

Homotopy limits in the category of dg-categories in terms of A_∞ -comodules

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Abstract

In this paper, we apply an explicit construction of a simplicial powering in dg-categories, due to Holstein (2016) and Arkhipov and Poliakova (2018), as well as our own results on homotopy ends (Arkhipov and Ørsted 2018), to obtain an explicit model for the homotopy limit of a cosimplicial system of dg-categories. We apply this to obtain a model for homotopy descent in terms of A_∞ -comodules, proving a conjecture by Block, Holstein, and Wei (2017) in the process.

1 Introduction

This is a preparatory paper covering homotopical details needed to define the derived category of \mathcal{H} -equivariant \mathcal{A} -dg-modules in the case where \mathcal{A} is a dg-algebra and \mathcal{H} is a dg-Hopf-algebra acting on \mathcal{A} . The example of interest is when X is a regular, affine scheme and G an algebraic group acting on X , and where $\mathcal{A} = \Omega(X)$ and $\mathcal{H} = \Omega(G)$, both equipped with the *zero* differential (*not* the de Rham differential). Compare this with the classical situation where $A = \mathcal{O}(X)$ is an ordinary algebra and $H = \mathcal{O}(G)$ is an ordinary Hopf algebra given by the functions on some algebraic group. Then we may define the category of H -equivariant A -modules by the *homotopy limit*

$$(A\text{-mod})^H = \mathop{\mathrm{holim}}_{\Delta} (H^{\otimes n} \otimes A)\text{-mod}$$

with respect to the model structure on categories described in Rezk (2000). In the case where $G \backslash X$ exists in schemes and the map $X \rightarrow G \backslash X$ is fully faithful, descent theory tells us that $(A\text{-mod})^H$ recovers $\mathrm{QCoh}(G \backslash X)$. If $G \backslash X$ exists only as a stack, it will instead recover quasi-coherent sheaves on that.

More generally, if $f: X \rightarrow Y$ is an fpqc morphism of schemes, we may consider its **descent groupoid**, the internal groupoid in schemes $X_1 \rightrightarrows X_0$ given by $X_0 = X$ and $X_1 = X \times_Y X$ (both maps in the fibre product being f). We may then consider its classifying space, the internal Kan complex in schemes given by

$$X_n = \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ factors}} = \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{n+1 \text{ factors}}$$

with the usual simplicial structure, the face maps $\partial_i: X_n \rightarrow X_{n-1}$ applying f at the i th step, and the degeneracy maps $\sigma_i: X_n \rightarrow X_{n+1}$ inserting the diagonal map at the i th step. Then Y becomes an augmentation of the simplicial scheme X :

$$Y \leftarrow \cdots X_0 \rightrightarrows X_1 \rightleftarrows X_2 \rightleftarrows \cdots.$$

Then descent theory tells us that we recover quasi-coherent sheaves on Y by gluing quasi-coherent sheaves on X_0 via gluing data stored in the categories $\mathrm{QCoh}(X_i)$ for $i > 0$. This may be formulated by saying that $\mathrm{QCoh}(Y)$ is given by the homotopy limit (see Corollary 2.2.4),

$$\mathrm{QCoh}(Y) = \mathop{\mathrm{holim}}_{\Delta} \mathrm{QCoh}(X).$$

The homotopy limit is the derived functor of the limit. It can be roughly formulated as a homotopy-invariant version of the limit where the usual squares only hold up to correction (in the case of Cat , by isomorphisms). In other words, up to correction, we have an augmented cosimplicial system of categories

$$\mathrm{QCoh}(Y) \dashrightarrow \mathrm{QCoh}(X_0) \rightrightarrows \mathrm{QCoh}(X_1) \rightleftarrows \mathrm{QCoh}(X_2) \rightleftarrows \cdots,$$

where the cosimplicial maps are given by pullbacks of the simplicial maps. We notice that $\mathop{\mathrm{holim}}_{\Delta} \mathrm{QCoh}(X)$ makes sense even if the scheme Y does not exist, as it depends only on the groupoid $X_1 \rightrightarrows X_0$.

Alternatively, the pullback and pushforward functors

$$f^*: \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*$$

yield a comonad $T = f^*f_*: \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X)$. Then the Barr–Beck theorem tells us that we recover $\mathrm{QCoh}(Y)$ as

$$\mathrm{QCoh}(Y) \cong T\text{-comod}(\mathrm{QCoh}(X)),$$

where the right-hand side the category of T -comodules in $\mathrm{QCoh}(X)$. In the affine situation, we may write $X = \mathrm{Spec}(A)$ and $X_1 = \mathrm{Spec}(C)$ and observe that C becomes a coalgebra in $A\text{-mod-}A$ with comultiplication $\Delta = \partial_1^\#: C \rightarrow C \otimes_A C$. Then $T\text{-comod}(\mathrm{QCoh}(X)) = C\text{-comod}(A\text{-mod})$ is just the category of C -comodules in $A\text{-mod}$. Again, this is definable using only the data of the groupoid $X_1 \rightrightarrows X_0$, even if Y does not exist (in schemes).

The purpose of this paper is to prove a homotopy version of the equivalence

$$\mathop{\mathrm{holim}}_{\Delta} \mathrm{QCoh}(X) \cong C\text{-comod}(A\text{-mod})$$

for affine dg-schemes. More precisely, we prove

4.1.1. Theorem. *Suppose that $\mathbf{X}_1 \rightrightarrows \mathbf{X}_0$ is a groupoid in affine dg-schemes, and consider the associated classifying space \mathbf{X} , given by*

$$\mathbf{X}_n = \mathbf{X}_1 \times_{\mathbf{X}_0} \mathbf{X}_1 \times_{\mathbf{X}_0} \cdots \times_{\mathbf{X}_0} \mathbf{X}_1.$$

Write $\mathcal{A}^n = \mathcal{A}_{\mathbf{X}_n}$ for the associated cosimplicial system of dg-algebras. Let $\mathcal{A} = \mathcal{A}^0$ and $\mathcal{C} = \mathcal{A}^1$, and note that \mathcal{C} is a counital coalgebra in $\mathcal{A}\text{-dgm-}\mathcal{A}$ via the map $\Delta = \partial_1^\#: \mathcal{C} \rightarrow \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$. Then we have an equivalence of dg-categories

$$\mathop{\mathrm{holim}}_{\Delta} \mathrm{QCoh}(\mathbf{X}) \cong \mathcal{C}\text{-comod}_{\infty}^{\mathrm{hcu,formal}}(\mathcal{A}),$$

where the right-hand side denotes the dg-category of formal, homotopy-counital A_∞ -comodules over \mathcal{C} in \mathcal{A} -dgm.

Much of the inspiration comes from Block, Holstein, and Wei (2017). In the process, we prove their Conjecture 1 and generalize their results.

In chapter 2, we set up classical descent theory, including the homotopy limit and Barr–Beck formulations. In chapter 3, we recall differential graded (co)algebras and (co)categories and their A_∞ -analogues. Finally, in chapter 4, we present our main results on homotopy limits of dg-categories, including the above theorem.

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2 Classical descent theory

Suppose that $f: X \rightarrow Y$ is an fpqc morphism of schemes. Then descent theory tells us that we recover quasi-coherent sheaves on Y by gluing quasi-coherent sheaves on X . Concretely, if we denote by $X_1 \rightrightarrows X_0$ the Čech groupoid (see the introduction), then the category $\mathrm{QCoh}(Y)$ is equivalent to the category of “descent data” on the groupoid X_\bullet , defined in the following manner:

Let $X_1 \rightrightarrows X_0$ be any internal groupoid in the category of schemes, and denote by X_\bullet its classifying space, the internal Kan complex in schemes given by $X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1$. The category $\mathrm{Desc}(X_\bullet)$ of **descent data** on X_\bullet has objects the pairs (M, θ) , where $M \in \mathrm{QCoh}(X_0)$, and $\theta: \partial_1^* M \rightarrow \partial_0^* M$ a map satisfying the cocycle and unit conditions

$$\partial_0^* \theta \circ \partial_2^* \theta = \partial_1^* \theta \quad \text{and} \quad \sigma_0^* \theta = \mathrm{id}.$$

A morphism $\alpha: (M, \theta) \rightarrow (N, \eta)$ is a morphism $\alpha: M \rightarrow N$ in $\mathrm{QCoh}(X_0)$ such that $\eta \circ \partial_1^* \alpha = \partial_0^* \alpha \circ \theta$.

2.0.1. Proposition. *Assuming the cocycle condition, the assumption $\sigma_0^* \theta = \mathrm{id}$ is equivalent to θ being an isomorphism.*

Proof. If θ is an isomorphism, we may apply $\sigma_0^* \sigma_0^* = \sigma_0^* \sigma_1^*$ to both sides of the cocycle condition and get $\sigma_0^* \theta \circ \sigma_0^* \theta = \sigma_0^* \theta$. Now $\sigma_0^* \theta$ is the image of an isomorphism and hence an isomorphism, so we obtain $\sigma_0^* \theta = \mathrm{id}$. Conversely, if $\sigma_0^* \theta = \mathrm{id}$, then we use the groupoid conditions

$$\partial_1(\iota \times \mathrm{id})\Delta = \sigma_0 \partial_1 \quad \text{and} \quad \partial_1(\mathrm{id} \times \iota)\Delta = \sigma_0 \partial_0$$

where Δ denotes the diagonal map; in the first case, it is the diagonal map

$$\Delta: X_1 \rightarrow X_1 \times_{X_0} X_1,$$

the fibre product being taken on both sides with respect to the source map ∂_1 . In the second equation, ∂_0 is used instead. We have $(\sigma_0 \partial_1)^* \theta = \partial_1^* \sigma_0^* \theta = \mathrm{id}$, and

hence

$$\begin{aligned} \text{id} &= ((\iota \times \text{id})\Delta)^* \partial_1^* \theta = ((\iota \times \text{id})\Delta)^* (\partial_0^* \theta \circ \partial_2^* \theta) \\ &= (\partial_0(\iota \times \text{id})\Delta)^* \theta \circ (\partial_1(\iota \times \text{id})\Delta)^* \theta = \iota^* \theta \circ \theta. \end{aligned}$$

Similarly, the other equation yields $\text{id} = \theta \circ \iota^* \theta$. \square

2.0.2. Example. Suppose that Y is a scheme and $\bigcup U_i \rightarrow Y$ an fpqc covering, and define $X = \coprod U_i$, so that the morphism $f: X \rightarrow Y$ is fpqc. Then the descent groupoid is exactly the **Čech groupoid** given by $X_0 = \coprod U_i$ and $X_1 = \coprod U_{ij}$ (here, we use the usual convention of letting $U_{i_0 \dots i_n} = U_{i_0} \cap \dots \cap U_{i_n}$). The source map $\partial_1: X_1 \rightarrow X_0$ is given by the embeddings $U_{ij} \hookrightarrow U_i$, the target map $\partial_0: X_1 \rightarrow X_0$ is given by the embeddings $U_{ij} \hookrightarrow U_j$, the unit $X_0 \rightarrow X_1$ is given by $U_i \xrightarrow{\cong} U_{ii}$, composition $\partial_1: X_1 \times_{X_0} X_1 \rightarrow X_1$ is given by $U_{ijk} \hookrightarrow U_{ik}$, and inversion $\iota: X_1 \rightarrow X_1$ is given by $U_{ij} \xrightarrow{\cong} U_{ji}$. More generally

$$X_n = X_1 \times_{X_0} X_1 \times_{X_0} \dots \times_{X_0} X_1 = \coprod U_{i_0 \dots i_n}.$$

If $\varphi: [m] \rightarrow [n]$ is a map in Δ , the map $\varphi^*: X_n \rightarrow X_m$ is given by the embedding $U_{i_0 \dots i_n} \hookrightarrow U_{i_{\varphi(0)} \dots i_{\varphi(m)}}$. Then a pair $(M, \theta) \in \text{Desc}(X)$ consists of collections $M = (M_i)$ of quasi-coherent sheaves $M_i \in \text{QCoh}(U_i)$ on each element in the covering, and collections $\theta = (\theta_{ij})$ of gluing morphisms $\theta_{ij}: M_i|_{U_{ij}} \rightarrow M_j|_{U_{ij}}$, subject to the cocycle and unit conditions

$$\theta_{jk} \circ \theta_{ij} = \theta_{ik} \quad \text{and} \quad \theta_{ii} = \text{id}_{M_i}.$$

As in the general case, this implies that all θ_{ij} are automatically isomorphisms. A morphism $\alpha: (M, \theta) \rightarrow (N, \eta)$ is a tuple $\alpha = (\alpha_i)$ of morphisms $\alpha_i: M_i \rightarrow N_i$ such that $\eta_{ij} \circ \alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}} \circ \theta_{ij}$ for all i, j . Then the descent statement from before simply translates to the fact that we recover quasi-coherent sheaves on Y from these data. \circ

2.1 Barr–Beck theorem and comodules

One classical way of rewriting the descent condition is via the Barr–Beck theorem. We state it in the generality we shall need it. Following Mac Lane (1997), a **comonad** on a category \mathcal{C} is an endofunctor $T: \mathcal{C} \rightarrow \mathcal{C}$ together with natural transformations $\Delta: T \rightarrow T^2$, called **comultiplication**, and $\varepsilon: T \rightarrow \text{id}_{\mathcal{C}}$, called **counit**, such that the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{\Delta} & T^2 \\ \Delta \downarrow & & \downarrow \Delta T \\ T^2 & \xrightarrow{T\Delta} & T^3 \end{array} \quad \begin{array}{ccc} & T & \\ & \swarrow \Delta & \searrow \Delta \\ T & \xleftarrow{\varepsilon T} & T^2 \xrightarrow{T\varepsilon} T. \end{array}$$

A **comodule** for a comonad T consists of an object $x \in \mathcal{C}$ together with a morphism $\text{ca}: x \rightarrow Tx$, called **coaction**, such that the following diagrams commute:

$$\begin{array}{ccc} x & \xrightarrow{\text{ca}} & Tx \\ \text{ca} \downarrow & & \downarrow \Delta \\ Tx & \xrightarrow{T\text{ca}} & T^2x \end{array} \quad \begin{array}{ccc} x & \xrightarrow{\text{ca}} & Tx \\ & \searrow & \downarrow \varepsilon \\ & & x. \end{array}$$

A map of comodules $f: (x, ca) \rightarrow (y, ca)$ is a map commuting with coaction. We thus obtain a category T -comod of comodules over T .

2.1.1. Example. Any pair of adjoint functors $F: \mathcal{D} \rightleftarrows \mathcal{C} : G$ defines a comonad in \mathcal{C} by letting $T = FG$ and defining the comultiplication $\Delta: T \rightarrow T^2$ by the unit of adjunction $FG \rightarrow F(GF)G$, and the counit by the counit of adjunction. We see that any object of the form $F(x) \in \mathcal{C}$ for $x \in \mathcal{D}$ is a comodule over T via $F(x) \rightarrow F(GF)(x)$. \circ

2.1.2. Barr–Beck theorem. Suppose that \mathcal{C} and \mathcal{D} are Abelian categories and $F: \mathcal{D} \rightleftarrows \mathcal{C} : G$ an adjunction with F full and exact. If $T = FG$ is the associated comonad, F descends to an equivalence of Abelian categories $F: \mathcal{C} \simeq T\text{-comod}$.

Suppose now that $f: X \rightarrow Y$ is a faithfully flat morphism of affine schemes $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, and consider the pullback square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\partial_1} & X \\ \partial_0 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y. \end{array}$$

Then base change shows that the comonad

$$T = f^* f_*: \text{QCoh}(X) \rightarrow \text{QCoh}(X), \quad M \mapsto A \otimes_B M,$$

is equal to

$$\partial_{1*} \partial_0^*: \text{QCoh}(X) \rightarrow \text{QCoh}(X), \quad M \mapsto (A \otimes_B A) \otimes_A M.$$

Now $C := A \otimes_B A = \mathcal{O}(X \times_Y X) = \mathcal{O}(X_1)$ is naturally a coalgebra in the monoidal category $(A\text{-mod}, A, \otimes_A)$ of A -bimodules via the comorphism $\partial_1^\#: C \rightarrow C \otimes_A C$ associated to $\partial_1: X_1 \times_{X_0} X_1 \rightarrow X_1$. The category of T -comodules is the same as the category $C\text{-comod}(A\text{-mod})$ of C -comodules in the category $A\text{-mod}$, so Barr–Beck becomes the statement that

2.1.3. Proposition. If $f: X = \text{Spec}(A) \rightarrow Y = \text{Spec}(B)$ is faithfully flat, consider the coalgebra $C = \mathcal{O}(X \times_Y X) = A \otimes_B A$. Then

$$f^*: \text{QCoh}(Y) = B\text{-mod} \xrightarrow{\sim} C\text{-comod}(A\text{-mod})$$

is an equivalence of Abelian categories.

Even if the descent groupoid X does not come from the covering of a scheme Y , the category of descend data still becomes equivalent to comodules over the coalgebra $C = \mathcal{O}(X_1)$. Suppose that $X_1 \rightrightarrows X_0$ is an internal groupoid in the category of affine schemes. By adjunction, the gluing map $\theta: \partial_1^* M \rightarrow \partial_0^* M$ is equivalent to a map $ca: M \rightarrow \partial_{1*} \partial_0^* M = C \otimes_A M$. We claim that this operation makes M into a comodule over C in the category $A\text{-mod}$.

2.1.4. Proposition. If $X_1 \rightrightarrows X_0$ is an internal groupoid in affine schemes, then we have an equivalence of categories $\text{Desc}(X) \cong C\text{-comod}(A\text{-mod})$, where the right-hand side denotes the category of C -comodules in $A\text{-mod}$.

For this, we need some technical lemmas that will come in handy later. For a simplicial object X , and any n , we shall use the notation ∂_{\max} for the morphism $\partial_n: X_n \rightarrow X_{n-1}$. This allows us to consider powers of these, e.g. $\partial_{\max}^l = \partial_1 \cdots \partial_{n-1} \partial_n$. Similarly, we write $\partial_0^l = \partial_0 \partial_0 \cdots \partial_0$. We use a similar convention in the cosimplicial case and hence write e.g. $(\partial^{\max})^l = \partial^n \partial^{n-1} \cdots \partial^1$ and $(\partial^0)^l = \partial^0 \partial^0 \cdots \partial^0$. Applying base change to the pullback diagram

$$\begin{array}{ccc} X_{i+j} & \xrightarrow{\partial_{\max}^j} & X_i \\ \partial_0^i \downarrow & & \downarrow \partial_0^i \\ X_j & \xrightarrow{\partial_{\max}^j} & X_0 \end{array}$$

we obtain $\partial_{\max}^j \partial_0^{i*} \cong \partial_0^{i*} \partial_{\max}^j$. This implies

2.1.5. Lemma. *Let $X_1 \rightrightarrows X_0$ be a monoid in the category of schemes, with X , the associated classifying space. Suppose we are given objects $M, N, P \in \text{QCoh}(X_0)$, along with two maps $\theta: \partial_{\max}^{i*} M \rightarrow \partial_0^{i*} N$ in $\text{QCoh}(X_i)$ and $\eta: \partial_{\max}^j N \rightarrow \partial_0^{j*} P$ in $\text{QCoh}(X_j)$, where $i, j \geq 0$. Then the composition*

$$\begin{aligned} M &\longrightarrow \partial_{\max}^i \partial_{\max}^{i*} M \xrightarrow{\theta} \partial_{\max}^i \partial_0^{i*} N \longrightarrow \partial_{\max}^i \partial_0^{i*} \partial_{\max}^j \partial_{\max}^{j*} N \\ &\xrightarrow{\eta} \partial_{\max}^i \partial_0^{i*} \partial_{\max}^j \partial_{\max}^{j*} P \cong \partial_{\max}^i \partial_{\max}^j \partial_0^{i*} \partial_0^{j*} P = \partial_{\max}^{i+j} \partial_0^{(i+j)*} P. \end{aligned}$$

is the same as the composition

$$\begin{aligned} M &\longrightarrow \partial_{\max}^{i+j} \partial_{\max}^{(i+j)*} M \xrightarrow{\partial_{\max}^{i+j} \theta} \partial_{\max}^{i+j} \partial_{\max}^{j*} \partial_0^{i*} N \\ &= \partial_{\max}^{i+j} \partial_0^{i*} \partial_{\max}^{j*} N \xrightarrow{\partial_0^{i*} \eta} \partial_{\max}^{i+j} \partial_0^{i*} \partial_0^{j*} P = \partial_{\max}^{i+j} \partial_0^{(i+j)*} P. \end{aligned}$$

In other words, the map is the unit of adjunction $M \rightarrow \partial_{\max}^{i+j} \partial_{\max}^{(i+j)*} M$ composed with the map $\partial_0^{i*} \eta \circ \partial_{\max}^{j*} \theta$.

Proof. The statement is local, so assume that $X_0 = \text{Spec}(A)$ and $X_1 = \text{Spec}(C)$ are affine. The two maps $\partial_0, \partial_1: X_1 \rightrightarrows X_0$ correspond to maps $\partial_0^\#, \partial_1^\#: A \rightrightarrows C$, which turn C into an A -bimodule, say, with $\partial_1^\#$ providing the left action and $\partial_0^\#$ the right one. Then θ and η become maps

$$\begin{aligned} \theta: M \otimes_A C^{\otimes i} &\longrightarrow C^{\otimes i} \otimes_A N && \text{in } C^{\otimes i}\text{-mod} \\ \text{and } \eta: N \otimes_A C^{\otimes j} &\longrightarrow C^{\otimes j} \otimes_A P && \text{in } C^{\otimes j}\text{-mod.} \end{aligned}$$

The unit of adjunction $M \rightarrow \partial_{\max}^i \partial_{\max}^{i*} M = M \otimes_A C^{\otimes i}$ is then given by $m \mapsto m \otimes 1 \otimes \cdots \otimes 1$, and similarly for the other units of adjunction. Then the statement we want simply says that

$$M \longrightarrow M \otimes_A C^{\otimes i} \xrightarrow{\theta} C^{\otimes i} \otimes_A N \longrightarrow C^{\otimes i} \otimes_A (N \otimes_A C^{\otimes j}) \xrightarrow{\eta} C^{\otimes i} \otimes_A (C^{\otimes j} \otimes_A P)$$

is equal to

$$M \longrightarrow M \otimes_A C^{\otimes(i+j)} \xrightarrow{\theta} C^{\otimes i} \otimes_A N \otimes_A C^{\otimes j} \xrightarrow{\eta} C^{\otimes(i+j)} \otimes_A P,$$

which boils down to associativity of the tensor product. \square

2.1.6. Lemma. *Let the assumptions be as in Lemma 2.1.5. Suppose that we are given objects $M, N \in \text{QCoh}(X_0)$ along with a map $\theta: \partial_{\max}^{n*} M \rightarrow \partial_0^{n*} M$. Let $0 < i < n$. Then we have a commutative diagram*

$$\begin{array}{ccccc} M & \longrightarrow & \partial_{\max}^n \partial_{\max}^{n*} M & \xrightarrow{\theta} & \partial_{\max}^n \partial_0^{n*} N \\ \downarrow & & \downarrow & & \downarrow \\ \partial_{\max}^{n+1} \partial_{\max}^{(n+1)*} M & = & \partial_{\max}^n \partial_{i*} \partial_i^* \partial_{\max}^{n*} M & \xrightarrow{\partial_i^* \theta} & \partial_{\max}^n \partial_{i*} \partial_i^* \partial_0^{n*} N = \partial_{\max}^{n+1} \partial_0^{(n+1)*} N. \end{array}$$

Proof. Clear from naturality of the unit of adjunction. \square

Proof of Proposition 2.1.4. The coassociativity diagram

$$M \xrightarrow{\text{ca}} C \otimes_A M \xrightarrow[\Delta \circ \text{id}]{\text{id} \otimes \text{ca}} C \otimes_A C \otimes_A M.$$

is the same as the diagram

$$\begin{array}{ccccc} & & \partial_{1*} \partial_0^* \partial_{1*} \partial_1^* M & \xrightarrow{\theta} & \partial_{1*} \partial_0^* \partial_{1*} \partial_0^* M \\ & & \uparrow & & \parallel \\ M & \longrightarrow & \partial_{1*} \partial_1^* M & \xrightarrow{\theta} & \partial_{1*} \partial_2^* \partial_0^* \partial_0^* M \\ & & \downarrow & \swarrow & \\ & & \partial_{1*} \partial_{1*} \partial_1^* \partial_0^* M & & \end{array}$$

The two lemmas show that this diagram is the same as the diagram

$$M \longrightarrow \partial_{1*} \partial_{2*} \partial_2^* \partial_1^* M \xrightarrow[\partial_1^* \theta]{\partial_0^* \theta \circ \partial_2^* \theta} \partial_{1*} \partial_{2*} \partial_0^* \partial_0^* M,$$

so commutativity boils down to the relation $\partial_1^* \theta = \partial_0^* \theta \circ \partial_2^* \theta$.

To verify the unit condition, notice that condition $(\varepsilon \otimes \text{id}_A) \circ \text{ca} = \text{id}_M$, note that the left-hand side is the composition

$$M \longrightarrow \partial_{1*} \partial_1^* M \xrightarrow{\theta} \partial_{1*} \partial_0^* M \longrightarrow \partial_{1*} \sigma_{0*} \sigma_0^* \partial_0^* M.$$

The fact that this is the identity on M follows from the commutative diagram

$$\begin{array}{ccccc} M & \longrightarrow & \partial_{1*} \partial_1^* M & \xrightarrow{\theta} & \partial_{1*} \partial_0^* M \\ \parallel & & \downarrow & & \downarrow \\ & & \partial_{1*} \sigma_{0*} \sigma_0^* \partial_1^* M & \xrightarrow{\sigma_0^* \theta} & \partial_{1*} \sigma_{0*} \sigma_0^* \partial_0^* M = M \end{array}$$

and the assumption that $\sigma_0^* \theta = \text{id}_M$. \square

2.2 Descent via homotopy limits

In this chapter, we give a small resume of the exposition of homotopy limits presented in Arkhipov and Ørsted (2018) and refer the reader to that paper for further details. If \mathcal{C} is a model category and Γ a category, we may consider the category of functors $\mathcal{C}^\Gamma = \text{Fun}(\Gamma, \mathcal{C})$ of functors $\Gamma \rightarrow \mathcal{C}$, which we shall also refer to as “diagrams”. It makes sense to call a map of diagrams $\alpha: F \rightarrow G$ a **weak equivalence** if $\alpha_\gamma: F(\gamma) \rightarrow G(\gamma)$ is a weak equivalence for all $\gamma \in \Gamma$. It is natural to refer to such weak equivalences as **componentwise** weak equivalences. However, we immediately run into the problem that the limit functor $\underline{\lim}: \mathcal{C}^\Gamma \rightarrow \mathcal{C}$ does not in general take componentwise weak equivalences to weak equivalences in \mathcal{C} . Since $\underline{\lim}$ is a right adjoint, this leads us into trying to *derive* it. The right derived functor of $\underline{\lim}$ is called the homotopy limit and is denoted $\underline{\text{holim}}: \mathcal{C}^\Gamma \rightarrow \mathcal{C}$. Dually, the left derived functor of $\underline{\lim}$ is called the **homotopy colimit** and is denoted $\underline{\text{hocolim}}: \mathcal{C}^\Gamma \rightarrow \mathcal{C}$.

Quillen’s model category machinery tells us how to derive the limit: We must equip the diagram category \mathcal{C}^Γ with a model structure with componentwise weak equivalences and in which the limit functor $\underline{\lim}: \mathcal{C}^\Gamma \rightarrow \mathcal{C}$ is a right Quillen functor. In this case, the derived functor is given by $\underline{\text{holim}} F = \underline{\lim} R(F)$ for some fibrant replacement $R(F)$ in \mathcal{C}^Γ . Indeed, such a model structure on \mathcal{C}^Γ exists e.g. if the model category \mathcal{C} is *combinatorial*. More precisely, we introduce

- The **projective model structure** $\mathcal{C}_{\text{Proj}}^\Gamma$ where weak equivalences and fibrations are calculated componentwise.
- The **injective model structure** $\mathcal{C}_{\text{Inj}}^\Gamma$ where weak equivalences and cofibrations are calculated componentwise.

Denoting by $\text{const}: \mathcal{C} \rightarrow \mathcal{C}^\Gamma$ the constant functor embedding, we clearly see that $\text{const}: \mathcal{C} \rightleftarrows \mathcal{C}_{\text{Inj}}^\Gamma: \underline{\lim}$ is a Quillen adjunction since const preserves (trivial) cofibrations. Dually, $\underline{\lim}: \mathcal{C}_{\text{Proj}}^\Gamma \rightleftarrows \mathcal{C}: \text{const}$ is a Quillen adjunction. The injective model structure being in general rather complicated, calculating such a replacement of a diagram in practice becomes very involved for all but the simplest shapes of the category Γ . Therefore, traditionally, other tools have been used.

We shall only be interested in homotopy limits over the simplex category Δ with objects finite ordered sets $[n] = \{0, 1, \dots, n\}$ and morphisms the order-preserving (i.e. non-decreasing) maps. In this case, one of the available formulas for the homotopy limit is the **fat totalization formula**:

2.2.1. Proposition (ibid., Example 6.4).

Suppose the model category \mathcal{C} is combinatorial and $X^\bullet: \Delta \rightarrow \mathcal{C}$ a cosimplicial diagram. Then the homotopy limit over the simplex category Δ may be calculated by the formula

$$\underline{\text{holim}}_\Delta X^\bullet = \int_{[n] \in \Delta_+} R(X^n)_n$$

where the integral refers to the **end** construction, and where $R: \mathcal{C} \rightarrow \mathcal{C}_{\text{Inj}}^{\Delta_+^{\text{op}}}$ is a functor that takes $x \in \mathcal{C}$ to an injectively fibrant replacement of the constant Δ_+^{op} -diagram at x .

Suppose that \mathcal{C} is in fact a *simplicial model category*, meaning that it is enriched, powered, and copowered over simplicial sets. The powering is given by a Quillen bifunctor

$$\mathbb{S}\text{Set}^{\text{op}} \times \mathcal{C} \longrightarrow \mathcal{C}, \quad (K, x) \longmapsto x^K,$$

where $\mathbb{S}\text{Set}$ denotes simplicial sets equipped with the Quillen model structure. Being a Quillen bifunctor means, among other things, that it takes (trivial) cofibrations in $\mathbb{S}\text{Set}$ to (trivial) fibrations. Since the standard simplex Δ^* is a projectively cofibrant Δ_+^{op} -diagram of simplicial sets, this implies that $\Delta_+^{\text{op}} \rightarrow \mathcal{C}, [n] \mapsto x^{\Delta^n}$, is a fibrant replacement functor like the one in the proposition.

We recall from Rezk (2000) the existence of a model structure on the category Cat of categories with

- weak equivalences given by equivalences of categories;
- cofibrations given by functors which are injective on objects;
- fibrations given by isofibrations, i.e. functors $F: \mathcal{C} \rightarrow \mathcal{D}$ such that any isomorphism of the form $h: F(c) \rightarrow d$ in \mathcal{D} there exists a morphism $g: c \rightarrow c'$ in \mathcal{C} such that $F(g) = h$.

This model structure is combinatorial and simplicial. The powering is given by $\mathcal{C}^{\Delta^n} = \text{Fun}(\text{Iso}^n, \mathcal{C})$, where Iso^n denotes the category with $n + 1$ objects, denoted by $0, 1, \dots, n$, and one unique arrow between any two objects (in particular, all arrows are isomorphisms, and we have a groupoid).

Thus Proposition 2.2.1 shows that

2.2.2. Proposition. *The homotopy limit $\mathop{\text{holim}}_{\Delta} \mathcal{C}^*$ of a Δ -diagram of categories is the category with objects pairs (M, θ) , where $M \in \mathcal{C}^0$ and θ is an isomorphism $\theta: \partial^1(M) \xrightarrow{\sim} \partial^0(M)$ satisfying the cocycle condition $\partial^0(\theta) \circ \partial^2(\theta) = \partial^1(\theta)$. A morphism $\alpha: (M, \theta) \rightarrow (N, \eta)$ consists of a map $\alpha: M \rightarrow N$ such that $\eta \circ \partial^1(\alpha) = \partial^0(\alpha) \circ \theta$.*

Proof. The object set of the homotopy limit is

$$\int_{[n] \in \Delta_+} \text{Hom}_{\text{Cat}}(\text{Iso}^n, \mathcal{C}^n),$$

which is the set of natural transformations $\theta: \text{Iso}^* \rightarrow \mathcal{C}^*$, i.e. maps $\theta^n: \text{Iso}^n \rightarrow \mathcal{C}^n$ commuting with all face maps. This amounts to an object $M = \theta^0(0) \in \mathcal{C}^0$ and an isomorphism $\theta = \theta^1: \partial^1(M) \xrightarrow{\sim} \partial^0(M)$ satisfying the mentioned cocycle condition. A morphism α consists of a collection of natural transformations $\alpha^n: \text{Iso}^n \rightarrow \mathcal{C}^n$ commuting with face maps. This boils down to a map $\alpha = \alpha^0: M \rightarrow N$ satisfying the mentioned condition. \square

2.2.3. Remark. We note that the conditions on θ imply that $\sigma^0(\theta) = \text{id}$: Indeed, applying $\sigma^0\sigma^1 = \sigma^0\sigma^0$ implies $\sigma^0(\theta) \circ \sigma^0(\theta) = \sigma^0(\theta)$, which together with the isomorphism condition implies $\sigma^0(\theta) = \text{id}$. This shows that the homotopy limit in Cat is always equal to the **totalization**

$$\int_{[n] \in \Delta} \text{Fun}(\text{Iso}^n, \mathcal{C}^n),$$

where the end is taken over the whole simplex category Δ , including the degeneracy maps. This is a special feature of Cat ; in other simplicial model categories, this only holds in some cases, e.g. if the diagram \mathcal{C}^\bullet is *Reedy-fibrant*. Δ

2.2.4. Corollary. *For a groupoid $X_1 \rightrightarrows X_0$ in schemes with classifying space X , we have an equivalence of categories $\text{Desc}(X) \cong \underline{\text{holim}}_\Delta \text{QCoh}(X)$.*

3 Differential graded categories and A_∞ -categories

In this chapter, we mainly follow the sources Lyubashenko (2008), Keller (2006) and Lefèvre-Hasegawa (2003), with a few generalizations. We follow the sign conventions of Lefèvre-Hasegawa (ibid., section 1.1). In this chapter, we fix a field k and denote by $\text{Vect} = \text{Vect}(k)$ the category of vector spaces over k .

3.1 Graded objects, complexes, and sign conventions

Consider the category Vect^{gr} of \mathbb{Z} -graded vector spaces, which we shall write as tuples $M = M^\bullet = (M^n)_{n \in \mathbb{Z}}$ with $M^n \in \text{Vect}$. The degree of a homogeneous element $x \in M^\bullet$ is denoted $|x|$. If M^\bullet and N^\bullet are graded vector spaces, their hom space $\text{Hom}_k^\bullet(M, N)$ is given by

$$\text{Hom}_k^n(M^\bullet, N^\bullet) = \prod_{m \in \mathbb{Z}} \text{Hom}_k(M^m, N^{m+n}).$$

The category Vect^{gr} is monoidal with tensor product given by

$$(M \otimes_k N)^n = \bigoplus_{p+q=n} M^p \otimes_k N^q.$$

If $f: M_1 \rightarrow M_2$ is a map of degree r and $g: N_1 \rightarrow N_2$ is a map of degree s , then $f \otimes g: M_1 \otimes_k N_1 \rightarrow M_2 \otimes_k N_2$ is a map of degree $r+s$ given on the (p, q) -component, where $p+q=n$, by

$$(-1)^{ps} f^p \otimes g^q: M_1^p \otimes N_1^q \rightarrow M_2^{p+r} \otimes N_2^{q+s}.$$

If M is a graded vector space, we write $M[p]$ for its shifted vector space given by $M[p]^n = M^{n+p}$. We denote by $s: M \rightarrow M[1]$ the map of degree -1 which is the identity in each component. The inverse map is written $\omega = s^{-1}$. Because of the above sign conventions, we note that $s^{\otimes n}: M^{\otimes n} \rightarrow M[1]^{\otimes n} = M^{\otimes n}[n]$ is given by

$$s^{\otimes n}(x_n \otimes \cdots \otimes x_1) = (-1)^{\sum_{i=1}^{n-1} (i-1)|x_i|} s(x_n) \otimes \cdots \otimes s(x_1)$$

and similarly for $\omega^{\otimes n}$. As a consequence, we have

$$\omega^{\otimes n} \circ s^{\otimes n} = (-1)^{n(n-1)/2} \text{id}_{M^{\otimes n}}.$$

If M and N are graded objects and $n \geq 1$, we shall make use of the bijections

$$(3.1.1) \quad \text{Hom}_k^{l+1-n}(M^{\otimes n}, N) \xrightarrow{\sim} \text{Hom}_k^l(M[1]^{\otimes n}, N[1])$$

$$f \mapsto f' = (-1)^l s \circ f \circ \omega^{\otimes n}$$

and

$$(3.1.2) \quad \begin{aligned} \mathrm{Hom}_k^{l+1-n}(M, N^{\otimes n}) &\xrightarrow{\sim} \mathrm{Hom}_k^l(M[-1], N[-1]^{\otimes n}) \\ f &\mapsto f' = (-1)^l \omega^{\otimes n} \circ f \circ s. \end{aligned}$$

We denote by $\mathrm{Com}(k)$ the category of complexes of k -vector spaces. We use cohomological notation, so to us, a complex is a graded vector space $M^\bullet \in \mathrm{Vect}^{\mathrm{gr}}$ equipped with a differential map $d: M \rightarrow M[1]$ with $d^2 = 0$. If M and N are complexes, the graded hom space $\mathrm{Hom}_k(M, N)$ becomes a complex with the differential $d(f) = d_N \circ f - (-1)^{|f|} f \circ d_M$. The tensor product $M \otimes_k N$ of graded objects becomes a complex via $d_{M \otimes_k N} = d_M \otimes \mathrm{id}_N + \mathrm{id}_M \otimes d_N$. The shifted object $M[n]$ becomes a complex via the differential $d_{M[n]^i} = (-1)^n d_{M^{n+i}}$.

Most of the following considerations make sense both in the graded and differential graded (dg-) case. To treat the two cases simultaneously, we let the symbol $*$ stand for any of the symbols gr and dg (and for the terms “graded” and “dg-”). We shall use the notation $k\text{-*mod}$ to refer to either $\mathrm{Vect}^{\mathrm{gr}}$ or $\mathrm{Com}(k)$.

3.2 Graded quivers and dg-quivers

If \mathbb{E} is a set, we may regard it as a discrete category. Even though $\mathbb{E}^{\mathrm{op}} = \mathbb{E}$, we shall use both \mathbb{E} and \mathbb{E}^{op} in the following. The category $k_{\mathbb{E}\text{-*mod}}$ of **left** $k_{\mathbb{E}\text{-*modules}}$ is the category of functors $\mathbb{E}^{\mathrm{op}} \rightarrow k\text{-*mod}$, and the category $\text{*mod-}k_{\mathbb{E}}$ of **right** $k_{\mathbb{E}\text{-*modules}}$ is the category of functors $\mathbb{E} \rightarrow k\text{-*mod}$ (the two categories are equal, for the time being). If \mathbb{F} is another set, the category $k_{\mathbb{E}\text{-*mod-}k_{\mathbb{F}}}$ of $k_{\mathbb{E}\text{-}k_{\mathbb{F}}\text{-*bimodules}}$ is the category of functors $\mathbb{E}^{\mathrm{op}} \times \mathbb{F} \rightarrow k\text{-*mod}$. We define a tensor product by convolution, i.e.

$$\otimes_{k_{\mathbb{E}}} : \text{*mod-}k_{\mathbb{E}} \times k_{\mathbb{E}\text{-*mod}} \longrightarrow k\text{-*mod}, \quad (\mathcal{C}, \mathcal{D}) \mapsto \bigoplus_{s \in \mathbb{E}} \mathcal{C}(s) \otimes_k \mathcal{D}(s).$$

This extends in a natural way to the case when one or both of \mathcal{C} or \mathcal{D} are $*$ -bimodules. In particular, $(k_{\mathbb{E}\text{-*mod-}k_{\mathbb{E}}, \otimes_{k_{\mathbb{E}}})$ becomes a monoidal category. It is also unital, with unit the bimodule $k_{\mathbb{E}}$ defined by $k_{\mathbb{E}}(s, s) = k$ and $k_{\mathbb{E}}(s, t) = 0$ if $s \neq t$. This category acts on $k_{\mathbb{E}\text{-*mod}}$ from the left and on $\text{*mod-}k_{\mathbb{E}}$ from the right. All of these allow a shift operation $\mathcal{C} \mapsto \mathcal{C}[n]$, $n \in \mathbb{Z}$, the shift being applied componentwise.

We shall also refer to the monoidal category $k_{\mathbb{E}\text{-*mod-}k_{\mathbb{E}}}$ as $*$ -**quivers** over k with object set \mathbb{E} and write it as $\text{*Quiv}(\mathbb{E})$. A **morphism** $\mathcal{C} \rightarrow \mathcal{D}$ of **degree** n is a morphism $\mathcal{C} \rightarrow \mathcal{D}[n]$. If \mathcal{C} and \mathcal{D} are augmented $*$ -quivers, the tensor product $\mathcal{C} \otimes_k \mathcal{D}$ has objects $\mathrm{Ob} \mathcal{C} \times \mathrm{Ob} \mathcal{D}$ and morphism spaces given by

$$(\mathcal{C} \otimes_k \mathcal{D})((c, c'), (d, d')) = \mathcal{C}(c, c') \otimes_k \mathcal{D}(d, d').$$

The $*$ -quiver \mathcal{C} is **augmented** if there is a chain of maps of $*$ -quivers $k_{\mathbb{E}} \xrightarrow{\eta} \mathcal{C} \xrightarrow{\epsilon} k_{\mathbb{E}}$ whose composition is the identity. In that case, its **reduction** $\overline{\mathcal{C}}$ is given by $\overline{\mathcal{C}} = \mathrm{Coker}(\eta) \cong \mathrm{Ker}(\epsilon)$, calculated in each degree. This gives us a canonical splitting $\mathcal{C} = \overline{\mathcal{C}} \oplus k_{\mathbb{E}}$. Conversely, the **augmentation** of a non-augmented $*$ -quiver is the augmented $*$ -quiver $\mathcal{C}^+ = \mathcal{C} \oplus k_{\mathbb{E}}$. If \mathcal{C} and \mathcal{D} are augmented $*$ -quivers with object set \mathbb{E} , a **morphism of augmented $*$ -quivers** is a morphism $\mathcal{C} \rightarrow \mathcal{D}$ that respects η and ϵ . This allows us to define a category $\text{*Quiv}_{\mathrm{aug}}(\mathbb{E})$ of augmented $*$ -quivers. It is equivalent to $\text{*Quiv}(\mathbb{E})$ via reduction, but this equivalence is not monoidal.

The collection $*\text{Quiv}$ of $*$ -quivers for all choices of the set \mathbb{E} also form a category. Namely, if $\mathcal{C} \in *\text{Quiv}(\mathbb{E})$ and $\mathcal{D} \in *\text{Quiv}(\mathbb{F})$, a morphism $f: \mathcal{C} \rightarrow \mathcal{D}$ consists of a map of sets $f: \mathbb{E} \rightarrow \mathbb{F}$ and a morphism $\mathcal{C} \rightarrow f^*\mathcal{D}$ in $*\text{Quiv}(\mathbb{E})$. Here $f^*\mathcal{D}$ denotes the composition $\mathcal{D} \circ (f \times f)$, with \mathcal{D} regarded as a functor $\mathbb{F}^{\text{op}} \times \mathbb{F} \rightarrow k\text{-mod}$. The category $*\text{Quiv}_{\text{aug}}$ of augmented $*$ -quivers for all choices of \mathbb{E} is defined analogously.

If f, g are morphisms $\mathcal{C} \rightarrow \mathcal{D}$ of augmented $*$ -quivers as above, a **natural transformation** of degree n consists of a morphism $\alpha: \mathcal{C} \rightarrow (f \times g)^*\overline{\mathcal{D}}[n]$ in $*\text{Quiv}(\mathbb{E})$, where $(f \times g)^*\overline{\mathcal{D}} = \overline{\mathcal{D}} \circ (f \times g)$. This allows us to define an internal hom in the category $*\text{Quiv}$, denoted $\text{Hom}_{*\text{Quiv}_{\text{aug}}}(\mathcal{C}, \mathcal{D})$, with objects the morphisms $\mathcal{C} \rightarrow \mathcal{D}$ and morphism space whose reduction consists of natural transformations, i.e.

$$\begin{aligned} \overline{\text{Hom}_{*\text{Quiv}_{\text{aug}}}(\mathcal{C}, \mathcal{D})}^n(f, g) &= \text{Hom}_{*\text{Quiv}(\mathbb{E})}(\mathcal{C}, (f \times g)^*\overline{\mathcal{D}}[n]) \\ &= \prod_{x, y \in \mathbb{E}} \text{Hom}_k(\mathcal{C}(x, y), \overline{\mathcal{D}}(f(x), g(y))). \end{aligned}$$

In the dg-case, the differential is applied componentwise. This turns $*\text{Quiv}_{\text{aug}}$ into a closed monoidal category, since we have

$$\text{Hom}_{*\text{Quiv}_{\text{aug}}}(\mathcal{C} \otimes_k \mathcal{D}, \mathcal{E}) \cong \text{Hom}_{*\text{Quiv}_{\text{aug}}}(\mathcal{C}, \text{Hom}_{*\text{Quiv}_{\text{aug}}}(\mathcal{D}, \mathcal{E}))$$

(Keller 2006, Lemma 5.1).

3.3 Graded categories and dg-categories

A **$*$ -category** with object set \mathbb{E} is an associative algebra in the monoidal category $(*\text{Quiv}(\mathbb{E}), \otimes_{k_{\mathbb{E}}}, k_{\mathbb{E}}) = (k_{\mathbb{E}}\text{-mod-}k_{\mathbb{E}}, \otimes_{k_{\mathbb{E}}}, k_{\mathbb{E}})$. This means exactly that we have a composition operation $m: \mathcal{A} \otimes_{k_{\mathbb{E}}} \mathcal{A} \rightarrow \mathcal{A}$ satisfying associativity. Notice that this splits into components $\mathcal{A}(t, u) \otimes \mathcal{A}(s, t) \rightarrow \mathcal{A}(s, u)$ and becomes a composition operation in the categorical sense. The $*$ -category \mathcal{A} is **unital** if it is unital as an algebra in the above sense, i.e. if it is also equipped with a unit map $\eta: k_{\mathbb{E}} \rightarrow \mathcal{A}$ satisfying the usual the unity axiom. By a “ $*$ -category”, we shall mean a unital $*$ -category. The category of such will be denoted $*\text{Alg}(k_{\mathbb{E}})$. The **opposite category** \mathcal{A}^{op} of $*$ -category \mathcal{A} has the same objects and morphism spaces as \mathcal{A} , but $f \circ g$ in \mathcal{A}^{op} is defined to be $(-1)^{|f||g|}g \circ f$ in \mathcal{A} .

We want to turn the category of all $*$ -categories, for varying sets \mathbb{E} , into a category. If $f: \mathbb{E} \rightarrow \mathbb{F}$ is a map of sets, we obtain a functor $f^*: k_{\mathbb{F}}\text{-mod-}k_{\mathbb{F}} \rightarrow k_{\mathbb{E}}\text{-mod-}k_{\mathbb{E}}$ given by restriction. This functor is lax monoidal: Indeed, we obtain a map $k_{\mathbb{E}} \rightarrow f^*k_{\mathbb{F}}$ in $*\text{Alg}(k_{\mathbb{E}})$, and if $M, N \in k_{\mathbb{F}}\text{-mod-}k_{\mathbb{F}}$, we obtain maps $f^*M \otimes_{k_{\mathbb{E}}} f^*N \rightarrow f^*(M \otimes_{k_{\mathbb{F}}} N)$. This implies that f^* induces a pullback functor $f^*: *\text{Alg}(k_{\mathbb{F}}) \rightarrow *\text{Alg}(k_{\mathbb{E}})$. Given $*$ -categories $\mathcal{A} \in *\text{Alg}(k_{\mathbb{E}})$ and $\mathcal{B} \in *\text{Alg}(k_{\mathbb{F}})$ with different sets of objects, we can thus define a **$*$ -functor** $F: \mathcal{A} \rightarrow \mathcal{B}$ between $*$ -categories to be a map of sets $F: \text{Ob } \mathcal{A} \rightarrow \text{Ob } \mathcal{B}$ along with a map $F: \mathcal{A} \rightarrow F^*\mathcal{B}$ in $*\text{Alg}(k_{\mathbb{E}})$. The $*$ -functors form the morphisms in the category $*\text{Cat} = *\text{Cat}(k)$ of all $*$ -categories over k .

The category of $*$ -categories is monoidal: The tensor product $\mathcal{A} \otimes_k \mathcal{B}$ has objects $\text{Ob } \mathcal{A} \times \text{Ob } \mathcal{B}$, and we write a double (a, b) as $a \otimes b$. The morphism space is given by

$$(\mathcal{A} \otimes_k \mathcal{B})(a \otimes a', b \otimes b') = \mathcal{A}(a, a') \otimes_k \mathcal{B}(b, b').$$

The composition is given by

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'| |f|} (f' \circ f) \otimes (g' \circ g).$$

It is in fact a closed monoidal category: The internal hom $*\text{Fun}(\mathcal{A}, \mathcal{B})$ has objects the set of $*$ -functors $F: \mathcal{A} \rightarrow \mathcal{B}$. The morphism space $*\text{Fun}(\mathcal{A}, \mathcal{B})(F, G)$ consists of **natural $*$ -transformations**. A natural $*$ -transformation $\alpha: F \rightarrow G$ of degree d consists of a collection $\alpha = (\alpha_x)_{x \in \mathcal{A}}$ of maps $\alpha_x \in \mathcal{B}^d(F(x), G(x))$ such that $\alpha_y \circ F(f) = (-1)^{dk} G(f) \circ \alpha_x$ for all $f \in \mathcal{A}^k(x, y)$. Composition is applied componentwise, $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$. In the dg-case, the differential is also applied componentwise, i.e. $(d\alpha)_x = d(\alpha_x)$.

A **left \mathcal{A} - $*$ -module** is a left module $M \in k_{\mathbb{E}}\text{-}*\text{mod}$ over the algebra $\mathcal{A} \in k_{\mathbb{E}}\text{-}*\text{mod-}k_{\mathbb{E}}$ in the categorical sense. Thus we require a bifunctor $\mathcal{A} \otimes_{k_{\mathbb{E}}} M \rightarrow M$ satisfying the usual associativity and unity conditions, and morphisms must respect this structure. By the closed monoidal structure, this is the same as a $*$ -functor $\mathcal{A} \rightarrow k\text{-}*\text{mod}$, which allows us to define the $*$ -category $\mathcal{A}\text{-}*\text{mod} = *\text{Fun}(\mathcal{A}, k\text{-}*\text{mod})$ of left \mathcal{A} -modules. Similarly, a **right \mathcal{A} - $*$ -module** is a right module $N \in *\text{mod-}k_{\mathbb{E}}$ in the categorical sense, and the $*$ -category of such is $*\text{mod-}\mathcal{A} = *\text{Fun}(\mathcal{A}^{\text{op}}, k\text{-}*\text{mod})$. If \mathcal{B} is another $*$ -category with object set \mathbb{F} , an **\mathcal{A} - \mathcal{B} - $*$ -bimodules** is a bimodule in the category $k_{\mathbb{E}}\text{-}*\text{mod-}k_{\mathbb{F}}$, and the $*$ -category of such is $\mathcal{A}\text{-}*\text{mod-}\mathcal{B} = *\text{Fun}(\mathcal{A} \otimes_k \mathcal{B}^{\text{op}}, k\text{-}*\text{mod})$. We obtain a paring

$$\begin{aligned} \otimes_{\mathcal{A}}: *\text{mod-}\mathcal{A} \otimes_k \mathcal{A}\text{-}*\text{mod} &\longrightarrow k\text{-}*\text{mod} \\ (M, N) &\longmapsto \text{Coeq}(M \otimes_{k_{\mathbb{E}}} \mathcal{A} \otimes_{k_{\mathbb{E}}} N \rightrightarrows M \otimes_{k_{\mathbb{E}}} N) \end{aligned}$$

which extends to a paring

$$\otimes_{\mathcal{A}}: \mathcal{C}\text{-}*\text{mod-}\mathcal{A} \otimes_k \mathcal{A}\text{-}*\text{mod-}\mathcal{B} \longrightarrow \mathcal{C}\text{-}*\text{mod-}\mathcal{B}.$$

In particular, we obtain a monoidal category $(\mathcal{A}\text{-}*\text{mod-}\mathcal{A}, \otimes_{\mathcal{A}})$. The unit is \mathcal{A} , regarded as an \mathcal{A} - \mathcal{A} -bimodule.

3.4 Graded algebras and dg-algebras over categories

Suppose as above that \mathcal{A} is a $*$ -category. A (unital) **$*$ -algebra** over \mathcal{A} is a (unital) associative algebra in the monoidal category $(\mathcal{A}\text{-}*\text{mod-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$. A $*$ -algebra is assumed to be unital unless explicitly stated otherwise. The category of \mathcal{A} - $*$ -algebras is denoted $*\text{Alg}(\mathcal{A})$. We note that the category of $*$ -categories with object \mathbb{E} is equal to $*\text{Alg}(k_{\mathbb{E}})$, so the notation is consistent with the one defined above. The category of non-unital $*$ -algebras over \mathcal{A} is denoted $*\text{Alg}_{\text{nu}}(\mathcal{A})$.

Above we defined morphisms between $*$ -categories with different objects. We can more generally define morphisms between $*$ -algebras over different $*$ -categories. If $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a $*$ -functor between $*$ -categories, we obtain a restriction functor $f^*: \mathcal{A}_2\text{-}*\text{mod-}\mathcal{A}_2 \rightarrow \mathcal{A}_1\text{-}*\text{mod-}\mathcal{A}_1$. We claim that this functor is lax monoidal, i.e. that we have maps $\mathcal{A}_1 \rightarrow f^* \mathcal{A}_2$ and $f^* M \otimes_{\mathcal{A}_1} f^* N \rightarrow f^*(M \otimes_{\mathcal{A}_2} N)$ in the category $\mathcal{A}_1\text{-}*\text{mod-}\mathcal{A}_1$ for all $M, N \in \mathcal{A}_2\text{-}*\text{mod-}\mathcal{A}_2$, satisfying the usual conditions. The first map comes from the definition of a $*$ -functor, noting that it is in fact a map in $\mathcal{A}_1\text{-}*\text{mod-}\mathcal{A}_1$. The construction of the second map follows from the universal property of the tensor product (we

write $\mathbb{E}_1 = \text{Ob } \mathcal{A}_1$ and $\mathbb{E}_2 = \text{Ob } \mathcal{A}_2$):

$$\begin{array}{ccccc}
f^*M \otimes_{k_{\mathbb{E}_1}} \mathcal{A}_1 \otimes_{k_{\mathbb{E}_1}} f^*N & \rightarrow & f^*M \otimes_{k_{\mathbb{E}_1}} f^* \mathcal{A}_2 \otimes_{k_{\mathbb{E}_1}} f^*N & \xrightarrow{\cong} & f^*M \otimes_{k_{\mathbb{E}_1}} f^*N & \rightarrow & f^*M \otimes_{\mathcal{A}_1} f^*N \\
& & \downarrow & & \downarrow & & \downarrow \\
& & f^*(M \otimes_{k_{\mathbb{E}_2}} \mathcal{B} \otimes_{k_{\mathbb{E}_2}} N) & \xrightarrow{\cong} & f^*(M \otimes_{k_{\mathbb{E}_2}} N) & \rightarrow & f^*(M \otimes_{\mathcal{A}_2} N).
\end{array}$$

Since the functor is lax monoidal, it induces a functor $f^*: * \text{Alg}(\mathcal{A}_2) \rightarrow * \text{Alg}(\mathcal{A}_1)$. If $\mathcal{B}_1 \in * \text{Alg}(\mathcal{A}_1)$ and $\mathcal{B}_2 \in * \text{Alg}(\mathcal{A}_2)$ are $*$ -algebras over two different $*$ -categories, a **morphism of $*$ -algebras** $f: (\mathcal{B}_1, \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$ consists of a $*$ -functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ along with a morphism of algebras $\mathcal{B}_1 \rightarrow f^*\mathcal{B}_2$ in $* \text{Alg}(\mathcal{A}_1)$. In particular, restricting along the unit $f = \eta: k_{\text{Ob } \mathcal{A}} \rightarrow \mathcal{A}$, we obtain an embedding $* \text{Alg}(\mathcal{A}) \hookrightarrow * \text{Alg}(k_{\text{Ob } \mathcal{A}})$. Thus we may equivalently regard the category of $*$ -categories as consisting of the collection of all $*$ -algebras over all $*$ -categories, each $\mathcal{B} \in * \text{Alg}(\mathcal{A})$ being identified with its image in $* \text{Alg}(k_{\text{Ob } \mathcal{A}})$. This somewhat circular-looking definition will be the one we shall later mimic in our other definitions.

A (unital) $*$ -algebra \mathcal{B} over a $*$ -category \mathcal{A} is **augmented** if there exists a morphism of \mathcal{A} - $*$ -algebras $\mathcal{B} \rightarrow \mathcal{A}$ such that $\mathcal{A} \xrightarrow{\eta} \mathcal{B} \xrightarrow{\varepsilon} \mathcal{A}$ is the identity. In this case, the reduction is given by $\overline{\mathcal{B}} = \text{Ker}(\varepsilon)$, and we obtain the splitting $\mathcal{B} = \overline{\mathcal{B}} \oplus \mathcal{A}$ in \mathcal{A} - $*$ -mod- \mathcal{A} . Conversely, if \mathcal{B} is a non-augmented, non-unital $*$ -algebra, its **augmentation** is the augmented $*$ -algebra $\mathcal{B}^+ = \mathcal{B} \oplus \mathcal{A}$ with augmentation given by the projection $\mathcal{B}^+ \rightarrow \mathcal{B}$. The composition map $m_{\mathcal{B}^+}: \mathcal{B}^+ \otimes_{\mathcal{A}} \mathcal{B}^+ \rightarrow \mathcal{B}^+$ is given by $m_{\mathcal{B}^+} = \varepsilon \otimes \text{id}_{\mathcal{B}} + m_{\mathcal{B}} + m_{\mathcal{A}} + \text{id}_{\mathcal{B}} \otimes \varepsilon$. A **morphism of augmented $*$ -algebras over \mathcal{A}** is a morphism of unital $*$ -algebras which commutes with the augmentation. Thus we obtain a category $* \text{Alg}_{\text{aug}}(\mathcal{A})$ of augmented \mathcal{A} - $*$ -algebras. Given $\mathcal{B}_1 \in * \text{Alg}_{\text{aug}}(\mathcal{A}_1)$ and $\mathcal{B}_2 \in * \text{Alg}_{\text{aug}}(\mathcal{A}_2)$, a morphism $f: (\mathcal{B}_1, \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$ of augmented $*$ -algebras over different $*$ -categories is a morphism of unital algebras making the square

$$\begin{array}{ccc}
\mathcal{B}_1 & \longrightarrow & f^*\mathcal{B}_2 \\
\downarrow & & \downarrow \\
\mathcal{A}_1 & \longrightarrow & f^*\mathcal{A}_2
\end{array}$$

commutative. Equivalently, it consists of a $*$ -functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ together with a morphism $f: \mathcal{B}_1 \rightarrow (f^*\overline{\mathcal{B}_2})^+$ in $* \text{Alg}_{\text{aug}}(\mathcal{A}_1)$.

In the case $\mathcal{A} = k_{\mathbb{E}}$, we obtain the definition of an augmented $*$ -category. If \mathcal{B} is an augmented $*$ -category, it is in particular augmented as a $*$ -quiver.

Tensor algebra

It is possible to freely generate a non-unital \mathcal{A} - $*$ -algebra from an arbitrary \mathcal{A} - $*$ -bimodule V , namely the **non-unital tensor category** $\overline{T}(V) \in * \text{Alg}(\mathcal{A})$ given by

$$\overline{T}(V) = \bigoplus_{n \geq 1} V \otimes_{\mathcal{A}} V \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} V \quad (n \text{ factors}),$$

equipped with the multiplication map $m: \overline{T}(V) \otimes_{\mathcal{A}} \overline{T}(V) \rightarrow \overline{T}(V)$ given by $m((x_n \otimes \cdots \otimes x_{i+1}) \otimes (x_i \otimes \cdots \otimes x_1)) = x_n \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_1$. A non-unital

*-algebra is called **free** if it is (isomorphic to a *-algebra) of this form. One may also define the **tensor algebra** $T(V) = \overline{T(V)}^+$. A unital *-algebra is called **free** if it is (isomorphic to a *-algebra) of this form. The proposition below shows why the term “free” makes sense.

If $f, g: (\mathcal{B}_1, \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$ are morphisms of *-algebras, we define an (f, g) -**derivation** of degree i to be a map $D: \mathcal{B}_1 \rightarrow (f \times g)^* \mathcal{B}_2[i]$ in \mathcal{A} -*mod- \mathcal{A} such that $D \circ m = (f \times g)^* m \circ (f \otimes D + D \otimes g)$. We write $\text{Der}^i(f, g)$ for the set of such. A **derivation** is an (id, id) -derivation. A dg-algebra may be considered as a graded algebra equipped with a derivation d satisfying $d^2 = 0$.

3.4.1. Proposition (Lefèvre-Hasegawa 2003, Lemme 1.1.2.1).

- (i) If \mathcal{A} is a *-category, $V \in \mathcal{A}$ -*mod- \mathcal{A} , and $\mathcal{B} \in * \text{Alg}_{\text{aug}}(\mathcal{A})$, we have an isomorphism of sets

$$\text{Hom}_{* \text{Alg}_{\text{aug}}(\mathcal{A})}(T(V), \mathcal{B}) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}\text{-*mod-}\mathcal{A}}(V, \overline{\mathcal{B}})$$

given by precomposition with the inclusion $V \hookrightarrow T(V)$. The inverse map takes the map of bimodules $f: V \rightarrow \overline{\mathcal{B}}$ to the map

$$\sum_{n \geq 0} m^{(n)} \circ f^{\otimes n}: T(V) \rightarrow \mathcal{B}.$$

Here $m^{(n)}$ denotes the n th iterate of m .

- (ii) If $\mathcal{A}_1, \mathcal{A}_2$ are *-categories, $\mathcal{B}_2 \in * \text{Alg}(\mathcal{A}_2)$, $V \in \mathcal{A}_1$ -*mod- \mathcal{A}_1 , and we are given two maps of *-algebras $f, g: (T(V), \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$, precomposition with the inclusion $V \hookrightarrow T(V)$ yields an isomorphism of sets

$$\text{Der}^i(f, g) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}\text{-*mod-}\mathcal{A}}(V, (f \times g)^* \mathcal{B}_2[i]).$$

The map in the opposite direction takes a map $h: V \rightarrow (f \times g)^* \mathcal{B}_2[i]$ to the (f, g) -derivation $T(V) \rightarrow \mathcal{B}_2[i]$ whose n th component is

$$m^{(n)} \circ \left(\sum_{i+1+j=n} f^{\otimes i} \otimes h \otimes g^{\otimes j} \right).$$

3.5 Graded coalgebras and dg-coalgebras

If \mathcal{A} is a *-category, the category of *-**coalgebras** over \mathcal{A} is the category of coalgebras (\mathcal{C}, Δ) in the monoidal category $(\mathcal{A}\text{-*mod-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$. In other words, we require a comultiplication map $\Delta: \mathcal{C} \rightarrow \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$ satisfying the usual coassociativity axiom. It is **counital** if we are also given a counit map $\varepsilon: \mathcal{C} \rightarrow \mathcal{A}$ satisfying the counit axiom. We shall assume by default that *-coalgebras are counital. A morphism of counital \mathcal{A} -*coalgebras is a morphism that commutes with the unit. We call \mathcal{C} **cocomplete** if any element is annihilated by sufficiently many iterations of Δ , where by an “element”, we mean any morphism in $\mathcal{C}(a, a')$ for $a, a' \in \text{Ob } \mathcal{A}$. The category of cocomplete, counital *-coalgebras over \mathcal{A} is denoted $* \text{Coalg}^c(\mathcal{A})$. The category of cocomplete, non-counital *-coalgebras over \mathcal{A} is denoted $* \text{Coalg}_{\text{ncu}}^c(\mathcal{A})$.

If $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a *-functor, the restriction functor $f^*: \mathcal{A}_2$ -*mod- $\mathcal{A}_2 \rightarrow \mathcal{A}_1$ -*mod- \mathcal{A}_1 has a left adjoint $f_!: \mathcal{A}_1$ -*mod- $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ -*mod- \mathcal{A}_2 given by

$$f_! M = \mathcal{A}_2 \otimes_{\mathcal{A}_1} M \otimes_{\mathcal{A}_1} \mathcal{A}_2.$$

It is oplax monoidal: Indeed, by adjunction, we obtain a map $f_! \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in $\mathcal{A}_2\text{-*mod-}\mathcal{A}_2$, and if $M, N \in \mathcal{A}_1\text{-*mod-}\mathcal{A}_1$, the unit of adjunction provides us with a map $M \otimes_{\mathcal{A}_1} N \rightarrow f^* f_! M \otimes_{\mathcal{A}_1} f^* f_! N \rightarrow f^*(f_! M \otimes_{\mathcal{A}_2} f_! N)$, which by adjunction is the same as a map $f_!(M \otimes_{\mathcal{A}_1} N) \rightarrow f_! M \otimes_{\mathcal{A}_2} f_! N$. This implies that $f_!$ descends to a functor $f_!: \text{*Coalg}^c(\mathcal{A}_1) \rightarrow \text{*Coalg}^c(\mathcal{A}_2)$. If $\mathcal{C}_1 \in \text{*Coalg}^c(\mathcal{A}_1)$ and $\mathcal{C}_2 \in \text{*Coalg}^c(\mathcal{A}_2)$ are $*$ -algebras over different $*$ -categories, we can define a **morphism of $*$ -algebras** $f: (\mathcal{C}_1, \mathcal{A}_1) \rightarrow (\mathcal{C}_2, \mathcal{A}_2)$ to be a $*$ -functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ together with a map $f_! \mathcal{C}_1 \rightarrow \mathcal{C}_2$ in $\text{*Coalg}^c(\mathcal{A}_2)$. By adjunction, when stating such map, one may as well state the map $f: \mathcal{C}_1 \rightarrow f^* \mathcal{C}_2$ in $\mathcal{A}_2\text{-*mod-}\mathcal{A}_2$, and we shall usually do so, even though this is only a map of $*$ -bimodules. One may similarly obtain a left adjoint $(f \times g)_!: \mathcal{A}_1\text{-*mod-}\mathcal{A}_1 \rightarrow \mathcal{A}_2\text{-*mod-}\mathcal{A}_2$ to the restriction functor $(f \times g)^*: \mathcal{A}_2\text{-*mod-}\mathcal{A}_2 \rightarrow \mathcal{A}_1\text{-*mod-}\mathcal{A}_1$ when $f, g: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ are $*$ -functors.

A (left) **$*$ -comodule** over a $*$ -coalgebra $\mathcal{C} \in \text{*Coalg}^c(\mathcal{A})$ is a left comodule in the categorical sense over \mathcal{C} in the category $\mathcal{A}\text{-*mod}$. This amounts to an object $M \in \mathcal{A}\text{-*mod}$ together with a coaction map $\text{ca}: M \rightarrow \mathcal{C} \otimes_{\mathcal{A}} M$ in $\mathcal{A}\text{-*mod}$, satisfying the usual coassociativity and counity conditions.

A counital $*$ -algebra is **coaugmented** if we are given a morphism of unital $\mathcal{A}\text{-*coalgebras} $\eta: \mathcal{A} \rightarrow \mathcal{C}$. It follows from the axioms of a unital $*$ -coalgebras that $\mathcal{A} \xrightarrow{\eta} \mathcal{C} \xrightarrow{\eta} \mathcal{A}$ is the identity. Here \mathcal{A} is regarded as an $\mathcal{A}\text{-*coalgebra}$, the multiplication being the identity map. In that case, we can define the **reduction** of \mathcal{C} as the non-counital $*$ -coalgebra $\overline{\mathcal{C}} = \text{Coker}(\eta)$. On the level of \mathcal{A} -bimodules, we then have the splitting $\mathcal{C} = \overline{\mathcal{C}} \oplus \mathcal{A}$. Conversely, if \mathcal{C} is a non-counital $*$ -coalgebra, its **coaugmentation** is the counital, coaugmented $*$ -algebra $\mathcal{C}^+ = \mathcal{C} \oplus \mathcal{A}$. We equip it with the comultiplication map $\Delta_{\mathcal{C}^+}: \mathcal{C}^+ \rightarrow \mathcal{C}^+ \otimes_{\mathcal{A}} \mathcal{C}^+$ given by $\Delta_{\mathcal{C}^+} = \eta \otimes \text{id}_{\mathcal{C}} + \Delta_{\mathcal{C}} + \Delta_{\mathcal{A}} + \text{id}_{\mathcal{C}} \otimes \eta$. A **morphism of coaugmented $*$ -coalgebras** is a morphism of counital $*$ -coalgebras that commutes with the coaugmentation. Thus we obtain a category $\text{*Coalg}_{\text{coaug}}(\mathcal{A})$ of coaugmented $*$ -coalgebras over \mathcal{A} . If $\mathcal{C}_1 \in \text{*Coalg}_{\text{coaug}}(\mathcal{A}_1)$ and $\mathcal{C}_2 \in \text{*Coalg}_{\text{coaug}}(\mathcal{A}_2)$, a morphism $f: (\mathcal{C}_1, \mathcal{A}_1) \rightarrow (\mathcal{C}_2, \mathcal{A}_2)$ of coaugmented $*$ -coalgebras over different $*$ -categories is a counital morphism making the square$

$$\begin{array}{ccc} f_! \mathcal{C}_1 & \longrightarrow & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ f_! \mathcal{A}_1 & \longrightarrow & \mathcal{A}_2 \end{array}$$

commutative. Equivalently is a $*$ -functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ together with a morphism $f: (f_! \overline{\mathcal{C}_1})^+ \rightarrow \mathcal{C}_2$ in $\text{*Coalg}_{\text{coaug}}(\mathcal{A}_2)$. A coaugmented $*$ -cocategory is **cocomplete** if its reduction is cocomplete in the above sense. Thus we obtain a category $\text{*Coalg}_{\text{coaug}}^c(\mathcal{A})$ of coaugmented, cocomplete $*$ -coalgebras over \mathcal{A} .

A **$*$ -category** with set of objects \mathbb{E} is a $*$ -coalgebra over $k_{\mathbb{E}}$. We denote by $\text{*Cocat}^c = \text{*Cocat}^c(k)$ the category of *cocomplete* $*$ -cocategories with arbitrary object sets and with morphisms the maps of coalgebras in the above sense.

Tensor coalgebra

It is possible to (co)freely (co)generate a non-counital $*$ -coalgebra from any $V \in \mathcal{A}\text{-*mod-}\mathcal{A}$, the **non-counital tensor coalgebra** $\overline{T}^c(V) \in \text{*Coalg}^c(\mathcal{A})$, which is

defined by the same formula as $\overline{T}(V)$ above, but is equipped with the comultiplication map $\Delta: \overline{T}^c(V) \rightarrow \overline{T}^c(V) \otimes_{\mathcal{A}} \overline{T}^c(V)$ given by

$$\Delta(x_n \otimes \cdots \otimes x_1) = \sum_i (x_n \otimes \cdots \otimes x_{i+1}) \otimes (x_i \otimes \cdots \otimes x_1)$$

for all $x_n, \dots, x_1 \in \mathcal{C}$. A non-counital $*$ -coalgebra is called **cofree** if it is (isomorphic to a $*$ -coalgebra) of this form. Note that a cofree non-counital $*$ -coalgebra is cocomplete since a string of $n + 1$ morphisms will be annihilated by n iterations of Δ . One may also define the **coaugmented tensor coalgebra** as $T^c(V) = \overline{T}^c(V)^+$. A counital $*$ -coalgebra is **cofree** if it is (isomorphic to a $*$ -coalgebra) of this form.

If $f, g: (\mathcal{C}_1, \mathcal{A}_1) \rightarrow (\mathcal{C}_2, \mathcal{A}_2)$ are maps of $*$ -coalgebras, an (f, g) -**coderivation** of degree i is a map $\alpha: (f \times g)_! \mathcal{C}_1 \rightarrow \mathcal{C}_2[i]$ in the category $\mathcal{A}_2\text{-*mod-}\mathcal{A}_2$ such that $\Delta_{\mathcal{C}_2} \circ \alpha = (f \otimes \alpha + \alpha \otimes g) \circ (f \times g)_! \Delta_{\mathcal{C}_1}$. By adjunction, we may equivalently regard an (f, g) -coderivation as a map $\alpha: \mathcal{C}_1 \rightarrow (f \times g)^* \mathcal{C}_2[i]$ in $\mathcal{A}_1\text{-*mod-}\mathcal{A}_1$, and we shall usually do so. We shall write $\text{Coder}^i(f, g)$ for the set of such. A **coderivation** is an (id, id) -coderivation. A dg-coalgebra may be considered as an algebra equipped with a coderivation d satisfying $d^2 = 0$.

3.5.1. Proposition (Lefèvre-Hasegawa 2003, Lemma 1.1.2.2).

- (i) If \mathcal{A} is a $*$ -category, $V \in \mathcal{A}\text{-*mod-}\mathcal{A}$, and $\mathcal{C} \in \text{Coalg}_{\text{coaug}}^c(\mathcal{A})$ then postcomposition with the projection $T^c(V) \rightarrow V$ yields an isomorphism of sets

$$\text{Hom}_{\text{Coalg}_{\text{coaug}}^c(\mathcal{A})}(\mathcal{C}, T^c(V)) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}\text{-*mod-}\mathcal{A}}(\overline{\mathcal{C}}, V).$$

The inverse map takes a map of bimodules $f: \overline{\mathcal{C}} \rightarrow V$ to the map

$$\sum_{n \geq 0} f^{\otimes n} \circ \Delta^{(n)}: \mathcal{C} \rightarrow T^c(V).$$

- (ii) If $\mathcal{A}_1, \mathcal{A}_2$ are $*$ -categories, $\mathcal{C} \in \text{Coalg}_{\text{coaug}}^c(\mathcal{A}_1)$, $V \in \mathcal{A}_2\text{-*mod-}\mathcal{A}_2$, and we are given two maps of coaugmented $*$ -coalgebras $f, g: (\mathcal{C}_1, \mathcal{A}_1) \rightarrow (T^c(V), \mathcal{A}_2)$, then postcomposition with the projection $T^c(V) \rightarrow V$ yields an isomorphism of sets

$$\text{Coder}^i(f, g) \cong \text{Hom}_{\mathcal{A}_1\text{-*mod-}\mathcal{A}_1}(\mathcal{C}, (f \times g)^* V[i]).$$

The inverse map takes a morphism $h: \mathcal{C} \rightarrow (f \times g)^* V[i]$, regards it as a map $h: (f \times g)_! \mathcal{C} \rightarrow V[i]$, and maps it to the (f, g) -coderivation $(f \times g)_! \mathcal{C} \rightarrow T^c(V)[i]$ whose n th component, $n \geq 1$, is

$$\left(\sum_{i+1+j=n} f^{\otimes i} \otimes h \otimes g^{\otimes j} \right) \circ \Delta^{(n)}.$$

3.6 A_∞ -algebras and A_∞ -categories

Let \mathcal{A} be a dg-category. We define a (non-unital) A_∞ -**algebra** \mathcal{B} over \mathcal{A} to be a cocomplete, coaugmented dg-coalgebra $(\mathcal{C}, d) \in \text{dgCoalg}_{\text{coaug}}^c(\mathcal{A})$ whose underlying graded coaugmented coalgebra is cofree in $\text{grCoalg}_{\text{coaug}}^c(\mathcal{A})$ and with d vanishing on $\mathcal{A} \subset \mathcal{C}$. We write it as $\mathcal{C} = T^c(\mathcal{B}[1])$ for some $\mathcal{B} \in \mathcal{A}\text{-grmod-}\mathcal{A}$. We shall usually focus on \mathcal{B} instead of \mathcal{C} and therefore refer to \mathcal{B} as an A_∞ -algebra over \mathcal{A} , while \mathcal{C} is called the **bar construction** of \mathcal{B}

and is written $\mathcal{C} = \text{Bar}(\mathcal{B}) = (T^c(\mathcal{B}[1]), d)$. Proposition 3.5.1(ii) shows that the coderivation d is determined by the projection $T^c(\mathcal{B}[1]) \rightarrow T^c(\mathcal{B}[1])[1] \rightarrow \mathcal{B}[2]$, which is the same as maps $m'_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^1(\mathcal{B}[1]^{\otimes i}, \mathcal{B}[1])$. Via the bijection (3.1.1), these m'_i are the same as maps $m_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^{2-i}(\mathcal{B}^{\otimes i}, \mathcal{B})$. Then the equation $d^2 = 0$ is equivalent to having for all $m \geq 1$ the equation

$$(3.6.1) \quad \sum (-1)^{ij+k} m_l \text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes k} = 0$$

where the sum runs over the integers $i, k \geq 0$ and $l, j \geq 1$ such that $l = i + 1 + k$ and $m = i + j + k$. Clearly, a dg-category is an A_∞ -category with m_1 the differential, m_2 the composition map, and all $m_i = 0$ for $i > 2$. A **unit** for the A_∞ -algebra \mathcal{B} consists of a map $\eta: \mathcal{A} \rightarrow \mathcal{B}$ in $\mathcal{A}\text{-grmod-}\mathcal{A}$ such that $m_2(\text{id} \otimes \eta) = \text{id} = m_2(\eta \otimes \text{id})$ and $m_m(\text{id}^{\otimes i} \otimes \eta \otimes \text{id}^{\otimes j}) = 0$ for all $m \neq 2$ and all $i, j \geq 0$ with $i + 1 + j = m$. By an “ A_∞ -algebra”, we shall in general mean a unital A_∞ -algebra. We are also going to consider the reduced bar construction $\overline{\text{Bar}}(\mathcal{A}) = \overline{\text{Bar}}(\mathcal{A})$.

Most definitions related to A_∞ -algebras carry over from the definitions on dg-coalgebras. A **morphism of A_∞ -algebras** $f: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ over \mathcal{A} is a morphism $f: \text{Bar}(\mathcal{B}_1) \rightarrow \text{Bar}(\mathcal{B}_2)$ in $\text{dgCoalg}_{\text{coaug}}^c(\mathcal{A})$. By Proposition 3.5.1(i), this amounts to a map $f \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^0(\text{Bar}(\mathcal{B}_1), \mathcal{B}_2[1])$, which is the same as a collection of maps $f'_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}(\mathcal{B}_1[1]^{\otimes i}, \mathcal{B}_2[1])$ for all $i \geq 1$. Via the bijection (3.1.1), this is the same as a collection of maps

$$f_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^{1-i}(\mathcal{B}_1^{\otimes i}, \mathcal{B}_2)$$

for all $i \geq 1$. These must be subject to some technical conditions that we shall not write (see e.g. Lefèvre-Hasegawa 2003, Définition 1.2.1.2), which are equivalent to f commuting with differentials. These form the maps in the category $A_\infty\text{Alg}_{\text{nu}}(\mathcal{A})$ of non-unital A_∞ -algebras over \mathcal{A} . If \mathcal{B}_1 and \mathcal{B}_2 are unital, the morphism f is called **unital** if $f_1 \circ \eta = \eta \circ f_1 = \text{id}$ and $f_n(\text{id}^{\otimes i} \otimes \eta \otimes \text{id}^{\otimes j}) = 0$ for all $n > 1$ and $i, j \geq 0$ with $i + 1 + j = n$. These form the morphisms of the category $A_\infty\text{Alg}(\mathcal{A})$ of (unital) A_∞ -algebras over \mathcal{A} .

Suppose that $\mathcal{B}_1 \in A_\infty\text{Alg}(\mathcal{A}_1)$ and $\mathcal{B}_2 \in A_\infty\text{Alg}(\mathcal{A}_2)$ are A_∞ -algebras over different dg-categories. A (unital) morphism $f: (\mathcal{B}_1, \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$ is a morphism of coaugmented dg-coalgebras $f: \text{Bar}(\mathcal{B}_1) \rightarrow \text{Bar}(\mathcal{B}_2)$ in the above sense, satisfying the unital condition. In other words, it is a dg-functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ together with a morphism $f: (f_! \overline{\text{Bar}}(\mathcal{B}_1))^+ \rightarrow \text{Bar}(\mathcal{B}_2)$ in $\text{dgCoalg}_{\text{coaug}}^c(\mathcal{A}_2)$, subject to the unital condition. Proposition 3.5.1(i) shows that such map is the same as a map $f_! \overline{\text{Bar}}(\mathcal{B}_1) \rightarrow \mathcal{B}_2[1]$ in $\mathcal{A}_2\text{-grmod-}\mathcal{A}_2$, which by adjunction is the same as a map $\overline{\text{Bar}}(\mathcal{B}_1) \rightarrow f^* \mathcal{B}_2[1]$ in $\mathcal{A}_1\text{-grmod-}\mathcal{A}_1$. Another application of Proposition 3.5.1(i) then shows that this is the same as a (unital) morphism $f: \text{Bar}(\mathcal{B}_1) \rightarrow \text{Bar}(f^* \mathcal{B}_2)$ of A_∞ -algebras in $A_\infty\text{Alg}(\mathcal{A}_1)$. Thus we may as well define a morphism $f: (\mathcal{B}_1, \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$ of A_∞ -algebras to be a dg-functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ together with a morphism $f: \mathcal{B}_1 \rightarrow f^* \mathcal{B}_2$ in $A_\infty\text{Alg}(\mathcal{A}_1)$; we notice that it makes sense to regard $f^* \mathcal{B}_2$ as an A_∞ -algebra this way because f^* is lax monoidal. (Remarkably, this shows that cofree dg-cocategories, unlike general dg-cocategories, allow a pullback operation.)

A unital A_∞ -algebra is **augmented** if it is equipped with a unital morphism of unital A_∞ -algebras $\varepsilon: \mathcal{B} \rightarrow \mathcal{A}$ in $A_\infty\text{Alg}(\mathcal{A})$. It is a consequence of the definition of unital morphisms that the composition $\mathcal{A} \xrightarrow{\eta} \mathcal{B} \xrightarrow{\varepsilon} \mathcal{A}$ is the

identity. Similarly to the dg-case, we may define the **reduction** of \mathcal{A} by $\overline{\mathcal{A}} = \text{Ker}(\varepsilon)$. Conversely, a non-unital A_∞ -algebra \mathcal{B} can be made into a unital, augmented A_∞ -algebra by $\mathcal{B}^+ = \mathcal{B} \oplus \mathcal{A}$. The A_∞ -operations on this will be given by $m_{1,\mathcal{B}^+} = m_{1,\mathcal{B}} + d_{\mathcal{A}}$, $m_{2,\mathcal{B}^+}(b+a, b'+a') = m_{2,\mathcal{B}}(b, b') + ab' + a'b + aa'$, and $m_{i,\mathcal{B}^+} = m_{i,\mathcal{B}}$ for $i > 2$. If \mathcal{B} is augmented, one may consider the **augmented bar construction** $\text{Bar}^+(\mathcal{B}) = \text{Bar}(\overline{\mathcal{B}})$.

An A_∞ -**category** with object set \mathbb{E} is a defined to be a (unital) A_∞ -algebra over the dg-category $k_{\mathbb{E}}$. Denote by $A_\infty \text{Cat} = A_\infty \text{Cat}(k)$ the category of (unital) A_∞ -categories with arbitrary object sets and morphisms the unital maps of A_∞ -algebras in the above sense, known as A_∞ -**functors**. The collection of non-unital A_∞ -categories also form a category $A_\infty \text{Cat}_{\text{nu}} = A_\infty \text{Cat}_{\text{nu}}(k)$.

3.7 A_∞ -coalgebras and A_∞ -cocategories

Let again \mathcal{A} be a dg-category. An A_∞ -**coalgebra** over \mathcal{A} is an augmented dg-algebra $(\mathcal{B}, d) \in \text{dgAlg}_{\text{aug}}(\mathcal{A})$ whose underlying graded augmented algebra is free in $\text{grAlg}_{\text{aug}}(\mathcal{A})$ and for which $\mathcal{B} \xrightarrow{d} \mathcal{B} \rightarrow \mathcal{A}$ is zero. We therefore write it as $\mathcal{B} = T(\mathcal{C}[-1])$ for some $\mathcal{C} \in \mathcal{A}\text{-grmod-}\mathcal{A}$. We shall usually focus on \mathcal{C} instead of \mathcal{B} and therefore refer to \mathcal{C} as an A_∞ -coalgebra over \mathcal{A} , while \mathcal{B} is called the **cobar construction** of \mathcal{C} and is written $\mathcal{B} = \text{Cob}(\mathcal{C}) = (T(\mathcal{C}[-1]), d)$. Therefore, Proposition 3.4.1(ii) shows that the derivation d on the free algebra $T(\mathcal{C}[-1])$ is determined by the inclusion

$$\mathcal{C}[-1] \hookrightarrow T^c(\mathcal{C}[-1]) \rightarrow T^c(\mathcal{C}[-1])[1],$$

which is the same as maps $\Delta'_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^1(\mathcal{C}[-1], \mathcal{C}[-1]^{\otimes i})$. Via the bijection (3.1.2), these Δ'_i are the same as maps $\Delta_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^{2-i}(\mathcal{C}, \mathcal{C}^{\otimes i})$. Then the equation $d^2 = 0$ is equivalent to having for all $m \geq 1$ the equation

$$(3.7.1) \quad \sum (-1)^{i+jk} (\text{id}^{\otimes i} \otimes \Delta_j \otimes \text{id}^{\otimes k}) \Delta_l = 0$$

for all i, j, k, l such that $i+j+k = m$ and $l = i+1+k$. Furthermore, for the summation in the cobar construction to be meaningful, we require that the product map $\prod_{i \geq 1} \Delta_i: \mathcal{C} \rightarrow \prod_{i \geq 1} \mathcal{C}^{\otimes i}$ must factor through the direct sum $\bigoplus_{i \geq 1} \mathcal{C}^{\otimes i}$. The A_∞ -algebra \mathcal{C} is **counital** if it is equipped with a map $\varepsilon: \mathcal{C} \rightarrow \mathcal{A}$ in the category $\mathcal{A}\text{-grmod-}\mathcal{A}$ satisfying $(\text{id} \otimes \varepsilon)\Delta_2 = \text{id} = (\varepsilon \otimes \text{id})\Delta_2$ and $(\text{id}^{\otimes i} \otimes \varepsilon \otimes \text{id}^{\otimes j})\Delta_m = 0$ for all $m \neq 2$ and all $i, j \geq 0$ with $i+1+j = m$. By an " A_∞ -coalgebra", we shall usually mean a counital one. We are also going to consider the non-augmented algebra $\overline{\text{Cob}}(\mathcal{C}) = \overline{\text{Cob}(\mathcal{C})}$.

Most definitions related to A_∞ -coalgebras carry over from the definitions on dg-algebras. A **morphism of A_∞ -coalgebras** $f: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ over \mathcal{A} is a morphism $f: \text{Cob}(\mathcal{C}_1) \rightarrow \text{Cob}(\mathcal{C}_2)$ in $\text{dgAlg}_{\text{aug}}(\mathcal{A})$. By Proposition 3.4.1(i), this amounts to a map $f \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^0(\mathcal{C}_1[-1], \overline{\text{Cob}}(\mathcal{C}_2))$, which is the same as a collection of maps $f'_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^0(\mathcal{C}_1[-1], \mathcal{C}_2[-1]^{\otimes i})$. Via the bijection (3.1.2), this is the same as a collection of maps

$$f_i \in \text{Hom}_{\mathcal{A}\text{-grmod-}\mathcal{A}}^{1-i}(\mathcal{C}_1, \mathcal{C}_2^{\otimes i}).$$

These must be subject to some technical conditions that we shall not write, which are equivalent to f commuting with the differential. These form the

morphisms in the category $A_\infty \text{Coalg}_{\text{nu}}(\mathcal{A})$ of non-counital A_∞ -coalgebras over the dg-category \mathcal{A} . If \mathcal{B}_1 and \mathcal{B}_2 are counital, a morphism f is called **counital** if $f_1 \circ \varepsilon = \varepsilon \circ f_1 = \text{id}$ and $f_n(\text{id}^{\otimes i} \otimes \varepsilon \otimes \text{id}^{\otimes j}) = 0$ for all $n > 1$ and $i, j \geq 0$ with $i + 1 + j = n$. These form the morphisms in the category $A_\infty \text{Coalg}(\mathcal{A})$ of (counital) A_∞ -coalgebras over \mathcal{A} .

Suppose that $\mathcal{C}_1 \in A_\infty \text{Coalg}(\mathcal{A}_1)$ and $\mathcal{C}_2 \in A_\infty \text{Coalg}(\mathcal{A}_2)$ are A_∞ -coalgebras over different dg-categories. A (unital) morphism $f: (\mathcal{C}_1, \mathcal{A}_1) \rightarrow (\mathcal{C}_2, \mathcal{A}_2)$ is a morphism of augmented dg-algebras $f: \text{Cob}(\mathcal{C}_1) \rightarrow \text{Cob}(\mathcal{C}_2)$ in the above sense, satisfying the counital condition. In other words, it consists of a dg-functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ together with a morphism $f: \text{Cob}(\mathcal{C}_1) \rightarrow (f^* \overline{\text{Cob}(\mathcal{C}_2)})^+$ in $\text{dgAlg}_{\text{aug}}(\mathcal{A}_1)$, subject to the counital condition. By Proposition 3.4.1(i), such map is the same as a map $\mathcal{C}_1[-1] \rightarrow f^* \overline{\text{Cob}(\mathcal{C}_2)}$ in $\mathcal{A}_1\text{-grmod-}\mathcal{A}_1$, which by adjunction is the same as a map $f_! \mathcal{C}_1[-1] \rightarrow \overline{\text{Cob}(\mathcal{C}_2)}$. Another application of Proposition 3.4.1(i) shows that this is the same as a morphism $f: \text{Cob}(f_! \mathcal{C}_1) \rightarrow \text{Cob}(\mathcal{C}_2)$ of A_∞ -coalgebras in $A_\infty \text{Coalg}(\mathcal{A}_2)$. In other words, we may as well define a morphism $f: (\mathcal{C}_1, \mathcal{A}_1) \rightarrow (\mathcal{C}_2, \mathcal{A}_2)$ to be a dg-functor $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ together with a morphism $f: f_! \mathcal{C}_1 \rightarrow \mathcal{C}_2$ in $A_\infty \text{Coalg}(\mathcal{A}_2)$; we notice that it makes sense to regard $f_! \mathcal{C}_1$ as an A_∞ -coalgebra because $f_!$ is oplax monoidal. (Remarkably, this shows that dg-algebras which are free as graded algebras, unlike general dg-algebras, allow a !-pushforward operation.)

A counital A_∞ -coalgebra \mathcal{C} is **coaugmented** if it is equipped with a counital morphism of counital A_∞ -coalgebras $\eta: \mathcal{A} \rightarrow \mathcal{C}$ in $A_\infty \text{Coalg}(\mathcal{A})$. It is a consequence of the definition of counital morphisms that the composition $\mathcal{A} \xrightarrow{\eta} \mathcal{C} \xrightarrow{\varepsilon} \mathcal{A}$ is the identity. Similarly to the dg-case, we may define the **reduction** of \mathcal{C} by $\overline{\mathcal{C}} = \text{Coker}(\eta)$. Conversely, a non-counital A_∞ -coalgebra \mathcal{C} can be made into a counital, coaugmented A_∞ -coalgebra by $\mathcal{C}^+ = \mathcal{C} \oplus \mathcal{A}$. The A_∞ -coalgebra structure on this is dual to the one we defined on augmented A_∞ -algebras. If \mathcal{C} is in augmented, we may define the **augmented cobar construction** by $\text{Cob}^+(\mathcal{C}) = \text{Cob}(\overline{\mathcal{C}})$.

An A_∞ -**cocategory** with set of objects \mathbb{E} is an A_∞ -coalgebra over the dg-category $k_{\mathbb{E}}$. Denote by $A_\infty \text{Cocat} = A_\infty \text{Cocat}(k)$ the category of A_∞ -cocategories with arbitrary object sets and morphisms the counital maps of A_∞ -coalgebras in the above sense. The collection of non-counital A_∞ -cocategories also form a category $A_\infty \text{Cocat}_{\text{nu}} = A_\infty \text{Cocat}_{\text{nu}}(k)$.

Restricting Bar^+ and Cob^+ to honest, (co)augmented dg-(co)categories, we obtain an adjunction

$$\text{Cob}^+ : \text{dgCocat}_{\text{coaug}}^c \rightleftarrows \text{dgCat}_{\text{aug}} : \text{Bar}^+,$$

see Lefèvre-Hasegawa (2003, Lemma 1.2.2.5).

3.7.2. Example. Denote by $[n]$ the poset consisting of $n + 1$ objects $0, 1, \dots, n$ and a unique morphisms $(i, j): i \rightarrow j$ for all $i \leq j$. Let $k[n]$ be its k -linearization, and regard it as an augmented dg-category with zero differential. The augmented bar construction

$$\text{Bar}^+(k[n]) = \overline{\text{Bar}}(k[n]) \oplus k_{[n]}$$

has set of objects $[n]$ and morphisms given by tensor products $(i_0, i_1, \dots, i_k) =$

$(i_{k-1}, i_k) \otimes (i_{k-2}, i_{k-1}) \otimes \cdots \otimes (i_0, i_1)$. The differential is given by

$$d(i_0, i_1, \dots, i_k) = \sum_{j=1}^{k-1} (-1)^{k-j+1} (i_0, \dots, \hat{i}_j, \dots, i_k)$$

Applying the cobar construction, we obtain the free dg-category

$$\text{Cob}^+(\text{Bar}^+(k[n])) = \overline{\text{Cob}(\overline{\text{Bar}(k[n])})} \oplus k_{[n]}.$$

This again has object set $[n]$ and with morphisms freely generated by morphisms $f_{i_0 i_1 \dots i_k} : i_0 \rightarrow i_k$ of degree $1 - k$ for all sequences $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, $k > 0$. The differential is given by

$$df_{i_0 i_1 \dots i_k} = \sum_{j=1}^{k-1} (-1)^{k-j} (f_{i_0 \dots \hat{i}_j \dots i_k} - f_{i_j \dots i_k} \circ f_{i_0 \dots i_j}).$$

In this formula, we use the convention that $f_{j_0 j_1 \dots j_l} = 0$ if the index contains repetitions and $l > 1$, and that $f_{j_0 j_0} = \text{id}$. \circlearrowright

A_∞ -comodules

Let \mathcal{C} be an A_∞ -coalgebra over a dg-category \mathcal{A} . A **left A_∞ -comodule** over \mathcal{C} is a dg-module in \mathcal{A} -dgmod over the dg-algebra $\text{Cob}(\mathcal{C})$ whose underlying graded module is *free*. By free, we mean that it has the form $\text{Cob}(\mathcal{C}) \otimes_{\mathcal{A}} M[-1]$ for some $M \in \mathcal{A}$ -grmod. As with coalgebras, we shall usually focus on M and refer to it as an A_∞ -comodule, while the dg-module $\text{Cob}(\mathcal{C}, M) := \text{Cob}(\mathcal{C}) \otimes_{\mathcal{A}} M[-1]$ is called the **cobar construction** of M . The action map

$$\text{Cob}(\mathcal{C}) \otimes_{\mathcal{A}} \text{Cob}(\mathcal{C}, M) \rightarrow \text{Cob}(\mathcal{C}, M)$$

amounts, by freeness and (3.1.2), to maps $\text{ca}_i : M \rightarrow C^{\otimes(1-i)} \otimes_{\mathcal{A}} M[2-i]$ for all $i \geq 1$, subject to the equation (3.7.1) where Δ_l is understood as ca_l , while Δ_j must be interpreted as ca_j in the case $k = 0$. Furthermore, in order for these maps to be definable on the cobar construction, we must require that the product $\prod_{i \geq 1} \text{ca}_i : M \rightarrow \prod_{i \geq 1} C^{\otimes(i-1)} \otimes_{\mathcal{A}} M$ factors through $\bigoplus_{i \geq 1} C^{\otimes(i-1)} \otimes_{\mathcal{A}} M$. We call M a **formal A_∞ -comodule** if we omit this last condition.

The collection of A_∞ -comodules over \mathcal{C} form a dg-category, namely the full dg-subcategory of $\text{Cob}(\mathcal{C})$ -dgmod consisting of dg-modules that are free as graded modules. A map $f : M \rightarrow N$ of degree $|f|$ is the same as a map of $\text{Cob}(\mathcal{C})$ -dg-modules $f : \text{Cob}(\mathcal{C}, M) \rightarrow \text{Cob}(\mathcal{C}, N)[|f|]$. This is equivalent via (3.1.2) to a collection of maps

$$f_i : M \rightarrow C^{\otimes(i-1)} \otimes_{\mathcal{A}} N[|f| + 1 - i]$$

for $i \geq 1$, such that the product $\prod_{i \geq 1} f_i : M \rightarrow \prod_{i \geq 1} C^{\otimes(i-1)} \otimes_{\mathcal{A}} N[|f| + 1 - i]$ factors through $\bigoplus_{i \geq 1} C^{\otimes(i-1)} \otimes_{\mathcal{A}} N[|f| + 1 - i]$. The differential

$$d(f) = d_{\text{Cob}(\mathcal{C}, N)} \circ f - (-1)^{|f|} f \circ d_{\text{Cob}(\mathcal{C}, M)}$$

is given by

$$d(f)_n = \sum (-1)^{i+jk} (\text{id}^{\otimes i} \otimes \Delta_j \otimes \text{id}^{\otimes k}) f_m - \sum (-1)^{p|f|} (\text{id}^{\otimes(p-1)} \otimes f_q) \text{ca}_p$$

where the first sum runs over $i, k \geq 0$ and $j, m > 1$ with $i+1+k = m$ and $i+j+k = n$ and the second over the $p, q \geq 1$ with $p+q-1 = n$. If $i = 0$, Δ_j should be understood as ca_j . If $f: M \rightarrow N$ and $g: N \rightarrow P$ are maps of A_∞ -comodules, their composition $g \circ f$ is the composition $\text{Cob}(\mathcal{C}, M) \xrightarrow{f} \text{Cob}(\mathcal{C}, N) \xrightarrow{g} \text{Cob}(\mathcal{C}, P)$. One checks via (3.1.2) that the n th component is

$$(g \circ f)_n = \sum_{n=l+k-1} (-1)^{|g|(l-1)} (\text{id}^{\otimes(l-1)} \otimes g_k) \circ f_l.$$

If \mathcal{C} is counital, the A_∞ -comodule M is called **counital** if $ca_2(\varepsilon \otimes \text{id}_M) = \text{id}_M$ and $ca_n(\text{id}^{\otimes i} \otimes \varepsilon \otimes \text{id}^{\otimes j} \otimes \text{id}_M) = 0$ for all $n > 2$ and all $i, j \geq 0$ with $i+j+2 = n$. It is called **homotopy-counital** if we have a homotopy $ca_2(\varepsilon \otimes \text{id}_M) \simeq \text{id}_M$ with respect to the differential ca_1 on M . We denote the dg-category of such by $\mathcal{C}\text{-comod}_\infty^{\text{hcu}}(\mathcal{A})$.

3.8 The A_∞ -category of A_∞ -functors

There exists an internal hom in the category of A_∞ -categories, known as the A_∞ -category of A_∞ -functors. It is in fact just a special case of an internal hom in the category of dg-cocategories. In this section, we stick mostly to the approach of Keller (2006). We define the tensor product $\mathcal{C} \otimes_k \mathcal{D}$ of two $*$ -categories over k as the tensor product of the underlying $*$ -quivers, equipped with the natural diagonal $*$ -category structure.

3.8.1. Proposition (ibid., Theorem 5.3, Lemma 5.2, Lemma 5.4).

- (i) The category $*\text{Cocat}_{\text{coaug}}^c$ of cocomplete, coaugmented $*$ -cocategories is a closed monoidal category with an internal hom which we denote by $*\text{Cofun}$. The $*$ -cocategory $*\text{Cofun}(\mathcal{C}, \mathcal{D})$ has objects the morphisms of coaugmented $*$ -cocategories $\mathcal{C} \rightarrow \mathcal{D}$. It is a $*$ -subquiver of $\text{Hom}_{*\text{Quiv}_{\text{aug}}}(\mathcal{C}, \mathcal{D})$, the internal hom in augmented $*$ -quivers.
- (ii) If \mathcal{C} and \mathcal{D} are dg-cocategories, $\text{dgCofun}(\mathcal{C}, \mathcal{D})$ is the full subcocategory of $\text{grCofun}(\mathcal{C}, \mathcal{D})$ consisting of morphisms of graded coaugmented cocategories commuting with the differential.
- (iii) If $\mathcal{D} = T^c(V)$ is cofree, then

$$*\text{Cofun}(\mathcal{C}, T^c(V)) \cong T^c(\overline{\text{Hom}_{*\text{Quiv}_{\text{aug}}}(\mathcal{C}, V)})$$

is also cofree. The isomorphism on objects is given as in Proposition 3.5.1(i). On morphisms, suppose that

$$f_n \xleftarrow{\alpha_n} f_{n-1} \xleftarrow{\alpha_{n-1}} \dots \xleftarrow{\alpha_1} f_0$$

is a composable string of natural transformations in $\text{Hom}_{*\text{Quiv}}(\mathcal{C}, V)$, that is, $\alpha_i \in \text{Hom}_{*\text{Quiv}}(\mathcal{C}, V)(f_{i-1}, f_i) \cong \text{Coder}(f_{i-1}, f_i)$. Then the leftwards arrow takes the tensor $\alpha_n \otimes \dots \otimes \alpha_1$ to the natural transformation $\langle \alpha_n, \dots, \alpha_1 \rangle$ whose N th component $\langle \alpha_n, \dots, \alpha_1 \rangle_N: \mathcal{C} \rightarrow (f_0 \times f_n)^* V^{\otimes N}$ is given by the sum of terms

$$(f_n^{\otimes r_n} \otimes \alpha_n \otimes f_{n-1}^{\otimes r_{n-1}} \otimes \dots \otimes \alpha_1 \otimes f_0^{\otimes r_0}) \circ \Delta^{(N)}$$

with $N = n + \sum r_i$. Here we regard each f_i as a natural transformation $f_i \rightarrow f_i$ of degree 1 via $\mathcal{C} \rightarrow \mathcal{C} \rightarrow (f_i \times f_i)^* V$.

Suppose now that \mathcal{A} and \mathcal{B} are non-unital A_∞ -categories, and let us show the existence of a non-unital A_∞ -category of non-unital A_∞ -functors $\mathcal{A} \rightarrow \mathcal{B}$. Since the underlying graded cocategory of $\text{Bar}(\mathcal{B})$ is cofree, we obtain from Proposition 3.8.1(iii) that the graded cocategory

$$\text{grCofun}(\text{Bar}(\mathcal{A}), \text{Bar}(\mathcal{B})) = T^c(\text{Hom}_{\text{grQuiv}}(\text{Bar}(\mathcal{A}), \mathcal{B}[1]))$$

is also cofree. Also, Proposition 3.8.1(ii) shows that, as a graded cocategory, $\text{dgCofun}(\text{Bar}(\mathcal{A}), \text{Bar}(\mathcal{B})) \subset \text{grCofun}(\text{Bar}(\mathcal{A}), \text{Bar}(\mathcal{B}))$ is the full subcocategory consisting of morphisms of graded cocategories commuting with the differential. Thus the underlying graded cocategory of $\text{dgCofun}(\text{Bar}(\mathcal{A}), \text{Bar}(\mathcal{B}))$ is also cofree, being a full subcocategory of a cofree graded cocategory. In other words, it is an A_∞ -category. Therefore, we may write it as $\text{Bar}(A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B}))$ and call $A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$ the (non-unital) A_∞ -**category of non-unital A_∞ -functors**. In other words, the underlying graded quiver of $A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$ is a full graded subquiver of the graded quiver $\text{Hom}_{\text{grQuiv}}(\text{Bar}(\mathcal{A}), \mathcal{B}[1])[-1]$, and hence we obtain that the morphism space between A_∞ -functors $f, g: \mathcal{A} \rightarrow \mathcal{B}$ is

$$\begin{aligned} A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})(f, g) &= \text{Hom}_{\text{grQuiv}(\text{Ob}\mathcal{A})}(\text{Bar}(\mathcal{A}), (f \times g)^* \mathcal{B}[1])[-1] \\ &= \text{Hom}_{\text{grQuiv}(\text{Ob}\mathcal{A})}(\text{Bar}(\mathcal{A}), (f \times g)^* \mathcal{B}). \end{aligned}$$

We shall refer to this as the space of **non-unital A_∞ -transformations** $f \rightarrow g$. In $\text{dgCofun}(\text{Bar}(\mathcal{A}), \text{Bar}(\mathcal{B}))$, the morphism space will instead be composable tensors $\langle s(\alpha_n), \dots, s(\alpha_1) \rangle$, where $\alpha_i \in A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})(f_{i-1}, f_i)$.

3.8.2. Example. Since a non-unital A_∞ -functor $f: \mathcal{A} \rightarrow \mathcal{B}$ is a map of graded quivers $\text{Bar}(\mathcal{A}) \rightarrow \mathcal{B}[1]$, we may regard it as a non-unital A_∞ -transformation $f \rightarrow f$ of degree 1 via the composition $\text{Bar}(\mathcal{A}) \rightarrow \overline{\text{Bar}}(\mathcal{A}) \rightarrow \mathcal{B}[1]$. Note that it is *not* an identity morphism on f , as it does not even have degree zero. \circ

We can describe the differential

$$d: \text{Bar}(A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})) \rightarrow \text{Bar}(A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B}))[1]$$

more explicitly. A morphism on the left-hand side consists of a sum of tensors

$$s^{\otimes n}(\alpha_n \otimes \dots \otimes \alpha_1) \in (A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})(f_{n-1}, f_n) \otimes \dots \otimes A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})(f_0, f_1))[n].$$

Then

$$\begin{aligned} ds^{\otimes n}(\alpha_n \otimes \dots \otimes \alpha_1) &= d_{\text{Bar}(\mathcal{B})} \circ s^{\otimes n}(\alpha_n \otimes \dots \otimes \alpha_1) \\ &\quad - (-1)^{|\alpha_n| + \dots + |\alpha_1| - n} s^{\otimes n}(\alpha_n \otimes \dots \otimes \alpha_1) \circ d_{\text{Bar}(\mathcal{A})}. \end{aligned}$$

Applying the projection $\text{pr}_1: \text{Bar}(\mathcal{B}) \rightarrow \mathcal{B}[1]$, the second term vanishes unless $n = 1$ (see Proposition 3.8.1(iii)). This allows us to calculate the n th component

$$m_n^{A_\infty\text{Fun}} = -\omega \circ \text{pr}_1 \circ d \circ (\omega^{\otimes n})^{-1}: A_\infty\text{Fun}_{\text{nu}}^{\otimes n} \rightarrow A_\infty\text{Fun}_{\text{nu}}[2-n].$$

Plugging in $(\omega^{\otimes n})^{-1} = (-1)^{n(n-1)/2} s^{\otimes n}$ and

$$s^{\otimes n}(\alpha_n \otimes \dots \otimes \alpha_1) = (-1)^{\sum (i-1)|\alpha_i|} s(\alpha_n) \otimes \dots \otimes s(\alpha_1) = (-1)^{\sum (i-1)|\alpha_i|} \langle s(\alpha_n), \dots, s(\alpha_1) \rangle$$

(see Proposition 3.8.1(iii) for the notation), we obtain that

3.8.3. Proposition. The A_∞ -operations $m_n^{\text{A}_\infty\text{Fun}}$ on $A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$ are given by

$$m_1^{\text{A}_\infty\text{Fun}}(\alpha_1) = \left(\bigoplus_{k \geq 1} m_k^{\mathcal{B}} \circ \omega^{\otimes k} \right) \circ \langle s(\alpha_1) \rangle - (-1)^{|\alpha_1|} \alpha_1 \circ d_{\text{Bar}(\mathcal{A})}$$

$$m_n^{\text{A}_\infty\text{Fun}}(\alpha_n, \dots, \alpha_1) = (-1)^{n(n-1)/2} (-1)^{\sum (i-1)|\alpha_i|} \left(\bigoplus_{k \geq n} m_k^{\mathcal{B}} \circ \omega^{\otimes k} \right) \circ \langle s(\alpha_n), \dots, s(\alpha_1) \rangle$$

for $n > 1$. The notation $\langle s(\alpha_n), \dots, s(\alpha_1) \rangle$ is explained in Proposition 3.8.1(iii). It must be evaluated using the sign conventions with the tensor product of maps of complexes.

3.8.4. Corollary.

- (i) The A_∞ -category $A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$ is unital if \mathcal{B} is. Indeed, the unit at $f \in A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$ is the composition $\text{Bar}(\mathcal{A}) \rightarrow k_{\text{Ob}\mathcal{A}} \xrightarrow{f^*\eta} f^*\mathcal{B}$.
- (ii) If \mathcal{B} is a dg-category, so is $A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$. Indeed, if $m_n^{\mathcal{B}} = 0$ for all $n > 2$, $m_n^{\text{A}_\infty\text{Fun}}$ vanishes, too.

Furthermore, if $f \in A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$ is a non-unital A_∞ -functor, we may regard it as a non-unital A_∞ -transformation f of degree 1, see Example 3.8.2. In this case, the equation $d_{\text{Bar}(\mathcal{B})} \circ f - f \circ d_{\text{Bar}(\mathcal{A})} = 0$ is equivalent to

$$m_1^{\text{A}_\infty\text{Fun}}(f) = m_2^{\text{A}_\infty\text{Fun}}(f, f).$$

Suppose now that both \mathcal{A} and \mathcal{B} are unital A_∞ -categories. The A_∞ -category of (unital) A_∞ -functors is the non-full subcategory

$$A_\infty\text{Fun}(\mathcal{A}, \mathcal{B}) \subset A_\infty\text{Fun}_{\text{nu}}(\mathcal{A}, \mathcal{B})$$

with objects the unital A_∞ -functors. The morphism space $A_\infty\text{Fun}(\mathcal{A}, \mathcal{B})(f, g)$ consists of the morphisms $\alpha \in \text{Hom}_{\text{grQuivOb}\mathcal{A}}(\text{Bar}(\mathcal{A}), (f \times g)^*\mathcal{B})$ satisfying the condition $\alpha(\text{id}^{\otimes i} \otimes \eta \otimes \text{id}^{\otimes j}) = 0$ for all $i, j \geq 0$. Equivalently, we have

$$(3.8.5) \quad A_\infty\text{Fun}(\mathcal{A}, \mathcal{B})(f, g) = \text{Hom}_{\text{grQuiv}(\text{Ob}\mathcal{A})}(T^c(\overline{\mathcal{A}}[1]), (f \times g)^*\mathcal{B})$$

where $\overline{\mathcal{A}}$ is the cokernel of the unit map $\eta: k_{\text{Ob}\mathcal{A}} \rightarrow \mathcal{A}$.

3.8.6. Example. Suppose that \mathcal{B} is a unital dg-category, and consider the unital dg-category $k[n]$, the k -linearization of the poset $[n]$, regarded as a category. We wish to calculate $A_\infty\text{Fun}(k[n], \mathcal{B})$. The object set is

$$\begin{aligned} \text{Hom}_{A_\infty\text{Cat}}(k[n], \mathcal{B}) &= \text{Hom}_{A_\infty\text{Cat}_{\text{nu}}}(\overline{k[n]}, \mathcal{B}) = \text{Hom}_{\text{Cocat}_{\text{dgncu}}}(\overline{\text{Bar}(k[n])}, \overline{\text{Bar}(\mathcal{B})}) \\ &= \text{Hom}_{\text{dgCocat}_{\text{coaug}}}(\text{Bar}^+(k[n]), \text{Bar}^+(\mathcal{B})) = \text{Hom}_{\text{dgCat}}(\text{Cob}^+(\text{Bar}^+(k[n])), \mathcal{B}). \end{aligned}$$

In other words, an object f consists of a collection $b_0 = f(0), \dots, b_n = f(n)$ of objects in \mathcal{B} together with a collection of morphism $f_{i_0 i_1 \dots i_k}: b_{i_0} \rightarrow b_{i_k}$ of degree $1 - k$ for all sequences $0 \leq i_0 < i_1 < \dots < i_k \leq n$, $k > 0$, and with differential

$$df_{i_0 i_1 \dots i_k} = \sum_{j=1}^{k-1} (-1)^{k-j} (f_{i_0 \dots i_j \dots i_k} - f_{i_j \dots i_k} \circ f_{i_0 \dots i_j})$$

with the convention that $f_{j_0 j_1 \dots j_l} = 0$ if the index contains repetitions and $l > 1$, and that $f_{j_0 j_0} = \text{id}$. If $f, g: \text{Cob}^+(\text{Bar}^+(k[n])) \rightarrow \mathcal{B}$ are two objects, (3.8.5) shows

that an A_∞ -transformation $\alpha: f \rightarrow g$ of degree $|\alpha|$ is a degree $|\alpha|$ map of graded quivers

$$\alpha: T^c(\overline{k[n]}[1]) \rightarrow (f \times g)^* \mathcal{B},$$

which is the data of a map

$$\alpha_{i_0 i_1 \dots i_k} = \alpha((i_0, \dots, i_k)) \in \text{Hom}_{\mathcal{B}}^{|\alpha|-k}(f(i_0), g(i_k))$$

for each sequence $0 \leq i_0 < i_1 < \dots < i_k \leq n$, $k \geq 0$. Also, we obtain that the composition map $\beta \circ \alpha := m_2(\beta \otimes \alpha)$ is

$$\begin{aligned} (\beta \circ \alpha)_{i_0 i_1 \dots i_k} &= -(-1)^{|\beta|} (m_2^{\mathcal{B}} \circ \omega^{\otimes 2}) \circ \langle s(\beta), s(\alpha) \rangle_{i_0, \dots, i_k} \\ &= -(-1)^{|\beta|} (m_2^{\mathcal{B}} \circ \omega^{\otimes 2}) \left(\sum_{j=0}^k (-1)^{|s(\alpha)|(k-j)} s(\beta)_{i_j, \dots, i_k} \otimes s(\alpha)_{i_0, \dots, i_j} \right) \\ &= -(-1)^{|\beta|} \sum_{j=0}^k (-1)^{|s(\alpha)|(k-j)} (-1)^{|s(\beta)_{i_j, \dots, i_k}|(-1)} \beta_{i_j, \dots, i_k} \circ \alpha_{i_0, \dots, i_j} \\ &= -(-1)^{|\beta|} \sum_{j=0}^k (-1)^{(|\alpha|-1)(k-j)} (-1)^{(|\beta|-(k-j)-1)(-1)} \beta_{i_j, \dots, i_k} \circ \alpha_{i_0, \dots, i_j} \\ &= \sum_{j=0}^k (-1)^{(k-j)|\alpha|} \beta_{i_j, \dots, i_k} \circ \alpha_{i_0, \dots, i_j}. \end{aligned}$$

A similar calculation shows that the differential $d\alpha = m_1(\alpha)$ is

$$\begin{aligned} (d\alpha)_{i_0 \dots i_k} &= d_{\mathcal{B}} \alpha_{i_0 \dots i_k} - (-1)^{|\alpha|} \alpha \left(d_{\text{Bar}(\overline{k[n]})}((i_0, \dots, i_k)) \right) \\ &\quad + (m_2^{\mathcal{B}} \circ \omega^{\otimes k}) \left((s(g) \otimes s(\alpha))_{i_0 \dots i_k} + (s(\alpha) \otimes s(f))_{i_0 \dots i_k} \right) \\ &= d_{\mathcal{B}} \alpha_{i_0 \dots i_k} + \sum_{j=1}^{k-1} \left((-1)^{|\alpha|} (-1)^{k-j} \alpha_{i_0 \dots i_j \dots i_k} \right. \\ &\quad \left. + (-1)^{(k-j)|\alpha|} g_{i_j \dots i_k} \circ \alpha_{i_0 \dots i_j} - (-1)^{|\alpha|} (-1)^{k-j} \alpha_{i_j \dots i_k} \circ f_{i_0 \dots i_j} \right). \end{aligned}$$

All higher A_∞ -operations vanish, so that we have an honest dg-category. We may simplify the last formula by regarding f and g as morphisms $f \rightarrow f$ resp. $g \rightarrow g$ of degree 1. Then the composition operation defined above makes sense for f and g as well, and we obtain

$$\begin{aligned} d\alpha &= d_{\mathcal{B}} \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_{\text{Bar}(\overline{k[n]})} + g \circ \alpha - (-1)^{|\alpha|} \alpha \circ f \\ 0 &= d_{\mathcal{B}} \circ f + f \circ d_{\text{Bar}(\overline{k[n]})} + f \circ f. \end{aligned}$$

In this case, as f factors through $\text{Bar}(\overline{k[n]}) \rightarrow \overline{\text{Bar}(\overline{k[n]})}$, expressions like f_{i_0} with a single index must be interpreted as zero. \circ

4 Homotopy limits in dg-categories

Given a dg-category \mathcal{A} , we denote by $Z^0 \mathcal{A}$ the dg-category with the same objects as \mathcal{A} , but with hom complexes given by $(Z^0 \mathcal{A})(x, y) = Z^0(\mathcal{A}(x, y))$, the set

of closed maps $x \rightarrow y$ in \mathcal{A} . We define $H^0\mathcal{A}$ analogously. A map $f \in \mathcal{A}^0(x, y)$ in \mathcal{A} is called a **homotopy equivalence** if its image in $H^0\mathcal{A}$ is an isomorphism. As in homological algebra, this amounts the existence of a map $g \in \mathcal{A}^0(y, x)$ in the other direction along with correcting morphisms $r_x \in \mathcal{A}^{-1}(x, x)$ and $r_y \in \mathcal{A}^{-1}(y, y)$ such that

$$dr_x = gf - \text{id}_x \quad \text{and} \quad dr_y = fg - \text{id}_y.$$

We recall from Tabuada (2005) and Tabuada (2010) the existence of a model structure on the 1-category dgCat with

- weak equivalences given by **quasi-equivalences**, i.e. functors $F: \mathcal{A} \rightarrow \mathcal{B}$ such that
 - (i) the induced functor $H^0F: H^0\mathcal{A} \rightarrow H^0\mathcal{B}$ is an equivalence of categories in the classical, non-enriched sense;
 - (ii) for all $x, y \in \mathcal{A}$, $F_{x,y}: \mathcal{A}(x, y) \rightarrow \mathcal{B}(F(x), F(y))$ is a quasi-isomorphism of chain complexes;
- fibrations given by dg-functors $F: \mathcal{A} \rightarrow \mathcal{B}$ such that
 - (i) the induced map $F: \mathcal{A}^*(x, y) \rightarrow \mathcal{B}^*(F(x), F(y))$ is surjective for all objects $x, y \in \mathcal{A}$, and
 - (ii) any map $f: F(x) \rightarrow y$ in \mathcal{B} that becomes an isomorphism in $H^0\mathcal{B}$ lifts as $f = F(g)$ for some morphism $g: x \rightarrow x'$ in \mathcal{A} that becomes an isomorphism in $H^0\mathcal{A}$.

This model structure is combinatorial, see Lurie (2017, Proposition 1.3.1.19).

We shall use the tools developed in the preceding chapters to develop a notion of homotopy descent of quasi-coherent sheaves on affine dg-schemes. Much of the inspiration is from Block, Holstein, and Wei (2017); however, using the tools developed earlier (specifically Example 3.8.6), we are able to solve their Conjecture 1 and prove their results in complete generality.

Recall from Proposition 2.2.1 that the homotopy limit of a Δ -diagram in the model category dgCat is given by

$$\underline{\text{holim}}_{\Delta} \mathcal{A}^n = \underline{\text{holim}}_{\Delta_+} \mathcal{A}^n = \int_{[n] \in \Delta_+} R_{\Delta_+}(\mathcal{A}^n)_n$$

where $R_{\Delta_+}: \text{dgCat} \rightarrow \text{dgCat}_{\text{Inj}}^{\Delta_+ \text{op}}$ is a fibrant replacement functor, taking a dg-category \mathcal{B} to an injectively fibrant replacement of the constant Δ_+ -diagram at \mathcal{B} . We obtain such a functor from Holstein, Poliakova and Arkhipov:

4.0.1. Theorem (Holstein 2016, section 3 and Arkhipov and Poliakova 2018). *If \mathcal{B} is a dg-category, then $\mathcal{B}_n \in \text{dgCat}_{\Delta_+}^{\Delta_+ \text{op}}$ given by*

$$\mathcal{B}_n = A_{\infty}\text{Fun}^{\circ}(k[n], \mathcal{B})$$

is an injectively fibrant replacement of the constant Δ_+ -diagram at \mathcal{B} . Here, the “ \circ ” means that the objects are A_{∞} -functors f such that $H^0f: k[n] \rightarrow H^0\mathcal{B}$ sends non-zero maps to isomorphisms. Equivalently, in the notation of Example 3.8.6, H^0f_{ij} is an isomorphism for all $i < j$.

In other words, if \mathcal{A}^\bullet is a cosimplicial system of dg-categories, then its homotopy limit is given by

$$\underline{\mathrm{holim}}_\Delta \mathcal{A}^n = \int_{[n] \in \Delta_+} \mathrm{A}_\infty \mathrm{Fun}^\circ(k[n], \mathcal{A}^n).$$

We wish to evaluate this expression more explicitly:

4.0.2. Proposition. *Suppose \mathcal{A}^\bullet is a cosimplicial system of dg-categories. Then the homotopy limit $\underline{\mathrm{holim}}_\Delta \mathcal{A}^\bullet$ is the dg-category with objects (M, θ) where $\theta = (\theta_n)_{n \geq 1}$ is a collection of morphisms $\theta_n \in \mathrm{Hom}_{\mathcal{A}^n}^{1-n}((\partial^{\max})^n M, (\partial^0)^n M)$, satisfying*

$$(4.0.3) \quad d\theta_n = -(\theta \circ \theta)_n + \sum_{i=1}^{n-1} (-1)^{n-i} \partial^i \theta_{n-1}$$

and with θ_1 invertible in $H^0 \mathcal{A}^1$. The composition $\theta \circ \theta$ is evaluated via (4.0.4) below with the conventions $|\theta| = 1$ and $\theta_0 = 0$. A morphism $\alpha: (M, \theta) \rightarrow (N, \eta)$ of degree $|\alpha|$ is a collection $\alpha = (\alpha_n)_{n \geq 0}$ of morphisms $\alpha_n \in \mathrm{Hom}_{\mathcal{A}^n}^{|\alpha|-n}((\partial^{\max})^n M, (\partial^0)^n N)$. The composition of two morphisms is

$$(4.0.4) \quad (\beta \circ \alpha)_n = \sum_{i=0}^n (-1)^{|\alpha|(n-i)} (\partial^0)^i \beta_{n-i} \circ (\partial^{\max})^{n-i} \alpha_i.$$

The differential on a morphism is

$$(4.0.5) \quad d(\alpha)_n = d(\alpha_n) + (\eta \circ \alpha)_n - (-1)^{|\alpha|} (\alpha \circ \theta)_n + (-1)^{|\alpha|} \sum_{j=1}^{n-1} (-1)^{n-j} \partial^j \alpha_{n-1}.$$

This proves Conjecture 1 in Block, Holstein, and Wei (2017).

Proof of Proposition 4.0.2. Using Holstein's theorem, we obtain the formula

$$\begin{aligned} \underline{\mathrm{holim}}_\Delta \mathcal{A}^n &= \int_{[n] \in \Delta_+} \mathrm{A}_\infty \mathrm{Fun}^\circ(k[n], \mathcal{A}^n) \\ &= \mathrm{Eq} \left(\prod_{[n] \in \Delta_+} \mathrm{A}_\infty \mathrm{Fun}^\circ(k[n], \mathcal{A}^n) \rightrightarrows \prod_{[n] \hookrightarrow [m]} \mathrm{A}_\infty \mathrm{Fun}^\circ(k[n], \mathcal{A}^m) \right). \end{aligned}$$

We initially note that the set of objects of $\mathrm{A}_\infty \mathrm{Fun}^\circ(k[n], \mathcal{A}^n)$ is the subset of the hom set $\mathrm{Hom}_{\mathrm{dgCat}}(\mathrm{Cob}^+(\mathrm{Bar}^+(k[n])), \mathcal{A}^n)$ of functors satisfying the homotopy invertibility condition. Thus an object of this equalizer is a collection $F = (F^n)_{[n] \in \Delta_+}$ of dg-functors $F^n: \mathrm{Cob}^+(\mathrm{Bar}^+(k[n])) \rightarrow \mathcal{A}^n$ satisfying $\varphi_* \circ F^n = F^m \circ \varphi_*$ for all $\varphi: [n] \hookrightarrow [m]$ in Δ_+ (in an enriched sense, one may say that it consists of natural transformations between the Δ_+ -diagrams $\mathrm{Cob}^+(\mathrm{Bar}^+(k[\cdot])) \rightarrow \mathcal{A}^\bullet$). Now if $0 \leq i \leq n$, we may consider the map $\varphi: [0] \hookrightarrow [n]$ taking 0 to i . Then $F^n(i) = F^n(\varphi_*(0)) = \varphi_*(F^0(0))$. Thus all the functors F^n are determined on the object level by the one object $M := F^0(0) \in \mathcal{A}^0$. Similarly, on the morphism level, $\mathrm{Cob}^+(\mathrm{Bar}^+(k[n]))$ is freely generated by the morphisms $f_{i_0 i_1 \dots i_l}: i_0 \rightarrow i_l$ with $0 \leq i_0 < i_1 < \dots < i_l \leq n$. But if we consider the map $\varphi: [l] \hookrightarrow [n]$ given by $\varphi(j) = i_j$, we have $f_{i_0 i_1 \dots i_l} = \varphi_*(f_{01 \dots l})$. In other words, F is determined on the object level by what it does to the non-degenerate morphism $f_{01 \dots l} \in$

$\text{Cob}^+(\text{Bar}^+(k[l]))$. Furthermore, these non-degenerate morphisms may be chosen freely. Summarizing, the objects of this equalizer consist of the data of an element $M \in \mathcal{A}^0$ and for each n a morphism

$$\theta_n := F^n(f_{01\dots n}): \partial_{\max^*}^n M \longrightarrow \partial_{0^*}^n M.$$

The differential is as stated since this is the image of the differential on $f_{01\dots n}$.

On the morphism level, suppose F^n and G^n are objects of the equalizer. A morphism $F^\bullet \rightarrow G^\bullet$ of degree d consists of a collection $\alpha^\bullet = (\alpha^n)_{[n] \in \Delta_+}$ of morphisms $\alpha^n: F^n \rightarrow G^n$ in $\text{A}_\infty \text{Fun}^\circ(k[n], \mathcal{A}^n)$ that simultaneously lie in the equalizer. From Example 3.8.6, we therefore get that the space of non-identity transformations $F^\bullet \rightarrow G^\bullet$ is

$$\begin{aligned} & \int \text{Hom}^\bullet(T^c(\overline{k[n]}[-1]), (F^n \times G^n)^* \mathcal{A}^n) = \int \text{Hom}^\bullet\left(\bigoplus_{l \geq 0} \overline{k[n]}^{\otimes l}[-l], (F^n \times G^n)^* \mathcal{A}^n\right) \\ & = \int \prod_{l \geq 0} \prod_{0 \leq i_0 < \dots < i_l \leq n} \text{Hom}_k^\bullet(k f_{i_{l-1}i_l} \otimes \dots \otimes f_{i_0 i_1}, \mathcal{A}^n(F^n i_0, G^n i_l)). \end{aligned}$$

Thus the transformation α^\bullet is freely determined by what it does to the non-degenerate elements $f_{n-1,n} \otimes \dots \otimes f_{0,1}$, i.e. by the elements

$$\alpha^n(f_{n-1,n} \otimes \dots \otimes f_{0,1}) \in \mathcal{A}^n(F^n i_0, G^n i_n).$$

Calling this element α_n , we obtain the desired description. \square

4.1 Homotopy descent of dg-schemes

If k is a field, the category of **affine dg-schemes** over k is the opposite category $\text{dgAff} = (\text{dgAlg}_{\text{com}}^{\leq 0})^{\text{op}}$ to the category of non-positively graded, graded commutative dg-algebras over k . If \mathbf{X} is a dg-scheme, the associated dg-algebra is denoted $\mathcal{A}_{\mathbf{X}}$. The category of **quasi-coherent dg-sheaves** on $(\mathbf{X}, \mathcal{A}_{\mathbf{X}})$ is the category $\mathcal{A}_{\mathbf{X}}\text{-dgmod}$ of $\mathcal{A}_{\mathbf{X}}$ -dg-modules.

4.1.1. Theorem. *Suppose that $\mathbf{X}_1 \rightrightarrows \mathbf{X}_0$ is a groupoid in affine dg-schemes, and consider the associated classifying space \mathbf{X} , given by*

$$\mathbf{X}_n = \mathbf{X}_1 \times_{\mathbf{X}_0} \mathbf{X}_1 \times_{\mathbf{X}_0} \dots \times_{\mathbf{X}_0} \mathbf{X}_1.$$

Write $\mathcal{A}^n = \mathcal{A}_{\mathbf{X}_n}$ for the associated cosimplicial system of dg-algebras. Let $\mathcal{A} = \mathcal{A}^0$ and $\mathcal{C} = \mathcal{A}^1$, and note that \mathcal{C} is a counital coalgebra in $\mathcal{A}\text{-dgmod}$ via the map $\Delta = \partial_1^\# : \mathcal{C} \rightarrow \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$. Then we have an equivalence of dg-categories

$$\underline{\text{holim}}_{\Delta} \text{QCoh}(\mathbf{X}_\bullet) \cong \mathcal{C}\text{-comod}_{\infty}^{\text{hcu, formal}}(\mathcal{A}),$$

where the right-hand side denotes the dg-category of formal, homotopy-counital A_∞ -comodules over \mathcal{C} in $\mathcal{A}\text{-dgmod}$.

Proof of Theorem 4.1.1. Proposition 4.0.2 gives us the general form of the homotopy limits of \mathcal{A}^\bullet , so we use the same notation. Thus an object of the homotopy limit is a pair (M, θ) , where $\theta = (\theta_n)_{n \geq 1}$ is a tuple of maps $\theta_n: \partial_{\max^*}^n M \rightarrow \partial_{0^*}^n M$ of degree $1 - n$, and a morphism $\alpha: (M, \theta) \rightarrow (N, \eta)$ is a collection $\alpha =$

$(\alpha_n)_{n \geq 0}$ of maps $\alpha_n: \partial_{\max}^{n*} M \rightarrow \partial_0^{n*} N$ of degree $|\alpha| - n$. For such a morphism, we scale it to define a new morphism $\tilde{\alpha}$ by $\tilde{\alpha}_n = (-1)^{|\alpha|(n+1)} \alpha_n$. Regarding this θ as a degree 1 morphism $(M, \theta) \rightarrow (M, \theta)$, we use the same notation as for morphisms and write $\tilde{\theta}_n = (-1)^{n+1} \theta_n$ for $n \geq 1$. We notice that (4.0.4) becomes

$$\widetilde{(\beta \circ \alpha)}_n = \sum_{i=0}^n (-1)^{|\beta|i} \partial_0^{i*} \tilde{\beta}_{n-i} \circ \partial_{\max}^{(n-i)*} \tilde{\alpha}_i.$$

Thus (4.0.3) becomes

$$(-1)^{n+1} d_{\partial_0^{n*} M} \circ \tilde{\theta}_n - \tilde{\theta}_n \circ d_{\partial_{\max}^{n*} M} = -(\widetilde{\theta \circ \theta})_n + \sum_{i=1}^{n-1} (-1)^i \partial_i^* \tilde{\theta}_{n-1}.$$

Since $\widetilde{d(\alpha)}_n = (-1)^{(|\alpha|+1)(n+1)} d(\alpha)_n$, (4.0.5) becomes

$$\begin{aligned} \widetilde{d(\alpha)}_n &= (-1)^{n+1} d_{\partial_0^{n*} N} \circ \tilde{\alpha}_n + (-1)^{|\alpha|} \tilde{\alpha}_n \circ d_{\partial_{\max}^{n*} M} + (\widetilde{\eta \circ \alpha})_n \\ &\quad - (-1)^{|\alpha|} (\widetilde{\alpha \circ \theta})_n + \sum_{j=1}^{n-1} (-1)^{j-1} \partial_j^* \tilde{\alpha}_{n-1} \\ &= (-1)^{n+1} d_{\partial_0^{n*} N} \circ \tilde{\alpha}_n + \sum_{i=0}^{n-1} (-1)^i \partial_0^{i*} \tilde{\eta}_{n-i} \circ \partial_{\max}^{(n-i)*} \tilde{\alpha}_i \\ &\quad + (-1)^{|\alpha|} \tilde{\alpha}_n \circ d_{\partial_{\max}^{n*} M} - \sum_{i=1}^n (-1)^{|\alpha|(i+1)} \partial_0^{i*} \tilde{\alpha}_{n-i} \circ \partial_{\max}^{(n-i)*} \tilde{\theta}_i \\ &\quad + \sum_{j=1}^{n-1} (-1)^{j-1} \partial_j^* \tilde{\alpha}_{n-1}, \end{aligned}$$

Now for an object (M, θ) , we add to the collection $(\theta_n)_{n \geq 1}$ the element $\theta_0 = -d_M$. We then define the comodule operations $\text{ca}_n: M \rightarrow \mathcal{C}^{\otimes(n-1)} \otimes M$ of degree $2 - n$ by

$$\text{ca}_n: M \longrightarrow \partial_{\max^*}^{n-1} \partial_{\max}^{(n-1)*} M \xrightarrow{\tilde{\theta}_{n-1}} \partial_{\max^*}^{n-1} \partial_0^{(n-1)*} M$$

where the first map is the unit of adjunction. In particular, $\text{ca}_1 = -d_M$. Similarly, if $\alpha: (M, \theta) \rightarrow (N, \eta)$ is a map of degree $|\alpha|$, we associate to it the map of A_∞ -comodules $f: M \rightarrow N$ of degree $|f| = |\alpha|$, where $f_n: M \rightarrow \mathcal{C}^{\otimes(n-1)} \otimes N$ is the map of degree $|\alpha| + 1 - n = |f| + 1 - n$ given by

$$f_n: M \longrightarrow \partial_{\max^*}^{n-1} \partial_{\max}^{(n-1)*} M \xrightarrow{\tilde{\alpha}_{n-1}} \partial_{\max^*}^{n-1} \partial_0^{(n-1)*} N.$$

It is immediate that the composition map above agrees with the composition of maps of A_∞ -comodules. Via Lemmata 2.1.5 and 2.1.6, the other two equa-

tions above now become

$$\begin{aligned}
& (-1)^{n+1} (d_{\mathcal{C} \otimes \dots \otimes \mathcal{C}} \otimes \text{id}_M - \text{id}_{\mathcal{C} \otimes \dots \otimes \mathcal{C}} \otimes \text{ca}_1) \text{ca}_{n+1} + \text{ca}_{n+1} \text{ca}_1 \\
&= - \sum_{i=1}^{n-1} (-1)^i (\text{id}_{\mathcal{C}^{\otimes i}} \otimes \text{ca}_{n-i+1}) \text{ca}_{i+1} \\
&\quad + \sum_{i=1}^{n-1} (-1)^i (\text{id}^{\otimes(i-1)} \otimes \Delta_{\mathcal{C}} \otimes \text{id}^{\otimes(n-i)}) \text{ca}_n
\end{aligned}$$

and

$$\begin{aligned}
(df)_n &= (-1)^{n+1} (d_{\mathcal{C} \otimes \dots \otimes \mathcal{C}} \otimes \text{id}_N - \text{id}_{\mathcal{C} \otimes \dots \otimes \mathcal{C}} \otimes \text{ca}_1) f_{n+1} + \sum_{i=0}^{n-1} (-1)^i \text{ca}_{n-i+1} f_{i+1} \\
&\quad + (-1)^{|\alpha|} f_{n+1} \text{ca}_1 - \sum_{i=1}^n (-1)^{|\alpha|(i+1)} f_{n-i+1} \text{ca}_{i+1} \\
&\quad + \sum_{j=1}^{n-1} (-1)^{j-1} (\text{id}^{\otimes(j-1)} \otimes \Delta_{\mathcal{C}} \otimes \text{id}^{\otimes(n-j)}) \text{ca}_n,
\end{aligned}$$

which are exactly the A_∞ -comodule equations from earlier.

To verify that we get exactly *homotopy-counital* A_∞ -comodules, we notice that the equation (4.0.3) in the case $n = 2$ yields that

$$d(\theta_2) = \partial_0^* \theta_1 \circ \partial_2^* \theta_1 - \partial_1^* \theta_1,$$

so that we have a homotopy $\partial_0^* \theta_1 \circ \partial_2^* \theta_1 \simeq \partial_1^* \theta_1$. Then one may adjust the proof of Proposition 2.0.1, replacing equalities by homotopies, shows that θ_1 being an isomorphism is equivalent to $\sigma_0^* \theta_1 \simeq \text{id}$. Similarly, one adjusts the last part of the proof of Proposition 2.1.4 to obtain that $(\varepsilon \otimes \text{id}_M) \text{ca}_2 \simeq \text{id}$, which means that M is homotopy-counital. \square

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