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Towards a definition of "mathematical digital competency"

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Abstract
Due to the embeddedness of digital technologies in mathematics education of today, we often see examples of students simultaneously using their mathematical competencies and digital competencies. In relevant literature, however, these are not seen as a connected whole. Based on reviewing existing competency frameworks, both mathematical and digital, and by exploring an empirical example, this article addresses the question of how to think about and understand a combined "mathematical digital competency" (MDC). In doing so, the article relies on the two theoretical frameworks of the instrumental approach and conceptual fields to "bridge" the mathematical and digital competency descriptions.

Keywords Mathematical competencies · Digital competencies · Mathematical digital competencies · Instrumental genesis · Conceptual field · Networking of theories

1 Introduction

Since the notion of *competency* (or competence) was first introduced in a psychological context as an alternative to that of intelligence (McClelland, 1973), it has gradually gained momentum, in societal and economic contexts as part of the so-called knowledge capitalism as well as in the educational paradigm. Han (2010) describes the 1990s' and 00s' change from a nation–state education to a global learning economy, where the notion of academic achievement was linked to (or replaced by) that of competency, not least due to OECD's DeSeCo (definition and selection of competencies) project[1] and the following Programme for International Student Assessment (PISA) and PIAAC efforts. In 2006, the European Parliament and the Council defined *competence* as a combination of knowledge, skills and attitudes relevant to

[1]http://www.oecd.org/education/skills-beyond-school/definitionandselectionofcompetenciesdeseco.htm

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the context in question. In 2008, they described competence in terms of responsibility and autonomy, whereas the European Qualifications Framework recommendation views competence as "the proven ability to use knowledge, skills and personal, social and/or methodological abilities, in work or study situations and in professional and personal development" (Ferrari, 2013, p. 37).

Today, the notion of competency has become a key construct in the educational paradigm and the current perception of literacy (e.g., Sadler, 2013; Stacey, 2010; Stacey & Turner, 2014), often overshadowing previously prevalent constructs such as knowledge and skill. Of course, the introduction of disciplinary competency descriptions has not gone uncriticized (e.g., Skovsmose, 2006; Jensen & Jankvist, 2018). Nevertheless, in some countries such as Denmark, competency descriptions are now an integrated part of the educational system from primary and secondary school through upper secondary school to tertiary educational programmes. More recently, the notion of digital competencies has heavily entered the scene. There are now various frameworks available, some of which are of a more generic and overarching nature. In the school context, we find, for example, the FutureLab project's report on digital literacy developed in the UK (Hague & Payton, 2010), the Norwegian National Framework for the use of ICT in schools (see Hatlevik & Christophersen, 2013), and the Welsh Digital Competence Framework for schools, implemented in 2016.[2] All these different frameworks have similarities in what skills students are required to gain, while the main difference is that for each framework, these skills are grouped in different overarching categories. Many countries now also include digital literacies in their school curriculum, although there is disagreement in terminology: e.g., digital competency (Norway), digital media literacy (Australia) and media literacy (UK) (Hatlevik & Christophersen, 2013).

Even though the research literature offers several descriptions of mathematical competencies and digital competencies, respectively, the two are rarely viewed as a connected whole. The heavy reliance on digital tools in the teaching and learning of mathematics today often calls for a simultaneous activation of both mathematical and digital competencies—what might be referred to as *mathematical digital competency* (MDC). Yet, from a theoretical point of view, it is still unclear how to think about and understand such a notion of MDC. In addition, the separate descriptions of mathematical and digital competencies do not seem to possess within them the potential *interplay* between the two sets of competencies.

As part of our research (e.g., Jankvist, Geraniou, & Misfeldt, 2018), we have come to find that such an interplay may be best described through two other theories from mathematics education research, namely, instrumental genesis (TIG) and conceptual fields (TCF). In this article, we address the question of *how the use of TIG and TCF can act as a "bridge" between the disciplinary competency descriptions of mathematics and the more generic descriptions of digital competencies.* We argue our case through an empirical example, which we believe displays students' activation and development of MDC. We begin with the presentation of this example, and as we later in the article present and discuss the theoretical perspectives of competencies, both digital and mathematical, and TIG and TCF, we shall continuously refer back to and analyse the example. While all these frameworks concern individual learning, our example is one of two students' collaboration in solving a mathematical task in order to get a window on their individual mathematical

[2] This is based on the DigComp framework (Ferrari, 2013), augmented with an area of computational thinking: http://learning.gov.wales/resources/browse-all/digital-competence-framework/framework?lang=en

learning. Drawing on our analytic discussions, we end the article with identifying components of a potential definition of MDC.

2 An empirical example

The following situation stems from the technical stream of Danish upper secondary school (called htx), where two second-year students (age 17–18 years) work on a mathematical modelling task under the observation of two maths counsellors (see Jankvist & Niss, 2015). As part of an intervention designed for the purpose of developing the two students' mathematical competencies, in particular related to mathematical modelling, they were to do a well-known task from the literature (Galbraith & Stillman, 2006), where they had to find the shortest distance of a run from gate 1 to gate 2 via 1 of 18 different stations (Fig. 1). The students were not familiar with this task beforehand. The role of the two "math counsellors", Tina and Henrik, during the intervention was somewhat secluded and more or less reduced to answering clarifying questions from the students.

Both students recognised that the Pythagorean theorem would reveal the missing lengths. They calculated the distance involving stations 1 and 8, and obtained 334.16 m and 286.74 m, respectively. They were quite surprised by this. Bob proposed an expression for the distance from gate 1 (point F in Fig. 1) to any given station, p, and wrote on his paper: $(120)^2 + (50 + (10 \cdot p - 10))^2 = \ldots$ (Fig. 2).

01 Katrine: Can you calculate the distance for station 9?
02 Bob: Can we use *MathCad*? Katrine, do you have MathCad on your computer?
03 Tina: Sure, you are welcome to do that.
04 Bob: Then we do not have to push in the expression on the calculator every time. We can also force out a graph from MathCad, if we want.
05 Katrine: So you just want to write in the expression and then change the values?
06 Bob: If we want to draw a graph...

Katrine began to type in the expression in the digital tool, while Bob dictated what to write.

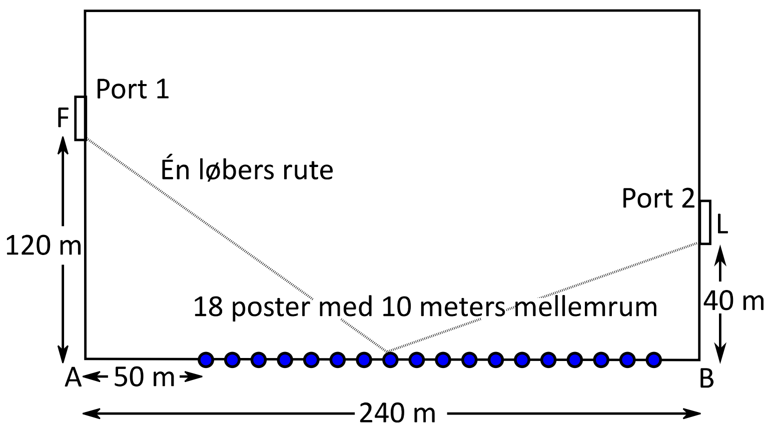


Fig. 1 The situation of the task. The text translates to "a runner's path" and "18 stations at 10-m intervals" (Rosendahl & Kærgaard, 2016, p. 296, based on Galbraith & Stillman, 2006, p. 145)

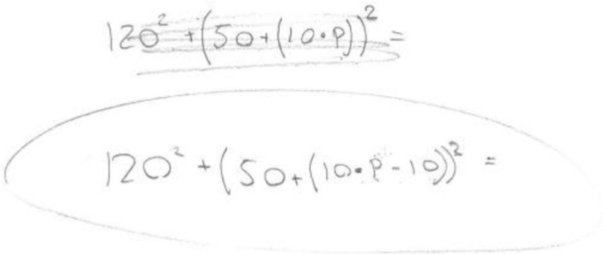


Fig. 2 Bob's expression (Rosendahl & Kærgaard, 2016, p. 298)

07 Bob: We need to find the total distance.
08 Katrine: Yeah, we need to find the shortest one.
09 Bob: Well, I am thinking about the total distance. So, you could call it "total" and then define it as this expression [Bob points to the expression in Fig. 2].
10 Katrine: No, I do not want to do fancy stuff like that. We just have to find the answer.
11 Bob: I am just thinking it would be easier… Because I was looking ahead and thought that if we want a graph, then we need to define it.
12 Katrine: Like a function or what?
13 Bob: Exactly.
14 Katrine: What do you wanna call it?
15 Bob: "Total." Just call it something. If you write *p* up here, then we can use that for the different stations—*p*—and then the station you want to begin with. You can type in 18, and then we should get a value (Fig. 3).

Katrine and Bob now realised that they had only calculated the first part of the route. They worked together to come up with an expression for the second part of the route from any *p* to *L* (Fig. 1). After some back and forth—where they also had difficulties with getting MathCad to draw the graph, because they typed in *x* instead of *p*—they ended up with the expression and associated graph in Fig. 4.

16 Katrine: Look, then you can find the slope of the tangent, where it is right here [points to the graph].
17 Bob: I am not really that familiar with the stuff on tangent slope…
18 Katrine: Good thing I have my notes then [finds her folder with notes from class].
19 Tina: Bob, do you follow that there is a horizontal tangent?

Fig. 3 Katrine and Bob's calculation for station 18 (Rosendahl & Kærgaard, 2016, p. 300)

$$p := 18$$

$$total_1 := 120^2 + [50 + (10\cdot p - 10)]^2$$

$$\sqrt{total_1} = 250.599$$

$$total_2 := \sqrt{total_1}$$

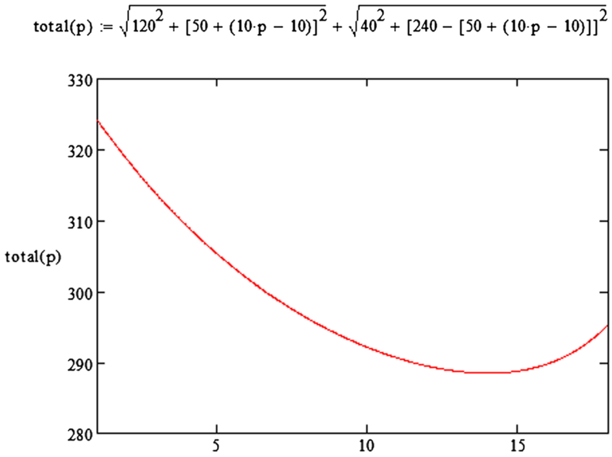


Fig. 4 Total distance and graph. Screenshot from MathCad (Rosendahl & Kærgaard, 2016, p. 300)

20 Bob: Yes, I get that there is a horizontal tangent, only I do not know how to do the calculations.
21 Katrine: You just need this one simple formula. First you must find the differential quotient, and then you set it equal to 0, because it is horizontal. So ours will be called *total′*(*p*)...

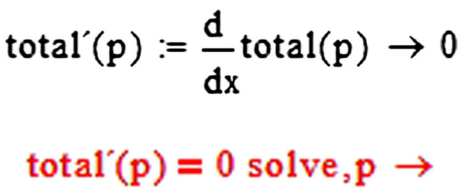
At this point, Katrine entered her formula into MathCad (Fig. 5), but it did not work. Katrine's example in her notes was one where the variable was named *x*.

22 Tina: You have made an elementary mistake here. You made it before also, when you did the graph. What does it say in the parentheses here? [points to *total′*(*p*)].
23 Bob: It should be *dp* there…
24 Katrine: Oops...

Katrine changed *dx* to *dp*, and the solution was finally revealed (Fig. 6).

25 Katrine: So it's at station 14.
26 Tina: Wow… so what is the total distance then?
27 Katrine: We need to find that or what?
28 Bob: We just go back and type 14 into our expression [and they do this].
29 Katrine: Now I can tell that Tina is happy. She is one big smile.

Fig. 5 Katrine's first attempt (Rosendahl & Kærgaard, 2016, p. 303)



$$\mathrm{total}'(p) := \frac{d}{dp}\mathrm{total}(p) \rightarrow \frac{5\cdot(2\cdot p + 8)}{\sqrt{\frac{(10\cdot p + 40)^2}{100} + 144}} + \frac{5\cdot(2\cdot p - 40)}{\sqrt{\frac{(10\cdot p - 200)^2}{100} + 16}}$$

$$\mathrm{total}'(p) = 0 \ \mathrm{solve}, p \ \rightarrow 14$$

Fig. 6 The optimal solution found (Rosendahl & Kærgaard, 2016, p. 304)

30 Tina: If you could place a station anywhere you wanted to on the "baseline", would you then be able to place it so that the route was even shorter than at station 14?
31 Katrine: No, because then it would not have been an integer.
32 Bob: I do not get that…
33 Katrine: Yes, if a station had been between these two points, it would not have been an integer. But it is. So it is exactly *on* a station.
34 Bob: Okay. Now I get it!

3 Mathematical competencies

Kilpatrick (2014) states that school mathematics sometimes "is portrayed as a simple contest between knowledge and skill" while "Competency frameworks are designed to demonstrate to the user that learning mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures" (p. 87). As examples of such frameworks, Kilpatrick mention three: the five strands of mathematical proficiency as identified by the Mathematics Learning Study of the US National Research Council, the five components of mathematical problem-solving ability identified in the Singapore mathematics framework, and the eight competencies of the Danish KOM framework. The latter was implemented as the basis of the PISA framework of mathematical competencies (e.g., Stacey & Turner, 2014).

KOM characterises mathematical competenc*e* as "…having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role" (Niss & Højgaard, 2011, p. 49). This overarching competence is spanned by eight distinct, yet mutually related, competenc*ies*—the distinction of competence and competency being fully deliberate.[3] A *mathematical competency* is (an individual's) "…well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p. 49). In addressing the question of what it means to master mathematics, KOM identifies eight competencies, grouped under two overall competenc*es* (cf. Fig. 7).

The *modelling competency* consists of the ability to analyse the foundations and properties of existing models and to assess their range and validity. It also involves being able to perform and utilise active modelling, including mathematising, which is what Bob and Katrine performed when transforming the extra-mathematical situation into a mathematical problem.

[3] Although not entirely aligned, there is some resemblance with Sadler's (2013) distinction between individual competencies and overall competence, which requires the consistent ability to orchestrate the former into an effective whole.

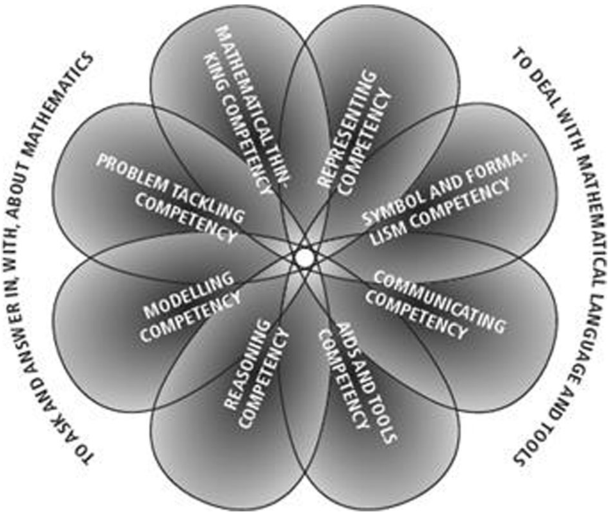


Fig. 7 The so-called "KOM flower" with its eight overlapping competencies (Niss & Højgaard (2011), p. 1)

The *problem tackling competency* involves the ability to detect, formulate, delimit, and specify different kinds of mathematical problems and solve them. The word "problem" is relative to the person who is trying to solve the task; what to one person is a routine task may be a problem to another. Both Katrine and Bob quickly realised that they could use the Pythagorean theorem. Bob was very good at devising a strategy for tackling the mathematised problem, but when it came to putting the concepts of tangent and differential quotient into play, he was more challenged (line 17).

The *aids and tools competency* first consists of having knowledge of the existence and properties of the diverse sorts of relevant aids and tools, including digital tools, employed in mathematics and of having an insight into their capabilities and limitations within different contexts. Secondly, it comprises the ability to reflectively use aids and tools. Bob possessed this competency to a larger extent than Katrine did. Not only did Bob immediately suggest to use MathCad, he also had an idea of why this was smart, i.e., that they did not "have to push in the expression" several times (line 04). It took a while for Katrine to grasp the effectiveness of this.

The *representing competency* comprises the ability to decode, interpret, and distinguish between and utilise different representations of mathematical objects, phenomena, problems, or situations. It also includes being able to understand the mutual relationships between different representational forms, their strengths and weaknesses, and to choose and switch between such in given situations. On several occasions, Bob suggested that they needed to make a graphical representation of the situation with the distances (lines 06, 11). He realised the strengths of one such, and once he convinced Katrine, this indeed paid off, since she immediately suggested to consider "the slope of the tangent" (line 16).

The *symbols and formalism competency* is about decoding symbolic and formal language, translating between mathematical symbolism and natural language, handling and utilising mathematical symbolism, and transforming symbolic expressions. It also focusses on the nature, role and meaning of symbols (and formal systems), and on the rules for their usage. Bob and Katrine were both able to apply symbols and appeared to grasp the efficacy of this, e.g., they defined the function named "total," and they named the variable p (line 15).

The competencies of representing and symbols and formalism are closely related to the *communication competency*. This consists of the ability to study and interpret others' written, oral or visual mathematical statements, explanations or texts as well as the ability to express oneself mathematically in such ways. Both Bob and Katrine seemed to possess this competency to sufficient degree for their educational level, as far as can be judged from the example.

The *reasoning competency* concerns the ability to follow and assess mathematical reasoning, i.e., a chain of arguments put forward—orally or in writing—in support of a claim (including the notion of a mathematical proof and a counter example). It is also about being able to carry out mathematical reasoning oneself. Towards the end of the example, where Bob and Katrine discussed the interpretation of their mathematical result (lines 25–34), we witness a nice example of mathematical reasoning in relation to problem tackling and de-mathematising.

The *mathematical thinking competency* is not to be confused with that of reasoning. It comprises an awareness of the types of questions that are typical of mathematics and an insight into the types of answers that can be expected. Also, the acts of abstracting and generalising (and much more) are part of this competency. Our example does not truly provide opportunities for the students to display possession of this competency.

As seen, the KOM framework does not address in detail the role of digital technologies in relation to the possession of mathematical competencies (Jankvist et al., 2018). A few constructs from mathematics education research do, however, consider this, but before we deal with those, let us first consider the notion of digital competency.

4 Digital competencies

According to Ferrari (2012), *digital competence* is "the set of knowledge, skills, attitudes (thus including abilities, strategies, values and awareness) that are required when using ICT and digital media to perform tasks; solve problems; communicate; manage information; collaborate; create and share content; and build knowledge effectively, efficiently, appropriately, critically, creatively, autonomously, flexibly, ethically, reflectively for work, leisure, participation, learning, socialising, consuming, and empowerment" (p. 43). There is a plethora of terms used to refer to digital competencies. Some also use the terms digital competency and digital literacy interchangeably (Hockly, 2012). However, referring to school students in particular, Hatlevik and Christophersen (2013) claim that there are differences: "A concept such as digital skills focuses on dealing with the technical conditions, whereas digital competence and literacy are broader terms that emphasise what kind of skills, understandings, and critical reflections students are able to use" (p. 241). Hague and Payton (2010) state that digital literacy is about "collaborating, staying safe and communicating effectively. It's about cultural and social awareness and understanding, and it's about being creative. Being digitally literate is about knowing when and why digital technologies are appropriate and helpful to the task at hand and when they are not. It's about thinking critically about all the opportunities and challenges digital technologies present" (p. 19). It is evident that Bob, in particular, identified the need of using a digital tool to trial different values (lines 02–06), when looking for the shortest distance, and he recognised that MathCad could allow them to enter a general expression to calculate all distances. He further demonstrated creative thinking by proposing to draw a graph, where the x-axis represented the different stations, and the y-axis represented the distance from gate 1 to each station and then to gate 2 (lines 04, 09–15). Bob was able to effectively communicate his ideas to Katrine, and in their collaborative efforts to create the graph, they, among others, recognised the need to store the created total function, which they then could retrieve at any future point (line 15). This enhanced both his and Katrine's learning experience. Hence, Bob and Katrine must be considered digitally literate.

Within mathematics education literature, a couple of constructs are closely related to the notion of digital competencies. One is that of *techno-mathematical literacies* (TmL), which is intended to describe functional mathematical knowledge as mediated by, usually digital, technologies within a given workplace practice (Kent et al., 2005; Van der Wal et al., 2017). Another is that of *techno-mathematical fluency* (Jacinto & Carreira, 2017), which builds on that of technological fluency (Papert & Resnick, 1995). Such fluency encompasses "the ability to combine two types of background knowledge and skills—mathematical and technological—constantly being intertwined to develop techno-mathematical thinking," emphasising also "the need to be fluent in a 'language' that entails both mathematical and technological knowledge" (Jacinto & Carreira, 2017, p. 1122). Focussing more directly on the notion of competency, Jasute and Dagiene (2012) discuss the need for digital competency in the teaching and learning of geometry, and in particular on students' and teachers' ability "to perform tasks effectively in a digital environment" (p. 1). In our example, it is clear that Bob may be more "fluent" in the language of MathCad than Katrine, yet both of the students managed to combine their mathematical and technological background knowledge to obtain new understanding and effectively perform the task at hand by engaging in the techno-mathematical discourse.

In the following sections, we turn to the two frameworks of TIG and TCF, which we find, as mentioned earlier, may act as a "bridge" in combining the sets of mathematical and digital competencies.

5 Theories of instrumental genesis and conceptual fields

TIG involves the process of transforming digital tools into mathematical instruments, which then become part of students' cognitive schemes (Vergnaud, 2009) and can be used to support learning of mathematical concepts (Artigue, 2002; Guin & Trouche, 1999). Drijvers, Godino, Font and Trouche (2013) present TIG in terms of three dualities.

The *artefact–instrument* duality describes the lengthy process of an artefact becoming an instrument in the hands of a user, which is referred to as *instrumental genesis*. In our example, Katrine prompted Bob to use a different station and calculate the distance using an algebraic expression (line 01). In his response, Bob seemed to reveal that MathCad was not only an artefact for him, but also an instrument for calculating the distance from any station and for *drawing the graph* (lines 02–06). For Katrine, MathCad, at this point in time, seemed to be only an artefact, but throughout the example, we witness a beginning instrumental genesis on her behalf.

The *instrumentation–instrumentalisation* duality refers to the relationship between the artefact and the user and can be applied to showcase how a student's knowledge directs the use of an artefact (instrumentalisation), and how a tool can shape and affect a student's thinking and actions (instrumentation). Bob used his previous experiences with MathCad and its functionalities to guide Katrine's and his thinking towards trialling different stations, but also drawing the graph (instrumentation—lines 02–06). He also suggested they input the derived algebraic expression in the digital tool to find the different distances (instrumentalisation—lines 09–15). The process of instrumentation is closely connected to the digital tool serving an *epistemic* purpose, meaning that it is used to create understanding or support learning within the user's cognitive system. By contrast, when a digital tool is used to create a difference in the world external to the user, it is said to serve a *pragmatic* purpose (e.g., Artigue, 2002). Digital tools serve of course both pragmatic and epistemic purposes, but any use which is only, or mainly, pragmatic is according to Artigue (2010) of little—or even negative—educational value. In the example, MathCad was firstly put to use for pragmatic purposes, i.e. calculate all distances (lines 02–04). Yet, towards the end of the example, the tool appeared to shift towards a more epistemic role, when it caused a situation where Katrine needed to understand that it had to be p instead of x (lines 15, 21–24, Fig. 5), and when Bob seemed to understand the use of the differentiated function (lines 17–21, 23).

The *scheme–technique* duality refers to "the relationships between thinking and gesture" (Drijvers et al., 2013, p. 26). From a practical perspective, techniques can be seen as "the observable part of a students' work on solving a given type of tasks (i.e. a set of organized gestures) and schemes as the cognitive foundations of these techniques that are not directly observable" (Drijvers et al., 2013, p. 27). A student's scheme is an adaptable resource that allows her to act in a certain way in situations she has already mastered, but at the same time, enables her to tackle new situations by applying her past knowledge and making inferences. Vergnaud (2009) states: "On the one hand, a scheme is *the invariant organization of activity for a certain class of situations*; on the other hand, its analytic definition must contain open concepts and possibilities of inference" (p. 88). This is to say that a scheme combines intentions and actions with conceptual knowledge. In our example, we should emphasise that Bob's and Katrine's collaborative work and dialogue to some extent reveal how their techniques and their individual schemes of knowledge came into play and may lead to their schemes' further development. In particular, both Bob's and Katrine's scheme on finding missing lengths of a right-angle triangle was revealed by their technique on using the Pythagorean theorem and deriving the expression presented in Fig. 2. Bob's scheme on algebraic language was showcased in his derived expression in Fig. 2, where he represented the distance from gate 1 (F) to any station, p, in an algebraic expression involving numerical and algebraic terms, but also later on when he suggested to Katrine how to input this in MathCad (lines 09–15). As part of his scheme, Bob appreciated the functionality of using MathCad to input their derived expression and trial different values for p, which was aligned with Katrine's scheme on what this expression represented and how it could lead to the solution (lines 04–06, 08, 12). In fact, Bob's technique involved helping Katrine input the expression for the total distance (Fig. 3) and creating the associated graph (Fig. 4). We agree with Drijvers' and colleagues' (2013) interpretation that techniques can be viewed as reflecting the schemes, and that they do have pragmatic as well as epistemic value (Lagrange, 2000). Pragmatic because students use these techniques to solve the task, and epistemic since they use them to learn and gain more mathematical knowledge, which is evident in the presented example.

Vergnaud (2009) states that, from a developmental point of view, a concept should never involve a single type of situation and therefore defines it as a triple of three sets: "A set of situations, a set of operational invariants (contained in schemes), and a set of linguistic and symbolic representations" (p. 94). The set of situations is meant to give meaning to the concept, and it acts as a point of reference. The operationality of the schemes, i.e. the context, is based on the set of invariants, which can be translated to concept-in-action and theorem-in-action. Their function is to allow a student, for example, to identify the relevant information and infer from its aims and rules. The set of linguistic and symbolic representations allow the concept, its properties, the situations and processing procedures to be represented symbolically. A scheme has also been defined as "a more or less stable way to deal with specific situations or tasks, guided by developing knowledge" (Drijvers et al., 2013, p. 27). Vergnaud claims that a concept is a living entity, "engaged in a genesis, personal or collective" (cited in Monaghan, Trouche, & Borwein, 2016, p. 235) and derives that a concept "takes sense in the frame of a *conceptual field*", which is "a set of situations and a set of concepts tied together" (Vergnaud, 2009, p. 86, our emphasis). In our example, Bob and Katrine in their conversations tied together mathematical concepts of, e.g., right-angle triangle (calculating the hypotenuse), function (total), variable (p), and differential quotient (d/dp), also through the graphical representations of these concepts (slope, tangent).

For Vergnaud (2009), a conceptual field, and the development of such, "involves situations, schemes and symbolic tools of representation" (p. 87). TCF distinguishes two forms of acquired knowledge: predicative and operative. The *predicative* knowledge enables students to do something, whereas the *operative* knowledge allows them to describe and give reasons for what has been done or is to be done (Monaghan et al., 2016). Vergnaud considers the predicative form of knowledge as a necessary resource in building new knowledge, but the operative form as more subtle and rich (Monaghan et al., 2016). In a given situation, students will rely on predicative and/or operative knowledge, their scheme and the conceptual field to decide upon actions and problem-solving strategies, including also relying on a digital tool. Bob and Katrine used their predicative knowledge to produce their graph (lines 06–15). Their schemes seemed to reveal a conceptual difficulty regarding the MathCad language, e.g., with recognising the artefact's syntax of the "draw a graph" command and the meaning of the variable used, p vs. x. Their operative knowledge of algebraic notation, but also their scheme on algebraic knowledge, prescribed them to consider x as the best representation to use in MathCad for the distance (instrumentalisation). Their collaboration and the involved conceptual field related to differentiation influenced their interaction with the digital tool (instrumentation). This may also have shaped and reformed their schemes of instrumented action, taking into consideration the further development in their algebraic knowledge, in particular; first by considering the use of a different letter i.e. p to represent the missing distance, and secondly, by recognising the necessity of having clarity in their solution in the digital tool. Later, when Katrine entered her formula into MathCad (Fig. 5), it did not work. In Katrine's example in her notes, the variable was named x. It is not clear if Katrine's mistake was due to a conceptual difficulty related to her operative mathematical knowledge (hence not knowing what the mathematical notation d/dx means, e.g., believing that d/dx is the symbol for differentiation of a function, or not recognising that the variable can be something else than x), or whether it was related to her predicative knowledge. Still, the response of the MathCad tool prompted both students to reconsider their actions and therefore their predicative knowledge and develop further their operative knowledge regarding the variable (instrumentation), as they reflected upon what their actions led to within MathCad. The math counsellor's intervention (line 22) helped Bob recognise the mistake and share the correct notation.

It is of course possible that predicative knowledge is used to solve a task with a digital tool without the students' scheme being developed, since development requires the use of operative knowledge. Yet, the various factors in this situation, i.e., the MathCad language, the formal mathematical notation, the math counsellor's intervention, both students' operative mathematical knowledge and predicative knowledge of the tool, and their instrumental genesis related to

MathCad, all seemed to have led to an enrichment and further development of the conceptual field of differentiation for both Bob and Katrine.

TIG and especially TCF offer sound theoretical frameworks to identify and analyse students' knowledge and mathematical competencies achieved during and after their interactions with a digital tool. TCF's aim is "to describe and analyse the progressive complexity, on a long- and medium-term basis, of the *mathematical competencies* that students develop inside and outside school" (Vergnaud, 2009, p. 83, our emphasis).

6 Discussion: bridging mathematical and digital competency descriptions

From the analyses in "Mathematical competencies" and "Digital competencies", it seems fair to say that Bob and Katrine do possess both digital and mathematical competencies. Nevertheless, what is also evident from the analyses is that the interplay between the two sets of competencies is not necessarily well articulated through the available competency frameworks. The descriptions of digital competencies are quite generic in nature and mainly concern citizenship, which, although sympathetic, often comes out as being too general for the situation of using digital technologies in mathematics education. The notions of TmL and techno-mathematical fluency do address aspects of what one would expect to be part of mathematical digital competencies. Yet, TmL does so mainly from a workplace perspective. The notion of techno-mathematical fluency appears to be closer to what one might expect, and is useful in framing some aspects of MDC. However, for more fine-grain, in-depth analyses of the role of digital tools and their impact on developing mathematical knowledge, TIG and TCF appear to us to offer more depth.

For example, analysing the example of Bob and Katrine using the three dualities of TIG (Drijvers et al., 2013) revealed how both students developed their scheme of instrumented action (Trouche, 2000) by using MathCad as a tool to think, to investigate and to evaluate their conjectures. All these could be considered as conceptual elements that guide students' actions and are developed through their activities and their reflections upon those activities. Unlike the more normative competency descriptions, TIG and TCF appear to capture more easily the developmental aspects of the students' digital and mathematical knowledge. For example, through the use of TIG, the students' instrumented action schemes, i.e., the cognitive foundations of the strategies they used, become visible. Also, by means of TCF, we notice how Bob and Katrine rely on both the predicative and the operative forms of knowledge during the instrumentalisation process. In other words, TIG and TCF provide us with means for describing and articulating the ongoing processes in play.

Although the KOM framework may be used to describe the students' possession (and development) of various mathematical competencies and ditto for the DigComp framework (see "Mathematical competencies" and "Digital competencies"), TIG and TCF may also serve as lenses through which to view the activation of competencies. For example, regarding TIG's third duality (scheme-technique), Bob and Katrine relied on their schemes of mathematical knowledge and showcased their thinking in the techniques (or gestures) they chose in solving the given mathematical task. Those techniques could in fact be characterised as a presentation of their mathematical competencies (e.g., *symbols and formalism mathematical competency* was evident by producing an algebraic expression for the total function) and their digital competencies (e.g., *developing content* and *programming* digital competency was evident by using the MathCad "language" to create and to program the total function). From a TCF perspective, the third duality in TIG is framed by the relationships between schemes and situations. In our example, both students relied on the predicative and/or the operative form of knowledge, their schemes and the conceptual field of differentiation to decide upon actions and certain strategies in solving the mathematical task—occasionally relying on the MathCad digital tool. Such strategies and their involved schemes are evidence for the possession of both mathematical and digital competencies, since as Vergnaud (1998) states: "What are competences composed of? They are composed of schemes aimed at facing situations" (p. 230).

Our analyses in the previous sections focussed on the techniques that Bob and Katrine used to solve the given mathematical task by collaborating and combining their knowledge of mathematics, but also the digital tool. TIG and TCF enabled us to observe and articulate how their knowledge and the artefacts came into play and how the students' discourse and their reasoning shaped different schemes and illustrated the use of various mathematical and digital competencies. Furthermore, it enabled us to describe how Bob and Katrine developed their scheme on enacting (or applying) mathematical knowledge to solve the given task using certain mathematical competencies, such as *problem tackling, representing and reasoning mathematical competencies*, as well as how they developed their scheme on enacting (or applying) their knowledge of the digital tool's constraints and possibilities showcasing a number of digital competencies, such as *problem-solving and content criterion digital competencies*. The combination of these two schemes and the enactment of their mathematical and digital competencies seem to have led the two students into their rich mathematical learning experience. Without the constructs from TIG and TCF, our description of this learning experience would have been poorer and more primitive.

Hence, in our view, TIG assists in deepening the digital competency description from a mathematics teaching and learning perspective, while TCF allows us amongst other things to articulate the role of conceptual knowledge in relation to the mathematical competency description. And since TIG and TCF are already linked through the use of Vergnaud's notion of scheme in the scheme-technique duality, TIG and TCF come to act as a bridge between the notions and descriptions of digital competencies and mathematical competencies. It thus seems only reasonable to us that a potential definition of MDC should draw on these two theoretical perspectives.

7 Conclusion: towards defining MDC

Returning to the outset of the article, as a mathematics educator, you are not in doubt when you see both mathematical competencies and digital competencies activated at the same time in a teaching and learning situation—which we believe is exactly what took place in the example of Bob and Katrine. Neither without mathematical competencies and mathematical knowledge nor without digital competencies and knowledge of the digital tool could the task have been solved in the same manner. Bob and Katrine displayed digital competencies in the sense that they knew "when and why digital technologies are appropriate and helpful to the task at hand and when they are not" (Hague & Payton, 2010, p. 19), and they also displayed and developed their mathematical tools and aids competency (see "Mathematical competencies"). In particular, they showed a *reflective use of the digital technology, also in regard to the tool's capabilities and limitations.* In the empirical example, this was also illustrated through the duality of instrumentation–instrumentalisation (e.g., in Bob's use of MathCad).

The notion of techno-mathematical fluency (Jacinto & Carreira, 2017) can be useful in our two students' display of their ability to "produce mathematical thinking by means of digital tools" (p. 1122) and their ability to combine and constantly intertwine mathematical and technological knowledge "to develop techno-mathematical thinking, i.e., new ways of knowing and understanding, and its effective communication" (p. 1134). In particular, they displayed *being able to engage in a techno-mathematical discourse*. We believe this aspect is evident not least in relation to the artefact-instrument duality (e.g., when Katrine—assisted by Bob—succeeded in finding the tangent using MathCad, including also the tool's language).

Worth noticing is that although the digital tool initially served a pragmatic purpose for the students, i.e., they did not have to calculate distances to all stations, the students wholeheartedly embraced its epistemic purpose as well. They showcased a *reflected use of digital technology for the purpose of (self-)learning*. Furthermore, Bob's and Katrine's recognition of the digital tool's epistemic purpose in the problem-solving situation allowed them to further develop both their digital and mathematical competencies in a reciprocal relationship between the two. The interplay of Bob's scheme on his knowledge of the digital tool and Katrine's scheme on mathematical knowledge through the lenses of TIG and TCF illustrates to us how powerful such interactions can be, and how the simultaneous activation of both types of schemes can offer students a deeper mathematical, yet digital, experience.

The work presented in this article could be considered a first step in reaching a greater degree of integration of the above-discussed frameworks and achieving a synthesising or integration of selected different theoretical constructs, as per a networking of theories approach (Bikner-Ahsbahs & Prediger, 2014). When different theoretical frameworks are combined—or networked—as done in this study, attention should of course be paid to the frameworks' reciprocal coherence, and in case this exists, then at which level the different frameworks are integrated. It seems to us that for the four theoretical frameworks of this study—KOM, DigComp, TIG and TCF—there is a reasonable level of coherence. TIG is conceived as a theoretical framework that allows us to consider students' processes from a developmental perspective, an issue which is always present when competencies are considered. On the one hand, we are able to provide a snapshot of what students have achieved so far. On the other hand, we are able to further the students' development in their learning processes and design teaching interventions that may trigger and support these. This aspect is particularly present in TIG and TCF. Furthermore, there is an embedded duality in many of the concepts of these four frameworks, not only in the three dualities of TIG and the forms of knowledge in TCF, but also in the definition of the mathematical competencies with their active and analysing sides, respectively. This "dual" aspect in all these definitions is important for grasping the meaning of such constructs: the construct is a product of a process, and cutting away one of them may change its meaning.

We of course realise that there may be limitations to our work. Our conclusions are derived using Bob and Katrine's collaborative work on the "runner's path" mathematical modelling task using a CAS environment as an illustrative case for demonstrating how MDC can be identified. Surely, further investigations may be needed to support the generalisability of our arguments. Still, based on our experience, we expect similar outcomes if we were to use, say, a DGS environment as opposed to the potential outcomes expected when using a programming tool, e.g., Logo or Scratch, which would bring other components, e.g., that of computational thinking. Another crucial aspect, when it comes to interpreting students' thinking processes, is that gestures can play a pivotal role. Hence, in a different setup—which might further support generalisability and applicability of our findings—considering more carefully students' gestures could be a window for interpreting and analysing students' thinking processes and learning outcomes.

We hope, though, to have illustrated that the frameworks of TIG and TCF hold a large potential for "bridging" the notions of mathematical competencies and digital competencies. We also find that these theories provide depth to the competency discussion. Hence, we propose that possessing mathematical digital competency includes at least the following three characteristics:

• *Being able to engage in a techno-mathematical discourse*. In particular, this involves aspects of the artefact-instrument duality in the sense that instrumentation has taken place and thereby initiated the process of becoming techno-mathematically fluent.

• *Being aware of which digital tools to apply within different mathematical situations and context, and being aware of the different tools' capabilities and limitations.* In particular, this involves aspects of the instrumentation–instrumentalisation duality.

• *Being able to use digital technology reflectively in problem solving and when learning mathematics*. This involves being aware and taking advantage of digital tools serving both pragmatic and epistemic purposes, and in particular, aspects of the scheme-technique duality, both in relation to one's predicative and operative form of knowledge.

We do not claim that the three TIG dualities can only be linked to one of the characteristics of MDC. Still, we find distinctive features between the different dualities and the three identified characteristics of MDC, as also illustrated through the analyses of the empirical example. In addition, we do not claim that the above three characteristics for MDC are the only possible ones. Rather, we present them as an elaboration of what a notion of mathematical digital competency might encompass, and as an illustration of how we might think about and approach such a notion. That is to say, the first initial steps towards a definition of MDC.

As pointed to by Hague and Payton, digital literacy "is about addressing the changing nature of subject knowledge and acknowledging that young people will need different kinds of skills, knowledge and understanding in order to develop their expertise in subjects (Hague & Payton, 2010, p.12). Due to its high reliance on digital technologies, this may be truer for the subject of mathematics than for many other subjects in the 21st century. The "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p. 49) is today, as opposed to previously, much more about one acting appropriately in one's relation with digital technologies either applied to a mathematical challenge or in a learning situation. As mentioned earlier (cf. "Theories of instrumental genesis and conceptual fields"), a conceptual field is "a set of situations and a set of concepts tied together" (Vergnaud, 2009, p. 86). For mathematics students of today, such situations may be embedded so deeply in a techno-mathematical discourse that, potentially, also their understanding of the mathematical concepts involved is almost inseparable from the digital tools and the students' instrumented techniques. In fact, such students' predicative form of knowledge may only enable them to "do something" within a digital environment. Hence, for such students, it is no longer only about either mathematical competency or digital competency. It becomes about *mathematical digital competency*.

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