NON-REDUCTIVE GEOMETRIC INVARIANT THEORY AND HYPERBOLICITY

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Abstract. The Green–Griffiths–Lang and Kobayashi hyperbolicity conjectures for generic hypersurfaces of polynomial degree are proved using intersection theory for non-reductive geometric invariant theoretic quotients and recent work of Riedl and Yang.

1. Introduction

The goal of the present paper is to apply a recently developed extension ([BDHK, BK]) of geometric invariant theory (GIT) to non-reductive actions to study hyperbolicity of generic hypersurfaces in a projective space. We use the results of [BDHK] to construct new compactifications of the invariant jet differentials bundles over complex manifolds. Intersection theory developed in [BK] for non-reductive GIT quotients, combined with the strategy of [DMR10], leads us to a proof of the Green–Griffiths–Lang conjecture for a generic projective hypersurface whose degree is bounded below by a polynomial in its dimension. A recent result of Riedl and Yang [RY] then implies the polynomial Kobayashi conjecture. These results are significant improvements of the earlier known degree bounds from $(\sqrt{n}\log n)^n$ to $16n^5(5n+4)$ in the GGL conjecture and from $(n\log n)^n$ to $16(2n-1)^5(10n-1)$ in the Kobayashi conjecture.

A projective variety X is called Brody hyperbolic if there is no non-constant entire holomorphic curve in X, i.e. any holomorphic map $f : \mathbb{C} \to X$ must be constant. Hyperbolic algebraic varieties have attracted considerable attention, in part because of their conjectured diophantine properties. For instance, Lang [Lan86] has conjectured that any hyperbolic complex projective variety over a number field K can contain only finitely many rational points over K. In 1970 Kobayashi [Kob70] formulated the following conjecture.

Conjecture 1.1 (**Kobayashi conjecture, 1970**). *A very general hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d is Brody hyperbolic if d is sufficiently large. Moreover, the complement $\mathbb{P}^{n+1} \setminus X$ is also Brody hyperbolic for large enough d.*

This conjecture has become a landmark in the field and has been the subject of intense study [Den, Bro17, B19, B18, Dar15]. For more details on recent results see the survey papers [Dem18, DR11]. Siu [Siu15] and Brotbek [Bro17] proved the Kobayashi hyperbolicity of projective hypersurfaces of sufficiently high (but not effective) degree, and effective degree bounds were worked out by Deng [Den] and Demailly [Dem18]. The conjectured optimal bound for d is $d \geq 2n+1$, but the best known bound was $(n\log n)^n$ by Merker and The-Anh Ta [MT].

A related, but stronger, conjecture is the Green–Griffiths–Lang (GGL) conjecture formulated in 1979 by Green and Griffiths [GG80] and in 1986 by Lang [Lan86].

Conjecture 1.2 (**Green-Griffiths-Lang conjecture, 1979**). *Any projective algebraic variety X of general type contains a proper algebraic subvariety $Y \subsetneq X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subseteq Y$.*

In particular, a projective hypersurface $X \subseteq \mathbb{P}^{n+1}$ is of general type if $\deg(X) \geq n+3$. A positive answer to the GGL conjecture has been given for surfaces by McQuillan [McQ98] under the assumption that the second Segre number $c_1^2 - c_2$ is positive. Siu [Siu02, Siu04, SY96, Siu15] and Demailly [Dem97] developed a powerful strategy to approach the conjecture for generic hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of high degree. Following this strategy, combined with techniques of Demailly [Dem97], the first effective lower bound for the degree of a generic hypersurface in the GGL conjecture was given by Diverio, Merker and Rousseau [DMR10], where the conjecture for generic projective hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of degree $\deg(X) > 2^{n^5}$ was confirmed. In Bérczi [B19] the first author introduced equivariant localisation on the Demailly–Semple tower and adapted the argument of [DMR10] to improve this lower bound to $\deg(X) > n^{8n}$. The residue formula of [B19] was later studied and further analyzed by Darondeau [Dar15]. The current best bound for the Green–Griffiths–Lang Conjecture, due to Demailly [Dem18], is $\deg(X) > \frac{n^4}{3}(n\log(n\log(24n)))^n$, and more recently Merker and The-Anh Ta [MT] achieved $\deg(X) > (\sqrt{n}\log n)^n$ by deeper study of the formula of [B19].

In this paper we replace the Demailly–Semple bundle with a computationally more efficient algebraic model coming from non-reductive geometric invariant theory [BDHK] and apply the equivariant intersection theory developed in [BK] to prove

Theorem 1.3 (**Polynomial Green-Griffiths-Lang theorem**). *Let $X \subseteq \mathbb{P}^{n+1}$ be a generic smooth projective hypersurface of degree $\deg(X) \geq 16n^5(5n+4)$. Then there is a proper algebraic subvariety $Y \subsetneq X$ containing all nonconstant entire holomorphic curves in X.*

Recently Riedl and Yang [RY] proved the following beautiful statement: if there are integers d_n for all positive n such that the GGL conjecture for hypersurfaces of dimension n holds for degree at least d_n then the Kobayashi conjecture is true for hypersurfaces with degree at least d_{2n-1}. Using this, Theorem 1.3 immediately implies

Theorem 1.4 (**Polynomial Kobayashi theorem**). *A generic smooth projective hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $\deg(X) \geq 16(2n-1)^5(10n-1)$ is Brody hyperbolic.*

The strategy of Demailly and Siu is based on first establishing algebraic degeneracy of holomorphic curves $f : \mathbb{C} \to X$, in the sense of proving the existence of certain polynomial differential equations $P(f, f', \ldots, f^{(k)}) = 0$ of some order k, and then finding enough such equations so that they cut out a proper algebraic locus $Y \subsetneq X$. The central tool for finding polynomial differential equations is the study of the bundle J_kX of k-jets of germs of holomorphic curves $f : \mathbb{C} \to X$ over X, and the associated Green–Griffiths bundles $E_{k,m}^{GG} = \mathcal{O}(J_kX)$ of algebraic differential operators whose elements are polynomial functions $Q(f', \ldots, f^{(k)})$ of weighted degree m in $f, f', \ldots, f^{(k)}$. In [Dem97] Demailly introduced the subbundles $E_{k,m} \subseteq E_{k,m}^{GG}$ of jet differentials which are (semi-)invariant under reparametrisation of the source $\mathbb{C}$. The group $\mathrm{Diff}_k(1)$ of k-jets of reparametrisation germs $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ at the origin acts fibrewise on J_kX and $\oplus_{m=1}^{\infty} E_{k,m} = \mathcal{O}(J_kX)^{U_k}$ is the graded algebra of jet differentials which are invariant under the maximal unipotent subgroup U_k of $\mathrm{Diff}_k(1)$. This bundle gives a better reflection of the geometry of entire curves in X, since it only depends on the images of such curves and not on their parametrisations. However, it also comes with a technical difficulty: the reparametrisation group $\mathrm{Diff}_k(1)$ is not reductive, and so the classical geometric invariant theory of Mumford [MFK94] cannot be applied to study the invariants and construct a compactification of a quotient of a nonempty open subset of J_kX by $\mathrm{Diff}_k(1)$ (see Bérczi and Kirwan [BK12], Doran and Kirwan [DK07]). Until recently, there existed only two different constructions for the compactification of these quotients.

(1) In [Dem97] Demailly describes a smooth compactification of an open subset of $J_kX/\mathrm{Diff}_k(1)$ as a tower of projectivised bundles on X — the Demailly–Semple bundle — endowed with tautological line bundles whose sections are $\mathrm{Diff}_k(1)$-invariants. Global sections of properly chosen twisted tautological line bundles over the Demailly–Semple bundle give algebraic differential equations of degree k. This model was extensively and successfully used in the past few decades, and it has a wast literature in hyperbolicity questions, see also [Dem18, DR11]. The main numerical breakthrough in the Green-Griffiths-Lang conjecture using the Demailly-Semple tower was achieved in [DMR10], where the first effective bound for the degree of a generic projective hypersurface was calculated. However, as it was pointed out in [B19], we cannot expect better than exponential bound in the GGL conjecture using the Demailly-Semple model.
(2) In [B18] the first author shows that the curvilinear component of the punctual Hilbert scheme of k points on $\mathbb{C}^n$ provides natural compactifications of the fibres of $J_kX/\mathrm{Diff}_k(1)$ over X. Sections of the tautological bundle give invariant jet differentials, and equivariant localisation developed in [BS12] gives information on the intersection theory of this curvilinear component. In [B18] the first author shows that the GGL conjecture for generic hypersurfaces with polynomial degree follows from a classical positivity conjecture of Rimányi [Rim01] for Thom polynomials. However, Rimányi's conjecture is currently out of reach.

The key idea of the present paper is to replace these existing models with a completely new construction, coming from the recently developed theory of non-reductive quotients [BDHK, BDHK18]. We construct a projective non-reductive GIT quotient $J_kX//\mathrm{Diff}_k(1)$ endowed with a tautological line bundle whose sections give (semi-) invariant jet differentials. Following Diverio–Merker–Rousseau [DMR10] (with the better pole order of Darondeau [Dar16] for slanted vector fields) and using holomorphic Morse inequalities we deduce the existence of these global sections from the positivity of a well-defined tautological integral over $J_kX//\mathrm{Diff}_k(1)$.

The second key ingredient of the present paper is the cohomological intersection theory developed in [BK], which allows us to prove the positivity of this integral at the critical order $k = n$ for hypersurfaces with polynomial degree.

2. Jet differentials

The central object of this paper is the algebra of (semi-) invariant jet differentials under reparametrisation of the source space $\mathbb{C}$. For more details see the survey papers of Demailly [Dem97] and Diverio–Rousseau [DR11].

2.1. **Jets of holomorphic maps.** If u, v are positive integers let $J_k(u, v)$ denote the vector space of k-jets of holomorphic maps $(\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$ at the origin; that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \ldots, k$. This is a finite-dimensional complex vector space, which can be identified with $J_k(u, 1) \otimes \mathbb{C}^v$; hence $\dim J_k(u, v) = v\binom{u+k}{k} - v$. We will call the elements of $J_k(u, v)$ *map-jets of order* k, or simply map-jets.

Eliminating the terms of degree $k + 1$ results in a surjective algebra homomorphism $J_k(u, 1) \twoheadrightarrow J_{k-1}(u, 1)$, and the sequence of such surjections $J_k(u, 1) \twoheadrightarrow J_{k-1}(u, 1) \twoheadrightarrow \ldots \twoheadrightarrow J_1(u, 1)$ induces an increasing filtration of $J_k(u,1)^*$:

$$J_1(u,1)^* \subseteq J_2(u,1)^* \subseteq \ldots \subseteq J_k(u,1)^*. \tag{1}$$

Using the standard coordinates on $\mathbb{C}^u$ and $\mathbb{C}^v$, a k-jet $f \in J_k(u,v)$ can be identified with its set of derivatives at the origin, the vector $(f'(0), f''(0), \ldots, f^{(k)}(0))$, where $f^{(j)}(0) \in \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v)$. Here Sym^l denotes the symmetric tensor product. In this way we get an isomorphism

$$J_k(u,v) \simeq J_k(u,1) \otimes \mathbb{C}^v \simeq \oplus_{j=1}^k \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v). \tag{2}$$

Map-jets can be composed via substitution and elimination of terms of degree greater than k, leading to the composition map

$$J_k(u,v) \times J_k(v,w) \to J_k(u,w), \quad (\Psi_1, \Psi_2) \mapsto \Psi_2 \circ \Psi_1 \text{ modulo terms of degree} > k\,. \tag{3}$$

When $k = 1$, we can identify $J_1(u,v)$ with the space of u-by-v matrices, and (3) reduces to multiplication of matrices.

We will call a jet $\gamma \in J_k(u,v)$ *regular* if $\gamma'(0)$ is has maximal rank, and we will use the notation $J_k^{\mathrm{reg}}(u,v)$ for the set of regular maps. When $u = v$ we get a group

$$\mathrm{Diff}_k(u) = J_k^{\mathrm{reg}}(u,u)$$

which we will call the *k-jet diffeomorphism group*.

2.2. **Jet bundles and differential operators.** Let X be a smooth projective variety of dimension n. Following Green and Griffiths [GG80] we let $J_kX \to X$ be the bundle of k-jets of germs of parametrised curves in X; that is, J_kX is the of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}, 0) \to (X, p)$, where the equivalence relation $\sim$ is such that $f \sim g$ if and only if the derivatives $f^{(j)}(0)$ and $g^{(j)}(0)$ are equal for $0 \le j \le k$ when computed in some local holomorphic coordinate system on an open neighbourhood of $p \in X$. The projection map $J_kX \to X$ is given by $f \mapsto f(0)$, and the elements of the fibre J_kX_p can be represented by Taylor expansions

$$f(t) = p + tf'(0) + \frac{t^2}{2!}f''(0) + \ldots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

up to order k at $t = 0$ of $\mathbb{C}^n$-valued maps $f = (f_1, f_2, \ldots, f_n)$ on open neighbourhoods of 0 in $\mathbb{C}$. Locally in these coordinates elements of the fibre J_kX_p can be identified with k-tuples of vectors $(f'(0), \ldots, f^{(k)}(0)/k!) \in (\mathbb{C}^n)^k$, so the fibre can be identified with $J_k(1,n)$.

Note that J_kX is not a vector bundle over X since the transition functions are polynomial but not linear, see §5 of Demailly [Dem97]. In fact, let Diff_X denote the principal $\mathrm{Diff}_k(n)$-bundle over X formed by all local polynomial coordinate systems on X. Then

$$J_kX = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} J_k(1,n).$$

is the associated bundle whose structure group is $\mathrm{Diff}_k(n)$.

Let $J_k^{\mathrm{reg}}X$ denote the bundle of k-jets of germs of parametrised curves $f : \mathbb{C} \to X$ in X which are regular in the sense that they have nonzero first derivative $f' \neq 0$. After fixing local coordinates near $p \in X$, the fibre $J_k^{\mathrm{reg}}X_p$ can be identified with $J_k^{\mathrm{reg}}(1,n)$ and

$$J_k^{\mathrm{reg}}X = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} J_k^{\mathrm{reg}}(1,n).$$

2.3. **Invariant jet differentials.** Let X be a complex n-dimensional manifold and let k be a positive integer. Recall that after choosing local coordinates on X near p we can identify $J_k^{\mathrm{reg}}X_p$ with $J_k^{\mathrm{reg}}(1,n)$. We can explicitly write out the reparametrisation action (defined in (3)) of $\mathrm{Diff}_k(1)$ on $J_k^{\mathrm{reg}}(1,n)$ as follows. Let $f_\xi(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) \in J_k^{\mathrm{reg}}(1,n)$ the k-jet of a germ at the origin (i.e no constant term) in $\mathbb{C}^n$ with $f^{(i)} \in \mathbb{C}^n$ such that $f' \neq 0$ and let $\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_k z^k \in J_k^{\mathrm{reg}}(1,1)$ with $\alpha_i \in \mathbb{C}, \alpha_1 \neq 0$. Then

$$f \circ \varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots + \left(\sum_{i_1+\ldots+i_l=k} \frac{f^{(l)}(0)}{l!}\alpha_{i_1}\ldots\alpha_{i_l}\right)z^k$$

$$= (f'(0),\ldots,f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1}+\ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2}+\ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} \quad (4)$$

where the (i,j) entry is $p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}$.

Remark 2.1. The linear representation of $\mathrm{Diff}_k(1)$ on $J_k^{\mathrm{reg}}(1,n)$ given by (4) embeds $\mathrm{Diff}_k(1)$ as a upper triangular subgroup of $\mathrm{GL}(n)$. This is a linear algebraic group but is not reductive, so Mumford's classical GIT cannot be used to construct compactifications of the orbit space $J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1)$ (cf. [BDHK, BDHK18]). This matrix group is parametrised along its first row with free parameters $\alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C}$, while the other entries are certain (weighted homogeneous) polynomials in these free parameters. It is a semidirect product

$$\mathrm{Diff}_k(1) = U_k \rtimes \mathbb{C}^*$$

of its unipotent radical U_k by a one-parameter subgroup $\mathbb{C}^*$ acting diagonally. Here U_k is the subgroup given by substituting $\alpha_1 = 1$, and the diagonal subgroup $\mathbb{C}^*$ acts with strictly positive weights $1\ldots,n-1$ on the Lie algebra $\mathrm{Lie}(U_k)$ of U_k. In Bérczi and Kirwan [BK] and Bérczi, Doran, Hawes and Kirwan [BDHK, BDHK18] we study actions of non-reductive groups of this type in a more general context.

The action of $\lambda \in \mathbb{C}^*$ on k-jets is thus described by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Following Demailly [Dem97], we introduce the Green-Griffiths vector bundle $E_{k,m}^{GG}$ whose fibres are complex-valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibres of J_kX of weighted degree m with respect to this $\mathbb{C}^*$ action; that is, they satisfy

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}).$$

The fibrewise $\mathrm{Diff}_k(1)$ action on J_kX induces an action on $E_{k,m}^{GG}$. Demailly in [Dem97] defined the bundle of invariant jet differentials of order k and weighted degree m as the subbundle $E_{k,m}^n \subseteq E_{k,m}^{GG}$ of polynomial differential operators $Q(f, f', \ldots, f^{(k)})$ which are invariant under U_k; that is for any $\varphi \in \mathrm{Diff}_k(1)$

$$Q((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}) = \varphi'(0)^m \cdot Q(f', f'', \ldots, f^{(k)}).$$

We call $E_k^n = \oplus_m E_{k,m}^n = (\oplus_m E_{k,m}^{GG})^{U_k}$ the Demailly–Semple bundle of invariant jet differentials.

2.4. **Compactifications of $J_kX/\mathrm{Diff}_k(1)$.** In order to find and describe invariant jet differentials we can try to construct projective completions of the quasi-projective fibrewise quotient

$$J_kX/\mathrm{Diff}_k(1) = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} (J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1)).$$

This quotient fibres over X (as $\mathrm{Diff}_k(X)$ acts fibrewise) and we can hope to detect invariant jet differentials as global sections of powers of ample line bundles on suitable fibrewise projective completions $\overline{J_kX/\mathrm{Diff}_k(1)}$. This strategy indeed works, and there exist two constructions in the literature.

(1) **The Demailly–Semple tower** The first construction goes back to Semple, and was studied and introduced into the study of hyperbolicity questions by the landmark paper of Demailly [Dem97]. The Demailly–Semple tower X_k is an iterated projective bundle over X

$$X_k \to X_{k-1} \to \ldots \to X_1 \to X_0 = X$$

endowed with projections $\pi_{i,k} : X_k \to X_i$ and canonical line bundles $\pi_{i,k}^* \mathcal{O}_{X_i}(1) \to X_k$ whose sections are global invariant jet differentials. The total space X_k is smooth of dimension $\dim(X_k) = n + k(n-1)$. For the details of the construction see [Dem97, DMR10]. In [B19] equivariant localisation was introduced on the Demailly-Semple tower, and following the strategy of [DMR10], the Green–Griffiths–Lang conjecture for generic hypersurfaces with degree at least n^{6n} was proved. In [B19] it was also proved that we cannot expect better than an exponential degree bound with this approach.

(2) **The curvilinear component of the Hilbert scheme of k points on $\mathbb{C}^n$** In [B17] the first author proves that the curvilinear component of the punctual Hilbert scheme $\mathrm{Hilb}_0^k(\mathbb{C}^n)$ supported at the origin is a compactification of the fibre $J_k(1,n)/\mathrm{Diff}_k(1)$ of $J_kX/\mathrm{Diff}_k(1)$. Using equivariant localisation, in [B18] we connect hyperbolicity of hypersurfaces with global singularity theory and Thom polynomials of A_n-singularities. Modulo a positivity conjecture of Rimányi, [B18] obtains the Green–Griffiths–Lang conjecture for hypersurfaces with polynomial ($> n^8$) degree. However, a complete proof of the positivity conjecture currently seems to be out of reach.

This paper introduces a third compactification coming from the recent development of non-reductive GIT. As we have seen, the reparametrisation group $\mathrm{Diff}_k(1)$ is not reductive, but it is a linear algebraic group with internally graded unipotent radical in the sense of [BDHK, BDHK18], and hence the construction and results of these papers apply. We use the fibrewise completion

$$J_kX//\mathrm{Diff}_k(1) := \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} (\mathbb{P}[\mathbb{C} \oplus J_k^{\mathrm{reg}}(1,n)]//\mathrm{Diff}_k(1))$$

where $\mathbb{P}[\mathbb{C} \oplus J_k^{\mathrm{reg}}(1,n)]//\mathrm{Diff}_k(1)) = \mathbb{P}[\mathbb{C} \oplus \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)]//\mathrm{Diff}_k(1)$ is a non-reductive GIT quotient. Then we apply the intersection theory and integration formulas for non-reductive GIT quotients proved in [BK] as our main computational tool.

3. Non-reductive geometric invariant theory

In [BDHK18] an extension of Mumford's classical GIT is developed for linear actions of a non-reductive linear algebraic group with internally graded unipotent radical over an algebraically closed field $\mathbf{k}$ of characteristic 0.

Definition 3.1. We say that a linear algebraic group $H = U \rtimes R$ has *internally graded unipotent radical* U if there is a central one-parameter subgroup $\lambda : \mathbb{G}_m \to Z(R)$ of the Levi subgroup R of H such that the adjoint action of $\mathbb{G}_m$ on the Lie algebra of U has all its weights strictly positive. Then $\hat{U} = U \rtimes \lambda(\mathbb{G}_m)$ is a normal subgroup of H and $H/\hat{U} \cong R/\lambda(\mathbb{G}_m)$ is reductive.

Let $H = U \rtimes R$ be a linear algebraic group with internally graded unipotent radical U acting linearly with respect to an ample line bundle L on a projective variety X; that is, the action of H on X lifts to an action on L via automorphisms of the line bundle. When $H = R$ is reductive, using Mumford's classical geometric invariant theory (GIT) [MFK94], we can define H-invariant open subsets $X^s \subseteq X^{ss}$ of X (the stable and semistable loci for the linearisation) with a geometric quotient X^s/H and projective completion $X/\!/H \supseteq X^s/H$ which is the projective variety associated to the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$. The variety $X/\!/H$ is the image of a surjective morphism ϕ from the open subset X^{ss} of X such that if $x, y \in X^{ss}$ then $\phi(x) = \phi(y)$ if and only if the closures of the H-orbits of x and y meet in X^{ss}. Furthermore the subsets X^s and X^{ss} can be described using the Hilbert–Mumford criteria for stability and semistability.

Mumford's GIT does not have an immediate extension to actions of non-reductive linear algebraic groups H, since the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$ is not necessarily finitely generated as a graded algebra when H is not reductive. It is still possible to define semistable and stable subsets X^{ss} and X^s, with a geometric quotient X^s/H which is an open subset of a so-called enveloping quotient $X \wr\!\wr H$ with an H-invariant morphism $\phi : X^{ss} \to X \wr\!\wr H$, and if the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated then $X \wr\!\wr H$ is the associated projective variety [BDHK, DK07]. But in general the enveloping quotient $X \wr\!\wr H$ is not necessarily projective, the morphism ϕ is not necessarily surjective (and its image may be only a constructible subset, not a subvariety, of $X \wr\!\wr H$). In addition there are in general no obvious analogues of the Hilbert–Mumford criteria.

However when $H = U \rtimes R$ has internally graded unipotent radical U and acts linearly on a projective variety X, then provided that we are willing to modify the linearisation of the action by replacing the line bundle L by a sufficiently divisible tensor power and multiplying by a suitable character of H (which will not change the action of H on X), many of the key features of classical GIT still apply.

Let such an H act linearly on an irreducible projective variety X with respect to a very ample line bundle L. Let $\chi : H \to \mathbb{G}_m$ be a character of H. Its kernel contains U, and its restriction to $\hat{U}$ can be identified with an integer so that the integer 1 corresponds to the character of $\hat{U}$ which fits into the exact sequence $U \hookrightarrow \hat{U} \to \lambda(\mathbb{G}_m)$. Let $\omega_{\min}$ be the minimal weight for the $\lambda(\mathbb{G}_m)$-action on $V := H^0(X, L)^*$ and let $V_{\min}$ be the weight space of weight $\omega_{\min}$ in V. Suppose that $\omega_{\min} = \omega_0 < \omega_1 < \cdots < \omega_{\max}$ are the weights with which the one-parameter subgroup $\lambda : \mathbb{G}_m \leq \hat{U} \leq H$ acts on the fibres of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(-1)$ over points of the connected components of the fixed point set $\mathbb{P}((H^0(X, L)^*)^{\mathbb{G}_m}$ for the action of $\mathbb{G}_m$ on $\mathbb{P}((H^0(X, L)^*)$; since L is very ample X embeds in $\mathbb{P}((H^0(X, L)^*)$ and the line bundle L extends to the dual $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(1)$ of the tautological line bundle on $\mathbb{P}((H^0(X, L)^*)$. Note that we can assume that there exist at least two distinct such weights since otherwise the action of the unipotent radical U of H on X is trivial, and so the action of H is via an action of the reductive group $R = H/U$.

Definition 3.2. Let c be a positive integer such that

$$\frac{\chi}{c} = \omega_{\min} + \epsilon$$

where $\epsilon > 0$ is sufficiently small; we will call rational characters χ/c with this property *well adapted* to the linear action of H, and we will call the linearisation well adapted if $\omega_{\min} < 0 \leq \omega_{\min} + \epsilon$ for sufficiently small $\epsilon > 0$. How small ϵ is required to be will depend on the situation; more precisely, we will say that some property P holds for well adapted linearisations if there exists $\epsilon(P) > 0$ such that property P holds for any linearisation for which $\omega_{\min} < 0 \leq \omega_{\min} + \epsilon(P)$.

Remark 3.3. In [BK17] it is shown that under hypotheses which will be satisfied in our situation it suffices to take $0 < \epsilon < 1$.

The linearisation of the action of H on X with respect to the ample line bundle $L^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $L_\chi^{\otimes c}$ denote this twisted linearisation. Let $X^{s,\mathbb{G}_m}_{\min +}$ denote the stable subset of X for the linear action of $\mathbb{G}_m$ with respect to the linearisation $L_\chi^{\otimes c}$; by the theory of variation of (classical) GIT [DH98, Tha96], if L is very ample then $X^{s,\mathbb{G}_m}_{\min +}$ is the stable set for the action of $\mathbb{G}_m$ with respect to any rational character χ/c such that $\omega_{\min} < \chi/c < \omega_{\min +1}$. Let

$$Z_{\min} := X \cap \mathbb{P}(V_{\min}) = \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{G}_m\text{-fixed point and} \\ \mathbb{G}_m \text{ acts on } L^*|_x \text{ with weight } \omega_{\min} \end{array} \right\}$$

and

$$X^0_{\min} := \{x \in X \mid p(x) \in Z_{\min}\} \quad \text{where} \quad p(x) = \lim_{\substack{t \to 0 \\ t \in \mathbb{G}_m}} t \cdot x \quad \text{for } x \in X.$$

Definition 3.4. (*cf.* [BDHK18]) With this notation, we define the following condition for the $\hat{U}$-action on X:

$$(*) \qquad \mathrm{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\min}.$$

Note that $(*)$ holds if and only if we have $\mathrm{Stab}_U(x) = \{e\}$ for all $x \in X^0_{\min}$. This is also referred to as the condition that 'semistability coincides with stability' for the action of $\hat{U}$ (or, when $\lambda : \mathbb{G}_m \to R$ is fixed, for the linear action of U); see Definition 3.7 below.

Definition 3.5. When $(*)$ holds for a well adapted action of $\hat{U}$ the min-stable locus for the $\hat{U}$-action is

$$X^{s,\hat{U}}_{\min +} = X^{ss,\hat{U}}_{\min +} = \bigcap_{u \in U} u X^{s,\lambda(\mathbb{G}_m)}_{\min +} = X^0_{\min} \setminus U Z_{\min}.$$

Definition 3.6. A *well-adapted linear action* of the linear algebraic group H on an irreducible projective variety consists of the data $(X, L, H, \hat{U}, \chi)$ where

(1) H is a linear algebraic group with internally graded unipotent radical U,
(2) H acts linearly on X with respect to a very ample line bundle L, while $\chi : H \to \mathbb{G}_m$ is a character of H and c is a positive integer such that the rational character χ/c is well adapted for the linear action of $\hat{U} = U \rtimes \mathbb{G}_m$ on X.

We will often refer to this set-up simply as a well-adapted action of H on X.

Theorem 3.1 ([BDHK18])**.** *Let $(X, L, H, \hat{U}, \chi)$ be a well-adapted linear action satisfying condition $(*)$. Then*

(1) the algebras of invariants

$$\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{U}} \text{ and } \oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{H} = (\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{U}})^{R}$$

are finitely generated;

(2) the enveloping quotient $X \wr\wr \hat{U}$ is the projective variety associated to the algebra of invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{U}}$ and is a geometric quotient of the open subset $X^{s,\hat{U}}_{\min +}$ of X by $\hat{U}$;

(3) the enveloping quotient $X \wr\wr H$ is the projective variety associated to the algebra of invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{H}$ and is the classical GIT quotient of $X \wr\wr \hat{U}$ by the induced action of $R/\lambda(\mathbb{G}_m)$ with respect to the linearisation induced by a sufficiently divisible tensor power of L.

Definition 3.7. Let X be a projective variety which has a well adapted linear action of a linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U. When $(*)$ holds we denote by $X^{s,H}_{\min +}$ and $X^{ss,H}_{\min +}$ the pre-images in $X^{s,\hat{U}}_{\min +} = X^{ss,\hat{U}}_{\min +}$ of the stable and semistable loci for the induced linear action of the reductive group $H/\hat{U} = R/\lambda(\mathbb{G}_m)$ on $X \wr\wr \hat{U} = X^{s,\hat{U}}_{\min +}/\hat{U}$.

By H-stability=H-semistability we mean that $(*)$ holds and $X^{s,H}_{\min +} = X^{ss,H}_{\min +}$. The latter is equivalent to the requirement that $\mathrm{Stab}_H(x)$ is finite for all $x \in X^{ss,H}_{\min +}$; then the projective variety $X \wr\wr H$ is a geometric quotient of the open subset $X^{s,H}_{\min +} = X^{ss,H}_{\min +}$ of X by the action of H.

Remark 3.8. When the conditions of Theorem 3.1 hold, we call $X \wr\wr H$ (respectively $X \wr\wr \hat{U}$) the GIT quotient and we denote it by $X//H$ (respectively $X//\hat{U}$).

It is shown in [BDHK] that if $(X, L, H, \hat{U}, \chi)$ is a well-adapted linear action satisfying H-stability=H-semistability, then

(1) there is a sequence of blow-ups of X along H-invariant projective subvarieties resulting in a projective variety $\hat{X}$ with a well adapted linear action of H which satisfies the condition $(*)$, so that Theorem 3.1 applies, giving us a projective geometric quotient

$$\hat{X}//\hat{U} = \hat{X}^{s,\hat{U}}_{\min +}/\hat{U}$$

and its (reductive) GIT quotient $\hat{X}//H = (\hat{X}//\hat{U})//R = (\hat{X}//\hat{U})//R$ where $R = H/U$;

(2) there is a sequence of further blow-ups along H-invariant projective subvarieties resulting in a projective variety $\tilde{X}$ satisfying the same conditions as $\hat{X}$ and in addition $\tilde{X}//H = \mathrm{Proj}(\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{H})$ is the geometric quotient by H of the H-invariant open subset $\tilde{X}^{s,H}_{\min +}$.

4. Moment maps and cohomology of non-reductive quotients

In this section we briefly summarise the results of [BK], which generalise results of the second author [Kir84] and Martin [Mar] to the cohomology of GIT quotients by non-reductive groups with internally graded unipotent radicals.

First let us recall the reductive picture. Let X be a nonsingular complex projective variety acted on by a complex reductive group G with respect to an ample linearisation. Then we can choose a maximal compact subgroup K of G and a K-invariant Fubini–Study Kähler metric on X with corresponding moment map $\mu : X \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{k}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{k}, \mathbb{R})$ is its dual. $\mathfrak{k}^*$ embeds naturally in the complex dual $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{k}, \mathbb{C})$ of the Lie algebra $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ of

G, as $\mathfrak{k}^* = \{\xi \in \mathfrak{g}^* : \xi(\mathfrak{k}) \subseteq \mathbb{R}\}$; using this identification we can regard $\mu : X \to \mathfrak{g}^*$ as a 'moment map' for the action of G, although of course it is not a moment map for G in the traditional sense of symplectic geometry.

In [Kir84] it is shown that the norm-square $f = ||\mu||^2$ of the moment map $\mu : X \to \mathfrak{k}^*$ induces an equivariantly perfect Morse stratification of X such that that the open stratum which retracts equivariantly onto the zero level set $\mu^{-1}(0)$ of the moment map coincides with the GIT semistable locus X^{ss} for the linear action of G on X. In particular this tells us that the restriction map

$$H^*_G(X;\mathbb{Q}) \to H^*_G(X^{ss};\mathbb{Q})$$

is surjective; we also have an isomorphism (of vector spaces though not of algebras) $H^*_G(X;\mathbb{Q}) \cong H^*(X;\mathbb{Q}) \otimes H^*(BG;\mathbb{Q})$. Moreover, $\mu^{-1}(0)$ is K-invariant and its inclusion in X^{ss} induces a homeomorphism

$$\mu^{-1}(0)/K \cong X//G. \tag{5}$$

When $X^s = X^{ss}$ the G-equivariant rational cohomology of X^{ss} coincides with the ordinary rational cohomology of its geometric quotient X^{ss}/G, which is the GIT quotient $X//G$, and we get expressions for the Betti numbers of $X//G$ in terms of the equivariant Betti numbers of the unstable GIT strata, which can be described inductively, and of X [Kir84]. In order to describe the ring structure on the rational cohomology of $X//G$, the surjectivity of the composition

$$\kappa : H^*_G(X;\mathbb{Q}) \to H^*_G(X^{ss};\mathbb{Q}) \cong H^*(X//G;\mathbb{Q})$$

can be combined with Poincaré duality on $X//G$ and the nonabelian localisation formulas for intersection pairings on $X//G$ given in [JK95].

Martin [Mar] used (5) to obtain formulas for the intersection pairings on the quotient $X//G$ in a different way, by relating these pairings to intersection pairings on the associated quotient $X//T_{\mathbb{C}}$, where $T_{\mathbb{C}} \subseteq G$ is a maximal torus. He proved a formula expressing the rational cohomology ring of $X//G$ in terms of the rational cohomology ring of $X//T_{\mathbb{C}}$ and an integration formula relating intersection pairings on the cohomology of $X//G$ to corresponding pairings on $X//T_{\mathbb{C}}$. This integration formula, combined with methods from abelian localisation, leads to residue formulas for pairings on $X//G$ which are closely related to those of [JK95] (see also [Ver96]).

In [BK] similar results are obtained for non-reductive actions. Let X be a nonsingular complex projective variety with a linear action of a complex linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U with respect to an ample line bundle L; then the Levi subgroup R is the complexification of a maximal compact subgroup Q of H. The unipotent radical U of H is internally graded by a central 1-parameter subgroup $\lambda : \mathbb{C}^* \to Z(R)$ of R. Let $\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \subseteq H$; then $\lambda(S^1) \subseteq \lambda(\mathbb{C}^*) \subseteq \hat{U}$ is a maximal compact subgroup of $\hat{U}$. Assume also that semistability coincides with stability for the $\hat{U}$-action, in the sense of Definition 3.7. Using the embedding $X \subseteq \mathbb{P}^n$ defined by a very ample tensor power of L, and a corresponding Fubini–Study Kähler metric invariant under the maximal compact subgroup Q of H, an H-moment map $\mu_H : X \to \mathrm{Lie}H^* = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Lie}H, \mathbb{C})$ is defined in [BK] by composing the $G = \mathrm{GL}(n+1)$-moment map $\mu_G : X \to \mathfrak{g}^*$ with the map of complex duals $\mathfrak{g}^* \to \mathrm{Lie}(H)^*$ coming from the representation $H \to \mathrm{GL}(n+1)$. It is shown in [BK] that if the linearisation of the action of H on X is well-adapted (which can be achieved by adding a suitable central constant to the moment map) and if H-stability=H-semistability (see Definition 3.7), then $H\mu_H^{-1}(0) = X^{s,H} = X^{ss,H}$ and the embedding of $\mu_H^{-1}(0)$ in $X^{ss,H}$ induces a homeomorphism

$$\mu_H^{-1}(0)/Q \simeq X//H = X^{s,H}/H.$$

In particular when $H = \hat{U} = U \rtimes \mathbb{C}^*$, this tells that the embedding $\mu_{\hat{U}}^{-1}(0) \hookrightarrow X^{ss,\hat{U}}$ induces a homeomorphism $\mu_{\hat{U}}^{-1}(0)/S^1 \simeq X//\hat{U}$. Indeed to have an embedding of $\mu_H^{-1}(0)$ in $X^{ss,H}$ and an induced homeomorphism $\mu_H^{-1}(0)/Q \simeq X//H$, the condition that H-stability=H-semistability can be weakened to the requirement that $\hat{U}$-stability=$\hat{U}$-semistability.

Similar results hold more generally when X is compact Kähler but not necessarily projective. Suppose that Y is a compact Kähler manifold acted on by a complex reductive Lie group G such that G is the complexification of a maximal compact subgroup K, so their Lie algebras satisfy $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Let $B \subseteq G$ be a Borel subgroup such that $G = KB$ and $K \cap B = T$ is a maximal torus in K. We fix $\hat{U} = U \rtimes \lambda(\mathbb{C})^* \subseteq B$ where $\lambda : \mathbb{C}^* \to T_{\mathbb{C}}$ grades the unipotent subgroup U of B; then $K \cap \hat{U} = S^1$ is a maximal compact subgroup of $\hat{U}$. The Lie algebra of $\hat{U}$ decomposes as a real vector space as

$$\hat{\mathfrak{u}} = \mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{u} \tag{6}$$

where $\mathrm{Lie}(K \cap \hat{U}) = \mathbb{R}$ and $\mathfrak{u}$ is the Lie algebra of the complex unipotent group U. The set of positive roots $\Delta^+ \subseteq \Delta$ contains the weights of the adjoint action of G on the Lie algebra of the unipotent radical of B, that is, the Cartan decomposition has the form

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}^+ \text{ where } \mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}^+ \text{ and } \mathfrak{g}^\pm = \oplus_{\alpha \in \Delta^\pm} e(\alpha).$$

Suppose that $H = U \rtimes R \subseteq G$ where R is the complexification of $Q = K \cap H$ and $\lambda(\mathbb{C}^*)$ is central in R, so that H has internally graded unipotent radical with $\hat{U} \subseteq H$. Suppose also that $X \subseteq Y$ is a compact complex submanifold invariant under the H action, and that K preserves the Kähler structure on Y, so $S^1 = K \cap \hat{U}$ and $Q = K \cap H$ preserve the induced Kähler structure on X. Then we can define (generalised) moment maps $\mu_{\hat{U}}$ and μ_H from X to the complex duals of the Lie algebras of $\hat{U}$ and H by composing the restriction maps from $\mathfrak{g}^*$ to these duals with the G-moment map on Y and the inclusion of X in Y.

In the present paper we will work with actions of the diffeomorphism group (see §2.3)

$$\hat{U} = \mathrm{Diff}_k(1) = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} : \begin{matrix} \alpha_1, \ldots, \alpha_k \in \mathbb{C}, \\ \alpha_1 \neq 0 \end{matrix} \right\}$$

on projective varieties. Therefore we only state the results of [BK] for the $H = \hat{U}$ case. In this situation the symplectic description of the GIT quotient $X//\hat{U}$ as $X//\hat{U} = \mu_{\hat{U}}^{-1}(0)/S^1$ fits into the diagram

$$\mu_K^{-1}(0)/S^1 \overset{j}{\hookrightarrow} \mu_{\hat{U}}^{-1}(0)/S^1 = X//\hat{U} \overset{i}{\hookrightarrow} \mu_{S^1}^{-1}(0)/S^1 = X//\mathbb{C}^* \tag{7}$$

Definition 4.1. For a weight α of $\mathbb{C}^* \subseteq \hat{U}$, let $\mathbb{C}_\alpha$ denote the corresponding 1-dimensional complex representation of $\mathbb{C}^*$ and let

$$L_\alpha := \mu_{S^1}^{-1}(0) \times_{S^1} \mathbb{C}_\alpha \to X//\mathbb{C}^*,$$

denote the associated line bundle whose Euler class is denoted by $e(\alpha) \in H^2(X//\mathbb{C}^*) \simeq H^2_{\mathbb{C}}(X)$. For a $\mathbb{C}^*$-invariant complex subspace $\mathfrak{a} \subseteq \mathfrak{b}$ let

$$V_{\mathfrak{a}} = \mu_{S^1}^{-1}(0) \times_{S^1} \mathfrak{a} \to X//\mathbb{C}^*$$

denote the corresponding vector bundle.

Then we have

Proposition 4.2 ([BK], Proposition 5.11)**.** *(1) The vector bundle $V_{\mathfrak{u}}^* \to X//\mathbb{C}^*$ has a C^∞-section s which is transverse to the zero section and whose zero set is the submanifold $\mu_{\hat{U}}^{-1}(0)//S^1 \subseteq X//\mathbb{C}^*$. Therefore the $\mathbb{C}^*$-equivariant normal bundle is*

$$\mathcal{N}(i) \simeq V_{\mathfrak{u}}^*.$$

(2) Let $\mathfrak{b} = \hat{\mathfrak{u}} \oplus \mathfrak{v}$ be a decomposition invariant under the adjoint $\mathbb{C}^*$ action. Then the complex vector bundle $V_{\mathfrak{v}}^* \to X//\mathbb{C}^*$ has a transversal section whose zero set is the submanifold $\mu_K^{-1}(0)//S^1$. Therefore the $\mathbb{C}^*$-equivariant normal bundles are

$$\mathcal{N}(j) \simeq V_{\mathfrak{v}}^* \text{ and } \mathcal{N}(i \circ j) \simeq V_{\mathfrak{v}\oplus\mathfrak{u}}^*$$

This leads us to the following theorems:

Theorem 4.3 ([BK], Theorem 5.12)**.** *Let X be a smooth projective variety endowed with a well-adapted action of $\hat{U} = U \rtimes \mathbb{C}^*$ such that $\hat{U}$-stability=$\hat{U}$-semistability holds. Then there is a natural ring isomorphism*

$$H^*(X//\hat{U}, \mathbb{Q}) \simeq \frac{H^*(X//\mathbb{C}^*, \mathbb{Q})}{ann(\mathrm{Euler}(V_{\mathfrak{u}})}.$$

Here $\mathrm{Euler}(V_{\mathfrak{u}}) \in H^*(X//\mathbb{C}^*)$ *is the Euler class of the bundle $V_{\mathfrak{u}}$ and*

$$\mathrm{ann}(\mathrm{Euler}(V_{\mathfrak{u}})) = \{c \in H^*(X//\mathbb{C}^*, \mathbb{Q}) | c \cup \mathrm{Euler}(V_{\mathfrak{u}}) = 0\} \subseteq H^*(X//\mathbb{C}^*, \mathbb{Q}).$$

is the annihilator ideal.

Theorem 4.4 ([BK], Theorem 5.13)**.** *Let X be a smooth projective variety endowed with a well-adapted action of $\hat{U} = U \rtimes \mathbb{C}^*$ such that $\hat{U}$-stability=$\hat{U}$-semistability holds. Assume that the stabiliser in $\hat{U}$ of a generic $x \in X$ is trivial. Given a cohomology class $a \in H^*(X//\hat{U})$ with a lift $\tilde{a} \in H^*(X//\mathbb{C}^*)$, then*

$$\int_{X//\hat{U}} a = \int_{X//\mathbb{C}^*} \tilde{a} \cup \mathrm{Euler}(V_{\mathfrak{u}}),$$

where $\mathrm{Euler}(V_{\mathfrak{u}})$ *is the cohomology class defined in Theorem 4.3. Here we say that $\tilde{a} \in H^*(X//\mathbb{C}^*)$ is a lift of $a \in H^*(X//\hat{U})$ if $a = i^*\tilde{a}$.*

Remark 4.5. Theorem 4.4 can be generalised to allow the triviality assumption for the stabiliser in $\hat{U}$ of a generic $x \in X$ to be omitted; then the sizes of the stabilisers in $\hat{U}$ and $\mathbb{C}^*$ of a generic $x \in X$ are included in the formula for $\int_{X//\hat{U}} a$.

Finally, we have residue formulas for the intersection pairings on the quotient $X//\mathbb{C}^*$. There are two surjective ring homomorphisms

$$\kappa_{\mathbb{C}^*} : H_{S^1}^*(X; \mathbb{Q}) \to H^*(X//\mathbb{C}^*; \mathbb{Q}) \text{ and } \kappa_{\hat{U}} : H_{\hat{U}}^*(X; \mathbb{Q}) = H_{S^1}^*(X; \mathbb{Q}) \to H^*(X//\hat{U}; \mathbb{Q})$$

from the S^1-equivariant cohomology of X to the ordinary cohomology of the corresponding GIT quotients. The fixed points of the maximal compact subgroup S^1 of $\hat{U}$ on $X \subseteq \mathbb{P}^n$ correspond to the weights of the $\mathbb{C}^*$ action on X, and since this action is well-adapted, these weights satisfy

$$\omega_{\min} = \omega_0 < 0 < \omega_1 < \ldots < \omega_n.$$

We can represent elements of $H^*_{\hat{U}}(X;\mathbb{Q}) = H^*_{S^1}(X;\mathbb{Q})$ as polynomial functions on the Lie algebra of $\mathbb{C}^*$ whose coefficients are differential forms on X and which are equivariantly closed.

Theorem 4.6 ([BK], Theorem 5.14 and Corollary 5.15)**.** *Let X be a smooth projective variety endowed with a well-adapted action of $\hat{U} = U \rtimes \mathbb{C}^*$ such that $\hat{U}$-stability=$\hat{U}$-semistability holds (in the sense of Definitions 3.4 and 3.6). Let z be the standard coordinate on the Lie algebra of $\mathbb{C}^*$. Given any $\hat{U}$-equivariant cohomology class η on X represented by an equivariant differential form $\eta(z)$ whose degree is the dimension of $X//\hat{U}$, we have*

$$\int_{X//\hat{U}} \kappa_{\hat{U}}(\eta) = n_{\mathbb{C}^*} \operatorname*{Res}_{z=\infty} \int_{F_{\min}} \frac{i^*_{F_{\min}}(\eta(z) \cup \mathrm{Euler}(V_{\mathfrak{u}})(z))}{\mathrm{Euler}(\mathcal{N}_{F_{\min}})(z)} dz$$

where $F_{\min}$ is the union of those connected components of the fixed point locus $X^{\mathbb{C}^*}$ on which the S^1-moment map takes its minimum value $\omega_{\min}$, and $n_{\hat{U}}$ is the positive integer which is the order of the stabiliser in $\hat{U}$ of a generic $x \in X$.

5. GIT compactification of the jet differentials bundle

Let X be a nonsingular complex projective variety of dimension n. In this section we describe a projective completion of the quasi-projective quotient $J_kX/\mathrm{Diff}_k(1)$, introduced in §2.3, using non-reductive GIT. Let Diff_X denote the principal $\mathrm{Diff}_k(n)$-bundle over X formed by all local polynomial coordinate systems on X. Note that this is not a vector bundle because the structure group is $\mathrm{Diff}_k(n)$. Then let

$$\mathcal{X}_k^{\mathrm{reg}} = J_k^{\mathrm{reg}}X/\mathrm{Diff}_k(1) \cong \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} X_k^{\mathrm{reg}}$$

where $X_k^{\mathrm{reg}} = J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1))$ is isomorphic to the fibre of $\mathcal{X}_k^{\mathrm{reg}}$ over X. In this section we will use the shorthand Diff_k for $\mathrm{Diff}_k(1)$ as introduced in §2.1. We will construct a projective completion

$$\mathcal{X}_k^{\mathrm{GIT}} \cong \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} X_k^{\mathrm{GIT}}$$

of $\mathcal{X}_k^{\mathrm{reg}}$ where the fibre X_k^{GIT} of $\mathcal{X}_k^{\mathrm{GIT}}$ over X is a non-reductive GIT quotient of the projective space

$$\mathbb{P} = \mathbb{P}(\mathbb{C} \oplus \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)) = \mathbb{P}[x : v_1 : v_2 : \ldots : v_k]$$

by

$$\hat{U} = \mathrm{Diff}_k = \mathrm{Diff}_k(1) = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} : \begin{array}{c} \alpha_1,\ldots,\alpha_k \in \mathbb{C}, \\ \alpha_1 \neq 0 \end{array} \right\}.$$

Here $v_1,\ldots,v_k \in \mathbb{C}^n$ are vectors representing the columns of a matrix $M \in \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$, and x is the compactifying coordinate, while $\hat{U}$ acts via the right action

$$[x : M]\cdot\hat{u} = [x : M\hat{u}] \text{ for } \hat{u} \in \hat{U}$$

or equivalently via the left action

$$\hat{u}\cdot[x : M] = [x : M(\hat{u})^{-1}] \text{ for } \hat{u} \in \hat{U}.$$

We want to apply the results of non-reductive GIT described in §3, which are stated for left actions. For this we need a one-parameter subgroup $\lambda : \mathbb{C}^* \to \hat{U}$ whose adjoint action on the Lie algebra of U has only strictly positive weights; we can take

$$\lambda(t) = \begin{pmatrix} t^{-1} & 0 & 0 & \dots & 0 \\ 0 & t^{-2} & 0 & \dots & 0 \\ 0 & 0 & t^{-3} & \dots & 0 \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & t^{-k} \end{pmatrix},$$

and then

$$\lambda(t) \cdot [x : M] = [x : M \begin{pmatrix} t & 0 & 0 & \dots & 0 \\ 0 & t^2 & 0 & \dots & 0 \\ 0 & 0 & t^3 & \dots & 0 \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & t^k \end{pmatrix}].$$

The weights of this (left) action of the one-parameter subgroup of $\hat{U}$ defined by λ are $\{0, 1, 2, \dots, k\}$. The minimal weight space is the point

$$Z_{\min} = \{[1 : 0 : \dots : 0]\}$$

and the U-stabiliser of this point is U. Thus the non-reductive GIT blow-up process described at the end of §3 starts with blowing up the projective space $\mathbb{P}$ along $Z_{\min}$ to get

$$\tilde{\mathbb{P}} = \mathrm{Bl}_{[1:0:\dots:0]}\mathbb{P} = \{([x : v_1, \dots, v_k], [w_1, \dots w_k]) : w_i \otimes v_j = w_j \otimes v_i \text{ for } 1 \le i < j \le k\}$$

embedded in $\mathbb{P}^{kn} \times \mathbb{P}^{kn-1} \subset \mathbb{P}^{(kn+1)kn-1}$. We fix the ample linearisation $L = \mathcal{O}_{\mathbb{P}^{kn}}(1) \otimes \mathcal{O}_{\mathbb{P}^{kn-1}}(1)$ on $\mathbb{P}^{kn} \times \mathbb{P}^{kn-1}$ and restrict it to $\tilde{\mathbb{P}}$. The minimal weight space for the action of $\lambda(\mathbb{C}^*)$ on $\tilde{\mathbb{P}}$ is the intersection $\tilde{Z}_{\min}$ of the exceptional divisor E and the strict transform of $\mathbb{P}[x : v_1 : 0 : \dots : 0] \subset \mathbb{P}$:

$$\tilde{Z}_{\min} = \{([1 : 0 : \dots : 0], [w_1 : 0 : \dots : 0]) : w_1 \in \mathbb{C}^n, \ w_1 \neq 0\} \subset E \subset \tilde{\mathbb{P}}.$$

The U-stabiliser of any point in $\tilde{Z}_{\min}$ is trivial, and hence stability coincides with semistability for the induced $\hat{U}$ action on $\tilde{\mathbb{P}}$. Thus we can apply Theorem 3.1 to obtain a non-reductive GIT quotient $\tilde{\mathbb{P}}/\!/\hat{U} = \tilde{\mathbb{P}}^{ss,\hat{U}}/\hat{U}$ with respect to a well-adapted shift of L. Here $\tilde{\mathbb{P}}/\!/\hat{U}$ is a projective variety and is a geometric quotient by $\hat{U}$ of the open subvariety $\tilde{\mathbb{P}}^{ss,\hat{U}} = \tilde{\mathbb{P}}^{s,\hat{U}}$ of $\tilde{\mathbb{P}}$, which contains $J_k^{\mathrm{reg}}(1, n)$. Thus $\tilde{\mathbb{P}}/\!/\hat{U}$ is a projective completion of $J_k^{\mathrm{reg}}X/\mathrm{Diff}_k(1)$.

5.1. **Equivariant cohomology of $\tilde{\mathbb{P}}$.** It is well-known (see eg [GH94]) that for a smooth projective subvariety $Y \subset X$ the rational cohomology ring $H^*(\mathrm{Bl}_Y X)$ of the blow-up $\pi : \mathrm{Bl}_Y X \to X$ with exceptional divisor E is isomorphic to

$$(\pi^* H^*(X) \oplus H^*(E))/\pi^* H^*(Y)$$

as a vector space. The multiplicative structure on $H^*(\tilde{\mathbb{P}})$ is given by the classical 'formula clef' of Lascu, Mumford and Scott [LMS75]. A more general framework to describe equivariant cohomology using fixed points and 1-dimensional invariant lines was introduced by Goresky, Kottwitz and MacPherson [GKM98]. In particular, for $\tilde{\mathbb{P}} = \mathrm{Bl}_{[1:0:\dots:0]}\mathbb{P}$ we have

$$H^*(\tilde{\mathbb{P}}) \cong (H^*(E) \oplus H^*(\mathbb{P}))/\mathbb{Q}$$

where $H^*(E) = \mathbb{Q}[\eta]/\eta^{nk-1}$ is the cohomology ring of $E = \mathbb{P}(T_{[1:0:\ldots:0]}\mathbb{P}) = \mathbb{P}^{nk-1}$, the projectivised tangent space at $Z_{\min} = [1 : 0 : \ldots : 0]$, and $H^*(\mathbb{P}) = \mathbb{Q}[\zeta]/\zeta^{nk}$ is the cohomology of the projective space $\mathbb{P}$. Here

$$\eta = c_1(\mathcal{O}_E(1) = c_1(N_{\tilde{\mathbb{P}}/E}) \text{ and } \zeta = c_1(\mathcal{O}_{\mathbb{P}}(1))$$

are the hyperplane classes where $N_{\tilde{\mathbb{P}}/E}$ is the normal bundle of E in $\tilde{\mathbb{P}}$. The space $\tilde{\mathbb{P}}$ is equivariantly formal for the action of $\lambda(\mathbb{C}^*) \subset \hat{U}$, and therefore

$$H_{\mathbb{C}^*}(\tilde{\mathbb{P}}) \cong H^*(\tilde{\mathbb{P}}) \otimes \mathbb{Q}[z] \cong [(H^*(E) \oplus H^*(\mathbb{P}))/\mathbb{Q}] \otimes \mathbb{Q}[z]$$

as $\mathbb{Q}[z] = H^*(B\mathbb{C}^*)$-modules. Here z is a generic coordinate on the Lie algebra of $\mathbb{C}^*$.

We will, in particular, use the $\mathbb{C}^*$-equivariant cohomology ring of the projective space E. The $\mathbb{C}^*$-weights on the tangent space $T_{[1:0:\ldots:0]}$ in $\mathbb{P}(\mathbb{C} \oplus \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n))$ are $z, 2z, \ldots, kz$ (all of these n times) and hence

$$H^*_{\hat{U}}(E) = H^*_{\mathbb{C}^*}(E) = \mathbb{Q}[\eta, z]/((\eta + z)^n(\eta + 2z)^n \ldots (\eta + kz)^n).$$

Embedding $\tilde{\mathbb{P}}$ in $\mathbb{P}^{kn} \times \mathbb{P}^{kn-1}$, we can identify ζ with the equivariant first Chern class of the hyperplane line bundle on P^{kn}, identify η with the equivariant first Chern class of the hyperplane line bundle on $\mathbb{P}^{kn-1}$, and identify z with the equivariant first Chern class of the hyperplane line bundle on $BS^1 = \mathbb{P}^\infty$, with all three pulled back to equivariant classes on $\tilde{\mathbb{P}}$. To apply Theorem 3.1 we twist the ample linearisation $L = \mathcal{O}_{\mathbb{P}^{kn}}(1) \otimes \mathcal{O}_{\mathbb{P}^{kn-1}}(1)$ on $\mathbb{P}^{kn} \times \mathbb{P}^{kn-1}$ by a rational character of $\hat{U}$ which can be identified with bz for some $b \in \mathbb{Q}$. The restriction to $\tilde{\mathbb{P}}$ of this ample linearisation is represented by

$$L_{\tilde{\mathbb{P}}} = \eta + bz + \zeta.$$

The minimal weight for the action of $\lambda(\mathbb{C}^*)$ on $L^*_{\tilde{\mathbb{P}}}$ is $-b - 1$ and for a well-adapted linearisation this needs to be $-\epsilon$ for some small positive ϵ. So $b = -1 + \epsilon$, and hence

$$L_{\tilde{\mathbb{P}}} = \eta - (1 - \epsilon)z + \zeta$$

with restriction to E given by

$$L_E = \eta - (1 - \epsilon)z.$$

In §5.2 of [BK17] it is shown that, because the action of $\hat{U}$ on $\tilde{\mathbb{P}}$ extends to an action of $\mathrm{GL}(k)$, we can, in fact, choose any rational $0 < \epsilon < 1$; we will make a choice of ϵ in this range later.

Recall that the non-reductive GIT quotient $\tilde{\mathbb{P}}/\hat{U} = \tilde{\mathbb{P}}/\!/\mathrm{Diff}_k$ given by Theorem 3.1 is a smooth projective variety of dimension $\dim(\tilde{\mathbb{P}}/\!/\mathrm{Diff}_k) = k(n-1)$. A sufficiently divisible power $L^{\otimes N}$ of L induces an ample line bundle $\mathcal{O}_{\tilde{\mathbb{P}}/\!/\mathrm{Diff}_k}(1)$ on this quotient; this line bundle pulls back to the restriction of $L^{\otimes N}$ to the (semi)stable locus in $\tilde{\mathbb{P}}$. The first Chern class of $\mathcal{O}_{\tilde{\mathbb{P}}/\!/\mathrm{Diff}_k}(1)$ has an equivariant lift to $\tilde{\mathbb{P}}$ which is the first Chern class of $L^{\otimes N}$, so its restriction to E is

$$c_1(i^*q^*(\mathcal{O}_{\tilde{\mathbb{P}}/\!/\mathrm{Diff}_k}(1)) = N(\eta - (1 - \epsilon)z)$$

Following the strategy of [DMR10] we wish to prove the positivity of a certain integral of the form

$$\int_{\tilde{\mathbb{P}}/\!/\mathrm{Diff}_k} c_1(\mathcal{O}_{\tilde{\mathbb{P}}/\!/\mathrm{Diff}_k}(1))^{k(n-1)}.$$

5.2. **Equivariant integration on $\tilde{\mathbb{P}}//\mathrm{Diff}_k$.** We start by collecting the localisation data needed to formulate the non-reductive integration formula of Theorem 4.6. In the previous section we fixed a linearisation for the action of $\hat{U} = \mathrm{Diff}_k$ on $\tilde{\mathbb{P}}$ with induced ample line bundle $O_{\tilde{\mathbb{P}}//\mathrm{Diff}_k}(1)$ on $\tilde{\mathbb{P}}//\mathrm{Diff}_k$ such that

$$c_1(\iota^* q^* O_{\tilde{\mathbb{P}}//\mathrm{Diff}_k}(1)) = N(\eta - (1-\epsilon)z).$$

According to Theorem 4.6 we can write

$$\int_{\tilde{\mathbb{P}}//\mathrm{Diff}_k} c_1(O_{\tilde{\mathbb{P}}//\mathrm{Diff}_k}(1))^{k(n-1)} = \operatorname*{Res}_{z=\infty} \int_{\tilde{Z}_{\min}} \frac{(N\eta + N(1-\epsilon)z)^{k(n-1)}|_{\tilde{Z}_{\min}} \cup \mathrm{Euler}^T(V_{\mathfrak{u}})dz}{\mathrm{Euler}^T(\mathcal{N}_{\tilde{Z}_{\min}/\tilde{\mathbb{P}}})}$$

The localisation data on the right hand side can be summarised as follows:

(1) Let $H^*_{\mathbb{C}^*}(\tilde{Z}_{\min}) = H^*_{\mathbb{C}^*}(\mathbb{P}^{n-1}) = \mathbb{C}[z,\theta]/((\theta+z)^n)$ be the equivariant cohomology ring of $\tilde{Z}_{\min}$. If $j : \tilde{Z}_{\min} \hookrightarrow E$ denotes the embedding then $j^* O_E(1) = O_E(1)|_{\tilde{Z}_{\min}} = O_{\tilde{Z}_{\min}}(1)$, so $j^*\eta = \theta$ and hence $a\eta + bz|_{\tilde{Z}_{\min}} = a\theta + bz$, and $\theta + z = c_1^{\mathbb{C}^*}(O_{\tilde{Z}_{\min}}(1))$ is the equivariant first Chern class.

(2) The weights of $\mathbb{C}^* \subset \mathrm{Diff}_k$ on $\mathfrak{u} = \mathrm{Lie}(U)$ are $1, 2, \ldots, (k-1)$, hence $\mathrm{Euler}(V_{\mathfrak{u}}) = z \cdot 2z \cdot \ldots \cdot (k-1)z = (k-1)!z^{k-1}$.

(3) The normal bundle of $\tilde{Z}_{\min}$ in E is $\mathrm{Euler}^T(N_{\tilde{Z}_{\min}/E}) = (\theta - z)^n(\theta - 2z)^n \cdot \ldots \cdot (\theta - (k-1)z)^n$. The weights here correspond to the second, third, ..., last column of $E = \mathbb{P}(\mathrm{Hom}(k,n))$. The normal bundle of E in $\tilde{\mathbb{P}}$ is the tautological bundle $O_E(-1)$ with equivariant first Chern class equal to $\theta - z$, and hence

$$\mathrm{Euler}^T(N_{\tilde{Z}_{\min}/\tilde{\mathbb{P}}}) = (\theta - z)(\theta - z)^n(\theta - 2z)^n \cdot \ldots \cdot (\theta - (k-1)z)^n.$$

Substituting this into the localisation formula we get

Proposition 5.1. *With the notation introduced above we have*

$$\int_{\tilde{\mathbb{P}}//\mathrm{Diff}_k} c_1(O_{\tilde{\mathbb{P}}//\mathrm{Diff}_k}(1))^{k(n-1)} = \operatorname*{Res}_{z=\infty} \int_{\tilde{Z}_{\min}} \frac{(k-1)!z^{k-1}(N\theta - N(1-\epsilon)z)^{k(n-1)}dz}{(\theta - z)^{n+1}(\theta - 2z)^n(\theta - 3z)^n \cdot \ldots \cdot (\theta - (k-1)z)^n}.$$

Remark 5.2. How does this formula work? Since $\tilde{Z}_{\min} = \mathbb{P}^{n-1}$ is a projective space with $c_1^{\mathbb{C}^*}(O_{\tilde{Z}_{\min}}(1)) = \theta + z$, we have $\int_{\tilde{Z}_{\min}} (\theta - z)^{n-1} = 1$. So we simply rewrite the right hand side in the new variable $\theta' = \theta - z$ and the residue will be the coefficient of $\frac{\theta'^{n-1}}{z}$.

5.3. **Second localisation on $\tilde{Z}_{\min}$.** The formula of Theorem 5.1 works perfectly well for integration on the fibres $X_k^{\mathrm{GIT}} = \tilde{\mathbb{P}}//\mathrm{Diff}_k$. In what follows, however, we want to apply Theorem 5.1 in a fibred situation, to integrate over a fibre bundle $\mathcal{X}_k^{\mathrm{GIT}}$ over X, whose fibres are isomorphic to $\tilde{\mathbb{P}}//\mathrm{Diff}_k$. For this purpose—as explained in Remark 6.3—we need to rewrite the formula using a second localisation on $\tilde{Z}_{\min} = \mathbb{P}^{n-1}$. The n-dimensional torus T acts on $\tilde{Z}_{\min} = \mathbb{P}^{n-1}$. The weights of the action of $\lambda(\mathbb{C}^*)$ on the tautological bundle are $\lambda_1, \ldots, \lambda_n$, and hence those of the hyperplane bundle are $-\lambda_1, \ldots, -\lambda_n$. Then the $T \times \mathbb{C}^*$-weights of L on $\tilde{Z}_{\min}$ are $-\lambda_1 - z, \ldots, -\lambda_n - z$ and the formula of Theorem 5.1 can be rewritten using Atiyah-Bott localisation on $\tilde{Z}_{\min}$ as

(8)
$$\int_{\tilde{\mathbb{P}}//\mathrm{Diff}_k} c_1(O_{\tilde{\mathbb{P}}//\mathrm{Diff}_k}(1))^{k(n-1)} = \operatorname*{Res}_{z=\infty} \sum_{j=1}^{n} \frac{(k-1)!(-z)^{k-1}(N\lambda_j + N(1-\epsilon)z)^{k(n-1)}dz}{\prod_{i\neq j}(\lambda_i - \lambda_j)(\lambda_j + z)\prod_{m=2}^{k}\prod_{i=1}^{n}((\lambda_i + mz) - (\lambda_j + z))}.$$

Note that here we multiplied all linear terms both in the denominator and the numerator with (-1), which does not change the sign of the rational expression because its total degree is 0. We can then apply the following iterated residue theorem on projective spaces:

Lemma 5.3 ([BS12], Prop 5.4)**.** *For a polynomial $P(u)$ on $\mathbb{C}$ we have*

$$\sum_{j=1}^{n} \frac{P(\lambda_j)}{\prod_{i \neq j}(\lambda_i - \lambda_j)} = \operatorname*{Res}_{w=\infty} \frac{P(w)}{\prod_{i=1}^{n}(\lambda_i - w)} dw. \tag{9}$$

Proof. We compute the residue on the right hand side of (9) using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. This residue is a contour integral, whose value is minus the sum of the w-residues of the form in (9). These poles are at $w = \lambda_i$, $i = 1 \ldots n$, and after cancelling the signs that arise, we obtain the left hand side of (9). □

We apply Lemma 5.3 to obtain the following result.

Proposition 5.4. *With the notations introduced above we have*

$$\int_{\tilde{\mathbb{P}}//\mathrm{Diff}_k} c_1(O_{\tilde{\mathbb{P}}//\mathrm{Diff}_k}(1))^{k(n-1)} = \operatorname*{Res}_{w=\infty} \operatorname*{Res}_{z=\infty} \frac{(k-1)!(-z)^{k-1}(Nw + N(1-\epsilon)z)^{k(n-1)} dwdz}{(w+z)\prod_{m=0}^{k-1}\prod_{i=1}^{n}((\lambda_i + mz - w)}.$$

Here the iterated residue means expansion on a contour where $z \gg w$, and taking the coefficient of $(zw)^{-1}$.

Proof. Apply Lemma 5.3 to the formula (8) with

$$P(u) = \operatorname*{Res}_{z=\infty} \frac{(k-1)!(-z)^{k-1}(Nu + N(1-\epsilon)z)^{k(n-1)} dz}{(u+z)\prod_{m=1}^{k-1}\prod_{i=1}^{n}((\lambda_i + mz) - u)}.$$

Then $P(u)$ is polynomial, because it is the coefficient of z^{-1} of the Taylor expansion of the rational expression when $z \gg \max\{u, \lambda_i\}$ (that is, expansion in $\frac{u}{z}, \frac{\lambda_i}{z}$). □

6. Equivariant localisation on $J_kX//\mathrm{Diff}_k(1)$

In this section we will use the original notation $\mathrm{Diff}_k(1)$ introduced in §2.1 instead of the shorthand notation Diff_k we used in the previous section. Recall the fibrewise quotient

$$\mathcal{X}_k^{\mathrm{GIT}} = J_kX//\mathrm{Diff}_k(1) = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} X_k^{\mathrm{GIT}}$$

where

$$X_k^{\mathrm{GIT}} = \tilde{\mathbb{P}}(\mathbb{C} \oplus \mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n))//\mathrm{Diff}_k(1)$$

is the GIT quotient constructed in §5 using the linearisation $L = \eta + (1-\epsilon)z + \zeta$ on the blow up $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}(\mathbb{C} \oplus \mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n))$ of $\mathbb{P}(\mathbb{C} \oplus \mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n))$.

$\mathcal{X}_k^{\mathrm{GIT}}$ fibres over X but is not a vector bundle, and hence it is not the pull-back of a universal vector bundle. To resolve this problem we will work with a linearised bundle $\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}$ associated to a principal $\mathrm{GL}(n)$-bundle over X, rather than $\mathcal{X}_k^{\mathrm{GIT}}$ which is associated to the principal $\mathrm{Diff}_k(1)$-bundle Diff_X. This way we reduce the structure group of $\mathcal{X}_k^{\mathrm{GIT}}$ to $\mathrm{GL}(n)$, and the linearised bundle has the same topological intersection numbers.

6.1. **The linearised bundle.** Recall that Diff_X stands for the principal $\mathrm{Diff}_k(n)$-bundle over X formed by all local polynomial coordinate systems on X. This is not a vector bundle—the structure group is $\mathrm{Diff}_k(n)$—but we can linearise it. The set $\mathrm{GL}(n)$ of linear coordinate changes forms a subgroup of $\mathrm{Diff}_k(n)$. Let GL_X denote the principal $\mathrm{GL}(n)$-bundle over X formed by all local linear coordinate systems on X. Then we can form the linearised bundle

$$\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}} = \mathrm{GL}_X \times_{\mathrm{GL}(n)} X_k^{\mathrm{GIT}}$$

which only remembers the linear action on the fibres. Note that the tangent bundle of X is the associated bundle

$$T_X = \mathrm{GL}_X \times_{\mathrm{GL}(n)} \mathbb{C}^n$$

and hence $\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}$ can be written as the GIT quotient

$$\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}} = \tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k})//\mathrm{Diff}_k(1) \tag{10}$$

where $\tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k})$ is the blow-up of the projectivised bundle $\mathbb{P}(\mathcal{O}_X \oplus (T_X)^{\oplus k})$ along the $\mathrm{Diff}_k(1)$-invariant subvariety $\mathbb{P}(\mathcal{O}_X)$. The following lemma tells us that we can replace integration over $\mathcal{X}_k^{\mathrm{GIT}}$ with integration over $\tilde{\mathcal{X}}_{k,\mathrm{GL}}^{\mathrm{GIT}}$.

Lemma 6.1. *Let $k \le n$ and let V be a $\mathrm{Diff}_k(n)$-equivariant vector bundle on $J_k(1,n)//\mathrm{Diff}_k(1)$. Let $\mathcal{V} = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} V$ and $\mathcal{V}^{\mathrm{GL}} = \mathrm{GL}_X \times_{\mathrm{GL}(n)} V$ the associated bundles. Let α be a polynomial in the Chern roots of $\mathcal{V}$ and α^{GL} the same polynomial in the Chern roots $\mathcal{V}^{\mathrm{GL}}$. Then*

$$\int_{\mathcal{X}_k^{\mathrm{GIT}}} \alpha = \int_{\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}} \alpha^{\mathrm{GL}}$$

Proof. $\mathrm{GL}(n)$ is a (strong) deformation retract of Diff_k via the homotopy

$$\mathrm{Diff}_k \times [0,1] \to \mathrm{Diff}_k$$

which sends the (ϕ, t) to ϕ_t whose linear part is identical to the linear part of ϕ but its quadratic and higher order terms are those of ϕ multiplied by t. This homotopy contracts the quadratic and higher order terms of ϕ to zero. This induces a retraction of the classifying spaces

$$\tau : B\mathrm{Diff}_k \to B\mathrm{GL}(n)$$

which is a homotopy equivalence.

Given a Diff_k-module V the embedding $\mathrm{GL}(n) \hookrightarrow \mathrm{Diff}_k$ also defines a $\mathrm{GL}(n)$-module structure on V and the corresponding universal bundles

$$E_{\mathrm{Diff}}V = E\mathrm{Diff}_k \times_{\mathrm{Diff}_k} V \text{ and } E_{\mathrm{GL}(n)}V = E\mathrm{GL}(n) \times_{\mathrm{GL}(n)} V$$

are homotopy equivalent. In particular,

$$E_{\mathrm{Diff}}X_k^{\mathrm{GIT}} = E\mathrm{Diff}_k \times_{\mathrm{Diff}_k} X_k^{\mathrm{GIT}} \text{ and } E_{\mathrm{GL}}X_k^{\mathrm{GIT}} = E\mathrm{GL}(n) \times_{\mathrm{GL}(n)} X_k^{\mathrm{GIT}}$$

are homotopy equivalent and therefore their pull-backs along the classifying map $\xi : X \to B\mathrm{Diff}_k$

$$\mathcal{X}_k^{\mathrm{GIT}} = \xi^* E_{\mathrm{Diff}}X_k^{\mathrm{GIT}} \text{ and } \mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}} = (\tau \circ \xi)^* E_{\mathrm{GL}}X_k^{\mathrm{GIT}}$$

are also homotopy equivalent. Let $\mathcal{V}^{\mathrm{univ}} = E\mathrm{GL}(n) \times_{\mathrm{GL}(n)} V$ denote the tautological bundle and α^{univ} the form we get by substituting the Chern roots of $\mathcal{V}^{\mathrm{univ}}$ into α. Then

$$\int_{\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}} (\tau \circ \xi)^* \alpha = \int_{\mathcal{X}_k^{\mathrm{GIT}}} \xi^* \alpha$$

gives the equality we wanted. □

To evaluate the integral $\int_{\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}} O_{\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}}(1)^{n+k(n-1)}$ we can first integrate (push forward) along the fibres of $\pi : \mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}} \to X$ followed by integration over X. In short, integration along the fibres can be done using localisation and is given by the residue formula of Theorem 5.1, then integration over X is equivalent to the substitution of the weights λ_i with the Chern roots of T_X.

More precisely, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}} & \longrightarrow & E_{\mathrm{GL}}X_k^{\mathrm{GIT}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau\circ\xi} & B\mathrm{GL}(n) \end{array}$$

which induces a diagram of cohomology maps

$$\begin{array}{ccc} H^*(\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}) & \longleftarrow & H^*(E_{\mathrm{GL}}X_k^{\mathrm{GIT}}) \\ \Big\downarrow \int_p & & \Big\downarrow \mathrm{Res} \\ H^*(X) & \underset{\mathrm{Sub}}{\longleftarrow} & H^*(B\mathrm{GL}(n)) \end{array}$$

Here

- Res is integration along the fibres, and these fibres are isomorphic to X_k^{GIT}. Theorem 5.4 tells that this residue operator sends $c_1(O_{E_{\mathrm{GL}}X_k^{\mathrm{GIT}}}(1))^{n+k(n-1)}$ to
$$\operatorname*{Res}_{w=\infty}\operatorname*{Res}_{z=\infty} \frac{(k-1)!(-z)^{k-1}(Nw+N(1-\epsilon)z)^{k(n-1)}dwdz}{(w+z)\prod_{m=0}^{k-1}\prod_{i=1}^{n}((\lambda_i+mz-w)}$$
- Sub is the substitution of the Chern roots of X into the weights $\lambda_1,\dots,\lambda_n$.
- $\int_p$ is integration along the fibre.

Commutativity tells us that integration along the fibre of a class pulled back from the tautological bundle over $E_{\mathrm{GL}}X_k^{\mathrm{GIT}}$ is given by applying the residue operation followed by the substitution of the Chern roots of X into the weights λ_i of the torus action.

The inverse of the total Chern class is the total Segre class of the bundle, so $c(X)^{-1} = s(X)$, and we can write
$$\frac{1}{\prod_{i=1}^{n}((\lambda_i+lz-w)} = \frac{1}{(lz-w)^n}s\left(\frac{1}{lz-w}\right) \text{ for } l=0,\dots,k-1$$
hence Proposition 5.4 gives the following integration formula on $\mathcal{X}_k^{\mathrm{GIT}}$.

Theorem 6.2. *With the notations introduced above we have*
$$\int_{\mathcal{X}_k^{\mathrm{GIT}}} c_1(O_{\mathcal{X}_k^{\mathrm{GIT}}}(1))^{n+k(n-1)} = \int_X \operatorname*{Res}_{w=\infty}\operatorname*{Res}_{z=\infty} \frac{(k-1)!(-z)^{k-1}(Nw+N(1-\epsilon)z)^{n+k(n-1)}dwdz}{(w+z)\prod_{l=0}^{k-1}(lz-w)^n}\prod_{l=0}^{k-1}s\left(\frac{1}{lz-w}\right)$$
Here the residue is a homogeneous degree $n = \dim(X)$ polynomial in the Segre classes of X, which is then integrated on X.

Remark 6.3. The key feature of this formula is that it separates the Segre classes from the residue variables, and hence the residue is a polynomial of degree n in the Segre classes. Without the second localisation on $\tilde{Z}_{\min}$, the formula involving z and θ would be more difficult to handle. This is because

$$H^*_{\mathbb{C}^*}(\tilde{Z}_{\min}) = H^*_{\mathbb{C}^*}(\mathbb{P}(TX)) = H^*(X)[\theta, z]/(\theta^n + c_1^{\mathbb{C}^*}(X)\theta^{n-1} + \ldots + c_n^{\mathbb{C}^*}(X))$$

and the relation $\theta^n + c_1^{\mathbb{C}^*}(X)\theta^{n-1} + \ldots + c_n^{\mathbb{C}^*}(X) = 0$ is hard to work with, being incompatible with our formula.

7. Proof of Theorem 1.3

Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree $\deg(X) = d$. Recall from (10) that

$$\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}} = \tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k})//\mathrm{Diff}_k(1)$$

is the GIT quotient of the blown-up projective bundle by $\mathrm{Diff}_k(1)$ defined by invariant sections of powers of the well-adapted ample linearisation L defined in §5:

$$\tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k})//\mathrm{Diff}_k(1) = \mathrm{Proj}(\oplus_{d\geq 0} H^0(\tilde{\mathbb{P}}_X, L^{\otimes d})^{\mathrm{Diff}_k(1)}).$$

The quotient map $q : \tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k})^{ss,\mathrm{Diff}_k(1)} \to \mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}$ is induced by the embedding of the invariant algebra

$$\rho : \oplus_{d\geq 0} H^0(\tilde{\mathbb{P}}_X, L^{\otimes d})^{\mathrm{Diff}_k(1)} \hookrightarrow \oplus_{d\geq 0} H^0(\tilde{\mathbb{P}}_X, L^{\otimes d})$$

Here we introduced the shorthand notation $\tilde{\mathbb{P}}_X = \tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k})$. Recall that we fixed N so that $L^{\otimes N}$ induces the ample line bundle $\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1)$ on the projective quotient which pulls back to the restriction of $L^{\otimes N}$ to the semistable locus.

Proposition 7.1. *Let $\pi : \mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}} \to X$ denote the projection. The direct image sheaf*

$$\pi_* \mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(m)) \subseteq \mathcal{O}(E_{k,\leq 2Nkm}))$$

is a subsheaf of the sheaf of holomorphic sections of $E_{k,\leq 2Nkm} = \oplus_{i=0}^{2Nkm} E_{k,i}$.

Proof. Let

$$\tilde{\mathbb{P}}_X = \tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k}) \subseteq \mathbb{P}((\mathcal{O}_X \oplus (T_X)^{\oplus k}) \otimes (T_X)^{\oplus k})$$

be the Segre embedding and $\tilde{\pi} : \tilde{\mathbb{P}}_X \to X$ be the fibration. Recall that we defined the line bundle L as the hyperplane line bundle $\mathcal{O}_{\mathbb{P}((\mathcal{O}_X \oplus (T_X)^{\oplus k}) \otimes (T_X)^{\oplus k})}(1)$ restricted to $\tilde{\mathbb{P}}$ (with a well-adapted shifted linearisation). Hence

$$\tilde{\pi}_* L^{\otimes N} \subseteq \mathcal{O}(((\mathcal{O}_X \oplus (T_X^*)^{\oplus k}) \otimes (T_X^*)^{\oplus k})^{\otimes N}) \subseteq \mathcal{O}(\oplus_{i=0}^{2N} (T_X^*)^{\otimes i}) \tag{11}$$

Sections of $\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1)$ pull-back and extend to $\mathrm{Diff}_k(1)$-invariant sections of $L^{\otimes N}$ on $\tilde{\mathbb{P}}$. Hence any section of the fibre $\pi_* \mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1))_x$ over $x \in X$ is some invariant polynomial $Q(f', f'', \ldots, f^{(k)})$ if we identify the jth copy of T_X^* in $(\mathcal{O}_X \oplus (T_X^*)^{\oplus k}) \otimes (T_X^*)^{\oplus k}$ with the jth derivative $f^{(j)}$ of a holomorphic jet f at x. The degree of $f^{(j)}$ in the jet differentials bundle is $j \leq k$, and by (11) the weighted homogeneous parts of Q have degree not bigger than $2Nk$. □

The following classical theorem connects global invariant jet differentials to the GGL conjecture.

Theorem 7.2 ((Fundamental vanishing theorem, Green–Griffiths [GG80], Demailly [Dem97], Siu [Siu02])). *Assume that there exist integers $k, m > 0$ and ample line bundle $A \to X$ such that there are nonzero global sections*

$$\emptyset \neq H^0(\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}, \mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(m) \otimes \pi^* A^{-1}) \hookrightarrow H^0(X, E_{k,\leq 2Nkm} \otimes A^{-1}).$$

Let $\sigma_1, \dots, \sigma_N$ be arbitrary nonzero sections and let $Z \subseteq J_k X$ be the base locus of these sections. Then every entire holomorphic curve $f : \mathbb{C} \to X$ necessarily satisfies $f_{[k]}(\mathbb{C}) \subseteq Z$. In other words, for every global $\mathrm{Diff}_k(1)$-invariant differential equation P vanishing on an ample divisor, every entire holomorphic curve f must satisfy the algebraic differential equation $P(f'(t), \dots, f^{(k)}(t)) \equiv 0$.

By Diverio [Div09, Theorem 1], for arbitrary ample A, $H^0(X, E_{k,m} \otimes A^{-1}) = 0$ holds for all $m \geq 1$ if $k < n$, so we can restrict our attention to the range $k \geq n$. Therefore, from now on **we will consider the $k = n$ case only.**

To control the order of vanishing of these differential forms along the ample divisor we choose A to be (as [DMR10]) a proper twist of the canonical bundle of X. Recall that the canonical bundle of the smooth, degree d hypersurface X is

$$K_X = \mathcal{O}_X(d - n - 2),$$

which is ample as soon as $d \geq n + 3$. The following theorem summarises the results of §3 in Diverio–Merker–Rousseau [DMR10] using the improved linear pole order for slanted vector fields by Darondeau [Dar16]

Theorem 7.3 ((Algebraic degeneracy of entire curves [DMR10] and [Dar16])). *Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $N = N(n, \delta)$ such that*

$$\emptyset \neq H^0(\mathcal{X}^{\mathrm{GIT}}_{n,\mathrm{GL}}, \mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{n,\mathrm{GL}}}(m) \otimes \pi^* K_X^{-2\delta Nnm}) \hookrightarrow H^0(X, E_{n,\leq 2Nnm} \otimes K_X^{-2\delta Nnm})$$

whenever $\deg(X) > N(n, \delta)$ *and* $m \gg 0$*. Then the Green-Griffiths-Lang conjecture holds whenever*

$$\deg(X) \geq \max(N(n, \delta), \frac{5n + 3}{\delta} + n + 2).$$

Following [DMR10], we choose A to be a proper twist of the canonical bundle of X, which is ample as soon as $d \geq n + 3$ and we prove the following theorem.

Theorem 7.4. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth complex hypersurface with ample canonical bundle, that is* $\deg X \geq n + 3$*. Then*

$$H^0(\mathcal{X}^{\mathrm{GIT}}_{n,\mathrm{GL}}, \mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{n,\mathrm{GL}}}(m) \otimes \pi^* K_X^{-2\delta Nnm}) \neq \emptyset$$

provided that $\delta = \frac{1}{16n^5}$, $\deg(X) > 2(4n)^5$, $2\delta Nnm \gg 0$ *is integer.*

Theorem 1.3 follows from Theorem 7.3 and Theorem 7.4.

To prove Theorem 7.4 we use the algebraic Morse inequalities of Demailly and Trapani to reduce the existence of global sections to the positivity of certain tautological integrals over $\mathcal{X}^{\mathrm{GIT}}_{n,\mathrm{GL}}$. Let $L \to X$ be a holomorphic line bundle over a compact Kähler manifold of dimension n and $E \to X$ a holomorphic vector bundle of rank r.

Theorem 7.5 ((Algebraic Morse inequalities, Demailly [Dem01], Trapani [Tra95])**.** *Suppose that $L = F \otimes G^{-1}$ is the difference of the nef line bundles F, G. Then for any nonnegative integer $q \in \mathbb{Z}_{\geq 0}$*

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes E) \leq r \frac{m^n}{n!} \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).$$

In particular, $q = 1$ asserts that $L^{\otimes m} \otimes E$ has a global section for m large provided that the intersection number $F^n - nF^{n-1}G$ is positive.

In order to apply this theorem we need an expression for $\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1) \otimes \pi^* K_X^{-2\delta N n}$ as a difference of nef bundles.

Proposition 7.6. *Let $d \geq n+3$ and therefore K_X ample. The following line bundles are nef on $\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}$:*

(1) $\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{n,\mathrm{GL}}}(1) \otimes \pi^* \mathcal{O}_X(4N)$

(2) $\pi^* \mathcal{O}_X(4N) \otimes \pi^* K_X^{2\delta N n}$ for any $\delta > 0$ and $2\delta N n$ integer.

Proof. First note that $T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}(2)$ is globally generated, and there is a surjective bundle map

$$(T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}(2))|_X^{\otimes m} \to T^*_X \otimes \mathcal{O}_X(2)^{\otimes m},$$

therefore $T^*_X \otimes \mathcal{O}_X(2)$ is globally generated. Consequently, the left hand side of the following surjective bundle map is globally generated for any D:

$$\Big(\mathcal{O}_X \oplus T^*_X \otimes \mathcal{O}_X(2) \oplus (T^*_X)^{\otimes 2} \otimes \mathcal{O}_X(4) \oplus \ldots \oplus (T^*_X)^{\otimes D} \otimes \mathcal{O}_X(2D)\Big) \to$$
$$\Big((\mathcal{O}_X \oplus T^*_X \oplus (T^*_X)^{\otimes 2} \oplus \ldots \oplus (T^*_X)^{\otimes D}) \otimes \mathcal{O}_X(2D)\Big)$$

Therefore the right hand side is also globally generated, so

$$\mathcal{O}_{\mathbb{P}(\oplus_{i=0}^{D}(T_X)^{\otimes i})}(1) \otimes \pi^* \mathcal{O}_X(2D) \tag{12}$$

is nef (note that in this paper we set $\mathbb{P}(V)$ to be the bundle of 1-dimensional subspaces in V, not the 1-dimensional quotients, hence sections of $\mathcal{O}_{\mathbb{P}(V)}(1)$ are points in V^*). Recall the Segre embedding

$$\tilde{\mathbb{P}}_X = \tilde{\mathbb{P}}(\mathcal{O}_X \oplus (T_X)^{\oplus k}) \subseteq \mathbb{P}((\mathcal{O}_X \oplus (T_X)^{\oplus k}) \otimes (T_X)^{\oplus k})$$

and that we defined the line bundle L as the hyperplane line bundle $\mathcal{O}_{\mathbb{P}((\mathcal{O}_X \oplus (T_X)^{\oplus k}) \otimes (T_X)^{\oplus k})}(1)$ restricted to $\tilde{\mathbb{P}}$ (with well-adapted shifted linearisation). Hence with $D = 2N$ (12) tells us that $L^{\otimes N} \otimes \tilde{\pi}^* \mathcal{O}(4N)$ is nef, hence the induced bundle $\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{n,\mathrm{GL}}}(1) \otimes \pi^* \mathcal{O}_X(4N)$ is also nef. The second part follows from the standard fact that the pull-back of an ample line bundle is nef. □

Consequently, we can express $\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1) \otimes \pi^* K_X^{-2\delta n N}$ as the following difference of two nef line bundles:

$$\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1) \otimes \pi^* K_X^{-2\delta n N} = (\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1) \otimes \pi^* \mathcal{O}_X(4N)) \otimes (\pi^* \mathcal{O}_X(4N) \otimes \pi^* K_X^{2\delta n N})^{-1}.$$

Theorem 7.4 follows from the Morse inequalities by proving that the following top form on $\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}$ is positive if $\delta = \frac{1}{16n^5}$ and $d > N(n,\delta) = 2(4n)^5$:

(13)
$$I(n,\delta) = c_1(\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1) \otimes \pi^* \mathcal{O}_X(2D))^{n^2} - n^2 c_1(\mathcal{O}_{\mathcal{X}^{\mathrm{GIT}}_{k,\mathrm{GL}}}(1) \otimes \pi^* \mathcal{O}_X(4N)^{(n^2-1)} c_1(\pi^* \mathcal{O}_X(4N) \otimes \pi^* K_X^{2\delta n N}).$$

Recall the notations $h = c_1(\mathcal{O}_X(1)), u = c_1(\mathcal{O}_{\mathcal{X}_{k,\mathrm{GL}}^{\mathrm{GIT}}}(1))$, and $c_1 = c_1(T_X)$ for the corresponding first Chern classes. Then $c_1(K_X) = -c_1 = (d-n-2)h$, and by dropping π^* from our formula (13) can be rewritten as

$$I(n,\delta) = (u+4Nh)^{n^2} - n^2(u+4Nh)^{n^2-1}(4Nh + 2\delta nN(d-n-2)h). \tag{14}$$

In the residue formula of Theorem 6.2 we substitute $u = Nw + N(1-\epsilon)z$ and we obtain

$$I_{n,\delta}(z,w,h) = (Nw+N(1-\epsilon)z+4Nh)^{n^2} - n^2(Nw+N(1-\epsilon)z+4Nh)^{n^2-1}(4Nh+2\delta nN(d-n-2)h). \tag{15}$$

Since $X \subseteq \mathbb{P}^{n+1}$ is a projective hypersurface we can express the Segre classes in Theorem 6.2 using that the Chern classes of X are expressible with $d = \deg(X)$ and h:

$$(1+h)^{n+2} = (1+dh)c(X),$$

where $c(X) = c(T_X)$ is the total Chern class of X. This gives

$$s(\frac{1}{x}) = \frac{1}{c(1/x)} = \left(1+\frac{dh}{x}\right)\left(1-\frac{h}{x}+\frac{h^2}{x^2}-\ldots\right)^{n+2}$$

and therefore Theorem 6.2 transforms into

$$\int_{\mathcal{X}_n^{\mathrm{GIT}}} I_{n,\delta} = \int_X \operatorname*{Res}_{w=\infty}\operatorname*{Res}_{z=\infty} \frac{(n-1)!(-z)^{n-1}I_{n,\delta}(z,w,h)dwdz}{(w+z)\prod_{l=0}^{n-1}(lz-w)^n}\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-w}\right)\left(1-\frac{h}{lz-w}+\ldots\right)^{n+2} \tag{16}$$

where from (15)

$$I_{n,\delta}(z,w,h) = N^{n^2}(w+(1-\epsilon)z+4h)^{n^2-1}\cdot\left(w+(1-\epsilon)z-2\delta n^3dh-\left(4n^2-2n^3\delta(n+2)-4\right)h\right).$$

It is easier to work with the residue formula after the following linear change of residue variables

$$z' = z+w, w' = w.$$

The determinant of this change is 1, and it does not change the homology class of the contour $z \gg w$, so it does not change the integral (see eg [Sze98]). Applying this and dropping the primes form the new variables we get

Proposition 7.7. *The integral* $\int_{\mathcal{X}_n^{\mathrm{GIT}}} I_{n,\delta}$ *is given by*

$$\int_X \operatorname*{Res}_{w=\infty}\operatorname*{Res}_{z=\infty}\frac{(n-1)!(z-w)^{n-1}I_{n,\delta}(z,w,h)dwdz}{(-1)^{n-1}z\prod_{l=0}^{n-1}(lz-(l+1)w)^n}\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)\left(1-\frac{h}{lz-(l+1)w}+\ldots\right)^{n+2}$$

where

$$I_{n,\delta}(z,w,h) = N^{n^2}((1-\epsilon)z+\epsilon w+4h)^{n^2-1}\cdot\left((1-\epsilon)z+\epsilon w-2\delta n^3dh-\left(4n^2-2n^3\delta(n+2)-4\right)h\right).$$

The main reason why the residue in Proposition 7.7 is computationally more suitable than its previous form in (16), is that the Taylor expansion of the first part of the rational expression has positive coefficients. Indeed, for $l \geq 1$

$$\frac{1}{lz-(l+1)w} = \frac{1}{lz}\left(1+\frac{(l+1)w}{lz}+\ldots\right) \text{ and } \frac{z-w}{z-2w} = 1+\frac{w}{z}\left(1+\frac{2w}{z}+\frac{4w^2}{z^2}+\ldots\right)$$

Hence the residue transforms into

(17)

$$\operatorname*{Res}_{w=\infty}\operatorname*{Res}_{z=\infty}\frac{-I_{n,\delta}(z,w,h)dwdz}{((n-1)!)^{n-1}z^{n(n-1)-1}(z-2w)w^n}\left(1+\frac{w}{z}\left(1+\frac{2w}{z}+\frac{4w^2}{z^2}+\ldots\right)\right)^{n-1}\prod_{l=2}^{n-1}\left(1+\frac{(l+1)w}{lz}+\ldots\right)^n$$
$$\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)\left(1-\frac{h}{lz-(l+1)w}+\ldots\right)^{n+2}.$$

7.1. **A first look at the iterated residue formula.** As a first step in analysing the residue formula (17) we write this integral as a polynomial in d and study its leading coefficient.

Notation 7.8. *For a nonnegative integer i and a partition $i = i_0 + i_1 + \ldots + i_{n-1}$ into integer vector $\mathbf{i} = (i_0, \ldots, i_{n-1})$ we introduce the shorthand notation*

$$C^{\mathbf{i}} = \operatorname*{Res}_{w=\infty}\operatorname*{Res}_{z=\infty}\frac{(n-1)!(z-w)^{n-1}((1-\epsilon)z+\epsilon w)^{n^2-i}}{z\prod_{l=0}^{n-1}(lz-(l+1)w)^{n+1-i_l}}.$$

Proposition 7.9. *(1) $\int_{X_n^{\mathrm{GIT}}} I_{n,\delta}$ is a polynomial in d of degree $n+1$ with zero constant term:*

$$\int_{X_n^{\mathrm{GIT}}} I_{n,\delta} = p_{n+1}(n,\delta)d^{n+1} + p_n(n,\delta)d^n + \ldots + p_1(n,\delta)d$$

where $p_i(n,\delta)$ is linear in δ and polynomial in n for all i.

(2) The leading coefficient is $p_{n+1}(n,\delta) > C^{\mathbf{0}}\left(1-\frac{2\delta n^5}{\epsilon(1-\epsilon)}\right)$ and $C^{\mathbf{0}} = C^{\mathbf{0}}(N,n,\epsilon) > 0$ is positive.

Proof. The residue in (17) is by definition the coefficient of $\frac{1}{zw}$ in the Laurent expansion of the rational expression in z, n, d, h and δ on the contour $z \gg w$, that is, in w/z. The result is a polynomial in n, d, h, δ, and in fact, a relatively easy argument shows that it is a polynomial in n, d, δ multiplied by h^n Indeed, setting degree 1 to z, w, h and 0 to n, d, δ, the rational expression in the residue has total degree $n-2$. Therefore the coefficient of $\frac{1}{zw}$ has degree n, so it has the form $h^n p(n,d,\delta)$ with a polynomial p. Since $\int_X h^n = d$, integration over X is simply a substitution $h^n = d$, resulting in the equation $\int_{X_n^{\mathrm{GIT}}} I_{n,\delta} = dp(n,\delta,d)$ for some polynomial $p(n,\delta,d)$. The highest power of d in $p(n,\delta,d)$ is d^n which proves the first part.

To prove the second part note that to get d^{n+1} in (17), we have two options.

(i) The first is to choose all the $\frac{dh}{lz-(l+1)w}$ terms in the product $\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)$, this contributes with

$$C^{\mathbf{0}} = \operatorname*{Res}_{w,z=\infty}\frac{N^{n^2}((1-\epsilon)z+\epsilon w)^{n^2}}{((n-1)!)^n z^{n(n-1)-1}(z-2w)^2w^{n+1}}\left(1+\frac{w}{2z}+\frac{w}{4z^2}+\ldots\right)^{n-1}\prod_{l=2}^{n-1}\left(1+\frac{(l+1)w}{lz}+\ldots\right)^{n+1}.$$

The rational expression here has Taylor expansion with positive coefficients, so $C^{\mathbf{0}}$ is automatically positive.

(ii) Alternatively we need to pick all but one $\frac{dh}{lz-(l+1)w}$ terms from the product $\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)$ and the $2\delta n^3 dh$ term from $I_{z,w,h}$. This way the contribution is $\sum_{s=0}^{n-1} 2\delta n^3 C^{\mathbf{e_s}}$ where $\mathbf{e}_s$ is the unit

vector with all but the s coordinate zero. We can write out these terms as follows.
For $2 \le s \le n-1$:

$$C^{\mathbf{e}_s} = -\operatorname*{Res}_{w,z=\infty} \frac{N^{n^2} s((1-\epsilon)z+\epsilon w)^{n^2-1}}{((n-1)!)^n z^{n(n-1)-2}(z-2w)^2 w^{n+1}} \left(1+\frac{w}{2z}+\frac{w}{4z^2}+\dots\right)^{n-1} \prod_{l=2}^{n-1}\left(1+\frac{(l+1)w}{lz}+\dots\right)^{n+1-\delta_{l,s}}$$

For $s=1$:

$$C^{\mathbf{e}_1} = -\operatorname*{Res}_{w,z=\infty} \frac{N^{n^2}((1-\epsilon)z+\epsilon w)^{n^2-1}}{((n-1)!)^n z^{n(n-1)-1}(z-2w) w^{n+1}} \left(1+\frac{w}{2z}+\frac{w}{4z^2}+\dots\right)^{n-1} \prod_{l=2}^{n-1}\left(1+\frac{(l+1)w}{lz}+\dots\right)^{n+1}$$

For $s=0$:

$$C^{\mathbf{e}_0} = \operatorname*{Res}_{w,z=\infty} \frac{N^{n^2}((1-\epsilon)z+\epsilon w)^{n^2-1}}{((n-1)!)^n z^{n(n-1)-1}(z-2w)^2 w^{n}} \left(1+\frac{w}{2z}+\frac{w}{4z^2}+\dots\right)^{n-1} \prod_{l=2}^{n-1}\left(1+\frac{(l+1)w}{lz}+\dots\right)^{n+1}$$

Here $\delta_{l,s}$ is 1 if $l=s$ and 0 otherwise. Due to the positivity of the Taylor expansion, $C^{\mathbf{e}_s} < \frac{nC^{\mathbf{0}}}{\epsilon(1-\epsilon)}$ holds and hence all these contributions are less than $\frac{2\delta n^4 C}{\epsilon(1-\epsilon)}$. Thus the second part is proved. ◻

We obtain the following corollary.

Proposition 7.10. *If $\delta < \frac{\epsilon(1-\epsilon)}{2n^5}$ then the leading coefficient $p_{n+1}(n,\delta) > 0$ is positive, and therefore $\int_{\mathcal{X}_n^{\mathrm{GIT}}} I_{n,\delta} > 0$ for $d \gg 0$.*

7.2. **Strategy for the proof of positivity of $I(n,\delta,d)$.** According to Proposition 7.9 we have to prove the positivity of the polynomial $\int_{\mathcal{X}_n^{\mathrm{GIT}}} = p_{n+1}(n,\delta)d^{n+1} + p_n(n,\delta)d^n + \dots + p_1(n,\delta)d$. The strategy is to show that for small enough δ the other coefficients satisfy

$$|p_{n+1-l}| < n^{4l} p_{n+1} \tag{18}$$

for $1 \le l \le n+1$. Then we can apply the following elementary statement.

Lemma 7.11 (Fujiwara bound)**.** *If $p(d) = p_{n+1}d^{n+1} + p_n d^n + \dots + p_1 d + p_0 \in \mathbb{R}[d]$ satisfies the inequalities*

$$p_{n+1} > 0; \quad |p_{n+1-l}| < D^l |p_{n+1}| \text{ for } l = 1, \dots n+1,$$

then $p(d) > 0$ for $d > 2D$.

7.3. **Estimation of the other coefficients.** We start with the study of the next coefficient, p_n. Similarly to the proof of Proposition 7.9 (2), here we can distinguish four cases how we can get d^n from the residue (17).

(i) If we take $n-1$ terms from the product $\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)$ and one h from $I_{z,w,h}$ then we get

$$\sum_{s=0}^{n-1} 2\delta n^3 (n+2) C^{\mathbf{e}_s}.$$

(ii) If we take $n-1$ terms from $\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)$ and one h from $\prod_{l=0}^{n-1}(1-\frac{h}{lz-(l+1)w}+\dots)^{n+2}$ then the contribution is

$$\sum_{s=0}^{n-1}\sum_{t=0}^{n-1}(n+2)C^{\mathbf{e}_s-\mathbf{e}_t}.$$

(iii) If we take $n-2$ terms from $\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)$, one dh from $I_{z,w,h}$ and one h from $\prod_{l=0}^{n-1}(1-\frac{h}{lz-(l+1)w}+\ldots)^{n+2}$ then the contribution is

$$\sum_{s=0}^{n-1}\sum_{t=0}^{n-1}\sum_{u=0}^{n-1} 2\delta n^3(n+2)C^{\mathbf{e}_s+\mathbf{e}_t-\mathbf{e}_u}.$$

(iv) Finally, if we take $n-2$ terms from $\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)$, one dh and one h from $I_{z,w,h}$ then the contribution is

$$\sum_{s=0}^{n-1}\sum_{t=0}^{n-1} 8\delta n^3(n^2-1)C^{\mathbf{e}_s+\mathbf{e}_t}.$$

Using the positivity of the Taylor expansion, the following estimations are easy to check:

$$C^{\mathbf{e}_s} < \frac{nC^{\mathbf{0}}}{\epsilon(1-\epsilon)},\quad C^{\mathbf{e}_s+\mathbf{e}_t} < \frac{n^2C^{\mathbf{0}}}{\epsilon^2(1-\epsilon)^2}.$$

$$C^{\mathbf{e}_s-\mathbf{e}_t} < \begin{cases} \frac{nC^{\mathbf{0}}}{\epsilon(1-\epsilon)} & \text{if } t>0 \\ \frac{n^2C^{\mathbf{0}}}{\epsilon(1-\epsilon)} & \text{if } t=0. \end{cases}$$

$$C^{\mathbf{e}_s+\mathbf{e}_t-\mathbf{e}_u} < \begin{cases} \frac{n^2C^{\mathbf{0}}}{\epsilon(1-\epsilon)} & \text{if } u>0 \\ \frac{n^3C^{\mathbf{0}}}{\epsilon(1-\epsilon)} & \text{if } u=0. \end{cases}$$

Moreover, if $\delta > \frac{1}{2n^5}$ holds, the dominant contributions of p_n are (iii) and (iv), and they give

$$|p_n| < \frac{10\delta n^9 C^{\mathbf{0}}}{\epsilon(1-\epsilon)}.$$

Similar computation shows that the dominant part in p_{n-s} for $0\le s\le n$ are the terms corresponding to the choice when we take $n-s-2$ terms from $\prod_{l=0}^{n-1}\left(1+\frac{dh}{lz-(l+1)w}\right)$ one dh and h^u from $I_{z,w,h}$ and h^{s+1-u} from $\prod_{l=0}^{n-1}(1-\frac{h}{lz-(l+1)w}+\ldots)^{n+2}$. This contribution is less than

$$\sum_{\alpha_1,\ldots,\alpha_{s+2}=0}^{n-1}\ \sum_{\beta_1,\ldots,\beta_{s+1-u}=0}^{n-1} 2\delta n^3(n+2)^{s+1-u}4^u\binom{n^2-1}{u}C^{(\mathbf{e}_{\alpha_1}+\ldots+\mathbf{e}_{\alpha_{s+2}})-(\mathbf{e}_{\beta_1}+\ldots+\mathbf{e}_{\beta_{s+1-u}})}$$

If the number of zeros among $\beta_1,\ldots,\beta_{s+1-u}$ is t then

$$C^{(\mathbf{e}_{\alpha_1}+\ldots+\mathbf{e}_{\alpha_{s+2}})-(\mathbf{e}_{\beta_1}+\ldots+\mathbf{e}_{\beta_{s+1-u}})} < \frac{n^{s+2+t}C^{\mathbf{0}}}{(\epsilon(1-\epsilon))^u}.$$

Hence for $\delta > \frac{1}{2n^5}$ we have

$$|p_{n-s}| < \frac{\delta 4^s n^{4s+9}C^{\mathbf{0}}}{s!(\epsilon(1-\epsilon))^{s+1}} < \frac{\delta n^{5s+9}C^{\mathbf{0}}}{(\epsilon(1-\epsilon))^{s+1}}.$$

7.4. **Proof of positivity.** To finish the proof, we calibrate ϵ and δ to give the best bound. We first fix $\epsilon = 1/2$, $\delta = \frac{\epsilon(1-\epsilon)}{4n^5} = \frac{1}{16n^5}$. With this choice we have:

- $p_{n+1}(n,\delta) > C^{\mathbf{0}}\left(1 - \frac{2\delta n^5}{\epsilon(1-\epsilon)}\right) = \frac{1}{2}C^{\mathbf{0}} > 0$;
- $|p_{n-s}| < \frac{\delta n^{5s+9}C^{\mathbf{0}}}{(\epsilon(1-\epsilon))^{s+1}} < (4n)^{5s+4}C^{\mathbf{0}}$.

The Fujiwara estimation of Lemma 7.11 works with $D = (4n)^5$. Hence for $\delta = \frac{1}{16n^5}$ and $d > 2(4n)^5$ the integral $\int_{\mathcal{X}_n^{\mathrm{GIT}}} I_{n,\delta} > 0$ is positive, hence it provides the existence of nonzero sections in Theorem 7.3. Then Theorem 7.4 applied with

$$d > \max(2(4n)^5, \frac{5n+3}{\delta} + n + 2) = 16n^5(5n+3) + n + 2$$

finishes the proof of Theorem 1.3.

References

[BDHK] Gergely Bérczi, Brent Doran, Thomas Hawes, and Frances Kirwan, *Projective completions of graded unipotent quotients*, arXiv:1607.04181.

[BDHK18] Gergely Brczi, Brent Doran, Thomas Hawes, and Frances Kirwan, *Geometric invariant theory for graded unipotent groups and applications*, Journal of Topology **11** (2018), no. 3, 826–855.

[BK] Gergely Bérczi and Frances Kirwan, *Moment maps and cohomology of non-reductive quotients*.

[BK12] ———, *A geometric construction for invariant jet differentials*, Surveys in differential geometry. Vol. XVII, Surv. Differ. Geom., vol. 17, Int. Press, Boston, MA, 2012, pp. 79–125. MR 3076059

[BK17] Gergely Bérczi and Frances Kirwan, *Graded unipotent groups and Grosshans theory*, Forum of Mathematics, Sigma **5** (2017), e21.

[Bro17] Damian Brotbek, *On the hyperbolicity of general hypersurfaces*, Publ. Math. Inst. Hautes Etudes Sci. **126** (2017), 1–34.

[BS12] Gergely Bérczi and András Szenes, *Thom polynomials of Morin singularities*, Ann. of Math. (2) **175** (2012), no. 2, 567–629. MR 2877067

[B17] Gergely Brczi, *Tautological integrals on curvilinear hilbert schemes*, Geom. Topol. **21** (2017), no. 5, 2897–2944.

[B18] ———, *Thom Polynomials and the GreenGriffithsLang Conjecture for Hypersurfaces with Polynomial Degree*, International Mathematics Research Notices (2018).

[B19] ———, *Towards the GreenGriffithsLang conjecture via equivariant localisation*, Proceedings of the London Mathematical Society **118** (2019), no. 5, 1057–1083.

[Dar15] Lionel Darondeau, *On the Logarithmic Green-Griffiths conjecture*, International Mathematics Research Notices **2016(6)** (2015), 1871–1923.

[Dar16] ———, *Slanted vector fields for jet spaces*, Math Z. **282** (2016), 547–575.

[Dem97] Jean-Pierre Demailly, *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 285–360. MR 1492539

[Dem01] ———, *Multiplier ideal sheaves and analytic methods in algebraic geometry*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), ICTP Lect. Notes, vol. 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 1–148. MR 1919457

[Dem18] ———, *Recent results on the Kobayashi and Green-Griffiths-Lang conjectures*, Contribution to the 16th Takagi lectures in celebration of the 100th anniversary of K.Kodaira.

[Den] Ya Deng, *On the Diverio-Trapani conjecture*, To appear in Annales Scientifiques de l'École Normale Supérieure, arXiv:1703.07560.

[DH98] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Publications Mathématiques de l'IHÉS **87** (1998), 5–51 (en). MR 2000b:14060

[Div09] Simone Diverio, *Existence of global invariant jet differentials on projective hypersurfaces of high degree*, Math. Ann. **344** (2009), no. 2, 293–315. MR 2495771

[DK07] Brent Doran and Frances Kirwan, *Towards non-reductive geometric invariant theory*, Pure Appl. Math. Q. **3** (2007), no. 1, part 3, 61–105. MR 2330155

[DMR10] Simone Diverio, Joël Merker, and Erwan Rousseau, *Effective algebraic degeneracy*, Invent. Math. **180** (2010), no. 1, 161–223. MR 2593279

[DR11] Simone Diverio and Erwan Rousseau, *A survey on hyperbolicity of projective hypersurfaces*, IMPA Mathematical Publications, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2011, [On the title page: A survey on hiperbolicity of projective hypersurfaces]. MR 2933164

[GG80] Mark Green and Phillip Griffiths, *Two applications of algebraic geometry to entire holomorphic mappings*, The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), Springer, New York-Berlin, 1980, pp. 41–74. MR 609557

[GH94] Phillip A Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley classics library, Wiley, New York, NY, 1994.

[GKM98] Mark Goresky, Robert Kottwitz, and Robert Macpherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math **131** (1998), 25–83.

[JK95] Lisa C. Jeffrey and Frances C. Kirwan, *Localization for nonabelian group actions*, Topology **34** (1995), no. 2, 291 – 327.

[Kir84] Frances Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, no. 31, Princeton University Press, Princeton, NJ, 1984.

[Kob70] Shoshichi Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure and Applied Mathematics, vol. 2, Marcel Dekker, Inc., New York, 1970.

[Lan86] Serge Lang, *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), no. 2, 159–205. MR 828820

[LMS75] A. T. Lascu, D. Mumford, and D. B. Scott, *The self-intersection formula and the formule-clef*, Mathematical Proceedings of the Cambridge Philosophical Society **78** (1975), no. 1, 117123.

[Mar] Shaun Martin, *Symplectic quotients by a nonabelian group and by its maximal torus*, arxiv:0001.002.

[McQ98] Michael McQuillan, *Diophantine approximations and foliations*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 87, 121–174. MR 1659270

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906

[MT] Joël Merker and The-Anh Ta, *Degrees* $(d \geq \sqrt{n}\log n)^n$ *and* $d \geq (n\log n)^n$ *in the conjectures of Green-Griffiths and of Kobayashi*, arXiv:1901.04042.

[Rim01] Richárd Rimányi, *Thom polynomials, symmetries and incidences of singularities*, Invent. Math. **143** (2001), no. 3, 499–521. MR 1817643

[RY] Eric Riedl and David Yang, *Applications of a Grassmannian technique in hypersurfaces*, arXiv:1806.02364.

[Siu02] Yum-Tong Siu, *Some recent transcendental techniques in algebraic and complex geometry*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 439–448. MR 1989197

[Siu04] ———, *Hyperbolicity in complex geometry*, The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 543–566. MR 2077584

[Siu15] ———, *Hyperbolicity of generic high-degree hypersurfaces in complex projective space*, Invent. Math. **202** (2015), no. 3, 1069–1166. MR 3425387

[SY96] Yum-Tong Siu and Sai-kee Yeung, *Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane*, Invent. Math. **124** (1996), no. 1-3, 573–618. MR 1369429

[Sze98] András Szenes, *Iterated residues and multiple Bernoulli polynomials*, Internat. Math. Res. Notices (1998), no. 18, 937–956. MR 1653791

[Tha96] Michael Thaddeus, *Geometric invariant theory and flips*, Jour. AMS (1996), 691–729.

[Tra95] Stefano Trapani, *Numerical criteria for the positivity of the difference of ample divisors*, Math. Z. **219** (1995), no. 3, 387–401. MR 1339712

[Ver96] Michle Vergne, *A note on the Jeffrey-Kirwan-Witten localisation formula*, Topology **35** (1996), no. 1, 243 – 266.

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