Homotopy (co)limits via homotopy (co)ends in general combinatorial model categories

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Abstract

We prove and explain several classical formulae for homotopy (co)limits in general (combinatorial) model categories which are not necessarily simplicially enriched. Importantly, we prove versions of the Bousfield–Kan formula and the fat totalization formula in this complete generality. We finish with a proof that homotopy-final functors preserve homotopy limits, again in complete generality.

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If $\mathscr{C}$ is a model category and Γ a category, we denote by $\mathscr{C}^\Gamma = \mathsf{Fun}(\Gamma, \mathscr{C})$ the category of functors $\Gamma \to \mathscr{C}$, which we shall also refer to as "diagrams" of shape Γ. It is natural to call a map of diagrams $\alpha\colon F \to G$ in $\mathscr{C}^\Gamma$ a *weak equivalence* if $\alpha_\gamma\colon F(\gamma) \to G(\gamma)$ is a weak equivalence in $\mathscr{C}$ for all objects $\gamma \in \Gamma$. We shall refer to such weak equivalences as **componentwise** weak equivalences. But then we immediately run into the problem that the limit functor $\varprojlim\colon \mathscr{C}^\Gamma \to \mathscr{C}$ does not in general preserve weak equivalences. Since $\varprojlim$ is a right adjoint, this leads us into trying to *derive* it. The right derived functor of $\varprojlim$ is called the **homotopy limit** and is denoted $\underleftarrow{\mathrm{holim}}\colon \mathscr{C}^\Gamma \to \mathscr{C}$. Dually, the left derived functor of $\varinjlim$ is called the **homotopy colimit** and is denoted $\underrightarrow{\mathrm{holim}}$.

For many purposes, the abstract existence of homotopy limits is all you need. However, there are also many cases where a concrete, minimalistic realization of them is useful for working with abstract notions. For instance, this paper grew out of an attempt to concretize a concept from derived algebraic geometry. More specifically, we wanted to develop a homological algebra model for the dg-category of quasi-coherent sheaves on a dg-scheme which are equivariant with respect to the action of a group dg-scheme. This question was addressed in Block, Holstein, and Wei (2017) where a partial result was obtained under serious restrictions (Proposition 13). The general case was stated as a conjecture, see Conjecture 1 in the same paper. In the companion paper to this one, Arkhipov and Ørsted (2018), we cover the general case and prove that conjecture (see Theorem 4.1.1), and the key result of homotopical nature is proved in the present note (see Example 6.4).

Quillen's model category machinery tells us how to derive the limit: We must equip the diagram category $\mathscr{C}^\Gamma$ with a model structure with componentwise weak equivalences and in which the limit functor $\varprojlim : \mathscr{C}^\Gamma \to \mathscr{C}$ is a right Quillen functor. In this case, the derived functor is given by $\operatorname{\underleftarrow{holim}} F = \varprojlim R(F)$ for some fibrant replacement $R(F)$ in $\mathscr{C}^\Gamma$. Indeed, Corollary 2.4 below shows that such a model structure on $\mathscr{C}^\Gamma$ exists e.g. if the model category $\mathscr{C}$ is *combinatorial*. More precisely, we introduce the *injective model structure* $\mathscr{C}^\Gamma_{\mathrm{Inj}}$ where weak equivalences and cofibrations are calculated componentwise. Denoting by $\mathrm{const}\colon \mathscr{C} \to \mathscr{C}^\Gamma$ the constant functor embedding, we clearly see that $\mathrm{const}\colon \mathscr{C} \rightleftarrows \mathscr{C}^\Gamma_{\mathrm{Inj}} : \varprojlim$ is a Quillen adjunction since const preserves (trivial) cofibrations.

The injective model structure being in general rather complicated, calculating such a replacement of a diagram in practice becomes very involved for all but the simplest shapes of the category Γ. Therefore, traditionally, other tools have been used. One of the most popular techniques involves *adding a parameter* to the limit functor $\varprojlim_\Gamma$ before deriving it. The result is the *end* bifunctor $\int_\Gamma \colon \Gamma^{\mathrm{op}} \times \Gamma \to \mathscr{C}$ (introduced below) which is in general much easier to derive.

One of the classical accounts of this technique is Hirschhorn (2003) who mainly works in the setting of *simplicial model categories*, which are model categories enriched over simplicial sets, and which furthermore are equipped with a *powering* functor

$$\mathsf{SSet}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathscr{C}, \qquad (K, c) \longmapsto c^K,$$

and a *copowering* functor

$$\mathsf{SSet} \times \mathscr{C} \longrightarrow \mathscr{C}, \qquad (K, c) \longmapsto K \otimes c,$$

satisfying some compatibility relations with the model structure. He then establishes the classical **Bousfield–Kan formula**

$$\operatorname{\underleftarrow{holim}} F = \int_{\gamma \in \Gamma} F(\gamma)^{N(\Gamma/\gamma)}, \tag{0.1}$$

where we write $N(\Gamma/\gamma)$ for the nerve of the comma category of maps in Γ with codomain γ. Or rather, he uses this formula as his *definition* of homotopy limits (see definition 18.1.8). A proof that this formula agrees with the general definition of homotopy limits is due to Gambino (2010, equation (2)) using the machinery of Quillen 2-functors.

Hirschhorn then generalizes this formula to arbitrary model categories in chapter 19, definition 19.1.5. He shows that even for non-simplicial model categories, one can replace simplicial powerings and copowerings by a weaker notion, unique up to homotopy in a certain sense. He then again takes the formula (0.1) to be his *definition* of a homotopy limit. This paper is devoted to proving why this formula agrees with the general definition of a homotopy limit (similarly to what Gambino did in the simplicial setting). To the authors' knowledge, such a proof has not been carried out in the literature before.

1 The end construction

Let Γ and $\mathscr{C}$ be categories, $\mathscr{C}$ complete and cocomplete, and let $H\colon \Gamma^{\mathrm{op}} \times \Gamma \to \mathscr{C}$ a bifunctor. The **end** of H is an object

$$\int_\Gamma H = \int_{\gamma\in\Gamma} H(\gamma,\gamma)$$

in $\mathscr{C}$, together with morphisms $\int_\Gamma H \to H(\gamma,\gamma)$ for all $\gamma \in \Gamma$, such that for any $f : \gamma \to \gamma'$, the following diagram commutes:

$$\begin{array}{ccc} \int_\Gamma H & \longrightarrow & H(\gamma,\gamma) \\ \downarrow & & \downarrow{\scriptstyle H(\gamma,f)} \\ H(\gamma',\gamma') & \xrightarrow[H(f,\gamma')]{} & H(\gamma,\gamma'). \end{array}$$

Furthermore, $\int_\Gamma H$ is universal with this property, meaning that if A is another object of $\mathscr{C}$ with a collection of arrows $A \to H(\gamma,\gamma)$ for all γ, subject to the same commutativity conditions, then these factor through a unique arrow $A \to \int_\Gamma H$:

$$\begin{array}{cccc} A & & & \\ & \searrow & & \\ & \int_\Gamma H & \longrightarrow & H(\gamma,\gamma) \\ & \downarrow & & \downarrow{\scriptstyle H(\gamma,f)} \\ & H(\gamma',\gamma') & \xrightarrow[H(f,\gamma')]{} & H(\gamma,\gamma'). \end{array}$$

Clearly, we may obtain the end by the formula

$$\int_\Gamma H = \mathrm{Eq}\Big(\prod_{\gamma\in\Gamma} H(\gamma,\gamma) \rightrightarrows \prod_{f:\,\gamma\to\gamma'} H(\gamma,\gamma')\Big).$$

Here, the second product runs over all morphisms $f : \gamma \to \gamma'$ in Γ, and the two arrows are given by $f_*\colon H(\gamma,\gamma) \to H(\gamma,\gamma')$ resp. $f^*\colon H(\gamma',\gamma') \to H(\gamma,\gamma')$. There is a dual notion of a *coend*, denoted instead by $\int^\Gamma H$, which we shall not spell out.

1.1. Example. Given functors $F,G\colon \Gamma \to \Gamma'$, we obtain a bifunctor

$$H = \mathrm{Hom}_{\Gamma'}(F(-),G(-))\colon \Gamma^{\mathrm{op}} \times \Gamma \to \mathsf{Set},$$

and the universal property shows that

$$\int_\Gamma H = \mathrm{Hom}_{\mathsf{Fun}(\Gamma,\Gamma')}(F,G)$$

is the set of natural transformations between F and G. ○

1.2. Example. A diagram $F \in \mathscr{C}^\Gamma$ may be regarded as a diagram in $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma}$ which is constant with respect to the first variable. In that case, it follows from the universal property of the end that $\int_\Gamma F = \varprojlim_\Gamma F$ recovers the limit of the diagram. ○

1.3. Proposition. *The end fits as the right adjoint of the adjunction*

$$\coprod_{\mathrm{Hom}_\Gamma} : \mathscr{C} \rightleftarrows \mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} : \int_\Gamma .$$

The left adjoint takes $A \in \mathscr{C}$ to the bifunctor $\coprod_{\mathrm{Hom}_\Gamma(-,-)} A \colon \Gamma^{\mathrm{op}} \times \Gamma \to \mathscr{C}$.

Proof. Clear from the definition. □

This is equivalent to the statement that we have an adjunction

$$\mathsf{Set}^{\Gamma^{\mathrm{op}}\times\Gamma}\Big(\mathrm{Hom}_\Gamma, \mathscr{C}(A,F)\Big) \cong \mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma}\Big(A, \int_\Gamma F\Big)$$

for $A \in \mathscr{C}$. This says that the end is the *weighted limit* $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} \to \mathscr{C}$ with weight Hom_Γ. The dual statement for *coends* is that the coend functor

$$\int^\Gamma : \mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} \longrightarrow \mathscr{C}$$

is left adjoint to $\prod_{\mathrm{Hom}_\Gamma}$.

2 The projective and injective model structures

If $\mathscr{C}$ is a model category and Γ any category, there is no completely general way to turn the functor category $\mathscr{C}^\Gamma = \mathsf{Fun}(\Gamma, \mathscr{C})$ into a model category. The naïve approach, calculating weak equivalences, cofibations, and fibrations componentwise, will not in general yield a model structure. It is natural to demand that at least the weak equivalences must be calculated componentwise for any model structure to be satisfactory. In general, however, at least one of the other two classes will in return become more complicated. The two most natural model structures one can hope for (which may or may not exist) are

- The **projective model structure** $\mathscr{C}^\Gamma_{\mathrm{Proj}}$ where weak equivalences and fibrations are calculated componentwise.
- The **injective model structure** $\mathscr{C}^\Gamma_{\mathrm{Inj}}$ where weak equivalences and cofibrations are calculated componentwise.

Existence of these model structures depends heavily on the structure of the target category $\mathscr{C}$ (see Proposition 2.5 below). We shall also use the attributes "projective(ly)" and "injective(ly)" when referring to these model structures, so e.g. "projectively cofibrant" means cofibrant in the projective model structure.

2.1. Proposition (Lurie 2009, Proposition A.2.8.7).
If $\mathscr{C}$ is a model category and $f\colon \Gamma \to \Gamma'$ a functor, denote by f^* the restriction functor $\mathscr{C}^{\Gamma'} \to \mathscr{C}^{\Gamma}$. Then f^* fits as the right and left adjoint of Quillen adjunctions

$$f_!\colon \mathscr{C}^{\Gamma}_{\mathrm{Proj}} \rightleftarrows \mathscr{C}^{\Gamma'}_{\mathrm{Proj}} : f^* \qquad \textit{resp.} \qquad f^*\colon \mathscr{C}^{\Gamma'}_{\mathrm{Inj}} \rightleftarrows \mathscr{C}^{\Gamma}_{\mathrm{Inj}} : f_*$$

whenever the model structures in question exist.

The adjoints $f_!$ and f_* are the usual **left** and **right Kan extensions** along f, which are given by limits

$$f_!F(\gamma') = \varinjlim_{f(\gamma)\to\gamma'} F(\gamma) \qquad \text{and} \qquad f_*F(\gamma') = \varprojlim_{\gamma'\to f(\gamma)} F(\gamma). \tag{2.2}$$

These limits are taken over the categories of maps $f(\gamma) \to \gamma'$ (resp. $\gamma' \to f(\gamma)$) in Γ' for varying $\gamma \in \Gamma$.

Proof. Since the adjunctions in question exist, their being Quillen follows from the observation that f^* clearly preserves (trivial) projective fibrations and (trivial) injective cofibrations. □

2.3. Corollary. *Assume in the following that the relevant model structures exist.*

(i) *If $\varphi\colon c \to c'$ is a (trivial) cofibration in $\mathscr{C}$ and $\gamma_0 \in \Gamma$ is an object, then the coproduct map $\coprod_{\Gamma(\gamma_0,-)} \varphi\colon \coprod_{\Gamma(\gamma_0,-)} c \to \coprod_{\Gamma(\gamma_0,-)} c'$ is a (trivial) cofibration in $\mathscr{C}^{\Gamma}_{\mathrm{Proj}}$. We shall refer to such (trivial) cofibrations as **simple projective cofibrations**.*

(ii) *If $f\colon \Gamma \to \Gamma'$ is a functor, then $f_!\colon \mathscr{C}^{\Gamma}_{\mathrm{Proj}} \to \mathscr{C}^{\Gamma'}_{\mathrm{Proj}}$ preserves simple (trivial) projective cofibrations, taking $\coprod_{\Gamma(\gamma_0,-)} \varphi$ to $\coprod_{\Gamma'(f(\gamma_0),-)} \varphi$.*

(iii) *If $\psi\colon c \to c'$ is a (trivial) fibration in $\mathscr{C}$ and $\gamma_0 \in \Gamma$ is an object, then the product map $\prod_{\Gamma(-,\gamma_0)} \psi\colon \prod_{\Gamma(-,\gamma_0)} c \to \prod_{\Gamma(-,\gamma_0)} c'$ is a (trivial) fibration in $\mathscr{C}^{\Gamma}_{\mathrm{Inj}}$. We shall refer to such (trivial) fibrations as **simple injective fibrations**.*

(iv) *If $f\colon \Gamma \to \Gamma'$ is a functor, then $f_*\colon \mathscr{C}^{\Gamma}_{\mathrm{Inj}} \to \mathscr{C}^{\Gamma'}_{\mathrm{Inj}}$ preserves simple (trivial) injective fibrations, taking $\prod_{\Gamma(-,\gamma_0)} \psi$ to $\prod_{\Gamma'(-,f(\gamma_0))} \psi$.*

Proof. Applying Proposition 2.1 to the embedding $\iota\colon \gamma_0 \hookrightarrow \Gamma$ of the full subcategory with γ_0 as the only object, we get that $\iota_!\varphi$ is a (trivial) cofibration. Now $\iota_!\varphi = \coprod_{\Gamma(\gamma_0,-)}\varphi$ by the above colimit formula for left Kan extension. The statement (ii) follows by applying Kan extensions to the diagram

$$\begin{array}{ccc} \gamma_0 & \hookrightarrow & \Gamma \\ \downarrow & & \downarrow f \\ f(\gamma_0) & \hookrightarrow & \Gamma' \end{array}$$

and using that Kan extensions, being adjoints to restriction, respect compositions. The other statements are dual. □

2.4. Corollary. *Denote by* $\mathrm{const}\colon \mathscr{C} \to \mathscr{C}^{\Gamma}$ *the functor taking $c \in \mathscr{C}$ to the constant diagram at c.*

(i) *If* $\mathscr{C}^{\Gamma}_{\mathrm{Proj}}$ *exists, then* $\varinjlim\colon \mathscr{C}^{\Gamma}_{\mathrm{Proj}} \rightleftarrows \mathscr{C} :\mathrm{const}$ *is a Quillen adjunction.*

(ii) *If* $\mathscr{C}^{\Gamma}_{\mathrm{Inj}}$ *exists, then* $\mathrm{const}\colon \mathscr{C} \rightleftarrows \mathscr{C}^{\Gamma}_{\mathrm{Inj}} :\varprojlim$ *is a Quillen adjunction.*

Proof. Apply Proposition 2.1 to the functor $\Gamma \to *$. □

2.5. Proposition (Lurie 2009, Proposition A.2.8.2).
If $\mathscr{C}$ *is a combinatorial model category, both the projective and injective model structures on* $\mathscr{C}^{\Gamma}$ *exist and are combinatorial.*

Given a generating set of (trivial) cofibrations in $\mathscr{C}$*, the corresponding* simple *(trivial) cofibrations in* $\mathscr{C}^{\Gamma}_{\mathrm{Proj}}$*, for all choices of* $\gamma_0 \in \Gamma$*, form a generating set of cofibrations.*

Sketch of proof. For the projective model structure, one checks by hand that simple (trivial) cofibrations have the left lifting property with respect to all degreewise fibrations (trivial fibrations). One then checks that the mentioned simple (trivial) cofibrations form a generating set. The injective model structure, on the other hand, requires more work and has a less explicit set of generating cofibrations. □

2.6. Proposition (ibid., Remark A.2.8.6).
A Quillen adunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :G$ *between combinatorial model categories induces Quillen adjunctions*

$$\mathscr{C}^{\Gamma}_{\mathrm{Proj}} \rightleftarrows \mathscr{D}^{\Gamma}_{\mathrm{Proj}} \qquad \textit{and} \qquad \mathscr{C}^{\Gamma}_{\mathrm{Inj}} \rightleftarrows \mathscr{D}^{\Gamma}_{\mathrm{Inj}}$$

which are Quillen equivalences if (F, G) *are.*

3 The Reedy model structure

A third approach exists to equip diagram categories $\mathscr{C}^{\Gamma}$ with a model structure, provided the category Γ has the structure of a Reedy category. Remarkably, unlike the projective and injective cases, this does not rely on any internal structure of $\mathscr{C}$.

A category Γ is called **Reedy** if it contains two subcategories $\Gamma_+, \Gamma_- \subset \Gamma$, each containing all objects, such that

- there exists a degree function $\mathrm{Ob}\,\Gamma \to \mathbb{Z}$, such that non-identity morphisms from Γ_+ strictly raise the degree and non-identity morphisms from Γ_- strictly lower the degree (more generally, an ordinal number can be used instead of $\mathbb{Z}$);
- each morphism $f \in \Gamma$ factors *uniquely* as $f = gh$ for $g \in \Gamma_+$ and $h \in \Gamma_-$.

We note that a direct category is Reedy with $\Gamma_+ = \Gamma$, and that an inverse category is Reedy with $\Gamma_- = \Gamma$.

3.1. Remark. If Γ is Reedy, then so is Γ^{op}, with $(\Gamma^{\mathrm{op}})_+ = (\Gamma_-)^{\mathrm{op}}$ and $(\Gamma^{\mathrm{op}})_- = (\Gamma_+)^{\mathrm{op}}$. △

3.2. Example. The simplex category $\mathbf{\Delta}$ is Reedy with $\mathbf{\Delta}_+$ consisting of injective maps and $\mathbf{\Delta}_-$ consisting of surjective maps. The degree function does the obvious thing, $[n] \mapsto n$. ○

If Γ is a Reedy category and $\mathscr{C}$ is any model category, and if $F \in \mathscr{C}^\Gamma$ is a diagram, we define the **latching** and **matching objects** by

$$L_\gamma F = \varinjlim_{(\alpha \underset{\neq}{\to} \gamma) \in \Gamma_+} F(\alpha) \qquad \text{and} \qquad M_\gamma F = \varprojlim_{(\gamma \underset{\neq}{\to} \alpha) \in \Gamma_-} F(\alpha).$$

In other words, the limit (resp., colimit) runs over the category of all *non-identity* maps $\alpha \to \gamma$ in Γ_+ (resp., $\gamma \to \alpha$ in Γ_-). The **latching map** is the canonical map $L_\gamma F \to F(\gamma)$, and the **matching map** is the canonical map $F(\gamma) \to M_\gamma F$.

If $f : F \to G$ is a map in $\mathscr{C}^\Gamma$, then the **relative latching map** is the map

$$F(\gamma) \coprod_{L_\gamma F} L_\gamma G \longrightarrow G(\gamma)$$

given by the universal property of the pushout. We say that f is a **(trivial) Reedy cofibration** if the relative latching map is a (trivial) cofibration in $\mathscr{C}$. If $F = \varnothing$, we recover the latching map. Dually, the **relative matching map** is the map

$$F(\gamma) \longrightarrow G(\gamma) \prod_{M_\gamma G} M_\gamma F$$

given by the universal property of the pullback. We say that f is a **(trivial) Reedy fibration** if the relative matching map is a (trivial) fibration in $\mathscr{C}$. If $F = *$, we recover the matching map.

3.3. Proposition (Hirschhorn 2003, Theorem 15.3.4).
*If $\mathscr{C}$ is an arbitrary model category and Γ is a Reedy category, then this defines a model structure on $\mathscr{C}^\Gamma$, called the **Reedy model structure**. The weak equivalences are componentwise weak equivalences. We shall write $\mathscr{C}^\Gamma_{\text{Reedy}}$ when we equip the diagram category with this model structure.*

3.4. Proposition (Lurie 2009, Example A.2.9.22).
Let $\mathscr{C}$ be a model category and Γ a Reedy category. Then

(i) *If $\Gamma = \Gamma_+$ is a direct category, the projective model structure $\mathscr{C}^\Gamma_{\text{Proj}}$ exists and coincides with the Reedy model structure.*

(ii) *If $\Gamma = \Gamma_-$ is an inverse category, the injective model structure $\mathscr{C}^\Gamma_{\text{Inj}}$ exists and coincides with the Reedy model structure.*

Furthermore, a map $f : F \to G$ in $\mathscr{C}^\Gamma$ is a

(iii) *(trivial) cofibration if and only if the restriction $f|_{\Gamma_+} : F|_{\Gamma_+} \to G|_{\Gamma_+}$ is a (trivial) projective cofibration in $\mathscr{C}^{\Gamma_+}_{\text{Proj}}$.*

(iv) *(trivial) fibration if and only if the restriction $f|_{\Gamma_-} : F|_{\Gamma_-} \to G|_{\Gamma_-}$ is a (trivial) injective fibration in $\mathscr{C}^{\Gamma_-}_{\text{Inj}}$.*

3.5. Proposition (Hirschhorn 2003, Theorem 15.5.2).
If Γ and Γ' are both Reedy categories, then so is $\Gamma \times \Gamma'$, and the three possible Reedy model structures one can put on $\mathscr{C}^{\Gamma\times\Gamma'}$ agree, i.e.

$$\mathscr{C}^{\Gamma\times\Gamma'}_{\mathrm{Reedy}} = \left(\mathscr{C}^{\Gamma}_{\mathrm{Reedy}}\right)^{\Gamma'}_{\mathrm{Reedy}} = \left(\mathscr{C}^{\Gamma'}_{\mathrm{Reedy}}\right)^{\Gamma}_{\mathrm{Reedy}}.$$

4 Homotopy limits

The following theorem is the basis for all our homotopy limit formulae:

4.1. Theorem. *Let $\mathscr{C}$ be a model category and Γ a category. Regard the functor category $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma}$ as a model category in any of the following ways:*

(i) *as $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} = (\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Proj}})^{\Gamma}_{\mathrm{Inj}}$ (assuming this model structure exists);*

(ii) *as $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} = (\mathscr{C}^{\Gamma}_{\mathrm{Proj}})^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}}$ (assuming this model structure exists);*

(iii) *as $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} = \mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma}_{\mathrm{Reedy}}$ (assuming Γ is Reedy).*

Then the end functor $\int_{\Gamma}\colon \mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} \to \mathscr{C}$ is right Quillen.

Proof. We initially prove the first statement and obtain the second one by duality. By Proposition 1.3, it suffices to check that the left adjoint $\coprod_{\mathrm{Hom}_\Gamma}$ takes (trivial) cofibations in $\mathscr{C}$ to (trivial) cofibations in $(\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Proj}})^{\Gamma}_{\mathrm{Inj}}$. If $c \to c'$ is a (trivial) cofibration in $\mathscr{C}$, then we must therefore consider the map $\coprod_{\Gamma(-,-)} c \to \coprod_{\Gamma(-,-)} c'$ in $(\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Proj}})^{\Gamma}_{\mathrm{Inj}}$. Checking that this is a (trivial) injective cofibration over Γ amounts, by definition, to checking this componentwise. But for a fixed $\gamma_0 \in \Gamma$, this component is $\coprod_{\Gamma(-,\gamma_0)} c \to \coprod_{\Gamma(-,\gamma_0)} c'$, which is a simple (trivial) projective cofibration in $\mathscr{C}^{\Gamma^{\mathrm{op}}}$.

For the Reedy case, we recall from Propositions 3.4 and 3.5 that being a (trivial) cofibration in the model category $\mathscr{C}^{\Gamma\times\Gamma^{\mathrm{op}}}_{\mathrm{Reedy}} = (\mathscr{C}^{\Gamma}_{\mathrm{Reedy}})^{\Gamma^{\mathrm{op}}}_{\mathrm{Reedy}}$ is equivalent to the restriction being a (trivial) cofibration in

$$\left(\mathscr{C}^{\Gamma_+}_{\mathrm{Proj}}\right)^{(\Gamma^{\mathrm{op}})_+}_{\mathrm{Proj}} = \left(\mathscr{C}^{\Gamma_+}_{\mathrm{Proj}}\right)^{(\Gamma_-)^{\mathrm{op}}}_{\mathrm{Proj}}.$$

But we have, by the unique factorization property of Reedy categories, that

$$\coprod_{\Gamma(-,-)} c = \coprod_{\gamma_0\in\Gamma} \coprod_{\Gamma_-(-,\gamma_0)} \coprod_{\Gamma_+(\gamma_0,-)} c$$

for any $c \in \mathscr{C}$. These consist of coproducts of exactly the same form as the ones appearing in the definition of simple (trivial) projective cofibrations (Corollary 2.3(i)). Thus we find that for any (trivial) cofibration $c \to c'$ in $\mathscr{C}$, the map $\coprod_{\Gamma(-,-)} c \to \coprod_{\Gamma(-,-)} c'$ is a (trivial) cofibration in $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma}_{\mathrm{Reedy}}$. □

Thus we can derive the end using any of these three model structures, when available. Write $\mathbb{R}\!\int_{\Gamma}\colon \mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma} \to \mathscr{C}$ for the derived functor, which we shall call the **homotopy end**.

4.2. Corollary. *If $\mathscr{C}$ is a combinatorial model category and Γ a category, then for a diagram $F \in \mathscr{C}^\Gamma$,*

$$\underleftarrow{\operatorname{holim}}_\Gamma F = \mathbb{R}\!\int_\Gamma F = \int_\Gamma R(F),$$

where R is a fibrant replacement with respect to the model structure $(\mathscr{C}^\Gamma_{\mathrm{Proj}})^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}}$ or, if Γ is Reedy, in $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma}_{\mathrm{Reedy}}$.

Proof. First write $\underleftarrow{\operatorname{holim}} F = \underleftarrow{\lim} R_\Gamma(F)$ for some fibrant replacement functor R_Γ in $\mathscr{C}^\Gamma_{\mathrm{Inj}}$. Now Corollary 2.4(i) and Proposition 2.6 show that the constant functor embedding $\mathscr{C}^\Gamma_{\mathrm{Inj}} \hookrightarrow (\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Proj}})^\Gamma_{\mathrm{Inj}}$ is right Quillen and thus preserves fibrant objects. Thus $R_\Gamma(F)$ is also fibrant in $(\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Proj}})^\Gamma_{\mathrm{Inj}}$. This proves the first equality sign. The second one is clear. □

Of course, even though F as a diagram in $\mathscr{C}^{\Gamma^{\mathrm{op}}\times\Gamma}$ was constant with respect to the first variable, $R(F)$ is in general not. Remarkably, since ends calculate naturality between the two variables, this often makes calculations of homotopy limits more manageable, compared to resolving the diagram inside $\mathscr{C}^\Gamma_{\mathrm{Inj}}$.

4.3. Corollary. *Suppose Γ is a direct category, and let $R\colon \mathscr{C} \to \mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}}$ be a functor that takes $c \in \mathscr{C}$ to a fibrant replacement of the constant diagram at c. Then*

$$\underleftarrow{\operatorname{holim}}_\Gamma F = \int_{\gamma\in\Gamma} R(F(\gamma))(\gamma).$$

Proof. Clearly, $R(F)$ is a fibrant replacement inside $(\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}})^\Gamma_{\mathrm{Proj}}$. By Propositions 3.4 and 3.5, this model category is equal to

$$\left(\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}}\right)^\Gamma_{\mathrm{Proj}} = \left(\mathscr{C}^{\Gamma^{\mathrm{op}}}_{\mathrm{Reedy}}\right)^\Gamma_{\mathrm{Reedy}} = \left(\mathscr{C}^{\Gamma}_{\mathrm{Reedy}}\right)^{\Gamma^{\mathrm{op}}}_{\mathrm{Reedy}} = \left(\mathscr{C}^{\Gamma}_{\mathrm{Proj}}\right)^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}},$$

so the result follows from Theorem 4.1. □

5 Bousfield–Kan formula

In Hirschhorn (ibid., chapter 19), homotopy limits are being developed for arbitrary model categories via a machinery of simplicial resolutions. In this section, we use Theorem 4.1/Corollary 4.2 to explain why this machinery works. Throughout, we denote by SSet the category of simplicial sets endowed with the Quillen model structure.

If $\mathscr{C}$ is a (complete) category and $X_\bullet \in \mathscr{C}^{\Delta^{\mathrm{op}}}$ a simplicial diagram in $\mathscr{C}$, we may extend $X_\bullet$ to a continuous functor $X\colon \mathsf{SSet}^{\mathrm{op}} \to \mathscr{C}$ via the right Kan extension along the Yoneda embedding $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathsf{SSet}^{\mathrm{op}}$:

$$X^K = \underleftarrow{\lim}_{\Delta^n\to K} X_n, \qquad K \in \mathsf{SSet}.$$

If $\mathscr{C}$ is a model category, the matching object at $[n]$ is $M_n X_\bullet = X^{\partial\Delta^n}$, and so $X_\bullet$ being Reedy-fibrant is equivalent to the map $X_n = X^{\Delta^n} \to X^{\partial\Delta^n}$ being a fibration in $\mathscr{C}$ for all n.

5.1. Theorem (Bousfield–Kan formula). *Suppose $\mathscr{C}$ is a combinatorial model category, Γ a category, and $F \in \mathscr{C}^\Gamma$. Let $R\colon \mathscr{C} \to \mathscr{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ be a functor that takes $c \in$*

$\mathscr{C}$ *to a Reedy-fibrant replacement of the constant* $\mathbf{\Delta}^{\mathrm{op}}$*-diagram at* c*. Let furthermore* $K \in \mathsf{SSet}^{\Gamma}_{\mathrm{Proj}}$ *be a projectively cofibrant resolution of the point. Then*

$$\underleftarrow{\mathrm{holim}}_{\Gamma} F = \int_{\gamma \in \Gamma} R(F(\gamma))^{K(\gamma)}.$$

One may prove (see e.g. Hirschhorn 2003, Proposition 14.8.9) that the diagram $K(-) = N(\Gamma/-) \in \mathsf{SSet}^{\Gamma}_{\mathrm{Proj}}$, taking γ to the nerve $N(\Gamma/\gamma)$ of the comma category Γ/γ of all maps in Γ with codomain γ, is a projectively cofibrant resolution of the point. Thus we have

$$\underleftarrow{\mathrm{holim}}_{\Gamma} F = \int_{\gamma \in \Gamma} R(F(\gamma))^{N(\Gamma/\gamma)}, \tag{5.2}$$

which is the classical form of the Bousfield–Kan formula.

The proof relies on the following standard lemma:

5.3. Lemma (Hovey 1999, Proposition 3.6.8).
Let $\mathscr{C}$ *be a model category and* $F\colon \mathsf{SSet} \to \mathscr{C}$ *a functor preserving colimits and cofibrations. Then* F *preserves trivial cofibrations if and only if* $F(\Delta^n) \to F(\Delta^0)$ *is a weak equivalence for all* n*.*

Proof of Theorem 5.1. Clearly, $R(F(-))_{\bullet}$ is a fibrant replacement of F with respect to the model structure $(\mathscr{C}^{\mathbf{\Delta}^{\mathrm{op}}}_{\mathrm{Reedy}})^{\Gamma}_{\mathrm{Proj}}$. The theorem will follow if we prove that $R(F(-))^{K(-)}$ is a fibrant replacements of F in $(\mathscr{C}^{\Gamma}_{\mathrm{Proj}})^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}}$. This will follow from Ken Brown's Lemma if we prove that the continuous functor

$$\left(\mathsf{SSet}^{\Gamma}_{\mathrm{Proj}}\right)^{\mathrm{op}} \longrightarrow \left(\mathscr{C}^{\Gamma}_{\mathrm{Proj}}\right)^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}}, \qquad K(-) \longmapsto R(F(-))^{K(-)},$$

takes opposites of (trivial) cofibrations to (trivial) fibrations. (Trivial) cofibrations in $\mathsf{SSet}^{\Gamma}_{\mathrm{Proj}}$ are generated from *simple* (trivial) projective cofibrations via pushouts and retracts, c.f. Proposition 2.5. Thus by continuity of the functor, it suffices to prove the statement for *simple* (trivial) cofibrations. We therefore let $\coprod_{\Gamma(\gamma_0,-)} K \hookrightarrow \coprod_{\Gamma(\gamma_0,-)} L$ be one such, where $K \hookrightarrow L$ is a (trivial) cofibration and $\gamma_0 \in \Gamma$. This is mapped to

$$\prod_{\Gamma(\gamma_0,-)} R(F(-))^L \to \prod_{\Gamma(\gamma_0,-)} R(F(-))^K.$$

Thus we must show that the composition

$$\mathsf{SSet}^{\mathrm{op}} \xrightarrow{\prod_{\Gamma(\gamma_0,-)}} \left(\mathsf{SSet}^{\Gamma^{\mathrm{op}}}_{\mathrm{Proj}}\right)^{\mathrm{op}} \longrightarrow \left(\mathscr{C}^{\Gamma}_{\mathrm{Proj}}\right)^{\Gamma^{\mathrm{op}}}_{\mathrm{Inj}}, \qquad K \longmapsto \prod_{\Gamma(\gamma_0,-)} R(F(-))^K,$$

takes (trivial) cofibrations to (trivial) fibrations. Checking that it takes cofibrations to fibrations amounts to checking this for the generating cofibrations $\partial\Delta^n \hookrightarrow \Delta^n$ in SSet. This holds by the assumption that $R(F(-))_{\bullet}$ is componentwise Reedy-fibrant. Since the functor takes colimits to limits, the claim now follows from the (dual of) the lemma. □

6 Homotopy-initial functors

A functor $f\colon \Gamma \to \Gamma'$ is called **homotopy-initial** if for all objects $\gamma' \in \Gamma'$, the nerve $N(f/\gamma')$ is contractible as a simplicial set; here f/γ' denotes the comma category whose objects are pairs (γ, α) where α is a map $f(\gamma) \to \gamma'$. A morphism $(\gamma_1, \alpha) \to (\gamma_2, \alpha)$ is a morphism $\gamma_1 \to \gamma_2$ in Γ making the diagram

$$\begin{array}{ccc} f(\gamma_1) & \searrow & \\ \downarrow & & \gamma' \\ f(\gamma_2) & \nearrow & \end{array}$$

commute. We aim to reprove the statement

6.1. Theorem (Hirschhorn 2003, Theorem 19.6.7).
Suppose $\mathscr{C}$ is a combinatorial model category and Γ, Γ' two categories. If $f: \Gamma \to \Gamma'$ is homotopy-initial, then we have

$$\underleftarrow{\operatorname{holim}}_{\Gamma'} F = \underleftarrow{\operatorname{holim}}_{\Gamma} f^* F$$

for all $F \in \mathscr{C}^{\Gamma'}$.

This relies on a few technical lemmas:

6.2. Lemma. *If $f: \Gamma \to \Gamma'$ is a functor, then $f_! N(\Gamma/-) = N(f/-) \in \mathsf{SSet}^{\Gamma'}$. In particular, since $N(\Gamma/-) \in \mathsf{SSet}^{\Gamma}_{\mathrm{Proj}}$ is cofibrant, $N(f/-) \in \mathsf{SSet}^{\Gamma'}_{\mathrm{Proj}}$ is cofibrant by Proposition 2.1.*

Proof. Since colimits in diagram categories over cocomplete categories can be checked componentwise, this boils down to the observation

$$\varinjlim_{f(\gamma) \to \gamma'} N(\Gamma/\gamma)_n = N(f/\gamma')_n. \qquad \square$$

The following lemma is inspired by Hirschhorn (ibid., Proposition 19.6.6). See also Riehl (2014, Lemma 8.1.4).

6.3. Lemma.
Suppose that $\mathscr{C}$ is a complete category and that Γ and Γ' are two categories with a functor $f: \Gamma \to \Gamma'$. Then we have

$$\int_{\gamma \in \Gamma} F(f(\gamma))^{N(\Gamma/\gamma)} = \int_{\gamma' \in \Gamma'} F(\gamma')^{N(f/\gamma')}$$

for $F \in (\mathscr{C}^{\Delta^{\mathrm{op}}})^{\Gamma}$ (see the previous chapter for an explanation of the power notation).

Proof. For the purpose of the proof, we recall that the Kan extension formulas in (2.2) may be equivalently written in terms of (co)ends:

$$f_! F(\gamma') = \int^{\gamma \in \Gamma} \Gamma'(f(\gamma), \gamma') \times F(\gamma) \qquad \text{and} \qquad f_* F(\gamma') = \int_{\gamma \in \Gamma} F(\gamma)^{\Gamma'(\gamma', f(\gamma))}.$$

Here we are using the natural copowering and powering of Set on $\mathscr{C}$, given by $S \times c = \coprod_S c$ and $c^S = \prod_S c$ for $S \in \mathsf{Set}$ and $c \in \mathscr{C}$, which make sense whenever $\mathscr{C}$ is complete resp. cocomplete. We shall furthermore make use of the so-called "co-Yoneda lemma" which says that

$$G(f(\gamma)) = \int_{\gamma' \in \Gamma'} G(\gamma')^{\Gamma'(f(\gamma), \gamma')} \qquad \text{for all } G \in \mathscr{C}^{\Gamma'}.$$

Finally, we use "Fubini's theorem" for ends, which says that ends, being limits, commute. This together yields

$$\begin{aligned}
\int_{\gamma\in\Gamma} F(f(\gamma))^{N(\Gamma/\gamma)} &= \int_{\gamma\in\Gamma}\int_{[n]\in\mathbf{\Delta}} F(f(\gamma))_n^{N(\Gamma/\gamma)_n}\\
&= \int_{\gamma\in\Gamma}\int_{[n]\in\mathbf{\Delta}}\int_{\gamma'\in\Gamma'} \Big(F(\gamma')_n^{\Gamma'(f(\gamma),\gamma')}\Big)^{N(\Gamma/\gamma)_n}\\
&= \int_{\gamma\in\Gamma}\int_{[n]\in\mathbf{\Delta}}\int_{\gamma'\in\Gamma'} F(\gamma')_n^{\Gamma'(f(\gamma),\gamma')\times N(\Gamma/\gamma)_n}\\
&= \int_{[n]\in\mathbf{\Delta}}\int_{\gamma'\in\Gamma'} F(\gamma')_n^{\int^{\gamma\in\Gamma}\Gamma'(f(\gamma),\gamma')\times N(\Gamma/\gamma)_n}\\
&= \int_{\gamma'\in\Gamma'} F(\gamma')_n^{f_!N(\Gamma/-)(\gamma')} = \int_{\gamma'\in\Gamma'} F(\gamma')^{N(f/\gamma')}
\end{aligned}$$

where the last equality sign is due to Lemma 6.2. □

Proof of Theorem 6.1. Theorem 5.1 and equation (5.2) show that

$$\underleftarrow{\operatorname{holim}}_{\Gamma} f^*F = \int_{\gamma'\in\Gamma'} R(F(\gamma'))^{N(f/\gamma')}.$$

Since $N(f/\gamma')$ is contractible for all γ', $N(f/-)$ is a projectively cofibrant resolution of the point by Lemma 6.2. Thus the right-hand side is exactly $\underleftarrow{\operatorname{holim}}_{\Gamma'} F$ by Theorem 5.1. □

6.4. Example: Fat totalization formula.
Recall from Example 3.2 that the simplex category $\mathbf{\Delta}$ is Reedy with $\mathbf{\Delta}_+$ being the subcategory containing only injective maps. The inclusion $\iota\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}$ is homotopy-initial (see e.g. Riehl 2014, Example 8.5.12 or Dugger 2008, Example 21.2), hence $\underleftarrow{\operatorname{holim}}_{\mathbf{\Delta}} X^\bullet = \underleftarrow{\operatorname{holim}}_{\mathbf{\Delta}_+} X^\bullet$ for all $X^\bullet \in \mathscr{C}^{\mathbf{\Delta}}$. As $\mathbf{\Delta}_+$ is a direct category, we obtain from Corollary 4.3 that we may calculate $\underleftarrow{\operatorname{holim}}_{\mathbf{\Delta}} X^\bullet$ as

$$\underleftarrow{\operatorname{holim}}_{\mathbf{\Delta}} X^\bullet = \int_{\mathbf{\Delta}_+} R(X^n)_n$$

for some functor $R\colon \mathscr{C} \to \mathscr{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ that takes x to an injectively (i.e. Reedy-) fibrant replacement of the constant diagram at x. This is the so-called **fat totalization** formula for homotopy limits over $\mathbf{\Delta}$. The dual formula for homotopy colimits over $\mathbf{\Delta}^{\mathrm{op}}$ is called the **fat geometric realization** formula. ○

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