for the case of $n$-player games, when $n \geq 3$.

In this paper we will not be concerned with the task of actually computing these Nash equilibrium refinements, but rather with the task of *verifying* their conditions. Note that verifying the conditions of a Nash equilibrium in a given strategic form game is computationally a trivial task. In contrast to this Hansen, Miltersen, and Sørensen [25] showed both NP-hardness and Sqrt-Sum-hardness for verifying the conditions of the standard refinements we consider here in $n$-player games, where $n \geq 3$. Here NP-hardness indicates that each of the verification problems are computationally intractable, whereas Sqrt-Sum-hardness indicate that the problems might not even be contained in NP. Here we show that the problems are $\exists\mathbb{R}$-complete, thus computationally equivalent to each other and to the decision problem for the existential theory of the reals.

The complexity of the corresponding problems for 2-player games is not completely settled. In strategic form games, trembling hand perfect equilibrium coincide with admissible equilibrium, which can be verified in polynomial time [40]. Recently Hansen and Lund [24] showed that verifying that a Nash equilibrium is proper is in fact NP-complete. Thus unless P = NP, the verification problems are *not* computationally equivalent for 2-player games. In extensive form games with perfect recall, Gatti and Panozzo [19] showed that verifying that a Nash equilibrium is a quasi-perfect equilibrium can be done in polynomial time and Gatti, Gilli, and Panozzo [32] showed the same for (the strategy part of) sequential equilibrium. The complexity of verifying that a Nash equilibrium in an extensive form game with perfect recall is trembling hand perfect remains to be settled.

In addition to Nash equilibrium refinements we also consider the solution concept of CURB sets of games in strategic form. For 2-player games Benisch et al. [2] gave polynomial time algorithms to compute all minimal CURB sets and to verify CURB sets. For $n$-player games Hansen, Miltersen, and Sørensen [25] showed both coNP-hardness and SQRT-SUM-hardness for verifying CURB sets. Here we show that the problem is $\forall\mathbb{R}$-complete.

Like the earlier NP-hardness and SQRT-SUM-hardness results by Hansen, Miltersen, and Sørensen [25], we prove $\exists\mathbb{R}$-hardness for verifying the conditions of a solution concept by reduction from the decision problem for computing the minmax value in 3-player games. Thus we first establish $\exists\mathbb{R}$-completeness for this problem. As a byproduct we are able to obtain a simpler and more direct proof of $\exists\mathbb{R}$-completeness of deciding the existence of a probability constrained Nash equilibrium compared to the original proof of Schaefer and Štefankovič.

# 2 Preliminaries

## 2.1 The existential theory of the reals

The decision problem for the existential theory of the reals, ETR, is that of deciding validity of sentences of the form $\exists x_1, \ldots, x_n \in \mathbb{R} : \phi(x_1, \ldots, x_n)$, where $\phi$ is a quantifier free formula with inequalities and equalities involving polynomials with rational coefficients as its atoms. It is easy to see that ETR is NP-hard (cf. [8]); on the other hand, the decision procedure by Canny [9] shows that ETR belongs to PSPACE.

The view of the decision problem of the existential theory of the reals as a complexity class existed only implicitly in the literature, e.g. [8, 37], until being studied more extensively under the name NPR by Bürgisser and Cucker [7] and then under the name $\exists\mathbb{R}$ by Schaefer and Štefankovič [33, 35]. In this paper we shall adopt the naming $\exists\mathbb{R}$. We shall also denote co$\exists\mathbb{R}$ by $\forall\mathbb{R}$.

Bürgisser and Cucker defined the class as being the constant-free Boolean part $BP^0(NP_\mathbb{R})$ of the class $NP_\mathbb{R}$, which is the analogue to NP in the Blum-Shub-Smale model of computation. They showed a large number of problems to be complete for $\exists\mathbb{R}$ or $\forall\mathbb{R}$. Interestingly, the corresponding problems with real-valued inputs are not known to be complete for $NP_\mathbb{R}$ or $coNP_\mathbb{R}$, but rather complete for classes derived from $NP_\mathbb{R}$ and $coNP_\mathbb{R}$ with additional "exotic" quantifiers, if classified as complete at all. Schaefer and Štefankovič simply defined the class $\exists\mathbb{R}$ as the closure of ETR under polynomial time many-one reductions, and proved in particular $\exists\mathbb{R}$-completeness for deciding the existence of a probability constrained Nash equilibrium in 3-player games. We will adopt this latter definition of $\exists\mathbb{R}$ in this paper since we will not make direct use of the Blum-Shub-Smale model of computation.

**Definition 1.** *$\exists\mathbb{R}$ is the class of languages that are reducible to* ETR *by a polynomial time reduction.*

## 2.2 Finite Games

We give here detailed but brief definitions of finite games in strategic form and extensive form and we define the notions of minmax value and Nash equilibrium. Definitions of Nash equilibrium *refinements* are given later.

Denote the standard $n$-simplex $\{x \in \mathbb{R}^{n+1} \mid x \geq 0 \wedge \sum_{i=1}^{n+1} x_i = 1\}$ by $\Delta^n$. We shall identify $\Delta^{n-1}$ with the set of probability distributions on $[n]$. We may further view $[n]$ as the subset of $\Delta^{n-1}$ consisting of the Dirac measures on $[n]$.

**Strategic Form Games.** A game $G$ in *strategic form* with $m$ players is defined as follows. Player $i$ has a finite set of *pure strategies* together with a utility function. We shall identify the pure strategies for Player $i$ with the set $[n_i]$. The utility function for Player $i$ is then a function $u_i : [n_1] \times \cdots \times [n_m] \to \mathbb{R}$.

A *(mixed) strategy* for Player $i$ is a probability distribution $x_i \in \Delta^{n_i-1}$ over pure strategies. It is *fully mixed* if it has full support. A strategy $x_i$ for each player $i$ together forms a *strategy profile* $x = (x_1, \ldots, x_m) \in \Delta^{n_1} \times \cdots \times \Delta^{n_m-1} \subseteq \mathbb{R}^n$, with $n = n_1 + \cdots + n_m$. The utility functions are naturally extended to strategy profiles by letting $u_1(x_1, \ldots, x_m) = \mathrm{E}_{a_i \sim x_i}\, u_1(a_1, \ldots, a_m)$. The value $u_i(x_1, \ldots, x_m)$ is called the *payoff* to Player $i$.

For a strategy profile $x$, denote $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. For $y \in \Delta^{n_i-1}$ and assuming that $i$ is given by the context, we define $x \setminus y$ to be the strategy profile $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m)$ and likewise write $(x_{-i}; y) = (x \setminus y)$. We say that $y$ is a *best reply* for Player $i$ against $x$ if $u_i(x \setminus y) \geq u_i(x \setminus y')$ for all $y' \in \Delta^{n_i-1}$. Note that Player $i$ always has a pure strategy best reply. A strategy profile $x$ is a *Nash equilibrium* if $x_i$ is a best reply against $x$ for all $i$. The *minmax value* of Player $i$ is defined as $\min_x \max_y u_i(x \setminus y)$. The minmax value is thus the minimum utility of a best reply. When $x$ attains the minimum, we call $x_{i-1}$ a minmax strategy profile.

**Extensive Form Games.** A game $G$ in *extensive form* of imperfect information with $m$ players is defined as follows. The structure of the game is described by an ordered tree $T$, i.e., a tree with a distinguished root and for each non-leaf node an ordering of its immediate successors. To each leaf node $\ell$ is associated a *utility vector* $u^{(\ell)} \in \mathbb{R}^m$. Let $P_0, \ldots, P_m$ be a partition of the non-leaf nodes of $T$ and let $L$ be the set of leaf nodes of $T$. For a non-leaf node $v$, denote by $S(v)$ the set of immediate successor nodes and define $n_v = |S(v)|$. The set $S(v)$ is ordered and we denote by $S(v, j)$ the $j$-th immediate successor of $v$. For nodes $v$ and $w$, write $v \preceq w$ if either $v = w$ or $w$ is a successor of $v$.

The nodes in $P_0$ are *chance nodes* and each such node $v$ is given a probability distribution in $\Delta^{n_v-1}$. The nodes in $P_i$ are the *decision points* of Player $i$. The set $P_i$ is further partitioned into a set $\mathcal{I}_i$ of *information sets*. The partition must satisfy that $n_v = n_w$ for all $v, w \in I$, all $I \in \mathcal{I}_i$ and all $i$. We denote this common number by $n_I$. Let $\mathcal{I} : \bigcup_{i=1}^m P_i \to \bigcup_{i=1}^m \mathcal{I}_i$ be the unique function satisfying that $v \in \mathcal{I}(v)$ for all non-leaf nodes $v$. In other words is $\mathcal{I}(v)$ the information set to which $v$ belong.

We say that Player $i$ has *perfect recall* in $G$ if for every $v \in P_i$, every $1 \le j \le n_{\mathcal{I}(v)}$, every $w \in P_i$ such that $S(v,j) \preceq w$, and every $w' \in \mathcal{I}(w)$ there exist $v' \in \mathcal{I}(v)$ such that $S(v',j) \preceq w'$. We say that $G$ is of perfect recall if all players have perfect recall.

Player $i$ has associated to each information set $I \in \mathcal{I}_i$ a finite set of actions of size $n_I$ which we shall identify with the set $[n_I]$. A pure strategy for Player $i$ specifies an action for each information set in $\mathcal{I}_i$, and is thus an element of the set $\bigtimes_{I \in \mathcal{I}_i}[n_I]$. A mixed strategy for Player $i$ is a probability distribution over pure strategies. It is fully mixed if it has full support. A *local strategy* at an information set $I \in \mathcal{I}_i$ is a probability distribution over the actions of $I$ and is thus an element of $\Delta^{n_I-1}$. A *behavior strategy* $b_i$ for Player $i$ specifies a local strategy $b_{i,I}$ for each information set in $\mathcal{I}_i$ and is thus an element of $\bigtimes_{I \in \mathcal{I}_i} \Delta^{n_I-1}$. A behavior strategy induces a mixed strategy that is the corresponding product distribution, and we will thus view a behavior strategy as a mixed strategy. Note that a behavior strategy is fully mixed if and only if the probability distribution of each information set has full support.

Like for strategic form games, a mixed strategy $x_i$ for each player $i$ together forms a strategy profile $x = (x_1, \dots, x_m)$. Similarly, define $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$ and $x \setminus y = (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_m)$, where $y$ is a mixed strategy $y$ for Player $i$. Sampling a pure strategy from $x$ for each player as well as sampling from each probability distribution given at every $w \in P_0$, defines a unique path from the root to a leaf. For a node $v$ of $G$, we define the *realization probability* $p(v,x)$ of $v$ given $x$ to be the probability that $v$ is on this path. Note in particular that this defines a probability distribution on $L$.

Kuhn [27] proved that if Player $i$ has perfect recall in $G$, then for every mixed strategy $y$ of Player $i$ there is a behavior strategy $y'$ of Player $i$ such that $p(v, x \setminus y) = p(v, x \setminus y')$ for all strategy profiles $x$ and all nodes $v$. Thus in games of perfect recall we may effectively restrict our attention to behavior strategies.

The utility function $u_i$ of Player $i$ is defined by the probability distribution induced on the leaf nodes by the given strategy profile. Namely, we define, $u_i(x) = \mathrm{E}_\ell \, u_i^{(\ell)}$. The definition of best reply and Nash equilibrium are then defined analogously to the case of strategic form games. We say $y$ is a best reply for Player $i$ against $x$ if $u_i(x \setminus y) \ge u_i(x \setminus y')$ for all Player $i$ strategies $y'$. A strategy profile is a Nash equilibrium if $x_i$ is a best reply against $x$ for all $i$.

Let $b$ be a behavior strategy profile, let $I \in \mathcal{I}_i$ be an information set for Player $i$, and let $d$ be a local strategy at $I$. Let $b_i \setminus_I d$ denote the behavior strategy for Player $i$ obtained from $b_i$ by replacing $b_{i,I}$ by $d$. We then further let $b \setminus_I d$ denote the strategy $b \setminus (b_i \setminus_I d)$. We say that $d$ is a *local best reply* at $I$ for Player $i$ if $u_i(b \setminus_I d) \ge u_i(b \setminus_I d')$ for all local strategies $d'$ at $I$.

# 3 $\exists\mathbb{R}$-completeness of minmax value in 3-player games

It is well-known that the problem of deciding whether a system of quadratic equations over the reals has a solution is complete for $\exists\mathbb{R}$. By results of Grigoriev and Vorobjov [21, 41], whenever a polynomial system has a solution, it has a solution whose coordinates are bounded in magnitude by a doubly-exponential function in the size of the encoding of the system. Combining this with repeated squaring, Schaefer [34] noted that deciding whether a system of quadratic equations has a solution remains $\exists\mathbb{R}$-hard even with a promise that a possible solution can be assumed to be contained in a constant bounded region, more precisely the unit ball $\mathrm{B}(\mathbf{0},1)$.

**Proposition 1** (Schaefer)**.** *It is $\exists\mathbb{R}$-hard to decide if a system of quadratic equations has a solution under the promise that either the system has no solutions or a solution exists in the unit ball $\mathrm{B}(\mathbf{0},1)$.*

For our purposes we need the similar statement for the interior of the standard corner simplex. While that is straightforward to derive from Proposition 1, for completeness we give the full reduction starting from any system of quadratic equations, closely following the structure of the proof by Schaefer.

Analogously with the standard simplex $\Delta^n$, we denote the standard corner $n$-simplex $\{x \in \mathbb{R}^n \mid x \geq 0 \wedge \sum_{i=1}^n x_i \leq 1\}$ by $\Delta_{\mathrm{c}}^n$.

**Proposition 2.** *It is $\exists\mathbb{R}$-hard to decide if a system of quadratic equations has a solution under the promise that either the system has no solutions or a solution $z$ exists that is in the interior of $\Delta_{\mathrm{c}}^n$ and also satisfies $z_i \leq \frac{1}{2}$ for all $i$ and that $\sum_{i=1}^n z_i \geq \frac{1}{2}$.*

*Proof.* Let $p_1,\ldots,p_s \in \mathbb{Z}[x_1,\ldots,x_k]$ be a system of quadratic polynomials with all coefficients having bit-length at most $\tau$. By [21, Lemma 10], if the system has a solution, a solution exists in $\mathrm{B}(\mathbf{0},R)$ for $R > 0$ satisfying $R = 2^{\tau 2^{O(k)}}$. Choose an integer $t = O(\log \tau + k)$ such that $2^{2^t} > 32kR$.

We now define a system of $s{+}t{+}1$ polynomials $q_1,\ldots,q_{s+t+1}$ in $n = 2k{+}t{+}1$ variables, $x_1^+,\ldots,x_k^+,x_1^-,\ldots,x_k^-,y_0,\ldots,y_t$. For $j \in \{1,\ldots,s\}$, let $q_j$ be the polynomial

$$y_0^2 \cdot p_j\left((x_1^+ - x_1^-)/y_0,\ldots,(x_k^+ - x_k^-)/y_0\right) \ ,$$

let $q_{s+1}$ be the polynomial $y_t - \frac{1}{2}$, and finally for $j \in \{1,\ldots,t\}$ let $q_{s+1+j}$ be the polynomial $y_{j-1} - y_j^2$.

Suppose first that $x \in \mathrm{B}(\mathbf{0},R)$ is a solution of the given system. In particular we have $-R \leq x_i \leq R$ for all $i \in \{1,\ldots,k\}$. Define $y_j = 2^{-2^{t-j}}$ for $j \in \{0,\ldots,t\}$, thereby making $q_{s+j}$ zero for $j \in \{0,\ldots,t\}$. Next define

$$x_i^+ = y_0 \cdot \max(R, R + x_i) \quad \text{and} \quad x_i^- = y_0 \cdot \max(R, R - x_i) \ .$$

It follows that $(x_i^+ - x_i^-)/y_0 = x_i$, thereby making $q_j$ zero for $j \in \{1,\ldots,s\}$. It remains to show that $z = (x_1^+,\ldots,x_k^+,x_1^-,\ldots,x_k^-,y_0,\ldots,y_t)$ satisfies the promise. Clearly by definition all variables are strictly positive and furthermore we have $y_j \leq \frac{1}{2}$ for all $j \in \{0,\ldots,t\}$. We now bound the sum of the variables from above. First,

$$\sum_{j=0}^{t} y_j < \sum_{j=1}^{\infty} 2^{-2^j} < \frac{1}{2} + \frac{1}{4} + \frac{1}{16}\sum_{j=0}^{\infty} 2^{-j} = \frac{7}{8} \ .$$

By the choice of $t$ we have $x_i^+, x_i^- \leq 2R2^{-2^t} < \frac{1}{16k}$ and it follows that

$$\sum_{i=1}^{k} x_i^+ + x_i^- < \frac{2k}{16k} = \frac{1}{8} \ ,$$

and the required promise is thus satisfied.

Conversely, suppose now that $z = (x_1^+, \dots, x_k^+, x_1^-, \dots, x_k^-, y_0, \dots, y_t)$ is a solution to the defined system. We must then have that $y_j = 2^{-2^{t-j}}$ for $j \in \{0, \dots, t\}$, and in particular $y_0 > 0$. Then we have a well-defined solution to the given system by letting $x_i = (x_i^+ - x_i^-)/y_0$ for $i \in \{1, \dots, k\}$. □

Next we will translate the resulting system of quadratic equations into a homogeneous bilinear system over the standard simplex.

**Proposition 3.** *It is $\exists\mathbb{R}$-complete to decide if a system of homogeneous bilinear equations $q_k(x, y) = 0$ has a solution $x, y \in \Delta^n$. It remains $\exists\mathbb{R}$-hard under the promise that either the system has no such solution or a solution $(x, x)$ exists where $x$ belong to the relative interior of $\Delta^n$ and further satisfies $x_i \leq \frac{1}{2}$ for all $i$.*

*Proof.* Containment in $\exists\mathbb{R}$ is straightforward. We show hardness by reduction from the promise problem of Proposition 2. Let $p_1, \dots, p_s \in \mathbb{Z}[x_1, \dots, x_n]$ be a system of quadratic polynomials. Write $p_k(x_1, \dots, x_n) = \sum_{\leq i \leq j} a_{ij}^{(k)} x_i x_j + \sum_i b_i^{(k)} x_i + c^{(k)}$ and define the homogeneous bilinear form $q_k(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1})$ by

$$q_k(x, y) = \sum_{1 \leq i \leq j \leq n} a_{ij}^{(k)} x_i y_j + \sum_{i=1}^{n} \sum_{j=1}^{n+1} b_i^{(k)} x_i y_j + \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} c^{(k)} x_i y_j \ ,$$

for all $k \in \{1, \dots, s\}$. Additionally, for $j = 1, \dots, n$, define

$$q_{s+j}(x, y) = \sum_{i=1}^{n+1} x_i y_j - x_j y_i \ .$$

Essentially we just have introduced the slack variable $x_{n+1} = 1 - \sum_{i=1}^{n} x_i$, replaced quadratic terms $x_i x_j$ by bilinear quadratic terms $x_i y_j$, homogenized using the equality $\sum_{i=1}^{n+1} x_i = 1$, and finally expressed that $x = y$.

Suppose now that $x$ belongs to the interior of $\Delta_{\mathrm{c}}^n$ and furthermore satisfies $x_i \leq \frac{1}{2}$ for all $i$ and that $\sum_{i=1}^{n} x_i \geq \frac{1}{2}$. Let $x_{n+1} = 1 - \sum_{i=1}^{n} x_i$. We then have $0 < x_{n+1} \leq \frac{1}{2}$ as well. Suppose now that $p_k(x_1, \dots, x_n) = 0$ for all $k \in \{1, \dots, s\}$. Clearly we then have $q_k(x_1, \dots, x_{n+1}, x_1, \dots, x_{n+1}) = 0$ for all $k \in \{1, \dots, s + n\}$ as well.

Conversely, if $x, y \in \Delta^n$ and $q_k(x, y) = 0$ for all $k \in \{1, \dots, s + n\}$ we must have $x_j = y_j$ using $q_{s+j}(x, y) = 0$ for $j = 1, \dots, n$. Then removing again the slack variable from $x$ we have $p_k(x_1, \dots, x_n) = 0$ for all $k = 1, \dots, s$, using that $q_k(x, x) = 0$ for all $k \in \{1, \dots, s\}$. □ □

We can now finally translate the homogeneous system into a 3-player game where each polynomial is translated into a pair of strategies for Player 1, and a minmax strategy profile for Player 2 and Player 3 obtaining minmax value 0 will correspond to a solution to system.

**Theorem 1.** *It is $\exists\mathbb{R}$-complete to decide for a given 3-player game in strategic form if the minmax value for Player 1 is at most 0. It remains $\exists\mathbb{R}$-hard under the promise that either the minmax value for Player 1 is strictly greater than 0 or the minmax value for Player 1 is equal to 0 and Player 2 and Player 3 can* *enforce this using a fully mixed strategy profile* $(x, y)$*, satisfying* $0 < x_i = y_i \leq \frac{1}{2}$ *for all* $i$*, and against which all strategies of Player 1 yield payoff 0. The game can furthermore be assumed to satisfy that* $u_1(2k-1, i, j) = -u_1(2k, i, j)$ *for all* $i, j, k$*.*

*Proof.* Containment in $\exists\mathbb{R}$ is straightforward. Namely, the question to decide is existence of $x \in \Delta^{n_2-1}$ and $y \in \Delta^{n_3-1}$ such that $\sum_{i,j} u_1(k, i, j)x_i y_j \leq 0$ for all $k$.

We will show hardness by reduction from the promise problem given by Proposition 3. Let $q_1(x, y), \ldots, q_m(x, y)$ be homogeneous bilinear polynomials in variables $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$. We form a 3-player game where Player 2 and Player 3 each have $n + 1$ strategies given by the set $\{1, \ldots, n+1\}$ and Player 1 has $2m$ strategies given by the set $\{1, \ldots, 2m\}$. Write $q_k(x, y) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij}^{(k)} x_i y_j$ for $k \in \{1, \ldots, m\}$. The payoff to Player 1 is then simply given by

$$u_1(2k-1, i, j) = -u_1(2k, i, j) = a_{ij}^{(k)} \ .$$

It follows that if $(x, y)$ is a strategy profile for Player 2 and Player 3, or equivalently $x, y \in \Delta^n$, then

$$\mathop{\mathrm{E}}_{i\sim x, j\sim y}[u_1(2k-1, i, j)] = -\mathop{\mathrm{E}}_{i\sim x, j\sim y}[u_1(2k, i, j, )] = q_k(x, y) \ ,$$

for all $k \in \{1, \ldots, m\}$.

Suppose now that $(x, x)$ is a solution to the given system, where $x$ belong to the relative interior of $\Delta^n$ and further satisfies $x_i \leq \frac{1}{2}$ for all $i$. Then $(x, x)$ is a fully mixed strategy such that $\mathrm{E}_{i\sim x, j\sim y}[u_1(k, i, j)] = 0$ for all $k$, which in particular means that $\max_k \mathrm{E}_{i\sim x, j\sim x}[u_1(k, i, j)] \leq 0$.

Conversely, if $(x, y)$ is a strategy profile such that $\max_k \mathrm{E}_{i\sim x, j\sim x}[u_1(k, i, j)] \leq 0$, it follows that $q_k(x, y) \leq 0$ and $-q_k(x, y) \leq 0$ for all $k$. Thus $q_k(x, y) = 0$ for all $k$, and we have a solution $x, y \in \Delta^n$ of the given system. □ □

## 3.1 $\exists\mathbb{R}$-completeness of constrained Nash equilibrium

The decision problem NashInABall is the following: Given a game $G$ in strategic form and a rational $r$, decide whether $G$ has a Nash equilibrium strategy profile in which all pure strategies of all players are played with probability at most $r$.

Schaefer and Štefankovič [35] showed that NashInABall is $\exists\mathbb{R}$-complete for 3-player games. Their hardness proof goes via Brouwer functions and uses the rather involved transformation of Etessami and Yannakakis [16] of Brouwer functions into 3-player games using an intermediate construction of 10-player games. Here we give a simpler and more direct proof of $\exists\mathbb{R}$-hardness of NashInABall as a consequence of Theorem 1.

**Theorem 2** (Schaefer and Štefankovič)**.** NashInABall $\exists\mathbb{R}$*-complete.*

*Proof.* We show $\exists\mathbb{R}$-hardness by reduction from the minmax problem in 3-player games. Let $G$ be the 3-player game given by Theorem 1. We next give each player an additional pure strategy $\perp$. The payoffs are as follows. The payoff to all players is 0 when at least one player plays $\perp$. For the strategy combinations where no player plays $\perp$, the payoff to Player 1 is the same as it would have been in $G$, and the payoffs to Player 2 and Player 3 is the *negative* of the payoff to Player 1.

Suppose first that the minmax value of Player 1 in $G$ is 0. Let $(\tau_2, \tau_3)$ be a minmax strategy profile for Player 2 and Player 3 in which each strategy is played with probability at most $\frac{1}{2}$, and against which all strategies of Player 1 yield payoff 0. Let $\tau_1$ be the uniform strategy profile for Player 1 in $G$. We claim that $\tau = (\tau_1, \tau_2, \tau_3)$ is a Nash equilibrium. Indeed, since Player 1 plays uniformly, all players receive payoff 0 regardless of the strategies of Player 2 and Player 3, since in $G$ we have $u_1(2k-1, i, j) = -u_1(2k, i, j, )$ for all $i, j, k$. Furthermore, since Player 2 and Player 3 play the minmax strategy profile $(\tau_2, \tau_3)$, all players receive payoff 0 regardless of the strategy of Player 1. In other words, no player can gain from deviating which means that $\tau$ is a Nash equilibrium. Finally note that every pure strategy is played with probability at most $\frac{1}{2}$ in $\tau$.

Suppose now that $\tau = (\tau_1, \tau_2, \tau_3)$ is a Nash equilibrium profile in which all pure strategies of all players are played with probability at most $\frac{1}{2}$. In particular each player plays $\perp$ with probability at most $\frac{1}{2}$. Let $\tau' = (\tau'_1, \tau'_2, \tau'_3)$ be the strategy profile where $\tau'_i$ is the conditional distribution obtained from $\tau_i$ given that Player $i$ does not play the strategy $\perp$. We claim that $(\tau'_2, \tau'_3)$ is a minmax strategy profile in $G$ ensuring minmax value 0 for Player 1. Indeed, since for a strategy combination where no player plays $\perp$, Player 1 receives the negative payoff of Player 2 and Player 3, the strategy profile $\tau'$, which is played with nonzero probability, must give all players payoff 0, since otherwise either Player 1 or Player 2 and Player 3 would gain from playing $\perp$ with probability 1. Also, Player 1 does not have a best reply in $G$ to $(\tau'_2, \tau'_3)$ ensuring payoff strictly greater than 0, which proves the claim. □ □

# 4 ∃ℝ-completeness of equilibrium refinements

In this section we prove ∃ℝ-completeness for decision problems associated with several equilibrium refinements. These results directly improve NP-hardness and SQRT-SUM-hardness results obtained by Hansen, Miltersen and Sørensen [25], and thereby settles their complexity exactly. The previous NP-hardness and SQRT-SUM-hardness results were proved by reduction from the minmax value problem we considered in Section 3. These reductions do not work directly for our purposes however, since they actually reduced from a *gap* version of the minmax value problem, deciding whether the minmax value is either *strictly less* or *strictly greater* than a given number $r$. A gap version of the problem is NP-hard even with any inverse polynomial gap [5, 23]. SQRT-SUM-hardness of the gap problem follows directly SQRT-SUM-hardness of the normal problem, using the the fact that a sum of square-roots can be compared for equality to a rational in polynomial time as shown by Blömer [4][1]. For ∃ℝ-hardness it is not possible to ensure such a gap; fortunately we are able to modify the reductions to circumvent this, instead relying on our ability to directly assume fully mixed minmax strategy profiles of Player 2 and Player 3.

[1]This crucial point of the reduction by Hansen, Miltersen and Sørensen was unfortunately omitted in the paper [25].

In contrast to the hardness results of Hansen, Miltersen and Sørensen we obtain *completeness* results. Showing that the problems that arise from Nash equilibrium refinements are contained in $\exists\mathbb{R}$ is *not* straightforward and hinges on the fact that one may construct a virtual infinitesimal by means of repeated squaring. As shown by Bürgisser and Cucker [7, Theorem 9.2] and Schaefer [34, Lemma 3.12] (cf. [35, Lemma 4.1]) this extends the power of the existential theory of the reals with universal quantification over an arbitrarily small $\varepsilon > 0$.

**Lemma 1.** *The problem of deciding validity of the sentence*

$$\exists \epsilon_0 > 0 \ \forall \varepsilon \in (0, \varepsilon_0) \ \exists x : \varphi(\varepsilon, x)$$

*when given a quantifier-free formula over the reals $\varphi(\varepsilon, x)$ belongs to $\exists\mathbb{R}$.*

The Nash equilibrium refinements concepts we consider can all be expressed as being limit points $x$ of a sequence $x^{(\varepsilon)}$ of points that can be expressed by an existentially quantified formula, which in many of the cases is straightforward to derive. Then containment in $\exists\mathbb{R}$ follows from Lemma 1.

## 4.1 Nash Equilibrium Refinements in Strategic Form Games

We first consider the concept of trembling hand perfect equilibrium, introduced by Selten [36]. We state an equivalent definition due to Myerson [30].

**Definition 2** (Trembling hand perfect equilibrium)**.** *Let $G$ be a $m$-player game in strategic form. A strategy profile $\sigma$ for $G$ is an $\varepsilon$-perfect equilibrium if and only if it is fully mixed and only pure strategies that are best replies get probability more than $\varepsilon$. A strategy profile $\sigma$ for $G$ is a trembling hand perfect equilibrium if and only if it is the limit point of a sequence of $\varepsilon$-perfect equilibria with $\varepsilon \to 0^+$.*

**Theorem 3.** *For any $m \geq 3$ it is $\exists\mathbb{R}$-complete to decide if a given pure strategy Nash equilibrium of a given $m$-player game in strategic form is trembling hand perfect.*

*Proof.* It is not hard to express that a given strategy profile in a $m$-player game is a trembling hand perfect equilibrium by a first-order formula over the reals. We reproduce below such a formulation by Etessami et al. [15] for completeness and illustration. Suppose that player $i$ has $n_i$ strategies. A strategy profile is thus a tuple $x \in \Delta^{n_1-1} \times \cdots \times \Delta^{n_m-1} \subseteq \mathbb{R}^n$, $n = n_1 + \cdots + n_m$. We first express the utility functions evaluated at a strategy profile of the form $x \setminus k$ as polynomials

$$u_i(x \setminus k) = \sum_{a_{-i}} u_i(a_{-i}; k) \prod_{j \neq i} x_{j,a_j} \ ,$$

where $a$ ranges over all pure strategy profiles and $k$ is a pure strategy of Player $i$. Let EPS-PE$(x', \varepsilon)$ be the quantifier-free formula defined by the conjunction of the following formulas, together expressing that $x'$ is an $\varepsilon$-perfect equilibrium.

$$x'_{i,k} > 0, \text{ for } i = 1, \ldots, m \text{ and } k = 1, \ldots, n_i$$

$$x'_{i,1} + \cdots + x'_{i,n_i} = 1, \text{ for } i = 1, \ldots, m$$

$$(u_i(x' \setminus k) \geq u_i(x' \setminus \ell)) \vee (x'_{i,k} \leq \varepsilon), \text{ for } i = 1, \ldots, m \text{ and } k, \ell = 1, \ldots, n_i$$

We can now express that $x$ is a trembling hand perfect equilibrium by

$$\forall \varepsilon > 0\ \exists x' : \text{EPS-PE}(x', \varepsilon) \wedge \|x - x'\|_2^2 \le \varepsilon \ .$$

Thus deciding whether a given strategy profile $x$ is a trembling hand perfect equilibrium is contained in $\exists\mathbb{R}$ by Lemma 1.

In order to show hardness for $\exists\mathbb{R}$ we reduce from the promise minmax problem given by Theorem 1. Let $G$ be the given 3-player game in strategic form. We define a new game $G'$ from $G$ where each player is given an additional pure strategy $\perp$. The payoffs to Player 2 and Player 3 are 0 for all pure strategy combinations. The payoff to Player 1 is 0 when at least one player plays $\perp$. For the pure strategy combinations where no player plays $\perp$ the payoff to Player 1 is the same as it would have been in $G$.

Clearly, in $G'$ the strategy profile $\mu = (\perp, \perp, \perp)$ is a Nash equilibrium. We claim that the minmax value for Player 1 in $G$ is 0 precisely when $\mu$ is a trembling hand perfect equilibrium in $G'$.

Suppose first that the minmax value for Player 1 in $G$ is 0 and let $(\tau_2, \tau_3)$ be a minmax strategy profile of Player 2 and Player 3, which we by assumption can assume is fully mixed and against which all strategies of Player 1 yield payoff 0. Let $\tau_1$ be the uniform strategy for Player 1 in $G$, and let $\tau = (\tau_1, \tau_2, \tau_3)$. Define for $k \ge 1$

$$\sigma^{(k)} = \left(1 - \frac{1}{k}\right)\mu + \frac{1}{k}\tau \ .$$

By assumption and construction $\sigma^{(k)}$ is a fully mixed strategy profile of $G'$ converging to $\mu$ as $k$ increases. Note that all strategies of all players are best replies to $\sigma^{(k)}$. Thus $\mu$ is trembling hand perfect.

Suppose next that the minmax value for Player 1 in $G$ is strictly greater than 0. Suppose $\sigma^{(k)} = (\sigma_1^{(k)}, \sigma_2^{(k)}, \sigma_3^{(k)})$ is a sequence of fully mixed strategy profiles converging to $\mu$. Since $\sigma_2^{(k)}$ and $\sigma_3^{(k)}$ do not place probability 1 on $\perp$, Player 1 has a reply to $(\sigma_2^{(k)}, \sigma_3^{(k)})$ with an expected payoff strictly larger than 0. Thus $\perp$ is not a best reply of Player 1 to $(\sigma_2^{(k)}, \sigma_3^{(k)})$. Since this holds for all sequences $\sigma^{(k)}$ it follows that $\mu$ is not trembling hand perfect. □ □

With minimal changes we obtain the same result for the concept of proper equilibrium of Myerson [30] that further refines that of a trembling hand perfect equilibrium.

**Definition 3** (Proper equilibrium)**.** *Let $G$ be a $m$-player game in strategic form. A strategy profile $\sigma$ for $G$ is an $\varepsilon$-proper equilibrium if and only if is fully mixed, and the following condition holds: Given two pure strategies, $k$ and $\ell$, of the same player, if $k$ is a worse reply against $\sigma$ than $\ell$, then $\sigma$ must assign a probability to $k$ that is at most $\varepsilon$ times the probability it assigns to $\ell$. A strategy profile $\sigma$ for $G$ is a proper equilibrium if and only if is the limit point of a sequence of $\varepsilon$-proper equilibria with $\varepsilon \to 0^+$.*

**Theorem 4.** *For any $m \ge 3$ it is $\exists\mathbb{R}$-complete to decide if a given pure strategy Nash equilibrium of a given $m$-player game in strategic form is proper.*

*Proof.* We can show containment in $\exists\mathbb{R}$ exactly in the same way as the proof of Theorem 3 by replacing the last set of formulas making up EPS-PE by

$$(u_i(x \setminus k) \ge u_i(x \setminus \ell)) \vee (x_{i,k} \le \varepsilon x_{i,\ell}), \text{ for } i = 1, \ldots, m \text{ and } k, \ell = 1, \ldots, n_i \ .$$

For establishing ∃ℝ-hardness we can simply observe that this follows already from the proof of Theorem 3. When the minmax value in $G$ is 0, the sequence $\sigma^{(k)}$ establishes that $\mu$ is even a proper equilibrium, since all strategies of all players are best replies to $\sigma^{(k)}$. When the minmax value in $G$ is strictly greater than 0, since $\mu$ is not trembling hand perfect $\mu$ is not proper as well. □ □

## 4.2 Nash Equilibrium Refinements in Extensive Form Games

For extensive form games we first consider the concept of sequential equilibrium, introduced by Kreps and Wilson [26]. Kreps and Wilson defined a sequential equilibrium to be a pair $(b, \mu)$, which is called an assessment, consisting of a behavior strategy profile $b$ together with a belief profile $\mu$. The belief profile $\mu$ specifies for each information set $I$ a probability distribution on the nodes contained in $I$. Here we will only be concerned with the strategy part $b$ of a sequential equilibrium. For this reason we will adopt an alternative definition of (the strategy part of) a sequential equilibrium, also given by Kreps and Wilson [26, Proposition 6]. Kreps and Wilson used the terminology *weakly perfect equilibrium* for this.

**Definition 4** (Sequential equilibrium)**.** *Let $G$ be a $m$-player game in extensive form with imperfect information but having perfect recall and let $u$ be the utilities of $G$. A behavior strategy $b$ is (the strategy part of) a sequential equilibrium if there is a sequence $(b^{(k)}, u^{(k)})$, where $b^{(k)}$ is a fully mixed behavior strategy profile and $u^{(k)}$ utilities with $\lim_{k\to\infty}(b^{(k)}, u^{(k)}) = (b, u)$ and for every $k$ and $i$, $b_i^{(k)}$ is a best reply to $b^{(k)}$ with respect to utilities $u_i^{(k)}$.*

**Theorem 5.** *For any $m \geq 3$ it is ∃ℝ-complete to decide if a given pure strategy profile of a given $m$-player game in extensive form is part of a sequential equilibrium.*

*Proof.* We first sketch the proof for containment in ∃ℝ. Let $u$ be the utilities of $G$. First we express by a quantifier free formula $\varphi(b', u')$ that $b'$ is a fully mixed behavior strategy profile and that $b'_i$ is a best response to $b'$ with respect to utilities $u'_i$ for all $i$. It is not hard to see that if $b'$ is fully mixed then $b'_i$ is a best reply to $b'$ with respect to $u'_i$ if and only if $b'_{i,I}$ is a local best reply to $b'$ with respect to $u'_i$ at every information set $I \in \mathcal{I}_i$ (cf. [40, Theorem 6.2.1]). Also, to verify that $b'_{i,I}$ is a local best reply we need only to compare against pure local strategies. We further note that $b' \setminus_I d$ may be written as a polynomial in variables $b'$, $d$ and $u'_i$ which is of size polynomially bounded in the size of the game. These observations now allows us to express $\varphi(b', u')$ in a simple way and we can finally express that $b$ is (the strategy part of) a sequential equilibrium by

$$\forall \varepsilon > 0\ \exists b', u' : \varphi(b', u') \wedge \|b - b'\|_2^2 \leq \varepsilon \wedge \|u - u'\|_2^2 \leq \varepsilon\ .$$

By Lemma 1 it then follows that deciding whether a given behavior strategy profile $b$ is (the strategy part of) a sequential equilibrium is contained in ∃ℝ.

For establishing ∃ℝ hardness we again reduce from the promise minmax problem given by Theorem 1. Let $G$ be the given 3-player game in strategic form. We define an extensive form game $G'$ from $G$ as follows. The players each get to choose an action of $G$ in turn, without revealing the choice to the other players. Player 2 chooses first, then Player 3, and finally Player 1. In $G'$ each player has a single information set thus making $G'$ strategically equivalent to $G$. The payoffs to Player 2 and Player 3 are 0 for all combinations of actions. The payoff to Player 1 is inherited from $G$. We now give each player an additional pure action $\perp$ in their respective information set. Choosing this action results in the game ending immediately with all players receiving payoff 0.

Clearly, in $G'$ the behavior strategy profile $b = (\perp, \perp, \perp)$ is a Nash equilibrium. We claim that the minmax value for Player 1 in $G$ is 0 precisely when $b$ is (the strategy part of) a sequential equilibrium.

Suppose first that the minmax value for Player 1 in $G$ is 0 and let $(\tau_2, \tau_3)$ be a minmax strategy profile of Player 2 and Player 3, which we by assumption can assume is fully mixed and against which all strategies of Player 1 yield payoff 0. Let $\tau_1$ be the uniform strategy for Player 1 in $G$ and let $\tau = (\tau_1, \tau_2, \tau_3)$. We now define for $k \geq 1$

$$b^{(k)} = \left(1 - \frac{1}{k}\right) b + \frac{1}{k}\tau \ .$$

By assumption and construction $b^{(k)}$ is a fully mixed behavior strategy profile converging to $b$ as $k$ increases. For the sequence of utilities we simply let $u^{(k)} = u$ for all $k$. Note that in $b^{(k)}$ all players are playing a best reply, and it follows that $b$ is (the strategy part of) a sequential equilibrium.

Suppose next that the minmax value for Player 1 in $G$ is strictly greater then 0. Thus let $\varepsilon > 0$ be such that the minmax value for Player 1 in $G$ is at least $\varepsilon$. Let $(b^{(k)}, u^{(k)})$ be a sequence of pairs of fully mixed behavior strategy profiles $b^{(k)}$ and utilities $u^{(k)}$ with $\lim_{k\to\infty}(b^{(k)}, u^{(k)}) = (b, u)$. Let $k$ be such that $\|u - u^{(k)}\|_\infty < \varepsilon/2$ and let $I$ denote the information set of Player 1. Since $b^{(k)}$ is fully mixed, $I$ is reached with non-zero probability. Conditioned on reaching $I$, the action $\perp$ gives Player 1 payoff 0, whereas the best reply gives payoff at least $\varepsilon$. Since $\|u - u^{(k)}\|_\infty < \varepsilon/2$ the payoff to Player 1 conditioned on reaching $I$ with respect to $u^{(k)}$ differs by less than $\varepsilon/2$ to the payoff with respect to $u$. Thus, conditioned on reaching $I$, the action $\perp$ gives Player 1 payoff strictly less than $\varepsilon/2$, whereas the best reply gives payoff at least $\varepsilon/2$. But $b^{(k)}$ is fully mixed and therefore is Player 1 not playing a best reply. Since this holds for all sequences $(b^{(k)}, u^{(k)})$ is follows that $b$ is not (the strategy part of) a sequential equilibrium. □ □

Selten [36] also defined the notion of trembling hand perfect equilibrium for extensive form games. This refines the notion of (the strategy part of) sequential equilibrium. As for the case of games in strategic form, we use an alternative definition stated by van Damme [39] analogous to the equivalent definition we used due to Myerson [30].

**Definition 5** (Trembling hand perfect equilibrium)**.** *Let $G$ be a $m$-player game in extensive form but having perfect recall. A behavior strategy profile $b$ for $G$ is an $\varepsilon$-perfect equilibrium if and only it is fully mixed and in every information set only actions that are local best replies get probability more than $\varepsilon$. A behavior strategy profile $b$ for $G$ is a trembling hand perfect equilibrium if and only if it is a limit point of a sequence of $\varepsilon$-perfect equilibria with $\varepsilon \to 0^+$.*

**Theorem 6.** *For any $m \geq 3$ it is $\exists\mathbb{R}$-complete to decide if a given pure strategy Nash equilibrium of a given $m$-player game in extensive form is trembling hand perfect.*

*Proof.* We first sketch the proof for containment in $\exists\mathbb{R}$. For every pure local strategy at $I$, we may write $u_i(b \setminus_I c)$ as a polynomial in variables $b$ which is of size polynomially bounded in the size of the game. It is then straightforward to express by a quantifier free formula $\varphi(b', \varepsilon)$ that $b'$ is an $\varepsilon$-perfect equilibrium and we can finally express that $b$ is a trembling hand perfect equilibrium by

$$\forall \varepsilon > 0 \exists b' : \varphi(b', \varepsilon) \wedge \|b - b'\|_2^2 \leq \varepsilon \ .$$

Thus by Lemma 1 it follows that deciding whether a given behavior strategy profile $b$ is a trembling hand perfect equilibrium is contained in $\exists\mathbb{R}$.

For establishing $\exists\mathbb{R}$-hardness we may reuse the construction of Theorem 5. When the minmax value is 0, the sequence $b^{(k)}$ establishes that $b$ is a trembling hand perfect equilibrium, since all actions of all players are local best replies to $b^{(k)}$. When the minmax value in $G$ is strictly greater than 0, since $b$ is not (the strategy part of) of sequential equilibrium, $b$ is not trembling hand perfect as well. □ □

It is possible to also to also consider a further refinement of extensive form trembling hand perfect equilibrium to what would be called extensive form proper equilibrium completely analogously to proper equilibrium refining trembling hand perfect equilibrium in strategic form games. We will not give formal statements concerning these, but just remark that the proof of Theorem 6 can be modified in the same way as the proof of Theorem 4 to give the same statement for extensive form proper equilibrium.

We finally consider the concept of quasi-perfect equilibrium of van Damme [39]. Let us note that Mertens [28] argued that the concept of quasi-perfect equilibrium is superior to that of trembling hand perfect equilibrium, and thus also to extensive form proper equilibrium.

Let $b$ be a behavior strategy profile, let $I \in \mathcal{I}_i$ be an information set for Player $i$, let $d$ be a local strategy at $I$ and let $b'_i$ be a behavior strategy for Player $i$. Let $b_i \setminus_I (d, b'_i)$ denote the behavior strategy for Player $i$ obtained from $b_i$ by replacing $b_{i,I}$ by $d$ as well as replacing $b_{i,I'}$ by $b'_{i,I'}$ in information sets $I' \in \mathcal{I}_i$ following $I$. We then further let $b \setminus_I (d, b'_i)$ denote the strategy $b \setminus (b_i \setminus_I (d, b'_i))$. We say that $d$ is a *quasi-best reply* at $I$ if $\max_{b'_i} u_i(b \setminus_I (d, b'_i)) \geq \max_{b'_i} u_i(b \setminus_I (d', b'_i))$ for all local strategies $d'$ at $I$.

**Definition 6** (Quasi-perfect equilibrium)**.** *Let $G$ be a $m$-player game in extensive form but having perfect recall. A behavior strategy profile $b$ for $G$ is an $\varepsilon$-quasi-perfect equilibrium if and only it is fully mixed and in every information set only actions that are quasi-best replies get probability more than $\varepsilon$. A behavior strategy profile $b$ for $G$ is a quasi-perfect equilibrium for $G$ if and only if it is a limit point of a sequence of $\varepsilon$-quasi-perfect equilibria with with $\varepsilon \to 0^+$.*

**Theorem 7.** *For any $m \geq 3$ it is $\exists\mathbb{R}$-complete to decide if a given pure strategy Nash equilibrium of a given $m$-player game in extensive form is quasi-perfect.*

*Proof.* We first sketch the proof for containment in $\exists\mathbb{R}$. First we note that given $b'$ the values $K_i^{I,c} := \max_{b''_i} u_i(b' \setminus_I (c, b''_i))$ may be computed by dynamic programming over all players $i$, information sets $I$, and pure strategies $c$, using polynomials of size polynomially bounded in the size of the game (cf. [14, Lemma 7]). Thus we may also express by a quantifier free formula $\kappa(b', k)$, that

$k_i^{I,c} = K_i^{I,c}$ for all $i$, $I$, and $c$. This is done by simply taking the conjunction of polynomial equalities stating that each step of the dynamic program is correctly computed. By a Tseitin-style transformation, using existential quantifying over $k$, we may thus access the values $K_i^{I,c}$ freely in further formulas. It is now simple to express by a quantifier free formula $\varphi(b', \varepsilon, k)$ that $b'$ is an $\varepsilon$-quasi-perfect equilibrium under the assumption that $\kappa(b', k)$ holds. We can now finally express that $b$ is a quasi-perfect equilibrium by

$$\forall \varepsilon > 0 \exists b', k : \kappa(b', k) \wedge \varphi(b', \varepsilon, k) \wedge \|b - b'\|_2^2 \leq \varepsilon \ .$$

Thus by Lemma 1 it follows that deciding whether a given behavior strategy profile $b$ is a quasi-perfect equilibrium is contained in $\exists\mathbb{R}$.

Lastly, $\exists\mathbb{R}$-hardness follows exactly in the same way as in the proof of Theorem 6. □

## 4.3 CURB Sets in Strategic Form Games

Here we consider the set valued solution concept of Closed Under Rational Behavior (CURB) strategy sets by Basu and Weibull [1].

**Definition 7** (CURB set)**.** *In a $m$-player game, a family of sets of pure strategies, $S_1, S_2, \dots, S_m$ with $S_i$ being a subset of the strategy set of player $i$, is* closed under rational behavior *(CURB) if and only if for all pure strategies $k$ of Player $i$ so that $k$ is a best reply to a product distribution $x$ on $S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_m$, we have that $k \in S_i$.*

**Theorem 8.** *For any $m \geq 3$ it is $\forall\mathbb{R}$-complete to decide whether a family of sets of pure strategies $S_1, S_2, \dots, S_m$ is CURB in a given $m$-player game in strategic form.*

*Proof.* We show that the problem of deciding whether a family of sets of pure strategies $S_1, S_2, \dots, S_m$ is *not* CURB is complete for $\exists\mathbb{R}$. Containment in $\exists\mathbb{R}$ is straightforward. We existentially quantify over a mixed strategy profile $x$ and have a disjunction for all $i$ and all $k \notin S_i$ over inequalities expressing that $k$ is a best reply to $x$.

In order to show hardness for $\exists\mathbb{R}$ we reduce from the promise minmax problem given by Theorem 1. Let $G$ be the given 3-player game in strategic form. We define a new game $G'$ from $G$ where Player 1 is given an additional pure strategy $\perp$. The payoffs to Player 2 and Player 3 are 0 for all strategy combinations. The payoff to Player 1 is 0 if he plays $\perp$ and otherwise is the same as it would have been in $G$.

By construction we now have that the minmax value of Player 1 in $G$ is 0 if and only if the set of all pure strategies except $\perp$ is not CURB in $G'$. Namely, if the minmax value of $G$ is 0, then $\perp$ is a best reply for Player 1 to the minmax strategy profile of Player 2 and Player 3, and the set of all pure strategies except $\perp$ is not CURB. On the other hand, if the minmax value of $G$ is strictly greater than 0, then $\perp$ is never a best reply in $G'$, and the set of all pure strategies except $\perp$ is therefore CURB. □ □

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