MOMENT MAPS AND COHOMOLOGY OF NON-REDUCTIVE QUOTIENTS

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Abstract. Let H be a complex linear algebraic group with internally graded unipotent radical acting on a complex projective variety X. Given an ample linearisation of the action and an associated Fubini–Study Kähler metric which is invariant for a maximal compact subgroup Q of H, we define a notion of moment map for the action of H, and show that when the (non-reductive) GIT quotient $X//H$ introduced in [4] exists it can be identified with the quotient by Q of a suitable level set for this moment map. When semistability coincides with stability for the action of H, we derive formulas for the Betti numbers of $X//H$ and we express the rational cohomology ring of $X//H$ in terms of the rational cohomology ring of the GIT quotient $X//T^H$, where T^H is a maximal torus in H. We relate intersection pairings on $X//H$ to intersection pairings on $X//T^H$, obtaining a residue formula for these pairings on $X//H$ analogous to the residue formula of [25]. As an application, we announce a proof of the Green–Griffiths–Lang and Kobayashi conjectures for projective hypersurfaces with polynomial degree.

1. Introduction

The aim of this paper is to extend the results of the second author [29] and Martin [37] in computing the rational cohomology of reductive GIT quotients to cases when the group which acts is not necessarily reductive. The methods developed in this paper are used in [6] to prove the Green–Griffiths–Lang conjecture and Kobayashi conjecture on hyperbolicity for generic projective hypersurfaces with polynomial degree.

In [2, 4] the existence was proved of projective GIT quotients of complex projective varieties by suitable linear actions of linear algebraic groups with internally graded unipotent radicals. Here we say that a linear algebraic group $H = U \rtimes R$ over $\mathbb{C}$, with unipotent radical U and Levi subgroup R, has internally graded unipotent radical if R has a central one-dimensional torus $\lambda : \mathbb{G}_m \to Z(R)$ whose weights for the adjoint action on the Lie algebra of U are all strictly positive. When H acts linearly on a projective variety and 'semistability coincides with stability' for the action of $\hat{U} = U \rtimes \lambda(\mathbb{G}_m)$, it is shown in [4] that — after proper adjustment of the linearisation — many of the key features and computational strengths of reductive GIT extend to this non-reductive situation. The present paper studies the rational cohomology of these non-reductive GIT quotients.

When X is a nonsingular complex projective variety acted on by a complex reductive group G with respect to an ample linearisation, we can choose a maximal compact subgroup K of G and a K-invariant Fubini–Study Kähler metric on X with corresponding moment map $\mu : X \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{k}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{k}, \mathbb{R})$ is its dual. Here $\mathfrak{k}^*$ embeds naturally in the complex dual $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{k}, \mathbb{C})$ of the Lie algebra $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ of G, as $\mathfrak{k}^* = \{\xi \in \mathfrak{g}^* : \xi(\mathfrak{k}) \subseteq \mathbb{R}\}$; using this identification we can regard $\mu : X \to \mathfrak{g}^*$ (or a G-equivariant family of such μ parametrised by all the maximal compact subgroups of G) as a 'moment map' for the action of G, although of course not a moment map for G in the traditional sense of symplectic geometry.

The second author in [29] used the norm-square $f = \|\mu\|^2$ of the moment map $\mu : X \to \mathfrak{k}^*$ as a ('minimally degenerate') Morse function and showed that the open stratum of the corresponding Morse stratification, which retracts K-equivariantly onto the zero level set $\mu^{-1}(0)$ of the moment map, coincides with the GIT semistable locus X^{ss} for the linear action of G on X. Moreover, $\mu^{-1}(0)$ is K-invariant and its inclusion in X^{ss} induces a homeomorphism

$$\mu^{-1}(0)/K \cong X/\!/G. \tag{1}$$

The equivariant Morse inequalities relate the equivariant Betti numbers of X to those of X^{ss} and the other (unstable) strata. In fact it is shown in [29] that the Morse stratification is equivariantly perfect, so that the equivariant Morse inequalities are equalities, and these Morse equalities provide explicit formulas for the G-equivariant Betti numbers of X^{ss} in terms of the G-equivariant Betti numbers of X and those of the other Morse strata, which can in turn be described inductively in terms of semistable loci for actions of reductive subgroups of G on nonsingular projective subvarieties of X. If $X^s = X^{ss}$, then the G-equivariant rational cohomology of X^{ss} coincides with the ordinary rational cohomology of its geometric quotient X^{ss}/G which is the GIT quotient $X/\!/G$, and therefore we get expressions for the Betti numbers of $X/\!/G$ in terms of the equivariant Betti numbers of the unstable GIT strata and of X.

Martin [37] used (1) to relate the topology of the quotient $X/\!/G$ and the associated quotient $X/\!/T$, where $T \subset G$ is a maximal torus. He proved a formula expressing the rational cohomology ring of $X/\!/G$ in terms of the rational cohomology ring of $X/\!/T$, and an integration formula which expresses intersection pairings on $X/\!/G$ in terms of intersection pairings on $X/\!/T$. This integration formula, combined with methods from abelian localisation, leads to residue formulas for cohomology pairings on $X/\!/G$ which are closely related to those of [25].

In this paper we follow a similar path for non-reductive actions. Let X be a nonsingular complex projective variety endowed with a linear action of a complex linear algebraic group $H = U \rtimes R$ with respect to an ample line bundle L. Here U is the unipotent radical of H, and R is the complexification of a maximal compact subgroup Q of H (so that R and Q are determined up to conjugation in H). Assume that the unipotent radical U of H is internally graded by a central 1-parameter subgroup $\lambda : \mathbb{C}^* \to R$ of the Levi subgroup R, and let $\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \subset H$; then $\lambda(S^1) \subset \lambda(\mathbb{C}^*) \subset \hat{U}$ is a maximal compact subgroup of $\hat{U}$. Assume also that semistability coincides with stability for the $\hat{U}$-action, in the sense of [4] (Definition 2.10 below). In [4] it is shown that then, after a suitable shift of the linearisation by a (rational) character of R — we call the resulting linearisation well-adapted — the algebra of invariants $\oplus_{k\geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated, and the corresponding projective variety $X/\!/H$ satisfies some of the key properties of Mumford's GIT: (i) it is the set of S-equivalence classes for the H action on a semistable locus $X^{ss,H}$, (ii) it contains as an open subvariety a geometric quotient of a stable locus $X^{s,H} \subset X^{ss,H}$, and (iii) the semistable and stable loci are described by Hilbert-Mumford type numerical criterion.

Using the embeddings $X \subset \mathbb{P}^n$ defined by very ample tensor powers of L, and corresponding Fubini–Study Kähler metrics invariant under maximal compact subgroups Q of H (or H-equivariant families of such metrics parametrised by the maximal compact subgroups of H), in this paper we define 'H-moment maps' $\mu_H : X \to \mathrm{Lie} H^* = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Lie} H, \mathbb{C})$ by composing $G = \mathrm{GL}(n+1)$-moment maps $\mu_G : X \to \mathfrak{g}^*$ with the maps of complex duals $\mathfrak{g}^* \to \mathrm{Lie}(H)^*$ coming from the representations $H \to \mathrm{GL}(n+1)$; cf. [35]. (When H is reductive this is compatible with the previous definition). Our main theorem is the following non-reductive analogue of (1).

Theorem 1.1. *Let X be a smooth complex projective variety endowed with a well-adapted action of a complex linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U and maximal compact subgroup $Q \leqslant R$. If H-stability=H-semistability (see Definition 2.10) then $H\mu_H^{-1}(0) = X^{s,H} = X^{ss,H}$ and the embedding of $\mu_H^{-1}(0)$ in $X^{ss,H}$ induces a homeomorphism*

$$\mu_H^{-1}(0)/Q \simeq X/\!/H = X^{s,H}/H.$$

In the special case when $H = \hat{U} = U \rtimes \mathbb{C}^*$, this tells that the embedding $\mu_{\hat{U}}^{-1}(0) \hookrightarrow X^{ss,\hat{U}}$ induces a homeomorphism $\mu_{\hat{U}}^{-1}(0)/S^1 \simeq X//\hat{U}$.

Remark 1.2. In fact, in order to have an embedding of $\mu_H^{-1}(0)$ in $X^{ss,H}$ and an induced homeomorphism $\mu_H^{-1}(0)/Q \simeq X//H$, the condition that H-stability=H-semistability can be weakened to the requirement that $\hat{U}$-stability=$\hat{U}$-semistability. For then there is an embedding of $\mu_{\hat{U}}^{-1}(0)$ in $X^{ss,\hat{U}}$ with an induced homeomorphism $\mu_{\hat{U}}^{-1}(0)/\lambda(S^1) \simeq X//\hat{U}$, and we can apply (1) to the induced action of $R/\lambda(\mathbb{C}^*)$ to obtain the corresponding result for H.

Let $\omega_0 < \omega_1 < \ldots < \omega_r$ be the weights of the linear action of $\lambda(\mathbb{C}^*)$ on X. Set

$$Z_{\min} = \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \lambda(\mathbb{C}^*)\text{-fixed point and} \\ \lambda(\mathbb{C}^*) \text{ acts on } L^*|_x \text{ with weight } \omega_0 \end{array} \right\}$$

and

$$X^0_{\min} = \left\{ x \in X \;\middle|\; \begin{array}{c} \lim_{t\to 0} \lambda(t)x \text{ lies} \\ \text{in } Z_{min} \end{array} \right\}.$$

Theorem 1.3 (Betti numbers). *Let X be a smooth complex projective variety endowed with a well-adapted action of a linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U. Let $\lambda : \mathbb{C}^* \to Z(R)$ be the central one-parameter subgroup of R which grades U, and let $\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \subset H$ as above. Assume that the condition that $\hat{U}$-stability=$\hat{U}$-semistability holds (see Definition 2.10).*

(1) The stratification $X^0_{\min} = X^{s,\hat{U}} \sqcup UZ_{\min}$ is equivariantly perfect, and the Poincaré series of the GIT quotient $X//\hat{U}$ satisfies

$$P_t(X//\hat{U}) = P_t(Z_{\min}) \frac{1 - t^{2(\dim(X) - \dim(Z_{\min}) - \dim(U))}}{1 - t^2}.$$

(2) Assume that stability=semistability holds for the $R/\lambda(\mathbb{C}^*)$ action on $Z_{\min}$. Then

$$P_t(X//H) = P_t(Z_{\min}//(R/\lambda(\mathbb{C}^*))) \frac{1 - t^{2(\dim(X) - \dim(Z_{\min}) - \dim(U))}}{1 - t^2}.$$

Note that a key difference from the reductive picture of [29] is that for well-adapted actions as above we have a distinguished subset $Z_{\min}$ which – as the formulas make clear – carries much of the cohomological information about $X//\hat{U}$ and $X//H$.

Remark 1.4. More generally if the centre of R has a subtorus which acts trivially on $Z_{\min}$ such that stability coincides with semistability for the induced action on $Z_{\min}$ of the quotient $\bar{R}$ of R by this subtorus, then $P_t(X//H)$ is the product of $P_t(Z_{\min}//\bar{R})$ by the Poincaré polynomial of a toric variety which is a GIT quotient by this subtorus of the fibre (an affine space) of the map $X^0_{\min} \to Z_{\min}$ which sends x to $\lim_{t\to 0} \lambda(t)x$.

In the second half of the paper we adapt the argument of Martin [37] to actions of complex linear algebraic groups with internally graded unipotent radicals, and extend his formulas to express the rational cohomology ring of $X//H$ in terms of the rational cohomology ring of the torus quotient $X//T^H$ where T^H is a maximal torus of R (and hence of H). We study the following key diagram, where T^Q is the compact torus which is a maximal compact subgroup of T^H (and so is a maximal torus of the maximal compact subgroup Q of R):

$$\begin{array}{ccc} \mu_H^{-1}(0)/T^Q & \xhookrightarrow{i} & \mu_{T^H}^{-1}(0)/T^Q = X//T^H \\ \downarrow \pi & & \\ \mu_H^{-1}(0)/Q = X//H. & & \end{array} \tag{2}$$

Theorem 1.5 (**Cohomology rings**)**.** *Let X be a smooth projective variety endowed with a well-adapted action of $H = U \rtimes R$ such that H-stability=H-semistability holds. Then there is a natural ring isomorphism*

$$H^*(X//H, \mathbb{Q}) \simeq \frac{H^*(X//T^H, \mathbb{Q})^W}{ann(e)}$$

where W denotes the Weyl group of R, which acts naturally on $X//T^H$, while $e = \mathrm{Euler}(V) \in H^*(X//T^H)^W$ is the Euler class of the bundle V associated to the roots of R and weights of the adjoint action of T^H on $\mathrm{Lie}(U)$ *(see Definition 5.11 for details). Finally*

$$\mathrm{ann}(e) = \{c \in H^*(X//T, \mathbb{Q}) | c \cup e = 0\} \subset H^*(X//T^H, \mathbb{Q}).$$

is the annihilator ideal.

Diagram (2) provides a natural way to define a lift of a cohomology class on $X//H$ to a class on $X//T^H$: we say that $\tilde{a} \in H^*(X//T^H)$ is a lift of $a \in H^*(X//H)$ if $\pi^* a = i^* \tilde{a}$.

Theorem 1.6 (**Integration formula**)**.** *Let X be a smooth projective variety endowed with a well-adapted action of $H = U \rtimes R$ such that H-stability=H-semistability holds. Assume that the stabiliser in H of a generic $x \in X$ is trivial. Given a cohomology class $a \in H^*(X//H)$ with a lift $\tilde{a} \in H^*(X//T^H)$, then*

$$\int_{X//H} a = \frac{1}{|W|} \int_{X//T^H} \tilde{a} \cup e$$

where we use the notation of Theorem 1.5.

Using these results, we prove a residue formula for intersection pairings on $X//H$; see Theorem 5.15.

Remark 1.7. Theorem 1.6 can be generalised to allow the triviality assumption for the stabiliser in H of a generic $x \in X$ to be omitted; then the sizes of the stabilisers in H and T^H of a generic $x \in X$ are included in the formula for $\int_{X//H} a$.

The layout of the paper is as follows. We start in §2 with a brief overview of non-reductive GIT and results of [4]. In §3 we define moment maps for linear actions of complex linear algebraic groups with internally graded unipotent radicals: first we focus on $\hat{U}$ actions and prove Theorem 1.1 for these, before turning to more general H actions. In §4 we study the Morse stratification of X and prove Theorem 1.3. §5 contains the topological argument of [37] and proves Theorem 1.5 and Theorem 1.6. We illustrate our methods in some simple examples in §6.

2. Non-reductive GIT

This section is a brief summary of results in [2, 3, 4], with the main focus on [4], which is most relevant to this paper.

Let $H = U \rtimes R$ be a linear algebraic group acting linearly with respect to an ample line bundle L on a projective variety X over an algebraically closed field $\Bbbk$ of characteristic 0. Here U is the unipotent radical of H and $R \cong H/U$ is a Levi subgroup of H. When $H = R$ is reductive, using classical geometric invariant theory (GIT) developed by Mumford in the 1960s [38], we can find H-invariant open subsets $X^s \subseteq X^{ss}$ of X with a geometric quotient X^s/H and projective completion $X//H \supseteq X^s/H$ which is the projective variety associated to the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$. The variety $X//H$ is the image of a surjective morphism ϕ from the open subset X^{ss} of X (consisting of the semistable points for the action, while X^s consists of the stable points) such that if $x, y \in X^{ss}$ then $\phi(x) = \phi(y)$ if and only if the closures of the H-orbits of x and y meet in X^{ss} (that is, x and y are 'S-equivalent'). Moreover the subsets X^s and X^{ss} can be described using the Hilbert–Mumford criteria for (semi)stability.

Some aspects of Mumford's GIT can be made to work when H is not reductive (cf. for example [3, 14, 16, 17, 19, 20, 33, 42]), although it cannot be extended directly to non-reductive linear algebraic group actions since the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$ is not necessarily finitely generated as a graded algebra when H is not reductive. We can still define (semi)stable subsets X^{ss} and X^s, the latter having a geometric quotient X^s/H which is an open subset of an 'enveloping quotient' $X \wr\wr H$ with an H-invariant morphism $\phi : X^{ss} \to X \wr\wr H$, and if the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^H$ is finitely generated then $X \wr\wr H$ is the associated projective variety [3, 14]. However in general the enveloping quotient $X \wr\wr H$ is not necessarily projective, the morphism ϕ is not necessarily surjective (indeed its image may not be a subvariety of $X \wr\wr H$, but only a constructible subset) and we have no obvious analogues of the Hilbert–Mumford criteria for (semi)stability.

2.1. **Enveloping quotients.** As before let H be a linear algebraic group acting on a variety X equipped with a linearisation with respect to an ample line bundle $L \to X$ (that is, a lift of the action of H to L). We let $S = \bigoplus_{r\geq 0} H^0(X, L^{\otimes r})$ be the graded $\Bbbk$-algebra of global sections of non-negative tensor powers of L, and let S^H be the subalgebra consisting of H-invariant sections. The inclusion $S^H \hookrightarrow S$ defines an H-invariant rational map of schemes

$$q : X \dashrightarrow \mathrm{Proj}(S^H), \tag{3}$$

which is well defined on the open subset of points where some invariant section of a positive tensor power of L does not vanish.

Definition 2.1. Let H be a linear algebraic group acting on a variety X and $L \to X$ a linearisation of the action. The *naively semistable locus* is the open subset

$$X^{\mathrm{nss}} := \bigcup_{f \in I^{\mathrm{nss}}} X_f$$

of X, where $I^{\mathrm{nss}} := \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is the set of invariant sections of positive tensor powers of L. The *finitely generated semistable locus* is the open subset

$$X^{\mathrm{ss,fg}} := \bigcup_{f \in I^{\mathrm{ss,fg}}} X_f$$

of X^{nss}, where

$$I^{\mathrm{ss,fg}} := \left\{ f \in \textstyle\bigcup_{r>0} H^0(X, L^{\otimes r})^H \mid (S^H)_{(f)} \text{ is a finitely generated } \Bbbk\text{-algebra} \right\}.$$

Remark 2.2. The finitely generated semistable locus $X^{\mathrm{ss,fg}}$ is in general strictly contained in X^{nss}, since the subalgebra of invariant sections can be non-noetherian (even if S is a finitely generated $\Bbbk$-algebra). When H acts on an irreducible affine variety $X = \mathrm{Spec}\, A$ and $L = \mathcal{O}_X$ is equipped with the canonical H-linearisation, defined by the trivial character $1 : H \to \mathbb{G}_m$, then $X^{\mathrm{nss}} = X$ while $X^{\mathrm{ss,fg}}$ is the union of all X_f where $(A^H)_f$ is finitely generated over $\Bbbk$. It follows from [11, Proposition 2.10] and [21] that if A^H is not finitely generated (as is the case for the famous Nagata example) then $\emptyset \neq X^{\mathrm{ss,fg}} \neq X^{\mathrm{nss}}$.

Definition 2.3. The *enveloping quotient* is the scheme

$$X \wr\!\wr H := \bigcup_{f \in I^{\mathrm{ss,fg}}} \mathrm{Spec}\left((S^H)_{(f)}\right) \subseteq \mathrm{Proj}(S^H)$$

together with the canonical map $q : X^{\mathrm{ss,fg}} \to X \wr\!\wr H$. The image $q(X^{\mathrm{ss,fg}})$ of this map is the *enveloped quotient*.

The enveloping quotient is a canonically defined reduced, separated scheme locally of finite type over $\Bbbk$. It is observed in [2] that, if the enveloping quotient is projective, then we have a finitely generated algebra of invariants and the enveloping quotient is the projective scheme associated to the algebra of invariants.

2.2. **Linear algebraic groups with graded unipotent radicals and the $\hat{U}$-theorem.** In general when a linear algebraic group H acts linearly on a projective variety X then $X \wr\!\wr H$ is not necessarily a projective variety, the morphism ϕ is not necessarily surjective (indeed its image may not be a subvariety of $X \wr\!\wr H$ but only a constructible subset) and we have no obvious analogues of the Hilbert–Mumford criteria for (semi)stability. There is, however, a class of non-reductive groups to which some of the key features and computational flexibility of reductive GIT can be extended [4, 2]. These are linear algebraic groups H whose unipotent radical U is graded either internally, by a central one-parameter subgroup of a Levi subgroup of H, or externally, by a one-parameter group of automorphisms of H whose induced action on H/U is trivial. In each case the weights of the action of the multiplicative group on the Lie algebra of U are required to be strictly positive.

Definition 2.4. We say that a linear algebraic group $H = U \rtimes R$ has *internally graded unipotent radical* U if there is a central one-parameter subgroup $\lambda : \mathbb{G}_m = \Bbbk^* \to Z(R)$ of the Levi subgroup R of H such that the adjoint action of $\mathbb{G}_m$ on the Lie algebra of U has all its weights strictly positive. Then $\hat{U} = U \rtimes \lambda(\mathbb{G}_m)$ is a normal subgroup of H and $H/\hat{U} \cong R/\lambda(\mathbb{G}_m)$ is reductive.

We say that H has *externally graded unipotent radical* if H is a linear algebraic group over $\Bbbk$ with unipotent radical U and there is a semidirect product $\hat{H} = H \rtimes \mathbb{G}_m$ such that the adjoint action of $\mathbb{G}_m$ on the Lie algebra of U has all weights strictly positive, and $\mathbb{G}_m$ commutes with a Levi subgroup $R \cong H/U$ of H. Thus $\hat{H}$ has internally graded unipotent radical, with the grading given by the one-parameter subgroup defining the semidirect product $H \rtimes \mathbb{G}_m$.

If $H = U \rtimes R$ has internally graded unipotent radical U with central one-parameter subgroup $\lambda : \mathbb{G}_m \to R$ defining the grading, then the conjugation action of $\lambda(\mathbb{G}_m)$ on H defines a one-parameter group of automorphisms of H which acts trivially on R and thus H becomes externally graded. Since the internally and externally grading one-parameter groups of automorphisms act in the same way on H, we have an isomorphism $\hat{H} \cong H \times \mathbb{G}_m$ where the internally grading subgroup is identified with $\lambda(\mathbb{G}_m) \leqslant H$ and the externally grading subgroup is identified with the diagonal in $\lambda(\mathbb{G}_m) \times \mathbb{G}_m$.

Let $H = U \rtimes R$ with internally graded unipotent radical U as above act linearly on an irreducible projective variety X with respect to an ample line bundle L. Let $\chi : H \to \mathbb{G}_m$ be a character of H. Its kernel contains U (since unipotent groups have no nontrivial characters), and its restriction to $\hat{U}$ can be identified with an integer in such a way that the integer 1 corresponds to the character of $\hat{U}$ which fits into the exact sequence $U \hookrightarrow \hat{U} \to \lambda(\mathbb{G}_m)$.

Let $\omega_{\min}$ be the minimal weight for the action of the one-parameter subgroup $\lambda : \mathbb{G}_m \leq \hat{U} \leq H$ on $V := H^0(X, L)^*$ and let $V_{\min}$ be the weight space of weight $\omega_{\min}$ in V. Then $\omega_{\min} = \omega_0$ where $\omega_0 < \omega_1 < \cdots < \omega_{\max}$ are the weights with which $\lambda(\mathbb{G}_m)$ acts on the fibres of L^* over the fixed point set for its action on X. We can assume that there exist at least two distinct such weights since otherwise the action of the unipotent radical U of H on X is trivial, and so the action of H is via an action of the reductive group $R = H/U$.

Definition 2.5. Let c be a positive integer such that

$$\frac{\chi}{c} = \omega_{\min} + \epsilon$$

where $\epsilon > 0$ is sufficiently small; we will call rational characters χ/c with this property *well adapted* to the linear action of H, and we will call the linearisation well adapted if $\omega_{\min} < 0 \leq \omega_{\min} + \epsilon$ for sufficiently small $\epsilon > 0$. How small ϵ is required to be will depend on the situation; more precisely, we will say that a property P holds for well adapted linearisations if there exists $\epsilon(P) > 0$ such that property P holds for any linearisation for which $\omega_{\min} < 0 \leq \omega_{\min} + \epsilon(P)$.

The linearisation of the action of H on X with respect to the ample line bundle $L^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $L_\chi^{\otimes c}$ denote this twisted linearisation. Let $X^{s,\mathbb{G}_m}_{\min+}$ denote the stable subset of X for the linear action of $\lambda(\mathbb{G}_m)$ with respect to the linearisation $L_\chi^{\otimes c}$; by the theory of variation of (classical) GIT [12, 41], if L is very ample then $X^{s,\mathbb{G}_m}_{\min+}$ is the stable set for the action of $\lambda(\mathbb{G}_m)$ with respect to any rational character χ/c such that $\omega_{\min} < \chi/c < \omega_1$. Let

$$Z_{\min} := X \cap \mathbb{P}(V_{\min}) = \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{G}_m\text{-fixed point and} \\ \mathbb{G}_m \text{ acts on } L^*|_x \text{ with weight } \omega_{\min} \end{array} \right\}$$

and

$$X^0_{\min} := \{x \in X \mid p(x) \in Z_{\min}\} \quad \text{where} \quad p(x) = \lim_{\substack{t\to 0\\ t\in\mathbb{G}_m}} t\cdot x \quad \text{for } x \in X.$$

Definition 2.6. (*cf.* [2]) With this notation, we define the following condition for the $\hat{U}$-action on X:

$$(*) \qquad \mathrm{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\min}.$$

Note that $(*)$ holds if and only if we have $\mathrm{Stab}_U(x) = \{e\}$ for all $x \in X^0_{\min}$. This is also referred to as the condition that 'semistability coincides with stability' for the action of $\hat{U}$ (or, when $\lambda : \mathbb{G}_m \to R$ is fixed, for the linear action of U); see Definition 2.10 below.

Definition 2.7. When $(*)$ holds for a well adapted action of $\hat{U}$ the min-stable locus for the $\hat{U}$-action is

$$X^{s,\hat{U}}_{\min+} = X^{ss,\hat{U}}_{\min+} = \bigcap_{u\in U} u X^{s,\lambda(\mathbb{G}_m)}_{\min+} = X^0_{\min} \setminus U Z_{\min}.$$

Definition 2.8. A *well-adapted linear action* of the linear algebraic group H on an irreducible projective variety is given by data $X, L, H, \hat{U}, \chi$ where

(1) H is a linear algebraic group with internally graded unipotent radical U,
(2) H acts linearly on X with respect to a very ample line bundle L (and by abuse of notation L will be used to denote both the line bundle and the linearisation given by the linear action of H on L), while $\chi : H \to \mathbb{G}_m$ is a character of H and c is a positive integer such that the rational character χ/c is well adapted for the linear action of $\hat{U} = U \rtimes \mathbb{G}_m$ on X.

We will often refer to this set-up simply as a well-adapted action of H on X.

Theorem 2.9 ([4]). *Let $(X, L, H, \hat{U}, \chi)$ be a well-adapted linear action satisfying condition $(*)$. Then*

(1) the algebras of invariants

$$\oplus_{m=0}^\infty H^0(X, L^{\otimes cm}_{m\chi})^{\hat{U}} \text{ and } \oplus_{m=0}^\infty H^0(X, L^{\otimes cm}_{m\chi})^H = (\oplus_{m=0}^\infty H^0(X, L^{\otimes cm}_{m\chi})^{\hat{U}})^R$$

are finitely generated;

(2) the enveloping quotient $X \wr\wr \hat{U}$ is the projective variety associated to the algebra of invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{U}}$ and is a geometric quotient of the open subset $X^{s,\hat{U}}_{\min +}$ of X by $\hat{U}$;

(3) the enveloping quotient $X \wr\wr H$ is the projective variety associated to the algebra of invariants $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{H}}$ and is the classical GIT quotient of $X \wr\wr \hat{U}$ by the induced action of $R/\lambda(\mathbb{G}_m)$ with respect to the linearisation induced by a sufficiently divisible tensor power of L.

Definition 2.10. Let X be a projective variety which has a well adapted linear action of a linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U. When $(*)$ holds we denote by $X^{s,H}_{\min +}$ and $X^{ss,H}_{\min +}$ the pre-images in $X^{s,\hat{U}}_{\min +} = X^{ss,\hat{U}}_{\min +}$ of the stable and semistable loci for the induced linear action of the reductive group $H/\hat{U} = R/\lambda(\mathbb{G}_m)$ on $X \wr\wr \hat{U} = X^{s,\hat{U}}_{\min +}/\hat{U}$.

By H-stability=H-semistability we mean that $(*)$ holds and $X^{s,H}_{\min +} = X^{ss,H}_{\min +}$. The latter is equivalent to the requirement that $\mathrm{Stab}_H(x)$ is finite for all $x \in X^{ss,H}_{\min +}$; then the projective variety $X \wr\wr H$ is a geometric quotient of the open subset $X^{s,H}_{\min +} = X^{ss,H}_{\min +}$ of X by the action of H.

2.3. **Blow-ups to ensure semistability coincides with stability.** Suppose now that a reductive group G acts linearly on a projective variety X which has some stable points, but also has semistable points which are not stable. In [30] it is described how one can blow X up along a sequence of G-invariant subvarieties to obtain a G-invariant morphism $\tilde{X} \to X$ where $\tilde{X}$ is an irreducible projective variety acted on linearly by G such that $\tilde{X}^{ss} = \tilde{X}^s$. The induced birational morphism $\tilde{X}//G \to X//G$ of the geometric invariant theoretic quotients is then an algorithmically determined partial desingularisation of $X//G$, in the sense that if X is nonsingular then the centres of the blow-ups can be taken to be nonsingular, and $\tilde{X}//G$ has only orbifold singularities (it is locally isomorphic to the quotient of a nonsingular variety by a finite group action) whereas the singularities of $X//G$ are in general much worse. Even when X is singular, we can regard the birational morphism $\tilde{X}//G \to X//G$ as resolving (most of) the contribution to the singularities of $X//G$ coming from the group action.

The set $\tilde{X}^{ss}$ can be obtained from X^{ss} as follows. There exist semistable points of X which are not stable if and only if there exists a non-trivial connected reductive subgroup of G fixing a semistable point. Let $r > 0$ be the maximal dimension of a reductive subgroup of G fixing a point of X^{ss} and let $\mathcal{R}(r)$ be a set of representatives of conjugacy classes of all connected reductive subgroups R of dimension r in G such that

$$Z_R^{ss} = \{x \in X^{ss} : R \text{ fixes } x\}$$

is non-empty. Then

$$\bigcup_{R \in \mathcal{R}(r)} GZ_R^{ss}$$

is a disjoint union of nonsingular closed subvarieties of X^{ss}. The action of G on X^{ss} lifts to an action on the blow-up $X_{(1)}$ of X^{ss} along $\bigcup_{R\in\mathcal{R}(r)} GZ_R^{ss}$ which can be linearised so that the complement of $X_{(1)}^{ss}$ in $X_{(1)}$ is the proper transform of the subset $\phi^{-1}(\phi(GZ_R^{ss}))$ of X^{ss} where $\phi : X^{ss} \to X//G$ is the quotient map (see [30] 7.17). Here we use the linearisation with respect to (a tensor power of) the pullback of the ample line bundle L on X perturbed by a sufficiently small multiple of the exceptional divisor $E_{(1)}$. This will give us an ample line bundle on the blow-up $\psi : X_{(1)} \to X$, and if the perturbation is sufficiently small it will have the property that

$$\psi^{-1}(X^s) \subseteq X_{(1)}^s \subseteq X_{(1)}^{ss} \subseteq \psi^{-1}(X^{ss}) = X_{(1)}.$$

Moreover no point of $X_{(1)}^{ss}$ is fixed by a reductive subgroup of G of dimension at least r, and a point in $X_{(1)}^{ss}$ is fixed by a reductive subgroup R of dimension less than r in G if and only if it belongs to the proper transform of the subvariety Z_R^{ss} of X^{ss}.

The construction of [30] in the reductive case can be modified in the non-reductive setting to prove the following result [2], which applies to any well adapted linear action of a linear algebraic group H with internally graded unipotent radical.

Theorem 2.11. *Let $X, L, H, \hat{U}, \chi$ define a well-adapted linear action as at Definition 2.8.*

(1) There is a sequence of blow-ups of X along H-invariant projective subvarieties (which appear as the closures of H-sweeps of intersections with $Z_{\mathbb{G}_m}$ of fixed point sets of subgroups of U) resulting in a projective variety $\hat{X}$ with a well adapted linear action of H (with respect to a power of an ample line bundle given by tensoring the pullback of L with small multiples of the exceptional divisors for the blow-ups) which satisfies the condition (∗) so that Theorem 2.9 applies, giving us a projective geometric quotient

$$\hat{X}//\hat{U} = \hat{X}^{s,\hat{U}}_{\min+}/\hat{U}$$

and its (reductive) GIT quotient $\hat{X} \wr\wr H = (\hat{X} \wr\wr \hat{U}) \wr\wr \bar{R} = (\hat{X} \wr\wr \hat{U})//\bar{R}$ where $\bar{R} \cong H/\hat{U} \cong R/\lambda(\mathbb{G}_m)$.

(2) Moreover there is a sequence of further blow-ups along H-invariant projective subvarieties (appearing as the closures of H-sweeps of connected components of pre-images of fixed point sets of reductive subgroups of $\bar{R}$ acting on $\hat{X}//\hat{U}$), resulting in a projective variety $\tilde{X}$ satisfying the same conditions as $\hat{X}$ and in addition H-stability=H-semistability, so that $\tilde{X}//H = \mathrm{Proj}(\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^H)$ is the geometric quotient by H of the H-invariant open subset $\tilde{X}^{s,H}_{\min+}$.

3. Moment maps for non-reductive groups

3.1. **Moment maps for $\hat{U}$ actions.** Suppose that Y is a compact Kähler manifold with a holomorphic action of a complex reductive Lie group G; then G is the complexification of any maximal compact subgroup K of G. Recall that the dual $\mathfrak{k}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{k}, \mathbb{R})$ of the Lie algebra $\mathfrak{k}$ of K embeds naturally in the complex dual $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{k}, \mathbb{C})$ of the Lie algebra $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ of G, as $\mathfrak{k}^* = \{\xi \in \mathfrak{g}^* : \xi(\mathfrak{k}) \subseteq \mathbb{R}\}$.

Remark 3.1. For a real or complex vector space V the notation $V^*_{(\mathbb{R})} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ will denote the real dual space, and for a complex V we set $V^*_{(\mathbb{C})} = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ for the complex dual space. By slightly abusing this notation, we will often write simply V^* for the dual when it is clear from the context whether we mean real or complex dual.

From the viewpoint of symplectic geometry it is customary to fix a maximal compact subgroup K of G and assume that the Kähler form on Y is K-invariant; then we can ask for a moment map for the K-action. For any maximal compact subgroup K of G and any Kähler form ω on Y we can average ω over K using Haar measure to obtain a K-invariant Kähler form ω_K, and ask for a K-moment map $\mu_K : Y \to \mathfrak{k}^*$ for this new Kähler form. Any other maximal compact subgroup of G is given by $K' = gKg^{-1}$ for some $g \in G$, and then $\omega_{K'} = g^*(\omega_K)$ is K'-invariant and if μ_K exists then $\mu_{K'} = Ad^*(g) \circ \mu_K \circ g^{-1}$ is a K'-moment map with respect to $\omega_{K'}$, where Ad^* denotes the co-adjoint action of G on the complex dual $\mathfrak{g}^*$ of its Lie algebra and $\mathfrak{k}^*$ is embedded in $\mathfrak{g}^*$ as above.

So to define a 'moment map' for the G-action on Y, instead of fixing a Kähler form it is natural to ask for a G-orbit Ω in

$$\{(K, \omega) \in \mathcal{K}_G \times \text{Kähler}(Y) : \omega \text{ is } K\text{-invariant}\}$$

where Kähler(Y) is the space of Kähler forms on the complex manifold Y and

$$\mathcal{K}_G = \{K | K \text{ is a maximal compact subgroup of } G\}.$$

We can call such an Ω a G-equivariant Kähler structure on Y, and we can define a moment map for the G-action on (Y,Ω) to be a G-equivariant map

$$\mu_G : \Omega \times Y \to \mathfrak{g}^*$$

such that $\mu_G(K,\omega,x) = \mu_{(K,\omega)}(x)$ for each $K \in \mathcal{K}_G$ and $x \in Y$, where $\mu_{(K,\omega)} : Y \to \mathfrak{g}^*$ is the composition of a moment map for the action of K on (Y,ω) with the canonical embedding described above of the dual of the Lie algebra of K in $\mathfrak{g}^*$. We can write the condition on the derivative of the moment map composed with this embedding as the requirement that if $a \in \mathfrak{g}$ and $\xi \in T_xY$ then the derivative of the complex-valued function given by evaluating this composition at a takes ξ to

$$(\overline{\eta_{\omega,x}(a_x,\xi)} - \eta_{\omega,x}(\iota_K(a)_x,\xi))/2i, \tag{4}$$

where $\eta_{\omega,x}$ is the value at x of the Hermitian metric on Y determined by the K-invariant Kähler form ω, while ι_K is the K-invariant involution on $\mathfrak{g}$ with fixed point set $\mathfrak{k}$, and $x \mapsto a_x$ is the holomorphic vector field on Y determined by the infinitesimal action of G. Note that if $a \in \mathfrak{k}$ then (4) is real and equal to $\omega_x(a_x,\xi)$. This is a natural point of view to take when we wish to extend the notion of a moment map to non-reductive linear algebraic groups with internally graded unipotent radicals, in particular if we wish to construct quotients of open subsets of Y by holomorphic actions of such groups.

Remark 3.2. When a reductive group G acts linearly on a complex projective variety $Y \subseteq \mathbb{P}^n$ via a representation $\rho : G \to \mathrm{GL}(n+1)$ then we can choose a G-equivariant Kähler structure $\Omega \subseteq \mathcal{K}_G \times$ Kähler(Y) in the sense just described, such that

i) for each maximal compact subgroup K of G and Kähler form ω on Y such that $(K,\omega) \in \Omega$ there is a K-invariant hermitian inner product $\langle\,,\rangle_K$ on $\mathbb{C}^{n+1}$ with respect to which $\rho(K)$ acts unitarily, and ω is the associated Fubini–Study form;

ii) there are moment maps $\mu_{(K,\omega)} : Y \to \mathrm{Lie}(K)^*$ for each $(K,\omega) \in \Omega$ giving $\mu_G : \Omega \times Y \to \mathfrak{g}^*$ as above with

$$\mu_{(K,\omega)}([y]).a = \frac{\langle y, \rho(a) y\rangle_K}{2\pi i \|y\|^2}$$

for $a \in \mathfrak{g}$ and $[y] \in Y \subseteq \mathbb{P}^n$;

iii) the G-stable and G-semistable loci of Y can be described in terms of the G-equivariant map μ_G : $\Omega \times Y \to \mathfrak{g}^*$ as

$$Y^s = \{x \in Y : 0 \in \mu_G(\Omega \times \{x\}) \text{ and } \dim \mathrm{Stab}_G(x) = 0\}$$

and

$$Y^{ss} = \{x \in Y : 0 \in \mu_G(\Omega \times \overline{Gx})\};$$

iv) we can define S-equivalence as usual on Y^{ss} as $x \sim y$ if and only if the closures of Gx and Gy meet in Y^{ss}, and then for any $(K,\omega) \in \Omega$ the inclusion of $\mu^{-1}_{(K,\omega)}(0)$ in Y^{ss} induces an identification

$$\mu^{-1}_{(K,\omega)}(0)/K \;\cong\; Y^{ss}/\sim \;\cong\; Y/\!/G.$$

Suppose now that Y is a compact Kähler manifold with a holomorphic action of $G = K_{\mathbb{C}}$ and that G has a subgroup $\hat{U} = U \rtimes \mathbb{C}^*$ with internally graded unipotent radical U, such that $K \cap \hat{U} = S^1$ is the maximal compact subgroup of the one-parameter subgroup $\mathbb{C}^*$ of $\hat{U}$. Then the Lie algebra of $\hat{U}$ decomposes as a real vector space as

$$\hat{\mathfrak{u}} = \mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{u} \tag{5}$$

where $\mathrm{Lie}(K \cap \hat{U}) = i\mathbb{R}$ and $\mathfrak{u}$ is the Lie algebra of the (complex) unipotent group U.

Suppose $X \subset Y$ is a compact complex submanifold invariant under the $\hat{U}$ action. Suppose that K preserves the Kähler form ω on Y and therefore $S^1 = K \cap \hat{U}$ preserves the Kähler structure on X given by its restriction. The Kähler form ω makes Y and X into symplectic manifolds acted on by K and S^1 respectively and in addition gives Y a K-invariant and X an S^1-invariant Riemannian metric. Assume that a moment map

$$\mu_K : Y \to \mathfrak{k}^* = \mathfrak{k}^*_{(\mathbb{R})}$$

exists for the action of K on Y, whose composition $\mu_G : Y \to \mathfrak{g}^* = \mathfrak{g}^*_{(\mathbb{C})}$ with the embedding $\mathfrak{k}^* = \mathfrak{k}^*_{(\mathbb{R})} \to \mathfrak{g}^*_{(\mathbb{C})} = \mathfrak{g}^*$ defines for us a 'moment map' for the action of G on Y. Composing this with the inclusion of X in Y and the restriction map $\mathfrak{g}^* = \mathfrak{g}^*_{(\mathbb{C})} \to \hat{\mathfrak{u}}^*_{(\mathbb{C})} = \hat{\mathfrak{u}}^* = \mathbb{C} \oplus \mathfrak{u}^* = \mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{u}^*$, where $\mathfrak{u}^* = \mathfrak{u}^*_{(\mathbb{C})}$, we get a diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_G} & \mathfrak{g}^* = \mathfrak{k}^* \oplus i\mathfrak{k}^* \\ & \searrow^{\mu_{\hat{U}}} & \downarrow \\ & & \hat{\mathfrak{u}}^* = \mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{u}^* \end{array} \quad . \tag{6}$$

We will allow the addition of a rational character (that is, a rational multiple of the derivative of a group homomorphism $\hat{U} \to \mathbb{C}^*$) to this 'moment map' $\mu_{\hat{U}} : X \to \hat{\mathfrak{u}}^*$.

Let $\mu_{\lambda(S^1)}$ be the moment map for the action of $\lambda(S^1)$ given by composing μ_K with the restriction map $\mathfrak{k}^* \to \mathrm{Lie}(S^1) = \mathbb{R}$ and adding the corresponding rational character.

Remark 3.3. (i) Indeed, in the Kähler setting there is no reason not to allow real multiples, rather than only rational ones, here.
(ii) Cf. [18].

Definition 3.4. In these circumstances let

$$Z_{\min}(X) = \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is fixed by } \lambda(\mathbb{G}_m) \text{ and} \\ \mu_{\lambda(S^1)}(x) \text{ is the minimal value taken by } \mu_{\lambda(S^1)} \text{ on } X \end{array} \right\}$$

and

$$X^0_{\min} := \{x \in X \mid p(x) \in Z_{\min}(X)\} \quad \text{where} \quad p(x) = \lim_{\substack{t \to 0 \\ t \in \mathbb{G}_m}} t \cdot x \quad \text{for } x \in X.$$

We will say that 'semistability coincides with stability' for the action of U on X if $Z_{\min}(X) \subseteq Z_{\min}(Y)$ and

$$\mathrm{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\min}(X). \tag{7}$$

As in the algebraic situation (cf. Definition 2.6), (7) holds if and only if we have $\mathrm{Stab}_U(x) = \{e\}$ for all $x \in X^0_{\min}$.

Definition 3.5. We will say that a 'moment map' $\mu_{\hat{U}}$, obtained as above, defines a well adapted Hamiltonian action of $\hat{U}$ on X if the minimal value $\omega_{\min}$ taken by $\mu_{\lambda(S^1)}$ on X satisfies $\omega_{\min} < 0 \leq \omega_{\min} + \epsilon$ for sufficiently small $\epsilon > 0$. As in Definition 2.8, how small ϵ is required to be will depend on the situation. We will say that a property P holds for well adapted Hamiltonian actions of $\hat{U}$ on X obtained as above (where $\mu_{\hat{U}} : X \to \hat{\mathfrak{u}}^*$ is the sum of a rational character of $\hat{U}$ and the restriction to X of the composition of a moment map for the K action on Y with the natural embedding $\mathfrak{k}^* \to \mathfrak{g}^*$ and the restriction map $\mathfrak{g}^* \to \hat{\mathfrak{u}}^*$), if there exists $\epsilon(P) > 0$ such that property P holds for any choice of rational character for which $\omega_{\min} < 0 \leq \omega_{\min} + \epsilon(P)$.

A *well-adapted Hamiltonian action* of a linear algebraic group H on a connected compact Kähler manifold X is given by ingredients $X, Y, \omega, \rho : H \to G, \hat{U}, \mu_H$ where

(1) (Y, ω) is a Kähler manifold with a holomorphic action of $G = K_{\mathbb{C}}$ and ω is K-invariant;
(2) X is a complex submanifold of Y;

(3) $H = U \rtimes R$ is a linear algebraic group and its unipotent radical U is internally graded by $\lambda : \mathbb{C}^* \to Z(R)$, giving $\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \leqslant H$;
(4) $\rho : H \to G$ is a homomorphism such that X is invariant under the induced action of H on Y;
(5) $\mu_H : X \to \mathrm{Lie}(H)^*$ is the sum of a rational character of H and the restriction to X of the composition of a moment map for the K action on Y with the natural embedding $\mathfrak{k}^* \to \mathfrak{g}^*$ and $\rho^* : \mathfrak{g}^* \to \mathrm{Lie}(H)^*$;
(6) the composition of $\mu_H : X \to \mathrm{Lie}(H)^*$ with the restriction map $\mathrm{Lie}(H)^* \to \hat{\mathfrak{u}}^*$ gives us a well adapted Hamiltonian action of $\hat{U}$ on X in the sense above.

The moment map for the action of $\lambda(\mathbb{C}^*)$ given by composing μ_H with the restriction map $\mathrm{Lie}(H)^* \to \mathrm{Lie}(\lambda(\mathbb{C}^*))^*$ takes the constant value $\omega_{\min}$ on $Z_{\min}(X)$. There is a moment map

$$\mu_{R/\lambda(\mathbb{C}^*)} : Z_{\min}(X) \to \mathrm{Lie}(R/\lambda(\mathbb{C}^*))^* \tag{8}$$

for the induced action of $R/\lambda(\mathbb{C}^*)$ on $Z_{\min}(X)$ given by composing the restriction of μ_H to $Z_{\min}(X)$ with $\mathrm{Lie}(H)^* \to \mathrm{Lie}(R)^*$ and identifying $\mathrm{Lie}(R/\lambda(\mathbb{C}^*))^*$ first with the kernel of $\mathrm{Lie}(R)^* \to \mathrm{Lie}(\lambda(\mathbb{C}^*))^*$ and then by translation with the pre-image of $\omega_{\min}$ in $\mathrm{Lie}(R)^*$.

Definition 3.6. We will say that 'semistability coincides with stability' for the action of H on X if semistability coincides with stability for the action of U on X in the sense of Definition 3.4, and in addition 0 is a regular value for the moment map $\mu_{R/\lambda(\mathbb{C}^*)} : Z_{\min}(X) \to \mathrm{Lie}(R/\lambda(\mathbb{C}^*))^*$ defined at (8) for the induced action of $R/\lambda(\mathbb{C}^*)$ on $Z_{\min}(X)$.

We are particularly interested in these 'moment maps' $\mu_{\hat{U}} : X \to \mathfrak{u}^*$ (or the corresponding $\hat{U}$-equivariant map $\mu_{\hat{U}} : \Omega \times X \to \hat{\mathfrak{u}}^*$, where Ω is the $\hat{U}$-orbit of $(S^1, \omega|_X)$ in $\mathcal{K}_{\hat{U}} \times \text{Kähler}(X)$), and more generally $\mu_H : X \to \mathrm{Lie}(H)^*$, in the special case when X is a nonsingular projective variety in $Y = \mathbb{P}(\mathbb{C}^{n+1})$ with a linear H action acting via a homomorphism $\rho : H \hookrightarrow \mathrm{GL}(n+1, \mathbb{C})$, and X is H-invariant. Here K is the unitary group $U(n+1)$ and its Lie algebra $\mathfrak{k} = u(n+1)$ consists of matrices A such that $A = -\bar{A}^T$. We assume that a maximal compact subgroup of H containing the circle $\lambda(S^1) \leqslant \hat{U}$ preserves the standard $U(n+1)$-invariant hermitian inner product $\langle\, ,\rangle$ on $\mathbb{C}^{n+1}$, and we use the corresponding Fubini-Study metric on $\mathbb{P}^n$ to define the $\hat{U}$-moment map $\mu_{\hat{U}} : X \to \hat{\mathfrak{u}}^*$ which satisfies

$$\mu_{\hat{U}}([x]) \cdot a = \frac{1}{2\pi i ||x||^2} \bar{x}^T \rho_*(a) x \in \mathbb{C} \text{ for all } a \in \hat{U}$$

where $\cdot$ denotes the natural pairing between the complex dual of the Lie algebra $\hat{\mathfrak{u}}$ and the Lie algebra itself. The real and imaginary parts can be written as

$$\mathrm{Re}(\mu_{\hat{U}}([x]) \cdot a) = \frac{1}{4\pi i ||x||^2} (\bar{x}^T(\rho_*(a) - \overline{\rho_*(a)}^T) x\,, \quad \mathrm{Im}(\mu_{\hat{U}}([x]) \cdot a) = \frac{1}{4\pi i ||x||^2} \bar{x}^T(\rho_*(a) + \overline{\rho_*(a)}^T) x.$$

Let $\xi \in T_x S^{2n+1}$ be a tangent vector to the unit sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ at $x \in S^{2n+1}$ representing the tangent vector $[\xi]$ at $[x] \in \mathbb{P}^n$. Then

$$2\pi i d\mu_{\hat{U}}([x])([\xi]) \cdot a = \bar{x}^T \rho_*(a) \xi + \bar{\xi}^T \rho_*(a) x = \langle \overline{\rho_*(a)}^T x, \xi \rangle + \langle \xi, \rho_*(a) x \rangle \tag{9}$$

where $\langle\, ,\rangle$ denotes the standard Hermitian inner product satisfying $\langle \xi, \chi \rangle = \bar{\xi}^T \chi = \overline{\langle \chi, \xi \rangle}$. Hence

$$4\pi i \mathrm{Re}(d\mu_{\hat{U}}([x])([\xi]) \cdot a) = \langle \overline{\rho_*(a)}^T x, \xi \rangle - \langle \xi, \overline{\rho_*(a)}^T x \rangle + \langle \xi, \rho_*(a) x \rangle - \langle \rho_*(a) x, \xi \rangle =$$
$$= \langle (\overline{\rho_*(a)}^T - \rho_*(a)) x, \xi \rangle + \langle (\xi, (\rho_*(a) - \overline{\rho_*(a)}^T) x \rangle.$$

Remark 3.7. These calculations generalise straightforwardly when $\hat{U}$ is replaced with H.

3.2. **A slice for the action of U.** Let X be a compact complex submanifold of Y, where (Y,ω) is a Kähler manifold with a holomorphic action of a complex reductive group $G = K_{\mathbb{C}}$ such that the Kähler form ω is invariant under the maximal compact subgroup K of G. Suppose that $\rho : H \to G$ is a homomorphism where the unipotent radical U of the linear algebraic group H is internally graded by $\lambda : \mathbb{C}^* \to H$ with $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$, and that X is H-invariant. Suppose also that $K \cap \hat{U} = \lambda(S^1)$ is the maximal compact subgroup of the one-parameter subgroup $\lambda(\mathbb{C}^*)$ of $\hat{U}$, and that H has a maximal compact subgroup Q containing $\lambda(S^1)$ whose image under ρ is contained in K. Assume that a moment map $\mu_K : Y \to \mathfrak{k}^*$ exists for the action of K on Y, and let $\mu_H : X \to \mathrm{Lie}(H)^*$ be the sum of a rational character of H and the restriction to X of the composition of μ_K with the natural embedding $\mathfrak{k}^* \to \mathfrak{g}^*$ and $\rho^* : \mathfrak{g}^* \to \mathrm{Lie}(H)^*$. Let $\mu_U : X \to \mathfrak{u}^*$ be the composition of μ_H with the restriction map $\mathrm{Lie}(H)^* \to \mathfrak{u}^*$.

Remark 3.8. Note that U has no nontrivial characters.

A crucial ingredient in the proof of Theorem 1.1 is the following result.

Proposition 3.9. *Suppose that $\mu_H : X \to \mathrm{Lie}(H)^*$ defines a well adapted Hamiltonian action of H on X, and that semistability coincides with stability for the action of H, in the sense of Definitions 3.5 and 3.6. Then there is some $\delta > 0$ such that*

i) the intersection $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ of $\mu_U^{-1}(0)$ with the open subset of X where $\mu_{\lambda(S^1)} < \delta$ is a Q-invariant symplectic submanifold of X;

ii) the U-sweep $W_\delta = U(\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta))$ of $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ is an open subset of X containing $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ as a closed submanifold;

iii) $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ is a slice for the action of U on the open subset W_δ of X, in the sense that if $u \in U$ and $ux \in \mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ for some $x \in \mu_U^{-1}(0)$ with $\mu_{\lambda(S^1)}(x) < \delta$, then $u = 1$;

iv) the inclusion of $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ in W_δ induces a homeomorphism $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta) \cong W_\delta/U$.

Remark 3.10. This complements [29] 7.2, which covers the case when $H = G = K_{\mathbb{C}}$ is reductive and does not require the action to be well adapted or to satisfy semistability=stability, showing that if $x \in \mu_K^{-1}(0)$ then $Gx \cap \mu_U^{-1}(0) = Kx$.

Proof. We can assume, for simplicity and without essential loss of generality, that $X = Y$, that H is a subgroup of G, that ρ is the inclusion of H in G, that $Q = H \cap K$, that $R = Q_{\mathbb{C}}$ is a Levi subgroup of H with $H = U \rtimes R$ and that $\lambda(\mathbb{C}^*)$ is a central one-parameter subgroup of R.

Let us fix a K-invariant inner product on $\mathfrak{k}$ (defined over the rationals), and use it and its complexification to identify the Lie algebras under consideration with their (real or complex) duals.

Suppose that $x \in X^0_{\min}$ and $u \in U$ are such that $x \in \mu_U^{-1}(0)$ and $ux \in \mu_U^{-1}(0)$. Then $u = \exp b$ where $b \in \mathfrak{u}$. We wish to show that $b = 0$.

For fixed $a, b \in \mathfrak{u}$ let $h^{(a,b)} : \mathbb{R} \to \mathbb{R}$ be defined by

$$h^{(a,b)}(t) = \mathrm{Re}(\mu_U(\exp(tb)x) \cdot a).$$

By (4) the derivative of $h^{(a,b)}(t)$ at t_0 is given by

$$\frac{dh^{(a,b)}}{dt}(t_0) = \mathrm{Re}(d\mu_U(y)(b_y) \cdot a) = \mathrm{Re}(\overline{\eta_{\omega,y}(a_y, b_y)} - \eta_{\omega,y}(\iota_K(a)_y, b_y))/2i)$$

where $y = \exp(t_0 b)x$. Here $\eta_{\omega,y}$ is the value at y of the Hermitian metric determined by the K-invariant Kähler form ω, while ι_K is the K-invariant involution on $\mathfrak{g}$ with fixed point set $\mathfrak{k}$, and $x \mapsto a_x$ is the holomorphic vector field on Y determined by the infinitesimal action of G.

To prove (iii) it suffices to show that if $b \neq 0$ and we choose a to be ib then $\mathrm{Re}(d\mu_U(y)(b_y)\cdot a) > 0$ whenever y is sufficiently close to $Z_{\min}(X)$. For then the assumption that the action is well adapted means that there is some $\delta > 0$ and some $\delta' > 0$ such that $\mathrm{Re}(d\mu_U(y)(b)\cdot ib) > \delta'$ whenever $b \in \mathfrak{u}$ has norm 1 and $\mu_{\lambda(S^1)}(y) < \delta$, and then we cannot have $x \in \mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ and $\exp(tb)x \in \mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ unless $t = 0$.

When $a = ib$ then

$$\mathrm{Re}(d\mu_H(y)(b_y)\cdot a) = \mathrm{Re}((\overline{\eta_{\omega,y}(ib_y,b_y)} - \eta_{\omega,y}(\iota_K(ib)_y,b_y))/2i) = \eta_{\omega,y}(b_y,b_y)/2 - \mathrm{Im}(\eta_{\omega,y}(\iota_K(ib)_y,b_y))/2.$$

We have $\eta_{\omega,y}(b_y,b_y) > 0$ unless $b_y = 0$, and the assumption that semistability coincides with stability for the H-action tells us that $b_y \neq 0$ for y sufficiently close to $Z_{\min}(X)$. Moreover the conditions that $\lambda(\mathbb{C}^*)$ acts on $\mathfrak{u}$ with only strictly positive weights and that $Z_{\min}(X) \subseteq Z_{\min}(Y)$ ensure that if $y \in Z_{\min}(X)$ then $\iota_K(ib)_y = 0$ and hence $\mathrm{Im}(\eta_{\omega,y}(\iota_K(ib)_y,b_y)) = 0$. Thus $\mathrm{Re}(d\mu_U(y)(b_y)\cdot a) > 0$ whenever y is sufficiently close to $Z_{\min}(X)$, as required.

Therefore we have $\delta > 0$ such that (iii) holds and $b_y \neq 0$ for every nonzero $b \in \mathfrak{u}$ and y with $\mu_{\lambda(S^1)}(y) < \delta$. (i) then follows from the definition of μ_U and the fact that U-orbits are complex submanifolds of X. (ii) and (iv) follow since the derivative of the map

$$U \times \mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta) \to X$$

given by $(u,x) \mapsto ux$ is bijective at every (u,x), so by (iii) this map is a homeomorphism onto an open subset of X.

□

We have the following corollary to the proof of Proposition 3.9.

Corollary 3.11. *Under the assumptions of Proposition 3.9, the $\hat{U}$-stabilizer $\mathrm{Stab}_{\hat{U}}(x)$ is finite for all $x \in \mu_{\hat{U}}^{-1}(0)$, and the inclusion of $\mu_{\hat{U}}^{-1}(0)$ in the open subset $\hat{U}\mu_{\hat{U}}^{-1}(0)$ of X induces a homeomorphism between the orbit space $\hat{U}\mu_{\hat{U}}^{-1}(0)/\hat{U}$ and the symplectic quotient $\mu_{\hat{U}}^{-1}(0)/S^1$ of $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ by the action of $\lambda(S^1)$.*

Proof. This follows from Proposition 3.9, together with the reductive case [29] 7.2 applied to the action of $\lambda(S^1)$ on the symplectic submanifold $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ of X, and the fact that $\mu_{\hat{U}}^{-1}(0) \subseteq \mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$. □

Remark 3.12. The same argument applied to the action of Q on $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ tells us, under the assumptions of Proposition 3.9, that $H\mu_H^{-1}(0)$ is an open subset of X, that the H-stabiliser of any $x \in H\mu_H^{-1}(0)$ is finite, and that the inclusion of $\mu_H^{-1}(0)$ in $H\mu_H^{-1}(0)$ induces a homeomorphism between the orbit spaces $H\mu_H^{-1}(0)/H$ and the symplectic quotient $\mu_H^{-1}(0)/Q$ of $\mu_U^{-1}(0) \cap \mu_{\lambda(S^1)}^{-1}(-\infty,\delta)$ by the action of Q.

Next we return to the algebraic situation and study the connection between $\hat{U}\mu_{\hat{U}}^{-1}(0)$ and $X^{s,\hat{U}} = X^0_{\min} \setminus \hat{U}Z_{\min}$. We first show that the former is an open subset of the latter.

Proposition 3.13. *Let X be a nonsingular complex projective variety in $\mathbb{P}(\mathbb{C}^{n+1})$ endowed with a well-adapted linear action of the graded unipotent group $\hat{U} = U \rtimes \mathbb{C}^*$ acting via the homomorphism $\rho : \hat{U} \hookrightarrow \mathrm{GL}(n+1,\mathbb{C})$ such that semistability coincides with stabililty for the action of $\hat{U}$. Then*

(1) $\mu_{\hat{U}}^{-1}(0)$ is a real submanifold of $X^{s,\hat{U}} = X^0_{\min} \setminus \hat{U}Z_{\min}$;

(2) $\hat{U}\mu_{\hat{U}}^{-1}(0)$ is an open subset of $X^{s,\hat{U}}$.

Proof. To prove (1) we recall that

$$X^{s,\hat{U}} = X^{ss,\hat{U}} = \{x \in X^0_{\min} : \dim \mathrm{Stab}_{\hat{U}}(x) = 0\}.$$

Since $\mu_{\mathbb{G}_m} = j^* \circ \mu_{\hat{U}}$ (where $j^* : \hat{\mathfrak{u}}^* \to \mathrm{Lie}(\mathbb{G}_m)^*$ is the restriction map), we have

$$\mu_{\hat{U}}^{-1}(0) \subseteq \mu_{\mathbb{G}_m}^{-1}(0) \subseteq X^{s,\mathbb{C}^*} = X^0_{\min} \setminus Z_{\min} \subset X^0_{\min}.$$

Corollary 3.11 now shows that $\mu_{\hat{U}}^{-1}(0) \subseteq X^{s,\hat{U}}$. Since $\dim \mathrm{Stab}_{\hat{U}}(x) = 0$ for every $x \in \mu_{\hat{U}}^{-1}(0)$, it follows from the definition (9) of $\mu_{\hat{U}}$ that its derivative is surjective on $\mu_{\hat{U}}^{-1}(0)$ and so $\mu_{\hat{U}}^{-1}(0)$ is a real submanifold of $X^{s,\hat{U}}$. For if its derivative were not surjective at $[x] \in \mu_{\hat{U}}^{-1}(0)$ represented by $x \in S^{2n+1}$ then by (9) there would exist $a \in \hat{\mathfrak{u}} \setminus \{0\}$ such that

$$0 = \bar{x}^T \rho_*(a)\xi + \bar{\xi}^T \rho_*(a)x = \langle \overline{\rho_*(a)}^T x, \xi\rangle + \langle \xi, \rho_*(a)x\rangle$$

for every ξ representing a tangent vector at $[x]$. Replacing ξ with $i\xi$ we deduce that $\langle \xi, \rho_*(a)x\rangle = 0$ for every ξ, and hence a would lie in the Lie algebra of the stabiliser of $[x]$ in $\hat{U}$, This contradiction completes the proof of (1).

For (2) note that $X^{s,\hat{U}}$ is $\hat{U}$-invariant and therefore by (1)

$$\hat{U}\mu_{\hat{U}}^{-1}(0) \subset X^{s,\hat{U}}.$$

We now show that $\hat{U}\mu_{\hat{U}}^{-1}(0) \subset X$ contains an open neighbourhood of $\mu_{\hat{U}}^{-1}(0)$ in X, and hence $\hat{U}\mu_{\hat{U}}^{-1}(0)$ is an open subset of $X^{s,\hat{U}}$. Consider the map

$$\alpha : \hat{\mathfrak{u}} \times \mu_{\hat{U}}^{-1}(0) \to X, \quad (b, [x]) \mapsto \exp(b)[x].$$

This is a smooth map of smooth manifolds, so it is enough to show that its derivative at $(1, [x])$ is surjective for any $[x] \in \mu_{\hat{U}}^{-1}(0)$ represented by $x \in S^{2n+1}$. If not, then there would exist a nonzero $[\xi] \in T_{[x]}X$ such that $\langle \xi, \chi\rangle = 0$ for all $\chi \in \mathrm{Im}(d\alpha(1, [x]))$. Taking $\chi = a_{[x]}$ for $a \in \hat{\mathfrak{u}}$ gives us

$$\xi \in \ker d\mu_{\hat{U}}(0) = T_x(\mu_{\hat{U}}^{-1}(0))$$

by the argument used to prove (1), and hence $\xi \in \mathrm{Im}(d\alpha(1, x))$, a contradiction.

□

We are now ready to prove the main theorem when $H = \hat{U}$.

Theorem 3.14. *Let X be a nonsingular complex projective variety in $\mathbb{P}(\mathbb{C}^{n+1})$ endowed with a well-adapted linear action of the graded unipotent group $\hat{U} = U \rtimes \mathbb{C}^*$ acting via the homomorphism $\rho : \hat{U} \hookrightarrow \mathrm{GL}(n+1, \mathbb{C})$ such that condition (∗) holds. Then the embedding $\hat{U}\mu_{\hat{U}}^{-1}(0) \hookrightarrow X^{s,\hat{U}}$ induces a homeomorphism*

$$\mu_{\hat{U}}^{-1}(0)/S^1 \simeq X/\!/\hat{U}.$$

Remark 3.15. Here X is by assumption connected.

Proof. According to Lemma 3.13 (2) the quotient $\hat{U}\mu_{\hat{U}}^{-1}(0)/\hat{U}$ is an open subset of the projective variety $X^{s,\hat{U}}/\hat{U} = X/\!/\hat{U}$, and $\hat{U}\mu_{\hat{U}}^{-1}(0)/\hat{U}$ is nonempty by the well adapted assumption. But by Proposition 3.9 $\hat{U}\mu_{\hat{U}}^{-1}(0)/\hat{U}$ is homeomorphic to $\mu_{\hat{U}}^{-1}(0)/S^1$, which is compact, while the complement of $X^{s,\hat{U}}$ in X has real codimension at least two and so $X^{s,\hat{U}}$ and its quotient $X^{s,\hat{U}}/\hat{U}$ are connected. Hence the embedding $\mu_{\hat{U}}^{-1}(0)/S^1 \hookrightarrow X/\!/\hat{U}$ is surjective. Moreover it is a continuous bijection from a compact space to a Hausdorff space, and therefore is a homeomorphism. □

Corollary 3.16. *Up to multiplication of the Kähler form by a positive constant, the homeomorphism* $\mu_{\hat{U}}^{-1}(0)/S^1 \cong \hat{U}\mu_{\hat{U}}^{-1}(0)/\hat{U} = X//\hat{U}$ *is an isomorphism of compact Kähler orbifolds with respect to the complex structure induced on the quotient* $\hat{U}\mu_{\hat{U}}^{-1}(0)/\hat{U}$*, the Kähler form induced on* $\mu_{\hat{U}}^{-1}(0)/S^1$ *by restricting the* S^1*-invariant Fubini–Study Kähler form* ω *on* X *to* $\mu_{\hat{U}}^{-1}(0)$ *and the Kähler structure on* $X//\hat{U}$ *given by the induced Fubini–Study form for a projective embedding defined by the* $\hat{U}$*-invariant sections of suitable powers of the line bundle* L*.*

Proof. The restriction of the S^1-invariant Fubini–Study Kähler form ω on X to $\mu_{\hat{U}}^{-1}(0)$ is a closed S^1-invariant 2-form which is degenerate precisely along the S^1-orbits in $\mu_{\hat{U}}^{-1}(0)$, and thus descends to a symplectic form on $\mu_{\hat{U}}^{-1}(0)/S^1$. This symplectic form is compatible with the complex structure induced on the quotient $\hat{U}\mu_{\hat{U}}^{-1}(0)/\hat{U}$, and the Kähler structure they define corresponds up to multiplication by a positive constant to the induced Fubini–Study form for a projective embedding defined by the $\hat{U}$-invariant sections of suitable powers of the line bundle L. □

3.3. **Moment maps for H actions.** Now suppose that Y is a compact Kähler manifold with a holomorphic action of a complex reductive Lie group $G = K_{\mathbb{C}}$, and that $H = U \rtimes R$ is a linear algebraic subgroup of G with internally graded unipotent radical U. As before suppose that $\lambda : \mathbb{C}^* \to R$ is a central one-parameter subgroup of the Levi subgroup R of H which grades U, with $\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \leqslant H$. Suppose also that $K \cap H = Q$ is a maximal compact subgroup of $R = Q_{\mathbb{C}}$. Then as before the Lie algebra of $\hat{U}$ decomposes as a real vector space as $\hat{\mathfrak{u}} = \mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{u}$ where $\mathrm{Lie}(K \cap \hat{U}) = i\mathbb{R}$ and $\mathfrak{u}$ is the Lie algebra of U, while the Lie algebra of H decomposes as

$$\mathrm{Lie}(H) = \mathrm{Lie}(Q) \oplus i\,\mathrm{Lie}(Q) \oplus \mathfrak{u} = (\mathrm{Lie}(\lambda(S^1)) \oplus \mathrm{Lie}(\bar{Q})) \oplus i(\mathrm{Lie}(\lambda(S^1)) \oplus \mathrm{Lie}(\bar{Q})) \oplus \mathfrak{u}, \tag{10}$$

where $\bar{Q}$ is a subgroup of Q such that (at least on the level of Lie algebras) Q is the product of $\bar{Q}$ and $\lambda(S^1)$.

Suppose that $X \subset Y$ is a compact complex submanifold of Y which is invariant under the H action. Assume that the maximal compact subgroup K of G preserves the Kähler structure on Y, so that the maximal compact subgroup $Q = K \cap H$ preserves the induced Kähler structure on X. Assume also that a moment map

$$\mu_K : Y \to \mathfrak{k}^*$$

exists for the action of K on Y. Composing this with the canonical embedding of the real dual $\mathfrak{k}^*$ into the complex dual $\mathfrak{g}^*$ and the restriction maps $\mathfrak{g}^* \to \mathrm{Lie}(H)^*$ and $\mathrm{Lie}(H)^* \to \hat{\mathfrak{u}}^*$, we get the following extended diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mu_G} & \mathfrak{g}^* = \mathrm{Lie}(K)^* \oplus i\,\mathrm{Lie}(K)^* & & \\ & \searrow^{\mu_H} & \downarrow p_1^* & & \\ & & \mathrm{Lie}(H)^* = \hat{\mathfrak{u}}^* \oplus \mathrm{Lie}(R/\mathbb{C}^*) & & \\ & & \downarrow p_2^* & \searrow^{p_3^*} & \\ & & \hat{\mathfrak{u}}^* & & \mathrm{Lie}(\bar{R}) = \mathrm{Lie}(\bar{Q}) \oplus i\,\mathrm{Lie}(\bar{Q}) \end{array} \tag{11}$$

where $\bar{R}$ is the complexification of $\bar{Q}$ and $\mu_{\hat{U}} = p_2^* \circ \mu_H$ while $\mu_{\bar{R}} = p_3^* \circ \mu_H$.

3.4. **Proof of Theorem 1.1 in full generality.** In this subsection we prove Theorem 3.14 for the more general situation when $(X, L, H, \hat{U}, \chi)$ is a well-adapted linear action satisfying H-stability=H-semistability (see Definition 2.10). The diagram above gives that

$$\mu_H^{-1}(0) = \mu_{\hat{U}}^{-1}(0) \cap \mu_{\bar{R}}^{-1}(0)$$

where both sides are Q-invariant, and hence

$$\mu_H^{-1}(0)/Q \cong ((\mu_{\hat{U}}^{-1}(0) \cap \mu_{\bar{R}}^{-1}(0))/S^1)/(Q/S^1). \tag{12}$$

Let

$$\mu_{R/\mathbb{C}^*} : \mu_{\hat{U}}^{-1}(0)/S^1 \cong X//\hat{U} \to \mathrm{Lie}(Q/S^1)^* \to \mathrm{Lie}(R/\mathbb{C}^*)^*$$

denote the moment map on the GIT quotient $X//\hat{U}$ with respect to the residual action of $R/\mathbb{C}^* = (Q/S^1)_{\mathbb{C}}$. There are canonical identifications of $\mathrm{Lie}(Q/S^1)$ with $\mathrm{Lie}\,\bar{Q}$ and of $\mathrm{Lie}(R/\mathbb{C}^*)$ with $\mathrm{Lie}\,\bar{R}$, such that $\mu_{R/\mathbb{C}^*}$ pulls back to $\mu_{\bar{R}}$ on $\mu_{\hat{U}}^{-1}(0)$. Then (12) together with Theorem 3.14 and its reductive version for the reductive group $R/\mathbb{C}^*$ gives

$$X//H \cong (X//\hat{U})//(R/\mathbb{C}^*) \cong \mu_{R/\mathbb{C}^*}^{-1}(0)/(Q/S^1) \cong \mu_H^{-1}(0)/Q.$$

Thus we have

Theorem 3.17. *Let $(X, L, H, \hat{U}, \chi)$ be a well-adapted linear action of a linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U on a projective variety X satisfying H-stability=H-semistability. Then there is an inclusion $\mu_H^{-1}(0) \hookrightarrow X^{s,H}$ which induces a homeomorphism*

$$\mu_H^{-1}(0)/Q \simeq X//H$$

where Q is a maximal compact subgroup of H.

4. Morse inequalities and Betti numbers

We recall some basic facts on perfect Morse stratifications, following [29]. For a topological space Y endowed with a G-action let

$$P_t(Y) = \sum_{i\geq 0} t^i \dim H^i(Y, \mathbb{Q}) \text{ and } P_t^G(Y) = \sum_{i\geq 0} t^i \dim H^i_G(Y, \mathbb{Q})$$

denote its rational Poincarié series and equivariant Poincaré series.

4.1. **Morse inequalities.** Given a smooth stratification $X = \cup_{\beta\in B} S_\beta$ of the manifold X in the sense of [29], we can build up the cohomology of X inductively from the cohomology of the strata. There is a Thom-Gysin sequence relating the cohomology of the stratum S_β and of the open subsets $\cup_{\gamma<\beta} S_\gamma$ and $\cup_{\gamma\leq\beta} S_\gamma$ of X. These give us the Morse inequalities which can be expressed as follows. Assume that each component of any stratum S_β has the same codimension, say $d(\beta)$, in X. Then

$$\sum_{\beta\in B} t^{d(\beta)} P_t(S_\beta) - P_t(X) = (1 + t)R(t)$$

where $R(t)$ is a series with non-negative integer coefficients. The stratification is called perfect if the Morse inequalities are equalities, that is, if

$$P_t(X) = \sum_{\beta\in B} t^{d(\beta)} P_t(S_\beta).$$

If the space X is acted on by a topological group G then the equivariant cohomology $H^*_G(X,\mathbb{Q})$ is define as the ordinary cohomology of the topological quotient $EG\times_G X$ where EG is a contractible space equipped with a free G-action (i.e. the total space of the universal G-bundle $EG \to BG$):

$$H^*_G(X;\mathbb{Q}) = H^*(EG \times_G X;\mathbb{Q}).$$

For any smooth G-invariant stratification $X = \cup_{\beta\in B} S_\beta$ we obtain equivariant Morse inequalities

$$\sum_{\beta\in B} t^{d(\beta)} P_t^G(S_\beta) - P_t^G(X) = (1+t)R_G(t)$$

where again $R_G(t)$ has non-negative coefficients. The stratification is called equivariantly perfect if these are equalities; that is, $R_G(t) = 0$.

In [29] it is shown that the normsquare of a moment map $f = \|\mu\|^2$ for a Hamiltonian action of a compact group K on a compact symplectic manifold X is equivariantly perfect. K is homotopy equivalent to its complexification $G = K_{\mathbb{C}}$ so G-equivariant cohomology is the same as K-equivariant cohomology for any G-space. In the GIT set-up the equivariant perfection of the normsquare of μ expresses the equivariant Betti numbers of the semistable locus X^{ss} inductively in terms of the equivariant Betti numbers of the unstable strata and those of X itself. When $X^s = X^{ss}$ holds, the action of G on X^{ss} is rationally free (finite stabilisers) and therefore

$$H^*_K(X^{ss};\mathbb{Q}) = H^*(X^{ss}/G;\mathbb{Q}) = H^*(X//G;\mathbb{Q})$$

holds which provides formulas for the Betti numbers of the GIT quotient.

We will use the following criterion due to Atiyah and Bott for a stratification to be equivariantly perfect.

Lemma 4.1 ([1] 1.4, [29] 2.18)**.** *Let $X = \cup_{\beta\in B} S_\beta$ be a smooth G-invariant stratification of X such that for each $\beta \in B$ the equivariant Euler class of the normal bundle to S_β in X is not a zero divisor in $H^*_G(S_\beta,\mathbb{Q})$. Then the stratification is equivariantly perfect over $\mathbb{Q}$.*

4.2. **Betti numbers for $\hat{U}$ actions.** Let X be a smooth projective variety endowed with a well-adapted action of $\hat{U}$ for which semistability coincides with stability. We aim to prove that the simple stratification

$$X^0_{\min} = X^{s,\hat{U}} \cup \hat{U} Z_{\min}$$

is $\hat{U}$-equivariantly perfect. We start with some technical lemmas.

Lemma 4.2 (See also [2])**.** *If $z \in Z_{\min}$ and $u \in U$, then $p(uz) = z$ and*

$$uz \in Z_{\min} \quad \textit{iff} \quad u \in \mathrm{Stab}_U(z).$$

Proof: If $u = \exp(\xi)$ for some $\xi \in \mathrm{Lie}\, U$ which is a weight vector for the action of $\mathbb{G}_m$, then this follows by choosing coordinates on $\mathbb{P}(H^0(X,L)^*)$ with respect to which the action of $\mathbb{G}_m$ is diagonal and the infinitesimal action of ξ is in Jordan form. Moreover p is $\mathbb{G}_m$-invariant and $Z_{\mathbb{G}_m}$ is fixed pointwise by $\mathbb{G}_m$, so both $\mathrm{Stab}_U(z)$ and $\{u \in U \mid uz \in Z_{\mathbb{G}_m}\}$ are invariant under conjugation by $\mathbb{G}_m$, and the general result follows. □

Lemma 4.3. *Suppose that the $\hat{U}$-semistability=$\hat{U}$-stability condition (*) holds for the action of $\hat{U}$ on X. Then*

(1) the U-sweep $UZ_{\min} = \hat{U}Z_{\min}$ is a closed subvariety of $X^0_{\min}$;
(2) the equivariant Euler class of the normal bundle to $UZ_{\min}$ in $X^0_{\min}$ is not a zero divisor in $H^*_{S^1}(UZ_{\min})$.

Proof. (1) It is well known (going back to Bialynicki-Birula [10] and beyond) that the restriction to $X^0_{\min} \setminus Z_{\min}$ of the $\mathbb{G}_m$-invariant morphism $p : X^0_{\min} \to Z_{\min}$ factors through a projective morphism from a geometric quotient $(X^0_{\min} \setminus Z_{\min})/\mathbb{G}_m$ to $Z_{\min}$, and the fibres of p can be identified with the affine cone associated to the fibre of the morphism from this geometric quotient to $Z_{\min}$; when X is nonsingular then the fibres of p are affine spaces and the fibres of the geometric quotient $(X^0_{\min} \setminus Z_{\min})/\mathbb{G}_m$ over $Z_{\min}$ are weighted projective spaces.

Lemma 4.2 tells us that the fibre over $x \in Z_{\min}$ of p is isomorphic to $U/\operatorname{Stab}_U(x) = U$. This is an affine space of dimension $\dim U$ on which $\mathbb{G}_m$ acts with strictly positive weights, so the induced morphism

$$(UZ_{\min} \setminus Z_{\min})/\mathbb{G}_m \to Z_{\min}$$

is a weighted projective bundle. In particular it follows that the embedding $(UZ_{\min} \setminus Z_{\min})/\mathbb{G}_m \to (X^0_{\min} \setminus Z_{\min})/\mathbb{G}_m$ is closed, and hence so is the corresponding embedding of affine cones $UZ_{\min} \to X^0_{\min}$.

(2) We want to show that the equivariant Euler class of the normal bundle N to $UZ_{\min}$ in $X^0_{\min}$ is not a zero-divisor in $H^*_{S^1}(UZ_{\min}, \mathbb{Q})$. We saw in the proof of (1) that $UZ_{\min}$ is an affine cone over $Z_{\min}$ and therefore the latter is a deformation retract of the former, giving us the isomorphism

$$H^*_{S^1}(UZ_{\min}, \mathbb{Q}) \simeq H^*_{S^1}(Z_{\min}, \mathbb{Q}).$$

Under this isomorphism the equivariant Euler class of N is identified with the equivariant Euler class of its restriction to $Z_{\min}$. However, $Z_{\min}$ is a union of fixed points components of S^1 and therefore the S^1-weights to the normal directions are all nonzero, proving that the S^1-equivariant Euler class, which is the product of these normal weights, is nonzero. □

Lemma 4.1 and Lemma 4.3 give us

Corollary 4.4. *The stratification $X^0_{\min} = X^{s,\hat{U}} \cup UZ_{\min}$ is equivariantly perfect over $\mathbb{Q}$.*

We are ready to prove

Theorem 4.5. *Let X be a smooth projective variety endowed with a well-adapted action of $\hat{U}$ such that condition (∗) holds. Then*

$$P_t(X//\hat{U}) = P_t(Z_{\min})\frac{1-t^{2d}}{1-t^2}.$$

where $d = \dim(X) - \dim(U) - \dim(Z_{\min})$.

Proof. We apply the Morse equalities for the S^1-equivariant Morse-stratification $X^0_{\min} = X^{s,\hat{U}} \cup UZ_{\min}$, where the codimension of the closed stratum $UZ_{\min}$ in $X^0_{\min}$ is $d = \dim(X) - \dim(U) - \dim(Z_{\min})$. Both strata retract equivariantly onto $Z_{\min}$, and hence we get

$$P_t^{S^1}(X^{s,\hat{U}}) = P_t^{S^1}(Z_{\min})(1-t^{2d}).$$

Finally we use that $P_t^{S^1}(Y) = P_t(Y)P_t(BS^1) = P_t(Y)\cdot\frac{1}{1-t^2}$ holds for any Y with a trivial S^1 action to get the final formula. □

4.3. **Stratifications and Betti numbers for H actions.** Let $(X, L, H, \hat{U}, \chi)$ be a well-adapted linear action of $H = U \rtimes R$ on a projective variety X satisfying H-stability=H-semistability (where as usual X is smooth and irreducible). The GIT quotient of X by this linear action of H is given by an iterated quotient

$$X//H = (X//\hat{U})//(R/\lambda(\mathbb{C}^*)),$$

where $\lambda : \mathbb{C}^* \to R$ is the central one-parameter subgroup of R which grades the unipotent radical U of H, and hence one way to compute Betti numbers of $X//H$ is to apply the machinery of [29] for $X//\hat{U}$

acted on by the reductive group $R/\lambda(\mathbb{C}^*)$. We work out an example in detail in §6. Another method is to generalise the simple Morse stratification of $X//\hat{U}$ described in the previous section to the H-action. Let $Z_{\min}^{s,R/\lambda(\mathbb{C}^*)} \subset Z_{\min}$ denote the stable locus (which coincides with the semistable locus) for the residual $R/\lambda(\mathbb{C}^*)$ action on $Z_{\min}$.

Lemma 4.6. *Assume that H-stability=H-semistability holds for the well-adapted H action on X, so that $Z_{\min}^{s,R/\lambda(\mathbb{C}^*)} = Z_{\min}^{ss,R/\lambda(\mathbb{C}^*)}$. Then*

(1) $p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}) \setminus UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)}$ is an open H-invariant subset of $X^{s,H}$;

(2) $p : p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}) \to Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}$ is H-equivariant, and the induced map $\bar{p} : (p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}) \setminus UZ_{\min}^{s,R/\lambda}(\mathbb{C}^*))/H \to Z_{\min}//(R/\lambda(\mathbb{C}^*))$ is a weighted projective bundle;

(3) $X//H = (p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}) \setminus UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)})/H$.

Proof. For (1) note that the pre-image under p of a point $z \in Z_{\min}$ with finite $R/\lambda(\mathbb{C}^*)$-stabiliser has finite H-stabiliser. The proof of (2) is similar to that of Lemma 4.3 (1). Finally, (3) is a consequence of the fact that a nonempty Zariski-open subset of an irreducible projective variety is dense. □

Lemma 4.7. *Suppose that H-semistablity=H-stability holds for a well-adapted action of $\hat{U}$ on X. Then*

(1) the U-sweep $UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)} = \hat{U}Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}$ is a closed subvariety of $p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)})$;

(2) the R-equivariant Euler class of the normal bundle to $UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)}$ in $p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)})$ is not a zero divisor in $H_R^*(UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)})$.

Proof. The proof of Lemma 4.3 applies here too. □

Theorem 4.8. *Let X be a smooth projective variety endowed with a well-adapted action of $H = U \rtimes R$ such that H-stability=H-semistability holds and $Z_{\min}^{s,R/\lambda(\mathbb{C}^*)} = Z_{\min}^{ss,R/\lambda(\mathbb{C}^*)}$. Then*

$$P_t(X//H) = P_t(Z_{\min}//(R/\lambda))\frac{1-t^{2d}}{1-t^2}$$

where $d = \dim(X) - \dim(U) - \dim(Z_{\min})$.

Proof. Let us introduce the shorthand notation $\tilde{X}^{s,H} = p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}) \setminus UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)}$. Again, apply the R-equivariant Morse equalities for the stratification $p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}) = \tilde{X}^{s,H} \cup UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)}$, where the codimension of the closed stratum $UZ_{\min}^{s,R/\lambda(\mathbb{C}^*)}$ in $p^{-1}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)})$ is $d = \dim(X) - \dim(U) - \dim(Z_{\min})$. Both strata retract onto $Z_{\min}^{s,R/\lambda(\mathbb{C}^*)}$, and so

$$P_t^R(\tilde{X}^{s,H}) = P_t^R(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)})(1-t^{2d}).$$

Note that the classifying space of R decomposes as $BR = B\lambda(\mathbb{C}^*) \times B(R/\lambda(\mathbb{C}^*))$, and thus $P_t^R(Y) = P_t^{R/\lambda(\mathbb{C}^*)}(Y)P(BS^1) = P_t^{R/\lambda(\mathbb{C}^*)}(Y)\frac{1}{1-t^2}$ holds for any Y acted on by $R/\lambda(\mathbb{C}^*)$. Therefore

$$P_t^R(\tilde{X}^{s,H}) = P_t^{R/\lambda(\mathbb{C}^*)}(Z_{\min}^{s,R/\lambda(\mathbb{C}^*)})\frac{1-t^{2d}}{1-t^2} = P_t(Z_{\min}//(R/\lambda(\mathbb{C}^*)))\frac{1-t^{2d}}{1-t^2}.$$

On the other hand, by Lemma 4.7

$$P_t^R(\tilde{X}^{s,H}) = P_t(\tilde{X}^{s,H}/H) = P_t(X//H).$$

□

5. Cohomology and intersection pairings on non-reductive quotients

Let X be a smooth projective variety endowed with a well-adapted action of the graded group $H = U \rtimes R$. Let $T_{\mathbb{C}}^H \subset R$ be the maximal complex torus in the reductive part. We aim to relate cohomology and intersection numbers of $X//H$ to those of its abelianisation $X//T_{\mathbb{C}}^H$. This section gives a non-reductive version of the results of Martin [37].

5.1. **Reductive abelianisation.** We start with a brief summary of Martin's results [37]. Suppose that X is a compact symplectic manifold acted on by a compact Lie group K with maximal torus T in a Hamiltonian fashion, with moment map $\mu_K : X \to \mathfrak{k}^* = \mathrm{Lie}(K)^*$ for the action of K and induced moment map $\mu_T : X \to \mathfrak{t}^*$ for the action of T. In the algebraic (or more generally Kähler) situation the symplectic quotient $\mu^{-1}(0)/K$ can be identified with the GIT (or Kähler) quotient $X//G$ of X by $G = K_{\mathbb{C}}$ [29]. Let T be a maximal torus of K and $T_{\mathbb{C}} \subset G = K \otimes \mathbb{C}$ the corresponding complex maximal torus in G.

Definition 5.1. Let $\alpha \in \mathfrak{t}^*$ be a weight of T, and let $\mathbb{C}_\alpha$ denote the corresponding 1-dimensional representation of T. Let $L_\alpha \to X//T_{\mathbb{C}}$ denote the associated bundle

$$L_\alpha := \mu_T^{-1}(0) \times_T \mathbb{C}_\alpha \to X//T_{\mathbb{C}},$$

whose Euler class is denoted by $e(\alpha) \in H^2(X//T_{\mathbb{C}}) \simeq H^2_T(X)$. The set of roots of G, that is, the set of nonzero weights of the adjoint action on $\mathfrak{g}$, is denoted by Δ. We fix a choice $\Delta^+ \subset \Delta$ of the set of positive roots, and denote by Δ^- the corresponding set of negative roots.

The following diagram of Martin [37] relates $X//G$ and $X//T_{\mathbb{C}}$ through a fibration and an inclusion:

$$\begin{array}{ccc} \mu_K^{-1}(0)/T & \xhookrightarrow{\ i\ } & \mu_T^{-1}(0)/T = X//T_{\mathbb{C}} \\ \downarrow{\scriptstyle \pi} & & \\ X//G = \mu_K^{-1}(0)/K & & \end{array} \tag{13}$$

Note that $X//G$ and $X//T_{\mathbb{C}}$ are symplectic manifolds, and hence possess compatible almost complex structures, unique up to homotopy. The following proposition is the main technical result of [37].

Proposition 5.2. *Suppose that 0 is a regular value of the moment map* $\mu_K : X \to \mathrm{Lie}(K)^*$.

(1) The vector bundle $\oplus_{\alpha\in\Delta^-} L_\alpha \to X//T_{\mathbb{C}}$ *has a section s, which is transverse to the zero section, and whose zero set is the submanifold* $\mu_K^{-1}(0)/T \subset X//T_{\mathbb{C}}$. *Therefore the normal bundle is*

$$\mathcal{N}(\mu_K^{-1}(0)/T \subset X//T_{\mathbb{C}}) \simeq \oplus_{\alpha\in\Delta^-} L_\alpha|_{\mu_K^{-1}(0)/T}$$

(2) Let $\mathrm{vert}(\pi) \to \mu_K^{-1}(0)/T$ *denote the relative tangent bundle for* π. *Then*

$$\mathrm{vert}(\pi) \simeq \oplus_{\alpha\in\Delta^+} L_\alpha|_{\mu_K^{-1}(0)/T}.$$

(3) There is a complex orientation of $\mu_K^{-1}(0)/T$ *such that the above isomorphisms are isomorphisms of complex-oriented spaces and vector bundles, with respect to the complex orientations of* $X//G$ *and* $X//T_{\mathbb{C}}$ *induced by their symplectic forms.*

Remark 5.3. Recall that 0 is a regular value of μ_K if and only if the stabiliser in K of every $x \in \mu_K^{-1}(0)$ is finite, and in the algebraic situation this happens if and only if semistability coincides with stability for the action of $G = K_{\mathbb{C}}$. If 0 is a regular value of μ_K it does not follow that 0 is a regular value of μ_T, but it does follow that the derivative of μ_T is surjective in a neighbourhood of $\mu_K^{-1}(0)$, and so the normal bundle to $\mu_K^{-1}(0)/T$ in $X//T_{\mathbb{C}}$ is well defined, at least as an orbi-bundle.

This proposition leads to a series of results relating the topology of $X//G$ and $X//T_{\mathbb{C}}$. We assume for simplicity that K acts freely on $\mu_K^{-1}(0)$, though the results can easily be generalised to the case when 0 is a regular value of μ_K.

Theorem 5.4 (**Cohomology rings** [37] Theorem A)**.** *Suppose that K acts freely on $\mu_K^{-1}(0)$. There is a natural ring isomorphism*

$$H^*(X//G,\mathbb{Q}) \simeq \frac{H^*(X//T,\mathbb{Q})^W}{ann(e)}.$$

Here W denotes the Weyl group of G which acts naturally on $X//T$; the class $e \in H^*(X//T)^W$ is the product of all roots $e = \prod_{\alpha\in\Delta} e(\alpha)$ and $\mathrm{ann}(e) \subset H^*(X//T,\mathbb{Q})^W$ is the annihilator ideal

$$\mathrm{ann}(e) = \{c \in H^*(X//T,\mathbb{Q})^W | c \cup e = 0\}.$$

Diagram (13) provides a natural lift of a cohomology class on X/G to a class on X/T, compatible with the above isomorphism. We say that $\tilde{a} \in H^*(X//T)$ is a lift of $a \in H^*(X//G)$ if $\pi^*a = i^*\tilde{a}$.

Theorem 5.5 (**Integration formula**, [37] Theorem B)**.** *Suppose that K acts freely on $\mu_K^{-1}(0)$. Given a cohomology class $a \in H^*(X//G)$ with lift $\tilde{a} \in H^*(X//T)$, then*

$$\int_{X//G} a = \frac{1}{|W|}\int_{X//T} \tilde{a} \cup e,$$

where $|W|$ is the order of the Weyl group of G, and e is the cohomology class defined in Theorem A.

5.2. **A nonabelian localisation formula.** Suppose as before that X is a compact symplectic manifold acted on by a compact Lie group K in a Hamiltonian fashion, with moment map $\mu_K : X \to \mathrm{Lie}(K)^*$. Recall that if the action of K on the zero level set $\mu_K^{-1}(0)$ is free (or has finite stabilisers) there is a surjective map $\kappa : H_K^*(X;\mathbb{Q}) \to H^*(\mu_K^{-1}(0)/K;\mathbb{Q})$ from the equivariant cohomology of X to the ordinary cohomology of $\mu_K^{-1}(0)/K$ given by composing the restriction map on rational equivariant cohomology to $\mu_K^{-1}(0)$ with an isomorphism between the rational equivariant cohomology of the latter and the ordinary rational cohomology of $\mu_K^{-1}(0)/K$ [29]. In [25] a formula is given for the evaluation on the fundamental class of $\mu_K^{-1}(0)/K$ of $\kappa(\eta)$ for $\eta \in H_K^*(X;\mathbb{Q})$ whose degree is the dimension of $\mu_K^{-1}(0)/K$, in terms of the fixed point locus of a maximal torus T of K on X.

Theorem 5.6 (**Localisation on reductive GIT quotients**, [25],[15])**.** *Suppose that* 0 *is a regular value of μ_K. Given any equivariant cohomology class η on X represented by an equivariant differential form $\eta(z)$ whose degree is the dimension of $\mu_K^{-1}(0)/K$, we have*

$$\int_{\mu_K^{-1}(0)/K} \kappa(\eta) = n_K \mathrm{JKRes}^{\Lambda}\left(e^2(z)\sum_{F\subset\mathcal{F}}\int_F \frac{i_F^*(\eta(z))}{\mathrm{Euler}^T(\mathcal{N}_F)(z)}[dz]\right)$$

In this formula z is a variable in $\mathfrak{t}_{\mathbb{C}}$ so that a T-equivariant cohomology class can be evaluated at z, and $e(z) = \prod_{\gamma\in\Delta^+}\gamma(z)$ is the product of the positive roots. JKRes^{Λ} *is a residue map which depends on a choice of a cone $\Lambda \subset \mathfrak{t}$, $\mathcal{F}$ is the set of components of the fixed point set of the maximal torus T on X. If $F \in \mathcal{F}$, i_F is the inclusion of F in X and $\mathrm{Euler}^T(\mathcal{N}_F)$ is the T-equivariant Euler class of the normal bundle to F in X. Finally, n_K is a rational constant depending on K and the size of the stabiliser in K of a generic $x \in X$.*

The precise definition of this residue map, whose domain is a suitable class of meromorphic differential forms on $\mathrm{Lie}(T)\otimes\mathbb{C}$), is given in Definition 8.5 of [25]. Here we give the description of Jeffrey and Kogan [26].

Let V be an r-dimensional real vector space and let $A = \{\alpha_1, \dots, \alpha_n\}$ be a collection of (not necessarily distinct) non-zero vectors in V^*. We consider α_is as linear functions on V . Let Λ be a connected component of $V \setminus \bigcup_{i=1}^n \alpha_i^\perp$, where $\alpha_i^\perp = \{v \in V | \alpha_i(v) = 0\}$. By this choice we have for all i that either $\alpha_i \in \Lambda^v$ or $-\alpha_i \in \Lambda^v$, where $\Lambda^v = \{\beta \in V^* | \beta(v) > 0, \forall v \in \Lambda\}$ is the dual cone of Λ. Let $\xi \in \Lambda$ and choose a basis $\{z_1, \dots, z_r\}$ of V^* such that $z_1(\xi) = 1$ and $z_2(\xi) = \dots = z_r(\xi) = 0$. Let $\varepsilon = \varepsilon_1 z_1 + \dots + \varepsilon_r z_r \in V^*$ and let $P \in \mathbb{R}[V]$ be a polynomial.

Definition 5.7. J-K residue [25, 26, 40] We define

$$\mathrm{Res}^+_{z_1} = \frac{P(\mathbf{z})e^{\varepsilon(\mathbf{z})}}{\prod_{i=1}^n \alpha_i(\mathbf{z})} dz_1 = \begin{cases} \mathrm{Res}_{z_1=\infty} \frac{P(\mathbf{z})e^{\varepsilon(\mathbf{z})}}{\prod_{i=1}^n \alpha_i(\mathbf{z})} dz_1 & \text{if } \varepsilon_1 \geq 0 \\ 0 & \text{if } \varepsilon < 0 \end{cases}$$

where $z_2, \dots, z_r$ are taken to be constants when we take the residue with respect to z_1. The residue map is then defined as

$$\mathrm{JKRes}^\Lambda \frac{P(\mathbf{z})e^{\varepsilon(\mathbf{z})}}{\prod_{i=1}^n \alpha_i(\mathbf{z})} d\mathbf{z} = \frac{1}{\det[(z_i, z_j)]_{i,j=1}^r} \mathrm{Res}^+_{z_r}\left(\dots\left(\mathrm{Res}^+_{z_1} \frac{P(\mathbf{z})e^{\varepsilon(\mathbf{z})}}{\prod_{i=1}^n \alpha_i(\mathbf{z})} dz_1\right)\dots\right) dz_r$$

where $\det[(x_i, x_j)]_{i,j=1}^r$ is the Gram determinant with respect to a fixed scalar product on V^*.

In short, after fixing coordinates and an inner product on $V = \mathfrak{t}$, and a cone $\Lambda \subset \mathfrak{t}$, the J-K residue in Theorem 5.6 can be interpreted as an iterated residue over those components of the torus-fixed point locus whose image under the torus moment map sit in Λ.

We will not need the definition in full generality in the folllowing examples, just the simple case when $G = \mathrm{GL}(k, \mathbb{C})$ with maximal compact $K = U(k)$ acting on $X = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ (or slightly more generally on $X = \mathrm{Hom}(\mathbb{C}^k, TX)$).

Example 5.8 (Localisation on $\mathrm{Grass}_k(\mathbb{C}^n)$, see also §7 of [37]). The Grassmannian $\mathrm{Grass}_k(\mathbb{C}^n)$ can be described as the symplectic quotient of the set of complex matrices $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ with n rows and k columns by the unitary group $U(k)$ where $g \in U(k)$ acts on a matrix $A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $A \circ g^{-1}$. The moment map $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \to u(k)$ is given by

$$\mu(A) = \bar{A}^T A - \mathrm{Id}_{n\times n}$$

if we identify the dual of the Lie algebra $u(k)$ with Hermitian matrices via the pairing $\langle H, \xi \rangle = \frac{i}{2}\mathrm{trace}(H\bar{\xi}^T)$.

The k column vectors of the matrix $A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ define vectors $v_1, \dots, v_k \in \mathbb{C}^n$, and the (i, j)-entry of $\bar{A}^T A$ is the Hermitian inner product (v_j, v_i). Hence $\mu^{-1}(0)$ consists of the unitary k-frames in $\mathbb{C}^n$, and taking the quotient by $U(k)$ gives the Grassmannian:

$$\mathrm{Grass}_k(\mathbb{C}^n) = \mu^{-1}(0)/U(k).$$

Let $u_1, \dots, u_k$ denote the weights of the maximal torus $T \subset U(k)$ acting on $\mathbb{C}^k$. The Weyl group of $U(k)$ is the symmetric group on k elements S_k, the roots of $U(k)$ can be enumerated by pairs of positive integers (i, j) with $1 \leq i, j \leq k$ and $i \neq j$, and the cohomology class corresponding to the root (i, j) is $z_j - z_i$. Let V denote the tautological rank k vector bundle over $\mathrm{Grass}_k(\mathbb{C}^n)$. There is only one fixed point, namely the origin, so Theorem 5.6 gives us (see also Proposition 7.2 in [37])

$$\int_{\mathrm{Grass}_k(\mathbb{C}^n)} \phi(V) = \frac{1}{k!} \mathrm{Res}_{\mathbf{z}=0} \frac{\phi(\mathbf{z}) \prod_{i\neq j}(z_i - z_j) d\mathbf{z}}{z_1^n \dots z_k^n}.$$

Example 5.9 (Localisation on $\mathrm{Flag}_k(\mathbb{C}^n)$). We expect a similar residue formula for Chern numbers of the tautological bundle over the complex flag variety $\mathrm{Flag}_k(\mathbb{C}^n) = \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathbb{C}^n)/B_k$ where B_k is the upper Borel in $\mathrm{GL}(k, \mathbb{C})$. The symplectic quotient description of $\mathrm{Flag}_k(\mathbb{C}^n)$ is a bit more delicate though, and our starting point is Proposition 1.2 of [37] and the corresponding diagram (13), which connects the different quotients $\mu_T^{-1}(0)/T, \mu_K^{-1}(0)/T$ and $\mu_K^{-1}(0)/K$.

Take again $X = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, $K = U(k), T = (S^1)^k$ as above, then $\mu_K^{-1}(0)$ consists of the unitary k-frames in $\mathbb{C}^n$ and a k-tuple $(v_1, \dots, v_k) \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ lies in $\mu_T^{-1}(0)$ precisely when each v_i has length 1. Therefore

$$\mu_K^{-1}(0)/K = \mathrm{Grass}_k(\mathbb{C}^n), \mu_K^{-1}(0)/T = \mathrm{Flag}_k(\mathbb{C}^n), \mu_T^{-1}(0)/T = (\mathbb{CP}^{n-1})^k$$

and according to Proposition 1.2 of [37] the equivariant Euler class of $\mu_K^{-1}(0)/T$ in $\mu_T^{-1}(0)/T$ is the product of the negative roots $\prod_{1\le i<j\le k}(z_i - z_j)$ of $U(k)$. This combined with Theorem 5.6 applied to $\mu_T^{-1}(0)/T = (\mathbb{CP}^{n-1})^k$ gives us the following

Proposition 5.10. *Let $V \to \mathrm{Flag}_k(\mathbb{C}^n)$ denote the tautological rank k bundle over the flag manifold. Then*

$$\int_{\mathrm{Flag}_k(\mathbb{C}^n)} \phi(V) = \mathrm{Res}_{\mathbf{z}=0} \frac{\phi(\mathbf{z}) \prod_{1\le i<j\le k}(z_i - z_j) d\mathbf{z}}{z_1^n \dots z_k^n}.$$

In [7] a different proof of this proposition is given (see [7] Proposition 5.4) which uses the Atiyah-Bott localisation instead of the localisation theorem from [25].

5.3. **Nonabelian localisation for non-redutive actions.** As before suppose that Y is a compact Kähler manifold acted on by a complex reductive Lie group $G = K_{\mathbb{C}}$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Let $H = U \rtimes R$ be a complex linear algebraic group, with unipotent radical U graded by a central one-parameter subgroup $\lambda : \mathbb{C}^* \to R$ of R, with $\hat{U} = U \rtimes \mathbb{C}^* \leqslant H$. Let Q be a maximal compact subgroup of R. Suppose that H acts on Y via $\rho : H \to G$, and that X is a complex submanifold of Y which is invariant under the action of X.

We can assume for simplicity and without significant loss of generality that H is a subgroup of G, that ρ is the inclusion of H in G, that $Q = H \cap K$, that $R = Q_{\mathbb{C}}$ is a Levi subgroup of H with $H = U \rtimes R$ and that $\lambda(\mathbb{C}^*)$ is a central one-parameter subgroup of R. Let T^H be a maximal torus for Q (and thus a maximal compact torus for H) with Lie algebra $\mathfrak{t}^H$, and let $U^R_{\max}$ be a maximal unipotent subgroup of $R = Q_{\mathbb{C}}$, with Lie algebra $\mathfrak{u}^R_{\max}$, which is normalised by T^H and corresponds to a positive Weyl chamber $\mathfrak{t}^H_+ \subseteq \mathfrak{t}^H$. Then $U^H_{\max} = U \rtimes U^R_{\max}$ is a maximal unipotent subgroup of H which is normalised by the maximal complex torus $T^H_{\mathbb{C}}$ of R (and of H). Since $\lambda(\mathbb{C}^*)$ is central in R and its adjoint action on $\mathfrak{u}$ has all weights strictly positive, there is a small perturbation of the point it represents in $\mathfrak{t}^H$ which acts on $\mathfrak{u}^H_{\max} = \mathrm{Lie}(U^H_{\max})$ with all weights strictly positive. This means that we can choose a maximal torus T of K containing T^H and a positive Weyl chamber $\mathfrak{t}_+ \subseteq \mathfrak{t}$ such that the corresponding maximal unipotent subgroup $U_{\max}$ of G contains $U^H_{\max}$, the corresponding Borel subgroup $B = U_{\max} T_{\mathbb{C}}$ contains the maximal solvable subgroup $B^H = U^H_{\max} T^R$ of H, and $G = KB = K \exp(i\mathfrak{t}) \exp(\mathfrak{u}_{\max})$.

Let $\mathfrak{r}^+$ denote the Lie algebra of the maximal unipotent subgroup $U^R_{\max}$ of R, and let $\mathfrak{r}^-$ denote the Lie algebra of the opposite maximal unipotent subgroup of R. We have the Cartan decomposition

$$\mathrm{Lie}(R) = \mathfrak{r}^- \oplus \mathfrak{t}^H_{\mathbb{C}} \oplus \mathfrak{r}^+$$

and the corresponding decomposition

$$\mathrm{Lie}(H) = \mathfrak{u} \oplus \mathfrak{r}^- \oplus \mathfrak{t}^H_{\mathbb{C}} \oplus \mathfrak{r}^+.$$

In the special case when $X \subset \mathbb{P}^n = Y$ is a projective variety endowed with a well-adapted linear action of H such that semistability coincides with stability for the action of H, we have seen in Theorem 3.17 that

$$X//H = \mu_H^{-1}(0)/Q.$$

This quotient fits into a diagram analogous to (13):

(14)
$$\begin{array}{ccc} \mu_H^{-1}(0)/T^H & \xhookrightarrow{\;i\;} & \mu_{T^H}^{-1}(0)/T^H = X//T^H_{\mathbb{C}} \\ \big\downarrow{\scriptstyle \pi} & & \\ \mu_H^{-1}(0)/Q \cong X//H & & \end{array}$$

The description of the corresponding normal bundles is just as in Martin's paper [37], using the line bundles corresponding to the weights for the action of T^H on $\mathrm{Lie}(H)/\mathfrak{t}^H_{\mathbb{C}}$.

Definition 5.11. (1) For a weight α of $T^H_{\mathbb{C}}$ let $\mathbb{C}_\alpha$ denote the corresponding 1-dimensional complex representation of T^H and let

$$L_\alpha := \mu_{T^H}^{-1}(0) \times_{T^H} \mathbb{C}_\alpha \to X//T^H_{\mathbb{C}},$$

denote the associated complex line bundle whose Euler class is denoted by $e(\alpha) \in H^2(X//T^H_{\mathbb{C}}) \simeq H^2_{T^H}(X)$.

(2) For a complex $T^H_{\mathbb{C}}$-module $\mathfrak{a}$ let $V_{\mathfrak{a}} = \mu_{T^H}^{-1}(0) \times_{T^H_{\mathbb{C}}} \mathfrak{a} \to X//T^H_{\mathbb{C}}$ denote the corresponding complex vector bundle.

Proposition 5.12. *Suppose that X is a nonsingular projective variety endowed with a well-adapted linear action of H such that semistability coincides with stability for the action of H.*

(1) The vector bundle $V_{\mathfrak{u}\oplus\mathfrak{r}^-} \to X//T^H_{\mathbb{C}}$ has a C^∞ section s which is transverse to the zero section and whose zero set is the submanifold $\mu_H^{-1}(0)/T^H \subset X//T^H_{\mathbb{C}}$, so the $T^H_{\mathbb{C}}$-equivariant normal bundle is

$$\mathcal{N}(i) \simeq V_{\mathfrak{u}\oplus\mathfrak{r}^-}.$$

(2) Let $\mathrm{vert}(\pi) \to \mu_H^{-1}(0)/T^H$ denote the relative tangent bundle for π. Then

$$\mathrm{vert}(\pi) \simeq V_{\mathfrak{r}^+}|_{\mu_H^{-1}(0)/T^H}.$$

Proof. The argument of [37] works with minor changes. For (1) and (2) we use the diagram

(15)
$$\begin{array}{ccl} X & \xrightarrow{\;\mu_G\;} & \mathfrak{g}^* = \mathrm{Lie}(K)^* \oplus i\,\mathrm{Lie}(K)^* \\ & \searrow^{\mu_H} & \big\downarrow{\scriptstyle p^*} \\ & & \mathrm{Lie}(H)^* = \mathfrak{u}^* \oplus \mathrm{Lie}(R)^* \\ & \searrow_{\mu_{T^H}} & \big\downarrow{\scriptstyle q^*} \\ & & \mathrm{Lie}(T^H_{\mathbb{C}})^* = \mathfrak{t}^H \oplus i\mathfrak{t}^H \end{array}$$

where q^* is the projection and p^* is the dual of the inclusion

$$p : \mathrm{Lie}(H) \hookrightarrow \mathfrak{k},\ A \mapsto A - \bar{A}^T,$$

see §3.4. Let $V = (q^*)^{-1}(0)$ denote the complex T^H-invariant subspace of $\mathfrak{g}^*$ sent to 0 by q^*. Note that μ_{T^H} is a T^H-equivariant map and the coadjoint action of T^H on H preserves the subspace V. Therefore V is isomorphic to $\mathfrak{u}^* \oplus (\mathfrak{r}^-)^*$. Then $\mu_{T^H}^{-1}(0) = \mu_H^{-1}(V)$, and the fact semistability coincides with stability for the H-action tells us that 0 is a regular value for μ_H, which implies that the subspace V is transverse to the map μ_H. The restriction of μ_H to $\mu_{T^H}^{-1}(0)$ defines an T^H-equivariant map $\tilde{s} : \mu_{T^H}^{-1}(0) \to V$ whose quotient by T^H defines a section s of the associated bundle $V_{\mathfrak{u}\oplus(\mathfrak{r}^-)} = \mu_{T^H}^{-1}(0) \times_{T^H} V \to X//T^H_{\mathbb{C}}$. Since

$0 \in \mathfrak{g}^*$ is a regular value of μ_H it follows that $0 \in V$ is a regular value of $\tilde{s}$, and hence s is transverse to the zero section, as required for (1).

(2) is a reductive statement; for the proof see Proposition 1.2 of [37].

□

The following statements on the relation between the topology of $X//H$ and $X//T_{\mathbb{C}}^H$ follow from Proposition 5.12 the same way as Theorem A and B of [37] follow from Proposition 1.2 of [37].

Theorem 5.13 (Cohomology rings). *Let X be a smooth projective variety endowed with a well-adapted action of $H = U \rtimes R$ such that H-stability=H-semistability holds. There is a natural ring isomorphism*

$$H^*(X//H, \mathbb{Q}) \simeq \frac{H^*(X//T_{\mathbb{C}}^H, \mathbb{Q})^W}{ann(\mathrm{Euler}(V_{\mathfrak{u}\oplus\mathfrak{r}^-\oplus\mathfrak{r}^+})}.$$

Here W denotes the Weyl group of R, which acts naturally on $X//T_{\mathbb{C}}^H$, while $\mathrm{Euler}(V_{\mathfrak{u}\oplus\mathfrak{r}^-\oplus\mathfrak{r}^+}) \in H^*(X//T_{\mathbb{C}}^H)^W$ is the Euler class of the bundle $V_{\mathfrak{u}\oplus\mathfrak{r}^-\oplus\mathfrak{r}^+}$ and

$$\mathrm{ann}(\mathrm{Euler}(V_{\mathfrak{u}\oplus\mathfrak{r}^-\oplus\mathfrak{r}^+})) = \{c \in H^*(X//T_{\mathbb{C}}^*, \mathbb{Q}) | c \cup \mathrm{Euler}(V_{\mathfrak{u}\oplus\mathfrak{r}^-\oplus\mathfrak{r}^+}) = 0\} \subset H^*(X//T_{\mathbb{C}}^H, \mathbb{Q}).$$

is the annihilator ideal.

Again, diagram (14) provides a natural way to define a lift of a cohomology class on $X//H$ to a class on $X//T_{\mathbb{C}}^H$: we say that $\tilde{a} \in H^*(X//T_{\mathbb{C}}^H)$ is a lift of $a \in H^*(X//H)$ if $\pi^* a = i^* \tilde{a}$.

Theorem 5.14 (Integration formula). *Let X be a smooth projective variety endowed with a well-adapted action of $H = U \rtimes R$ such that H-stability=H-semistability holds. Given a cohomology class $a \in H^*(X//H)$ with a lift $\tilde{a} \in H^*(X//T_{\mathbb{C}}^H)$, then*

$$\int_{X//H} a = \frac{1}{|W|} \int_{X//T_{\mathbb{C}}^H} \tilde{a} \cup \mathrm{Euler}(V_{\mathfrak{u}\oplus\mathfrak{r}^-\oplus\mathfrak{r}^+}),$$

where we use the notation of Theorem 5.13.

5.4. **Localization for quotients by non-reductive groups with internally graded unipotent radicals.** Let X be a smooth projective variety with a well-adapted linear action of $H = U \rtimes R$ such that H-stability=H-semistability holds. We follow the notation and conventions of the previous section. We have two surjective ring homomorphisms

$$\kappa_{T_{\mathbb{C}}^H} : H^*_{T_{\mathbb{C}}^H}(X) \to H^*(X//T_{\mathbb{C}}^H) \text{ and } \kappa_H : H^*_R(X) = H^*_{T_{\mathbb{C}}^H}(X)^W \to H^*(X//H)$$

from the equivariant cohomology of X to the ordinary cohomology of the corresponding GIT quotient. The fixed points of the maximal torus $T_{\mathbb{C}}^H$ on $X \subset \mathbb{P}^n$ correspond to the weights of the $T_{\mathbb{C}}^H$ action on X, and since this action is well-adapted, these weights satisfy

$$\omega_{\min} = \omega_0 < 0 < \omega_1 < \ldots < \omega_r.$$

Theorem 5.15 (Localisation for non-reductive quotients). *Let X be a smooth projective variety endowed with a well-adapted linear action of $H = U \rtimes R$ such that H-stability=H-semistability holds. For any equivariant cohomology class η on X represented by an equivariant differential form $\eta(z)$ whose degree is the dimension of $X//H$ we have*

$$\int_{X//H} \kappa(\eta) = n_K \mathrm{JKRes}^\Lambda \left(\sum_{F \subset \mathcal{F}} \int_F \frac{i_F^*(\eta(z))\mathrm{Euler}(V_{\mathfrak{u}\oplus\mathfrak{r}^-\oplus\mathfrak{r}^+})(z)}{\mathrm{Euler}^T(\mathcal{N}_F)(z)} [dz] \right).$$

In this formula z is a variable in $\mathfrak{t}^H_{\mathbb{C}}$ so that a T^H-equivariant cohomology class can be evaluated at z, while JKRes^Λ is the JK residue map which depends on a choice of a cone $\Lambda \subset \mathfrak{t}^H$. The positive integer n_K depends on K and the size of the stabiliser in K of a generic $x \in X$.

In the special case when $H = \hat{U} = U \rtimes \mathbb{C}^*$ the theorem has a particularly nice form.

Corollary 5.16 (**Localisation for $\hat{U}$ quotients**). *Let X be a smooth projective variety with a well-adapted action of $\hat{U} = U \rtimes \mathbb{C}^*$ such that $\hat{U}$-stability=$\hat{U}$-semistability holds. Let z be the standard coordinate on the Lie algebra of $\mathbb{C}^*$.. For any equivariant cohomology class η on X represented by an equivariant differential form $\eta(z)$ whose degree is the dimension of $X//\hat{U}$ we have*

$$\int_{X//\hat{U}} \kappa(\eta) = n_{\mathbb{C}^*} \mathrm{Res}_{z=\infty} \int_{F_{\min}} \frac{i^*_{F_{\min}}(\eta(z) \cup \mathrm{Euler}(V_{\mathfrak{u}})(z))}{\mathrm{Euler}^T(\mathcal{N}_{F_{\min}})(z)} dz$$

where $F_{\min}$ is the union of those connected components of the fixed point locus $X^{\mathbb{C}^*}$ on which the S^1 moment map takes its minimum value $\omega_{\min}$.

6. Examples

In this section we illustrate the formulas of this paper with some examples.

6.1. Two points and a line in $\mathbb{P}^2$.

Let $X = (\mathbb{P}^2)^2 \times (\mathbb{P}^2)^*$ with elements (p, q, L) where p and q are points in $\mathbb{P}^2$ and L is a line in $\mathbb{P}^2$. Let X have the usual (left) action of the standard Borel subgroup $B = U \rtimes T$ of SL(3) consisting of upper triangular matrices. This action is linear with respect to the product of $\mathcal{O}(1)$ for each of the three projective planes whose product is X. We represent points in $\mathbb{P}^2$ by column vectors, with the action of a matrix A given by pre-multiplication by A; we represent lines

$$L_{(a,b,c)} = \{[x : y : z] \in \mathbb{P}^2 | ax + by + cz = 0\}$$

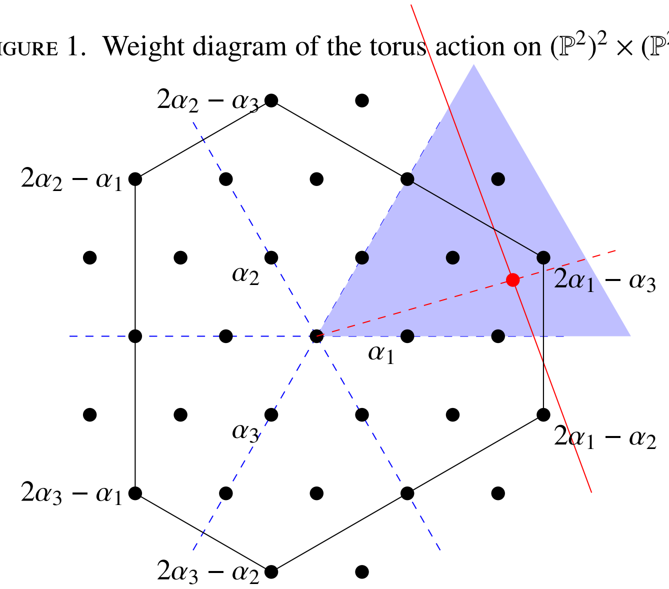
by row vectors (a, b, c) and the action of A is then given by post-multiplication by A^{-1}. The weights of the action of the maximal torus T of B on the image under the Segre embedding of X lie in an irregular hexagon (which is their convex hull) in the dual of the Lie algebra of the maximal compact subgroup of T (see Figure (6.1)). These weights are the elements of the set $\{\alpha_i + \alpha_j - \alpha_k : 1 \le i, j, k \le 3\}$ (where $\alpha_1, \alpha_2, \alpha_3$ are the restrictions to SL(3) of the standard basis of the weight lattice for GL(3) so that $\alpha_1 + \alpha_2 + \alpha_3 = 0$). Their multiplicities, meaning the numbers of fixed points on X with the given weights, are given in the following table:

Weight type	$2\alpha_i - \alpha_j$	$\alpha_i + \alpha_j - \alpha_k$	α_i
Multiplicity	1	2	5

In order for a one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ to satisfy the condition that all its weights for the adjoint action on Lie U should be positive, we require its derivative to lie in the interior of the standard positive Weyl chamber $\mathfrak{t}_+$ for SL(3). We use the standard invariant inner product on the Lie algebra of the unitary group $U(3)$ to identify the Lie algebra of T with its dual. We can then choose this derivative to be close to the maximal weight $2\alpha_1 - \alpha_3$ in the hexagon and then the corresponding one-parameter subgroup λ is close to $\lambda(t) = \mathrm{diag}(t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3})$ where

$$\lambda_1 = 5/3, \ \lambda_2 = -1/3, \ \lambda_3 = -4/3. \tag{16}$$

FIGURE 1. Weight diagram of the torus action on $(\mathbb{P}^2)^2 \times (\mathbb{P}^2)^*$



Note that the stratification we describe below does not depend on a small perturbation of λ (this is what we mean by "close to λ"), but for the action of the residual $R = T/\lambda(\mathbb{C}^*)$ on $X//\hat{U}$ this perturbation will be essential below. Then λ defines the subgroup

$$\hat{U} = U \rtimes \lambda(\mathbb{C}^*) = \left\{ \begin{pmatrix} t^{\lambda_1} & \alpha & \beta \\ 0 & t^{\lambda_2} & \gamma \\ 0 & 0 & t^{\lambda_3} \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

In Figure 6.1 the red dashed line represents the ray in $\mathfrak{t}^*$ dual to λ. The weights for λ are the orthogonal projections of the weights for SL(3) onto this red dashed line. Therefore the minimal weight for λ is the image of $2\alpha_1 - \alpha_3$ under this projection, and it corresponds to the T-fixed point

$$Z_{\min} = \{z_{\min}\} = \{([0,0,1],[0,0,1],L_{(1,0,0)})\}$$

where the points p and q coincide at $[0,0,1]$ and both lie on the line L which is defined by $x = 0$. The stabiliser in U of $z_{\min}$ is trivial, and

$$Uz_{\min} = \{[x,y,z],[x,y,z],[a,b,c] | ax + by + cz = 0, z \neq 0 \neq a\}$$

whose closure is

$$\overline{Uz_{\min}} = \{(p,q,L) \in X | p = q \in L\},$$

Moreover,

$$X^0_{\min} = \{([x_1,y_1,z_1],[x_2,y_2,z_2],L_{(a,b,c)}) \in X | z_1 \neq 0 \neq z_2, a \neq 0\}$$

and by Lemma 4.3

$$X^0_{\min} \setminus Uz_{\min}$$

has a projective geometric quotient $X//\hat{U} = (X^0_{\min} \setminus Uz_{\min})/\hat{U}$ when the linearisation is twisted by a rational character χ in the interior of the hexagon near to the T-weight for $z_{\min}$. In Figure (6.1) the red dot on the dashed red line indicates the shifted origin. Finally, $X//B = (X//\hat{U})//R$ is the reductive GIT quotient of the projective variety $X//\hat{U}$ by the residual torus action where $R = T/\lambda(\mathbb{C})$.

According to Theorem 4.5, the Betti numbers of the GIT quotient $X//\hat{U}$ are

$$P_t(X//\hat{U}) = P_t(X_{\min}^{\mathbb{C}^*})\frac{1-t^{2d}}{1-t^2} = \frac{1-t^6}{1-t^2} = 1+t^2+t^4.$$

The Betti numbers of the projective variety $X//\hat{U}$ suggest that $X//\hat{U}$ is a weighted projective space $\mathbb{P}[n_1,n_2,n_3]$ for some integers n_i determined by the weights of $\mathbb{C}^*$ in $\hat{U}$. Before we check this by hand, and study the residual $\mathbb{C}^*_R = T/\lambda(\mathbb{C})$ action on $X//\hat{U}$, we make an observation. Since $Z_{\min} = z_{\min}$ is a point, $\mathbb{C}^*_R$ stabilizes it, and $Z_{\min}^{s,\mathbb{C}^*_R} = Z_{\min}^{ss,\mathbb{C}^*_R} = \emptyset$, so strictly speaking, the conditions of Theorem 1.3 do not hold. However, the argument of Lemma 4.6 shows that $X^{ss,H}/B \to z_{\min}$ is a projective fibration again over a point, so $X//H = \mathbb{P}^1$ and hence

$$P_t(X//B) = 1+t^2.$$

We now check by hand what $X//B$ is as follows. Using the U action, every element of $X^0_{\min}$ can be moved into the slice

$$\mathcal{S} = X^0_{\min}/U \cong \{([0,\alpha,1],[\beta,0,1],[1,0,\gamma]) \mid \alpha,\beta,\gamma\in\mathbb{C}\} \cong \mathbb{C}^3$$

where the residual $\mathbb{C}^* \subset \hat{U}$-weights on the coordinates α,β,γ are given by

$$w(\alpha) = \lambda_2-\lambda_1, w(\beta) = \lambda_3-\lambda_1, w(\gamma) = \lambda_1-\lambda_3.$$

The points in $Uz_{\min}$ correspond to the origin $\alpha=\beta=\gamma=0$ and therefore

$$(X^0_{\min}\setminus Uz_{\min})/\hat{U} = (\mathbb{C}^3\setminus\{0\})/\mathbb{C}^* = \mathbb{P}[\lambda_2-\lambda_1,\lambda_3-\lambda_1,\lambda_1-\lambda_3].$$

The weights for the residual $R = T/\lambda(\mathbb{C}) \simeq \mathbb{C}^*$ action on $X//\hat{U}$ are represented by the orthogonal projections of the original weights onto the red line in Figure 6.1, and we have to drop the image of the maximal weight $2\alpha_1-\alpha_3$ from these. The origin sits between $\pi(\alpha_1)$ and $\pi(2\alpha_1-\alpha_3)$. We choose the positive Weyl chamber to be the lower half line. With this choice the order of the positive weights are

$$\pi(2\alpha_2-\alpha_3) < \pi(2\alpha_2-\alpha_1) < \pi(\alpha_1+\alpha_2-\alpha_3) < \pi(\alpha_2) < \pi(\alpha_2+\alpha_3-\alpha_1) < \pi(2\alpha_1-\alpha_3).$$

In particular, this means that

$$\pi(\alpha_2) < \pi(\alpha_1) < \pi(\alpha_3)$$

so R corresponds to the one-parameter subgroup

$$R = \left\{\begin{pmatrix} t^{\pi(\alpha_1)} & 0 & 0 \\ 0 & t^{\pi(\alpha_2)} & 0 \\ 0 & 0 & t^{\pi(\alpha_3)} \end{pmatrix} : \pi(\alpha_2) < \pi(\alpha_1) < \pi(\alpha_3) \in \mathbb{C}\right\}.$$

where the weight of the the second coordinate on $\mathbb{P}^2$ is the highest weight followed by the weight of the first and finally the third coordinate.

The nonzero coordinates of the image of the slice $\mathcal{S}$ under the Segre embedding

$$\mathbb{P}^2\times\mathbb{P}^2\times(\mathbb{P}^2)^* \supset \mathcal{S} = \mathbb{P}(\alpha:\beta:\gamma) \hookrightarrow \mathbb{P}^7$$

correspond to the 3-term products of nonzero coordinates on $\mathcal{S}$: one nonzero coordinates from each copy of $\mathbb{P}^2$. These are the weights shown in Figure 6.1:

$$w(1\cdot 1\cdot\gamma) = \alpha_1,\ \ w(1\cdot\beta\cdot\gamma) = \alpha_3,\ \ w(1\cdot\beta\cdot 1) = \alpha_1$$

$$w(\alpha\cdot 1\cdot 1) = \alpha_1+\alpha_2-\alpha_3,\ \ w(\alpha\cdot 1\cdot\gamma) = \alpha_2,\ \ w(\alpha\cdot\beta\cdot 1) = \alpha_2,\ \ w(\alpha\cdot\beta\cdot\gamma) = \alpha_2$$

The convex hull of the projections of these weights to the red line contains the red origin. The image of the line

$$\mathbb{P}^1 = \{[\alpha:\beta:\gamma]\in\mathcal{S}|\alpha=0\} \hookrightarrow \mathbb{P}^7$$

has the following nonzero coordinates:

$$w(1\cdot 1\cdot \gamma)=\alpha_1,\;\; w(1\cdot \beta\cdot \gamma)=\alpha_3,\;\; w(1\cdot \beta \cdot 1)=\alpha_1$$

which all sit on the positive half line, and it represents an unstable stratum of codimension 1 and Poincaré polynomial $1+t^2$. Similarly, the image of the point

$$[1:0:0]\hookrightarrow \mathbb{P}^7$$

has only the following nonzero coordinate:

$$w(\alpha\cdot 1\cdot 1)=\alpha_1+\alpha_2-\alpha_3$$

which again represents an unstable stratum of codimension 2. The unstable strata contribute to the Poincaré polynomial by $t^2(1+t^2)+t^4=t^2+2t^4$. Let $q: X^{ss,\hat{U}}=X_{\min}\setminus \hat{U}Z_{\min}\to X/\!/\hat{U}$ denote the quotient map. The Poincaré series is then given by

$$P_t(X/\!/B)=P_t((X/\!/\hat{U})/\!/R)=P_t(X/\!/\hat{U})P_t(B\mathbb{C}^*)-t^{\mathrm{codim}(q(S_{2\alpha_1-\alpha_3}))}P_t(C_{2\alpha_1-\alpha_3})P_t(B\mathbb{C}^*)=$$
$$=\frac{1}{1-t^2}\left(\frac{1-t^6}{1-t^2}-t^2-2t^4\right)=1+t^2.$$

6.2. **n points and a line on $\mathbb{P}^2$.** We can easily generalise the argument of the previous section to give a complete description of the GIT quotient $X/\!/\hat{U}$ where $X=(\mathbb{P}^2)^n\times(\mathbb{P}^2)^*$ is the set of tuples $(p_1,\ldots,p_n,L)$ with n points and a line L in $\mathbb{P}^2$. Let X have the usual (left) action of the standard Borel subgroup $B=U\rtimes T$ of SL(3), consisting of upper triangular matrices, which is linear with respect to the product of $\mathcal{O}(1)$ for each of the $n+1$ projective planes whose product is X.

Proposition 6.1. *The Poincaré polynomial of the GIT quotients $X/\!/\hat{U}$ and $X/\!/H$ are*

$$P_t(X/\!/\hat{U})=P_t(z_{\min})\frac{1-t^{4n}}{1-t^2}=1+t^2+\ldots+t^{4n-4}.$$
$$P_t(X/\!/H)=P_t(z_{\min})\frac{1-t^{4n-2}}{1-t^2}=1+t^2+\ldots+t^{4n-6}.$$

Hence the Betti numbers of these quotients are those of (weighted) projective spaces, as expected.

References

[1] M. Atiyah, R. Bott, The moment map and equivariant cohomology, Topology, 23 (1984), 1-28.

[2] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, Projective completions of graded unipotent quotients, arXiv:1607.04181.

[3] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, Constructing quotients of algebraic varieties by linear algebraic group actions, in Handbook of Group Actions (Editors: Lizhen Ji, Athanase Papadopoulos, Shing-Tung Yau), Advanced Lectures in Mathematics (ALM), Higher Education Press and International Press, 2018.

[4] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, Geometric invariant theory for graded unipotent groups and applications, Journal of Topology 11 (2018), 826-855.

[5] G. Bérczi, J. Jackson, F. Kirwan, Variation of non-reductive geometric invariant theory, Surveys in Differential Geometry 22 (2017), 49-69.

[6] G. Bérczi, F. Kirwan, Non-reductive geometric invariant theory and hyperbolicity, in preparation.

[7] Bérczi, G., Szenes, A.: Thom polynomials of Morin singularities. Ann. Math. 175 (2012), 567-629.

[8] N. Berline and M. Vergne, Zéros d'un champ de vecteurs et classes caractéristic équivariantes, Duke Math. J. 50 (1983), 539-549.

[9] R. Bott, A Residue Formula for Holomorphic Vector Fields, J. of Diff. Geom.1 (1967), 311-330.

[10] A. Bialynicki-Birula, Some theorems on actions of algebraic groups, Ann. Math. 2 (1973), 480- 497.

[11] H. Derksen and G. Kemper, Computing invariants of algebraic groups in arbitrary characteristic, Advances in Mathematics 217 (2008), 2089-2129.

[12] I. Dolgachev, Lectures on invariant theory, London Mathematical Society Lecture Note Series 296, Cambridge University Press, 2003.
[13] I. Dolgachev and Y. Hu, Variation of geometric invariant theory quotients, Publications Mathématiques de L'Institut des Hautes Scientifiques 87 (1998), 5-51. doi:10.1007/BF02698859.
[14] B. Doran and F. Kirwan, Towards non-reductive geometric invariant theory, Pure Appl. Math. Q. 3 (2007), 61-105.
[15] J. Duistermaat and G. Heckman, On the variation in the cohomology of the symplectic form of the reduced phase space, Invent. Math., 69 (1982), 259-268; Addendum, 72 (1983), 153-158.
[16] A. Fauntleroy, Categorical quotients of certain algebraic group actions, Illinois Journal Math. 27 (1983), 115-124.
[17] A. Fauntleroy, Geometric invariant theory for general algebraic groups, Compositio Mathematica 55 (1985), 63-87.
[18] D. Greb and C. Miebach, Hamiltonian actions of unipotent groups on compact Kähler manifolds, Épijournal de Géométrie Algébrique 2 (2018), epiga:4962.
[19] G.-M. Greuel and G. Pfister, Geometric quotients of unipotent group actions, Proc. London Math. Soc. (3) 67 (1993) 75-105.
[20] G.-M. Greuel and G. Pfister, Geometric quotients of unipotent group actions II, Singularities (Oberwolfach 1996), 27-36, Progress in Math. 162, Birkhauser, Basel 1998.
[21] F. Grosshans, Localization and invariant theory, Advances in Mathematics 21 (1976), 50-60.
[22] F. Grosshans, The invariants of unipotent radicals of parabolic subgroups, Invent. Math. 73 (1983), 1–9.
[23] V. Guillemin and J. Kalkman, The Jeffrey-Kirwan localization theorem and residue operations in equivariant cohomology, J. Reine Angew. Math., 470 (1996), 123-142.
[24] V. Guillemin, L. Jeffrey and R. Sjamaar, Symplectic implosion, Transformation Groups 7 (2002), 155-184.
[25] L. Jeffrey and F. Kirwan, Localization for nonabelian group actions, Topology, 34 (1995), 219-268.
[26] L. Jeffrey and M. Kogan, Localization theorems by symplectic cuts, in The Breadth of Symplectic and Poisson Geometry, Progr. Math. 232, Birkhäser, Boston, MA, 2005, 303-326.
[27] E. Lerman, Symplectic cuts, Mathematical Research Letters 2 (1995), 247-258
[28] Y.-H. Kiem, Intersection cohomology of quotients of nonsingular varieties, Invent. Math. 155 (2004), 163-202.
[29] F. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Mathematical Notes 31, Princeton University Press, Princeton, NJ, 1984.
[30] F. Kirwan, Partial desingularisations of quotients of nonsingular varieties and their Betti numbers, Ann. Math. (2) 122 (1985), 41-85.
[31] F. Kirwan, Rational intersection cohomology of quotient varieties, Invent. Math. 86 (1986), 471-505.
[32] F. Kirwan, Rational intersection cohomology of quotient varieties II, Invent. Math. 90 (1987), 153-167.
[33] F. Kirwan, Quotients by non-reductive algebraic group actions, arXiv:0801.4607, in 'Moduli Spaces and Vector Bundles', Cambridge University Press.
[34] F. Kirwan, Symplectic implosion and non-reductive quotients, in 'Geometric Aspects of Analysis and Mechanics', Progress in Mathematics Vol 292, 2011, 213-256.
[35] F. Kirwan, Symplectic quotients of unstable Morse strata for normsquares of moment maps, arXiv:1802.09237.
[36] J. Kollár, Quotient spaces modulo algebraic groups, *Ann. Math. (2)* **145** (1997), 33-79.
[37] S. Martin, Symplectic quotients by a nonabelian group and by its maximal torus, arxiv:0001.002.
[38] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory, 3rd edition, Springer, 1994.
[39] P.E. Newstead, Introduction to moduli problems and orbit spaces, Tata Institute Lecture Notes, Springer, 1978.
[40] Zs. Szilagyi, Computation of Jeffrey-Kirwan residue using Gröbner basis, arXiv:1210.5155.
[41] M. Thaddeus, Geometric invariant theory and flips, Journal of Amer. Math. Soc. 1996.
[42] J. Winkelmann, Invariant rings and quasiaffine quotients, Math. Z. 244 (2003), 163-174.

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