Extending Single Tolerances to Set Tolerances

Gerold Jäger[a], Marcel Turkensteen[b]

[a] *Department of Mathematics and Mathematical Statistics,*
University of Umeå,
SE-901-87 Umeå, Sweden,
gerold.jaeger@math.umu.se
[b] *Department of Economics and Business,*
School of Business and Social Sciences,
University of Aarhus,
Fuglesangs Alle 4,
DK-8210 Aarhus, Denmark,
matu@econ.au.dk

Abstract

The theory of single upper and lower tolerances for combinatorial minimization problems was formalized in 2005 for the three types of cost functions sum, product, and maximum, and since then it has shown to be rather useful in creating heuristics and exact algorithms. However, such single tolerances are often used because the assessment of multiple cost changes is considered too complicated. This paper addresses that issue. In this paper we extend this theory from single to set tolerances for these three types of cost functions. In particular, we characterize specific values of set upper and lower tolerances as positive and infinite, and we show a criterion for the uniqueness of an optimal solution to a combinatorial minimization problem. Furthermore, we present one exact formula and several bounds for computing set upper and lower tolerances using the relation to their corresponding single tolerance counterparts.

Keywords: Sensitivity analysis, Combinatorial optimization, Single tolerance, Set tolerance

1. Introduction

The notion of tolerances originates from *sensitivity analysis* of combinatorial minimization problems [7, 8, 15, 32], which is a well-established topic in linear programming [8] and mixed integer programming [15]. The notion of *single tolerance* corresponds to the most elementary topic of sensitivity analysis, namely the special case when the value of a single element in a feasible solution is subject to an additive change. More precisely, for an element *in* a given optimal solution, its *single upper tolerance* determines the maximum additive increase of the individual cost of this given element preserving the optimality of this solution, while keeping the costs of other elements unchanged. Analogously, for an element *not in* a given optimal solution, its *single lower tolerance* determines the maximum additive decrease of the individual cost of this given element preserving the optimality of this solution, while keeping the costs of other elements unchanged. So the tolerance is a measure of stability of optimal solutions. Efficient methods for computing (single) tolerances have been presented for the following combinatorial minimization problems: for the Minimum Spanning Tree Problem (MSTP) [5, 19, 33], the TSP [23], the Linear Assignment Problem (LAP) [36], network flow problems [16, 30], shortest path problems [27, 30], scheduling problems [18], the Maximum Capacity Problem [27], and linear forms [31]. The first successful implicit application of (upper) tolerances in algorithm design has appeared in the so-called Vogel's Approximation Method for the Transportation Simplex Problem [28]. Furthermore, tolerances have been used for a straightforward enumeration of the k-best solutions for some natural k for the Linear Assignment Problem [25] and the TSP [26] as well as a base of the Max-Regret heuristic for solving the Three-Index Assignment Problem [1].

The theory of single tolerances has been formalized by Goldengorin, Jäger, Molitor [11, 12] for three different types of cost functions, namely of types sum, product and maximum. Based on this theory, effective heuristics and exact algorithms have been created and implemented [6, 9, 10, 13, 14, 20, 29, 34] for the TSP as well as for related problems [2, 17, 22], proving the usefulness of the concept of tolerances. In 2010, Libura introduced and investigated the so-called *robustness tolerance* [24] where one not only determines for which changes a solution remains optimal, but also where it remains robust, i.e., gives the lowest regret. However, this is not a topic of this paper.

One disadvantage of using single tolerance values is that they only apply to changes in single parameter values. We cannot consider the effect of multiple parameter changes. Such multiple parameter changes are relevant in sensitivity analysis, as we illustrate in Section 3, but also in the above mentioned algorithms. Currently, these algorithms consider the deletion or inclusion of one element at a time. However, if we know that multiple elements are to be removed (included), a tool that computes the joint effect of the removal (inclusion) of multiple elements would be very useful indeed.

The purpose of this work is to extend the theory of single tolerances to so-called *set tolerances* where the set upper tolerances are defined for a set of elements *in* a given optimal solution, and the set lower tolerances are defined for a set of elements *not in* a given optimal solution. The *set tolerance* is defined as the maximum sum of values that can be be added to the elements of the set so that the given optimal solution stays optimal.

The main question in this paper is whether and how set tolerances can be computed using single tolerances and the values they can attain.

We reach the following results where some of them are only valid for some of the three types of cost functions:

- The set upper and lower tolerances are well defined, i. e., do not depend on the corresponding optimal solution.
- The sets overlapping with each feasible solution are exactly the sets with infinite set upper tolerance.

- The sets not contained in the union of all feasible solutions are exactly the sets with infinite set lower tolerance.
- The uniqueness of an optimal solution can be described by the set upper and lower tolerances.
- The set upper and lower tolerances can be bounded by their corresponding single tolerance counterparts (i.e., the single tolerances of all elements in the set). The relations are completely different for the three types of cost function.

Some of the theorems in this paper were presented in a previous work [21], but proofs and interpretations of the results were not presented.

This paper is organized as follows. In Section 2, we give the notions of combinatorial minimization problem and of the single upper and lower tolerances. In Section 3, we highlight the shortcomings of single tolerances and motivate the usage of set tolerances. In Section 4, we present the theory of set upper tolerances and in Section 5, we proceed with the theory of set lower tolerances. In Section 6, we provide several basic computational examples and in Section 7, we give two applications of this theory, namely the Linear Assignment Problem and the Asymmetric Bottleneck Traveling Salesman Problem. Finally, Section 8 provides the conclusions and some suggestions for future research.

2. Notations and Definitions

In this section, we formally present the notation and key results on combinatorial minimization problems and single upper and lower tolerances (see [11, 12]). Note that we add the adjective 'single' to distinguish these from set tolerances.

2.1. Combinatorial Minimization Problems

A *combinatorial minimization problem* $\mathcal{P}$ is given by a tuple $(\mathcal{E}, D, c, f_c)$ where $\mathcal{E}$ is a finite *ground set of elements*, $D \subseteq 2^{\mathcal{E}} \setminus \{\emptyset\}$ is the set of *feasible solutions*, $c : \mathcal{E} \to \mathbb{R}$ is the *cost function*, which assigns costs to each single element of $\mathcal{E}$, $f_c : D \to \mathbb{R}$ is the objective (cost) function, which depends on the function c and assigns costs to each feasible solution D.
Then the problem is to find a feasible solution with a cost as small as possible. Of course, analogous considerations can be made if the costs have to be maximized, i. e., for combinatorial maximization problems.
$S^\star \subseteq \mathcal{E}$ is called an *optimal solution* of $\mathcal{P}$ if $S^\star$ is a feasible solution and the cost $f_c(S^\star)$ of $S^\star$ is minimum, i. e., $S^\star \in D$ and $f_c(S^\star) = \min\{f_c(S) \mid S \in D\}$. We denote the cost of an optimal solution $S^\star$ of $\mathcal{P}$ by $f_c(\mathcal{P})$ and the set of optimal solutions by $D^\star$. There are some particular cost functions which often occur in practice, namely (cf. Examples 1 and 2):

- The cost function $f_c : D \to \mathbb{R}$ is of **type** $\sum$ if for all $S \in D$: $f_c(S) = \sum_{e \in S} c(e)$ holds.

- The cost function $f_c : D \to \mathbb{R}$ is of **type** $\prod$ if for all $S \in D$: $f_c(S) = \prod_{e \in S} c(e)$ holds and for all $e \in \mathcal{E} : c(e) > 0$ holds.
- The cost function $f_c : D \to \mathbb{R}$ is of **type MAX** if for all $S \in D$: $f_c(S) = \max \{c(e) \mid e \in S\}$ holds. Such a cost function is also called *bottleneck function*.

Cost functions of type $\sum$, $\prod$, MAX are *monotonically increasing* in a single element $e \in \mathcal{E}$, i.e., the cost of a subset of $\mathcal{E}$ does not become cheaper, if the cost of e increases.
Furthermore, cost functions of type $\sum$, $\prod$, MAX are *continuous* when changing cost values. As in [11, 12], we only consider combinatorial minimization problems $\mathcal{P} = (\mathcal{E}, D, c, f_c)$ that fulfill the following three conditions:

Condition 1. *The set D of feasible solutions of $\mathcal{P}$ is independent of the cost function c.*

Condition 2. *The cost function $f_c : D \to \mathbb{R}$ is of type $\sum$, $\prod$, or MAX.*

Condition 3. *There is at least one optimal solution of $\mathcal{P}$, i. e., $D^\star \neq \emptyset$.*

Let a combinatorial minimization problem $\mathcal{P} = (\mathcal{E}, D, c, f_c)$ be given. We obtain a new combinatorial minimization problem if we add some constant $\alpha \in \mathbb{R}$ to the cost of a single element $e \in \mathcal{E}$. We denote the new problem by $\mathcal{P}_{c_{\alpha,e}} = (\mathcal{E}, D, c_{\alpha,e}, f_{c_{\alpha,e}})$, which is formally defined as $c_{\alpha,e}(\bar{e}) = \begin{cases} c(\bar{e}), & \text{if } \bar{e} \neq e \\ c(\bar{e}) + \alpha, & \text{if } \bar{e} = e \end{cases}$ for all $\bar{e} \in \mathcal{E}$. Note that $f_{c_{\alpha,e}}$ is of the same type as f_c, unless the cost function is of type $\prod$ and $\alpha \leq -c(e)$.
For $M \subseteq D$, we denote the cost of the best solution included in M by $f_c(M)$. The cost $f_c(M)$ for $M = \emptyset$ is defined as infinite, i. e., $+\infty$. Obviously, for all $M \subseteq D$ it holds that $f_c(\mathcal{P}) \leq f_c(M)$.
Let $e \in \mathcal{E}$. We denote the set of feasible solutions of D such that each of them does not contain $e \in \mathcal{E}$ by $D_-(e)$, i. e., $D_-(e) = \{ S \in D \mid e \in \mathcal{E} \setminus S\}$. Analogously, we denote the set of feasible solutions of D such that each of them contains $e \in \mathcal{E}$ by $D_+(e)$, i. e., $D_+(e) = \{ S \in D \mid e \in S\}$.
Now we generalize our considerations from a single element $e \in \mathcal{E}$ to a subset $E \subseteq \mathcal{E}$ with $E = \{e_1, e_2, \ldots, e_k\}$ and $k \geq 1$ where $e_1, e_2, \ldots, e_k$ are in a fixed order.
Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k$. We also obtain a new combinatorial minimization problem if for all $l \in \mathbb{N}$ with $1 \leq l \leq k$ we add $\alpha_l \in \mathbb{R}$ to the cost of e_l. We denote the new problem by $\mathcal{P}_{c_{\vec{\alpha},E}} = (\mathcal{E}, D, c_{\vec{\alpha},E}, f_{c_{\vec{\alpha},E}})$, which is formally defined as $c_{\vec{\alpha},e}(\bar{e}) = \begin{cases} c(\bar{e}), & \text{if } \bar{e} \in \mathcal{E} \setminus E \\ c(\bar{e}) + \alpha_l, & \text{if } \bar{e} = e_l \end{cases}$ for all $\bar{e} \in \mathcal{E}$. Note that $f_{c_{\vec{\alpha},E}}$ is of the same type as f_c, unless the cost function is of type $\prod$ and $\alpha_l \leq -c(e_l)$ for at least one $l \in \{1, 2, \ldots, k\}$.
We now define the single tolerances briefly, as these definitions are needed in the following. The corresponding theory of single tolerances can be found in [11, 12].

2.2. Single Upper Tolerances

Let $\mathcal{P}$ be an instance, $S^\star$ an optimal solution of $\mathcal{P}$, and $e \in S^\star$. Then the *single upper tolerance* $u_{S^\star}(e)$ of e with respect to $S^\star$ is defined as the supremum by which the cost of e can be increased such that $S^\star$ remains optimal, provided that the costs of all other elements $\bar{e} \in \mathcal{E} \setminus \{e\}$ remain unchanged, i. e., the single upper tolerance of e is defined as follows:

$$u_{S^\star}(e) \quad := \quad \sup\,\{\alpha \in \mathbb{R}_0^+ \mid \; S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{\alpha,e}}\}.$$

Because of the monotonicity and the continuity of cost functions of type $\sum$, $\prod$, MAX, it holds that:

$$u_{S^\star}(e) = \inf\,\{\alpha \in \mathbb{R}_0^+ \mid S^\star \text{ is not an optimal solution of } \mathcal{P}_{c_{\alpha,e}}\}. \qquad (1)$$

It holds that $u_{S^\star}(e)$ is either an element of $\mathbb{R}_0^+$ or infinite. Because of the continuity of the cost function, for all $e \in S^\star$ with $0 \leq u_{S^\star}(e) < +\infty$, it holds that:

$$u_{S^\star}(e) \quad = \quad \max\,\{\alpha \in \mathbb{R}_0^+ \mid S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{\alpha,e}}\}. \qquad (2)$$

By [11, 12], it holds for any instance $\mathcal{P}$ that the single upper tolerance does not depend on a particular optimal solution of $\mathcal{P}$. Therefore, we refer to the upper tolerance of e with respect to an optimal solution $S^\star$ as the upper tolerance of e with respect to $\mathcal{P}$, $u_{\mathcal{P}}(e)$. Let $UTE_{\mathcal{P}} := \{e \in \mathcal{E} \mid \exists S^\star \in D^\star : e \in S^\star\}$ be the set of elements in $\mathcal{E}$ for which the upper tolerance is defined with respect to $\mathcal{P}$. Obviously, it holds that:

$$UTE_{\mathcal{P}} \quad = \quad \bigcup_{S^\star \in D^\star} S^\star. \qquad (3)$$

Later we will need the following theorems from [11, 12] about single upper tolerances.

Theorem 1. *Let $\mathcal{P}$ be an instance and $e \in UTE_{\mathcal{P}}$. Then the following statements are equivalent:*

a) $e \in \bigcap_{S \in D} S$.

b) $u_{\mathcal{P}}(e) = +\infty$.

Theorem 2. *Let $\mathcal{P}$ be an instance and $e \in UTE_{\mathcal{P}}$. Then the following statements hold:*

a) $u_{\mathcal{P}}(e) = f_c(D_-(e)) - f_c(\mathcal{P})$, *if the cost function is of type* $\sum$.

b) $u_{\mathcal{P}}(e) = \frac{f_c(D_-(e)) - f_c(\mathcal{P})}{f_c(\mathcal{P})} \cdot c(e)$, *if the cost function is of type* $\prod$.

c) $u_{\mathcal{P}}(e) = f_c(D_-(e)) - c(e)$, *if the cost function is of type MAX.*

Theorem 3. *Let $\mathcal{P}$ be an instance, and let $e \in UTE_{\mathcal{P}}$.*

a) *Let the cost function be of type $\sum$ or $\prod$. Then it holds that:*

$$e \in \bigcap_{S^\star \in D^\star} S \quad \Leftrightarrow \quad u_{\mathcal{P}}(e) > 0.$$

b) *Let the cost function be of type MAX. Then it holds that:*

$$e \in \bigcap_{S^\star \in D^\star} S \quad \Rightarrow \quad u_{\mathcal{P}}(e) > 0.$$

2.3. Single Lower Tolerances

Let $\mathcal{P}$ be an instance, $S^\star$ an optimal solution of $\mathcal{P}$, and $e \in \mathcal{E} \setminus S^\star$. We seek the supremum by which the cost of $e \in \mathcal{E}$ can be decreased such that $S^\star$ remains optimal, provided that the costs of all other elements remain unchanged. Note that if the cost function is of type $\prod$, the costs of the elements are larger than 0. Define $\delta(e) := \begin{cases} +\infty, & \text{if the cost function is of type } \sum \text{ or MAX} \\ c(e), & \text{if the cost function is of type } \prod \end{cases}$. $\delta(e)$ is the supremum by which $e \in \mathcal{E}$ can be decreased such that the cost function remains of type $\sum$, $\prod$, or MAX, respectively. The *single lower tolerance* of e with respect to $S^\star$ is defined as follows:

$$l_{S^\star}(e) \quad := \quad \sup \{\alpha \in \mathbb{R}_0^+ \mid S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{-\alpha,e}}\}. \tag{4}$$

Because of the monotonicity and the continuity of the cost function, it holds that:

$$l_{S^\star}(e) \quad = \quad \inf \{\alpha \in \mathbb{R}_0^+ \mid S^\star \text{ is not an optimal solution of } \mathcal{P}_{c_{-\alpha,e}}\}. \tag{5}$$

For all $e \in \mathcal{E} \setminus S^\star$, it holds that $l_{S^\star}(e)$ is either an element of $\mathbb{R}_0^+$ or infinite. More precisely, it holds that: $0 \le l_{S^\star}(e) \le \delta(e)$. Because of the continuity of the cost function, for all $e \in \mathcal{E} \setminus S^\star$ and each $0 \le l_{S^\star}(e) < \delta(e)$, it holds that:

$$l_{S^\star}(e) \quad = \quad \max \{\alpha \in \mathbb{R}_0^+ \mid S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{-\alpha,e}}\}.$$

As by [11, 12], for an instance $\mathcal{P}$ the single lower tolerance does not depend on a particular optimal solution of $\mathcal{P}$, we refer to the lower tolerance of e with respect to an optimal solution $S^\star$ as the lower tolerance of e with respect to $\mathcal{P}$, $l_{\mathcal{P}}(e)$. Let $LTE_{\mathcal{P}} := \{e \in \mathcal{E} \mid \exists\, S^\star \in D^\star : e \in \mathcal{E} \setminus S^\star\}$ be the set of elements in $\mathcal{E}$ for which the lower tolerance is defined with respect to $\mathcal{P}$. Obviously, it holds that:

$$LTE_{\mathcal{P}} \quad = \quad \mathcal{E} \setminus \bigcap_{S^\star \in D^\star} S^\star. \tag{6}$$

Later we will need the following theorems from [11, 12] about single lower tolerances.

Theorem 4. *Let $\mathcal{P}$ be an instance where the cost function is of type $\sum$ or $\prod$. Furthermore, let $e \in LTE_{\mathcal{P}}$. Then the following statements are equivalent:*

a) $e \in \mathcal{E} \setminus \bigcup_{S \in D} S$.

b) $l_{\mathcal{P}}(e) = \delta(e)$.

Theorem 5. *Let $\mathcal{P}$ be an instance and $e \in LTE_{\mathcal{P}}$. Then the following statements hold:*

a) $l_{\mathcal{P}}(e) = f_c(D_+(e)) - f_c(\mathcal{P})$, *if the cost function is of type* $\sum$.

b) $l_{\mathcal{P}}(e) = \frac{f_c(D_+(e)) - f_c(\mathcal{P})}{f_c(D_+(e))} \cdot c(e)$, *if the cost function is of type* $\prod$.

c) *Let*

$$g(e) \quad := \quad \begin{cases} \min_{S \in D_+(e)} \max_{a \in S \setminus \{e\}} \{c(a)\}, & \text{if } D_+(e) \neq \emptyset \\ +\infty, & \text{if } D_+(e) = \emptyset \end{cases} .$$

$l_{\mathcal{P}}(e) = \begin{cases} c(e) - f_c(\mathcal{P}), & \text{if } g(e) < f_c(\mathcal{P}) \\ +\infty, & \text{otherwise} \end{cases}$, *if the cost function is of type MAX.*

Theorem 6. *Let $\mathcal{P}$ be an instance, and let $e \in LTE_{\mathcal{P}}$.*

a) *Let the cost function be of type $\sum$ or $\prod$. Then it holds that:*

$$e \in \mathcal{E} \setminus \bigcup_{S^\star \in D^\star} S \quad \Leftrightarrow \quad l_{\mathcal{P}}(e) > 0.$$

b) *Let the cost function be of type MAX. Then it holds that:*

$$e \in \mathcal{E} \setminus \bigcup_{S^\star \in D^\star} S \quad \Rightarrow \quad l_{\mathcal{P}}(e) > 0.$$

3. Motivating Examples

A set tolerance measures the largest change in the cost of the elements in a given set such that the current optimal solution remains optimal; see Eqs. (7), (13). The size of the change is measured as the sum of the changes in the individual parameter values. We provide two numerical examples to illustrate the usage of set upper and lower tolerances in Examples 1 and 2, and we compute the single upper and lower tolerances based on the results presented in Section 2. In Section 7 we then provide the set upper and lower tolerances and their interpretation.
In Example 1 we consider the Linear Assignment Problem (LAP); LAPs are discussed in [3]. Formally, the LAP is defined as follows. Let $n \in \mathbb{N}$ and $V := \{v_1, v_2, \ldots, v_n\}$. Furthermore, let $c : V \times V \to \mathbb{R}$ be a cost function. Then the aim is to find a one-to-one function $\phi : V \to V$ such that $\sum_{i=1}^{n} c(v_i, \phi(v_i))$ is minimized. Clearly the LAP has a cost function of type $\sum$.

Example 1. *Assume that we have three workers available for three jobs such that each worker can perform precisely one job. The cost of assigning a worker to a job is described by the following LAP instance with* $n = 3$ *where the cost function* $c : V \times V \to \mathbb{R}$ *is defined as follows:*

	v_1	v_2	v_3
v_1	0	4	5
v_2	7	0	6
v_3	8	9	0

The purpose is to find a cost minimizing assignment of jobs to the workers.
Each feasible solution contains exactly three elements from $V \times V$. $S_1 := \{(v_1, v_1), (v_2, v_2), (v_3, v_3)\}$ *is the only optimal solution with cost* 0*, i.e., this corresponds to assigning worker* 1 *to job* 1*, assigning worker* 2 *to job* 2*, and assigning worker* 3 *to job* 3.
As $\{(v_1, v_2), (v_2, v_1), (v_3, v_3)\}$ *with cost* 11 *is the best solution not containing* (v_1, v_1) *and also the best solution not containing* (v_2, v_2) *and as* $\{(v_1, v_3), (v_2, v_2), (v_3, v_1)\}$ *with cost* 13 *is the best solution not containing* (v_3, v_3)*, it holds because of Theorem 2a) that:*

$$u_{\mathcal{P}}((v_1, v_1)) = u_{\mathcal{P}}((v_2, v_2)) = 11, \; u_{\mathcal{P}}((v_3, v_3)) = 13.$$

Similarly, it holds because of Theorem 5a) that:

$$\begin{aligned} l_{\mathcal{P}}((v_1, v_2)) &= l_{\mathcal{P}}((v_2, v_1)) = 11, \\ l_{\mathcal{P}}((v_2, v_3)) &= l_{\mathcal{P}}((v_3, v_2)) = 15, \\ l_{\mathcal{P}}((v_1, v_3)) &= l_{\mathcal{P}}((v_3, v_1)) = 13. \end{aligned}$$

These upper and lower tolerance values tell us whether the current optimal assignment remains optimal if one cost parameter changes. This is, for example, the case if the cost of (v_1, v_1) increases by at most 11 units. Single upper tolerance values, however, cannot be used to assess the effect of such multiple cost changes on the optimality of a solution.
Using set tolerances we consider the impact of the following cost changes in Example 8 in Section 7:

- Suppose that we can distribute a cost increase α over the assignments $E_1 = \{(v_1, v_1), (v_3, v_3)\} \subseteq S_1$. What is the largest value of α such that S_1 can remain optimal?
- Suppose that the costs of the assignments $E_2 = \{(v_1, v_2), (v_3, v_2)\} \subseteq \mathcal{E} \setminus S_1$ can be reduced. What is the maximum decrease α in these element costs that still allows S_1 to be optimal?

A practical application of this is that the workers may wish to negotiate an increase in their rewards for performing the current jobs 1 and 3 (thereby increasing the current assignment costs), but they wish to keep the current assignments of workers to jobs. What increase can they achieve?

Moreover, the LAP is used as a relaxation to the ATSP; see [9, 17, 34]. The sets of cycles that one typically obtains as a solution to the LAP contains multiple cycles which are to be combined into a single tour through all locations. Set tolerances can be used to evaluate the costs of removing multiple arcs from the different cycles.
In Example 2 we consider a problem with objective MAX, namely the problem of determining a tour in a network through all locations where the length of the longest connection is minimized. This is an Asymmetric Bottleneck Traveling Salesman Problem (ABTSP); Bottleneck TSPs are discussed in [4, 35]. Formally, the ABTSP is defined as follows. Let $G = (V, E)$ be a directed graph with vertex set $V := \{v_1, v_2, \ldots, v_n\}$ and arc set E. Furthermore, let $c : E \to \mathbb{R}$ be a cost function on the set of arcs. Then the aim is to find a tour $(v_{j_1}, v_{j_2}, \ldots, v_{j_n}, v_{j_1})$ such that

$$\max\left\{c(v_{j_n}, v_{j_1}), \max_{i=1}^{n-1}\{c(v_{j_i}, v_{j_{i+1}})\}\right\}$$

is minimized. Clearly, the ABTSP has a cost function of type MAX.

Example 2. *Let $n = 4$ and the cost function $c : E \to \mathbb{R}$ be defined as follows:*

	v_1	v_2	v_3	v_4
v_1	–	1	2	11
v_2	3	–	4	5
v_3	12	6	–	7
v_4	8	9	10	–

$S_1 := \{(v_1, v_2, v_3, v_4, v_1)\}$ and $S_2 := \{(v_1, v_3, v_2, v_4, v_1)\}$ are the optimal solutions with cost 8. For the sake of convenience, we denote a solution by the arcs that are contained in the tour. It holds because of Theorem 2c) that:

$$u_{\mathcal{P}}((v_1, v_2)) = 7,\ u_{\mathcal{P}}((v_2, v_3)) = 4,\ u_{\mathcal{P}}((v_3, v_4)) = 1,\ u_{\mathcal{P}}((v_4, v_1)) = 1,$$
$$u_{\mathcal{P}}((v_1, v_3)) = 6,\ u_{\mathcal{P}}((v_3, v_2)) = 2,\ u_{\mathcal{P}}((v_2, v_4)) = 3.$$

It holds because of Theorem 5c) that:

$$l_{\mathcal{P}}((v_1, v_2)) = l_{\mathcal{P}}((v_1, v_3)) = l_{\mathcal{P}}((v_1, v_4)) = +\infty,$$
$$l_{\mathcal{P}}((v_2, v_1)) = l_{\mathcal{P}}((v_2, v_3)) = l_{\mathcal{P}}((v_2, v_4)) = +\infty,$$
$$l_{\mathcal{P}}((v_3, v_1)) = l_{\mathcal{P}}((v_3, v_2)) = l_{\mathcal{P}}((v_3, v_4)) = +\infty,$$
$$l_{\mathcal{P}}((v_4, v_2)) = 1,\ l_{\mathcal{P}}((v_4, v_3)) = +\infty.$$

Using set tolerances we consider the impact of the following cost changes in Example 9 in Section 7:

- Suppose that we can distribute a cost increase α over the connections $E_1 = \{(v_1, v_3), (v_4, v_1)\} \subseteq S_2$. What is the largest value of α such that S_2 can remain optimal?

- Suppose that the costs of the connections $E_2 = \{(v_1, v_4), (v_4, v_3)\} \subseteq \mathcal{E} \setminus S_1$ can be reduced. What is the maximum decrease α in these element costs that still allows S_1 to be optimal?

Here we have the following practical application. Suppose that a government organization wishes to discourage transporters from using certain connections in a road network through the use of tariffs or speed impediments. Suppose further that electric cars will be used frequently in this area. As there are usually only a few fuel stations for electric cars, those cars have to fuel very often and for avoiding to fuel for long distances they prefer routes, where the maximum distance between cities is minimized It holds that if the cost increases from the incentives are less than α, a driver of an electric car may still use the current paths, and the tariffs or the speed impediments only lead to cost or driving time increases.
If we use single upper or lower tolerance values, we can only measure whether a given solution remains optimal for one single parameter change at a time. However, as illustrated, there are situations where it is useful to know the largest joint cost change α on a set of elements such that the current optimal solution remains optimal.
In the following two sections we introduce set tolerances formally and determine the values they can attain.

4. Set Upper Tolerances

This section introduces set upper tolerances. First we provide a definition, and then we derive bounds on the values that set upper tolerances can attain and finally we consider set upper tolerances in special cases, such as unique optimal solutions.
Let $\mathcal{P}$ be an instance, $S^\star$ an optimal solution of $\mathcal{P}$, and $E = \{e_1, e_2, \dots, e_k\} \subseteq S^\star$. Extending the single upper tolerance, define the *set upper tolerance* $u_{S^\star}(E)$ of E with respect to $S^\star$ as the supremum of all those α such that the costs of all elements $e \in E$ are not decreased, the sum of all increases equals α, and $S^\star$ remains optimal, provided that the costs of all elements $\bar{e} \in \mathcal{E} \setminus E$ remain unchanged, i. e., the set upper tolerance of E is defined as follows:

$$u_{S^\star}(E) \;:=\; \sup \Big\{ \alpha \in \mathbb{R} \;\Big|\; \exists\, \alpha_1, \alpha_2, \dots, \alpha_k \geq 0 \text{ with } \alpha = \sum_{l=1}^{k} \alpha_l, \tag{7}$$

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k),\; S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{\vec{\alpha},E}} \Big\}.$$

By definition, it holds that:

$$u_{S^\star}(\{e\}) \;=\; u_{S^\star}(e). \tag{8}$$

It holds that $u_{S^\star}(E)$ is either an element of $\mathbb{R}_0^+$ or infinite. Because of the continuity of the cost function, for all $E \subseteq S^\star$ with $u_{S^\star}(E) < +\infty$, it holds that:

$$u_{S^\star}(E) \;:=\; \max\left\{\alpha \in \mathbb{R} \;\middle|\; \exists\, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 0 \text{ with } \alpha = \sum_{l=1}^{k} \alpha_l, \right. \tag{9}$$
$$\left. \vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k),\ S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{\vec{\alpha},E}} \right\}.$$

Lemma 1. *Let $\mathcal{P}$ be an instance. Let $S_1, S_2 \in D$, $E = \{e_1, e_2, \ldots, e_k\} \subseteq S_1 \cap S_2$. Furthermore, let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $\alpha_l \geq 0$ for $l = 1, 2 \ldots, k$. Then it holds that:*

a) $f_c(S_1) \leq f_c(S_2) \Rightarrow f_{c_{\vec{\alpha},E}}(S_1) \leq f_{c_{\vec{\alpha},E}}(S_2)$.

b) $f_c(S_1) = f_c(S_2) \Rightarrow f_{c_{\vec{\alpha},E}}(S_1) = f_{c_{\vec{\alpha},E}}(S_2)$.

Proof:

a) Let $f_c(S_1) \leq f_c(S_2)$. Then it holds for the following cases:

Case 1: The cost function is of type $\sum$.

$$\begin{aligned} f_{c_{\vec{\alpha},E}}(S_1) &= \sum_{l=1}^{k}(c(e_l) + \alpha_l) + \sum_{e \in S_1 \setminus E} c(e) \;=\; \sum_{l=1}^{k} \alpha_l + f_c(S_1) \\ &\leq \sum_{l=1}^{k} \alpha_l + f_c(S_2) \;=\; \sum_{l=1}^{k}(c(e_l) + \alpha_l) + \sum_{e \in S_2 \setminus E} c(e) \\ &= f_{c_{\vec{\alpha},E}}(S_2). \end{aligned}$$

Case 2: The cost function is of type $\prod$.

$$\begin{aligned} f_{c_{\vec{\alpha},E}}(S_1) &= \prod_{l=1}^{k}(c(e_l) + \alpha_l) \cdot \prod_{e \in S_1 \setminus E} c(e) \;=\; \prod_{l=1}^{k}(c(e_l) + \alpha_l) \cdot \frac{f_c(S_1)}{\prod_{l=1}^{k} c(e_l)} \\ &\leq \prod_{l=1}^{k}(c(e_l) + \alpha_l) \cdot \frac{f_c(S_2)}{\prod_{l=1}^{k} c(e_l)} \;=\; \prod_{l=1}^{k}(c(e_l) + \alpha_l) \cdot \prod_{e \in S_2 \setminus E} c(e) \\ &= f_{c_{\vec{\alpha},E}}(S_2). \end{aligned}$$

Case 3: The cost function is of type MAX.

Let $t \in \{1, 2, \ldots, k\}$ with $c(e_t) + \alpha_t = \max_{l=1}^{k} \{c(e_l) + \alpha_l\}$.

Case 3.1: $f_c(S_1) \leq f_c(S_2) < c(e_t) + \alpha_t$.

$$f_{c_{\vec{\alpha},E}}(S_1) \;=\; c(e_t) + \alpha_t \;=\; f_{c_{\vec{\alpha},E}}(S_2).$$

Case 3.2: $c(e_t) + \alpha_t \leq f_c(S_1) \leq f_c(S_2)$.

$$f_{c_{\vec{\alpha},E}}(S_1) \;=\; f_c(S_1) \;\leq\; f_c(S_2) \;=\; f_{c_{\vec{\alpha},E}}(S_2).$$

Case 3.3 $f_c(S_1) < c(e_t) + \alpha_t \le f_c(S_2)$.

$$f_{c_{\vec{\alpha},E}}(S_1) \;=\; c(e_t) + \alpha_t \;\le\; f_c(S_2) \;=\; f_{c_{\vec{\alpha},E}}(S_2).$$

b) This follows directly from **a)** by switching the roles of S_1 and S_2. □

In Theorem 7 we show that set upper tolerances are independent of the optimal solution to which the considered set of edges belongs.

Theorem 7. *Let $\mathcal{P}$ be an instance. It holds that:*

$$\forall S_1, S_2 \in D^\star \; \forall E \subseteq S_1 \cap S_2 : \qquad u_{S_1}(E) = u_{S_2}(E).$$

Proof: Let $E = \{e_1, e_2, \dots, e_k\}$ and $S_1, S_2 \in D^\star$ with $E \subseteq S_1 \cap S_2$. Because of $S_1, S_2 \in D^\star$, it holds that $f_c(S_1) = f_c(S_2)$. By Lemma 1b), $f_{c_{\vec{\alpha},E}}(S_1) = f_{c_{\vec{\alpha},E}}(S_2)$ for all $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with $\alpha_l \ge 0$ for $l = 1, 2 \dots, k$. By definition, it follows that $u_{S_1}(E) = u_{S_2}(E)$. □

From Theorem 7 it follows that for any optimal solution S^* such that $E \subseteq S^*$, $u_{S^*}(E)$ has the same value. Thus, we can refer to the set upper tolerance of E with respect to any optimal solution $S^\star$ as the set upper tolerance of E with respect to $\mathcal{P}$, and denote it by $u_{\mathcal{P}}(E) = u_{S^*}(E)$. In other words, if any set E belongs to more than one optimal solution, the upper tolerance of the instance is achieved on any of the optimal solutions S^* that contain E.
Let $UTS_{\mathcal{P}} := \{E \subseteq \mathcal{E} \;\big|\; \exists S^\star \in D^\star : E \subseteq S^\star\}$ be the set of subsets of $\mathcal{E}$ for which the set upper tolerance is defined with respect to $\mathcal{P}$. By definition, it holds that:

$$\begin{aligned} e \in UTE_{\mathcal{P}} \;&\Leftrightarrow\; \{e\} \in UTS_{\mathcal{P}}, \\ E \in UTS_{\mathcal{P}} \;&\Rightarrow\; \forall e \in E:\; e \in UTE_{\mathcal{P}}. \end{aligned} \tag{10}$$

The following theorem shows that if the joint costs α of a set of elements E increase by more than the upper tolerance value of the set, then no optimal solution exists that contains all elements of E.

Theorem 8. *Let $\mathcal{P}$ be an instance, $E = \{e_1, e_2, \dots, e_k\} \in UTS_{\mathcal{P}}$ with $u_{\mathcal{P}}(E) \ne +\infty$. Furthermore, let $\vec{\alpha} = (\alpha_1, \alpha_2, \dots,\; \alpha_k)$, $\alpha = \sum_{l=1}^{k} \alpha_l$ with $\alpha_l \ge 0$ for $l = 1, 2, \dots, k$ and $\alpha > u_{\mathcal{P}}(E)$. Then E is not a subset of any optimal solution of $\mathcal{P}_{c_{\vec{\alpha},E}}$.*

Proof: Assume that $E \subseteq S$ for an optimal solution S of $\mathcal{P}_{c_{\vec{\alpha},E}}$. Because of Condition 1, S is a feasible solution of $\mathcal{P}$. As $E \in UTS_{\mathcal{P}}$ holds, an optimal solution $S^\star$ of $\mathcal{P}$ with $E \subseteq S^\star$ exists. It holds that $f_c(S^\star) \le f_c(S)$. As $E \subseteq S \cap S^\star$, it follows from Lemma 1a) that:

$$f_{c_{\vec{\alpha},E}}(S^\star) \quad\le\quad f_{c_{\vec{\alpha},E}}(S). \tag{11}$$

As $\alpha > u_{\mathcal{P}}(E) = u_{S^\star}(E)$ and because of Eq. (9), $S^\star$ is not an optimal solution of $\mathcal{P}_{c_{\vec{\alpha},E}}$. Because of Eq. (11), S is also not an optimal solution of $\mathcal{P}_{c_{\vec{\alpha},E}}$. Thus, we have a contradiction, and E is not a subset of any optimal solution of $\mathcal{P}_{c_{\vec{\alpha},E}}$. □

One challenge is the computation of set upper tolerances, i.e., the largest value of α for which the current solution remains optimal. This computation is quite complicated as it requires the implicit consideration of all combinations of $(\alpha_1, \ldots, \alpha_k)$ such that $\alpha_1 + \ldots + \alpha_k = \alpha$. However, Theorem 9 shows that we can bound the set upper tolerance values of any set E using the single upper tolerance values of the elements in the set.

Theorem 9. *Let $\mathcal{P}$ be an instance and $E = \{e_1, e_2, \ldots, e_k\} \in UTS_{\mathcal{P}}$. Then the following inequalities hold:*

a) $\max_{l=1}^{k} \{u_{\mathcal{P}}(e_l)\} \ \leq \ u_{\mathcal{P}}(E)$.

b) *If the cost function is of type $\sum$ or $\prod$, $u_{\mathcal{P}}(E) \ \leq \ \sum_{l=1}^{k} u_{\mathcal{P}}(e_l)$.*

c) *If the cost function is of type MAX, $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l) \ \leq \ u_{\mathcal{P}}(E)$.*

Proof: Let $S^\star$ be an optimal solution of $\mathcal{P}$ with $E \subseteq S^\star$.

a) Trivially, the inequality is true if $u_{\mathcal{P}}(E) = +\infty$. In the following let $u_{\mathcal{P}}(E) \neq +\infty$.

Assume that $\max_{l=1}^{k} \{u_{\mathcal{P}}(e_l)\} > u_{\mathcal{P}}(E)$. Let $t \in \{1, 2, \ldots, k\}$ with $u_{\mathcal{P}}(e_t) = \max_{l=1}^{k} \{u_{\mathcal{P}}(e_l)\}$. By Eq. (9), for all $\alpha \in \mathbb{R}$ with $u_{\mathcal{P}}(E) < \alpha < u_{\mathcal{P}}(e_t)$ with $\alpha = \sum_{l=1}^{k} \alpha_l$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\alpha_l \geq 0$ for $l = 1, 2, \ldots, k$, $S^\star$ is not an optimal solution of $\mathcal{P}_{c_{\vec{\alpha},E}}$. In particular, this holds for the following definition where $l = 1, 2, \ldots, k$: $\alpha_l = \begin{cases} (u_{\mathcal{P}}(E) + u_{\mathcal{P}}(e_t))/2, & \text{if } l = t \\ 0, & \text{otherwise} \end{cases}$.

With $(u_{\mathcal{P}}(E) + u_{\mathcal{P}}(e_t))/2 < u_{\mathcal{P}}(e_t)$ we receive a contradiction to Eq. (1).

b) Trivially, the inequality is true if $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l) = +\infty$. In the following let $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l) \neq +\infty$.

Assume that $u_{\mathcal{P}}(E) > \sum_{l=1}^{k} u_{\mathcal{P}}(e_l)$. Choose $\alpha \in \mathbb{R}$ with $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l) < \alpha \leq u_{\mathcal{P}}(E)$, $\alpha = \sum_{l=1}^{k} \alpha_l$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $\alpha_l \geq 0$ for $l = 1, 2, \ldots, k$ so that $S^\star$ is an optimal solution of $\mathcal{P}_{c_{\vec{\alpha},E}}$. Then a $t \in \{1, 2, \ldots, k\}$ exists with $\alpha_t > u_{\mathcal{P}}(e_t)$. By Eq. (2), a feasible solution S exists with $f_{c_{\alpha_t,e_t}}(S) < f_{c_{\alpha_t,e_t}}(S^\star)$. We distinguish between the following cases:

Case 1: The cost function is of type $\sum$.

$$\begin{aligned} f_{c_{\vec{\alpha},E}}(S) &= f_{c_{\alpha_t,e_t}}(S) + \sum_{l=1, l\neq t, e_l \in S}^{k} \alpha_l \\ &< f_{c_{\alpha_t,e_t}}(S^\star) + \sum_{l=1, l\neq t}^{k} \alpha_l \\ &= f_{c_{\vec{\alpha},E}}(S^\star). \end{aligned}$$

Case 2: The cost function is of type $\prod$.

$$\begin{aligned} f_{c_{\vec{\alpha},E}}(S) &= f_{c_{\alpha_t,e_t}}(S) \cdot \prod_{l=1, l\neq t, e_l \in S}^{k} \frac{c(e_l)+\alpha_l}{c(e_l)} \\ &< f_{c_{\alpha_t,e_t}}(S^\star) \cdot \prod_{l=1, l\neq t}^{k} \frac{c(e_l)+\alpha_l}{c(e_l)} \\ &= f_{c_{\vec{\alpha},E}}(S^\star). \end{aligned}$$

Thus, $S^\star$ is not an optimal solution of $\mathcal{P}_{c_{\vec{\alpha},E}}$, which is a contradiction.

c) Trivially, the inequality is true if $u_{\mathcal{P}}(E) = +\infty$. If $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l) = +\infty$, then it also holds that $\max_{l=1}^{k}\{u_{\mathcal{P}}(e_l)\} = +\infty$. It follows from a) that $u_{\mathcal{P}}(E) = +\infty$, and the inequality is also true. In the following let both terms $u_{\mathcal{P}}(E)$ and $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l)$ be not equal $+\infty$.

Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\alpha_l = u_{\mathcal{P}}(e_l)$ for $l = 1, 2, \ldots, k$. A $t \in \{1, 2, \ldots, k\}$ exists with

$$f_{c_{\vec{\alpha},E}}(S^\star) = c(e_t) + \alpha_t. \tag{12}$$

Then it holds for each feasible solution S that:

$$\begin{aligned} f_{c_{\vec{\alpha},E}}(S^\star) &= c(e_t) + \alpha_t, && \text{because of Eq. (12)} \\ &\leq f_{c_{\alpha_t,e_t}}(S^\star) \\ &\leq f_{c_{\alpha_t,e_t}}(S), && \text{because of } \alpha_t = u_{\mathcal{P}}(e_t) \\ &\leq f_{c_{\vec{\alpha},E}}(S). \end{aligned}$$

Then, $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l) \leq u_{\mathcal{P}}(E)$. □

Examples with strict inequality in Theorem 9c) are provided later (see Example 7 on Page 26 and Example 9 on Page 29).
As a consequence of Theorem 7, it holds that $u_{S^*}(E) = u_{\mathcal{P}}(E)$ for each $E \subseteq S^*$. Therefore, it is not necessary to know all optimal solutions to use Theorem 9. It suffices to have an optimal solution that contains E. It is easy to see that this also holds for the subsequent theorems.
The following two theorems are generalizations of Theorem 1 and Theorem 3, respectively, and show the values of the set upper tolerances if elements in the considered set belong to all feasible solutions and all optimal solutions, respectively.

Theorem 10. *Let $\mathcal{P}$ be an instance where the cost function is of type $\sum$ or $\prod$. Let $E = \{e_1, e_2, \ldots, e_k\} \in UTS_{\mathcal{P}}$. Then the following statements are equivalent:*

a) $E \cap \bigcap_{S \in D} S \neq \emptyset$.

b) $u_{\mathcal{P}}(E) = +\infty$.

c) $\max_{l=1}^{k}\{u_{\mathcal{P}}(e_l)\} = +\infty$.

Proof:

" a) $\Leftrightarrow$ c)" The following statements are equivalent:

$$E \cap \bigcap_{S \in D} S \neq \emptyset,$$
$$\exists\, l \in \{1, 2, \ldots, k\} : \; e_l \in \bigcap_{S \in D} S,$$
$$\exists\, l \in \{1, 2, \ldots, k\} : \; u_{\mathcal{P}}(e_l) = +\infty, \qquad \text{because of Theorem 1}$$
$$\max_{l=1}^{k} \{u_{\mathcal{P}}(e_l)\} = +\infty.$$

" b) $\Rightarrow$ c)" This follows from Theorem 9b), because if $\sum_{l=1}^{k} u_{\mathcal{P}}(e_l) = +\infty$, then it also holds that $\max_{l=1}^{k} \{u_{\mathcal{P}}(e_l)\} = +\infty$.

" c) $\Rightarrow$ b)" This follows from Theorem 9a). □

In practice, one would use part c) of Theorem 10, $\max_{l=1}^{k} \{u_{\mathcal{P}}(e_l)\} = +\infty$, to show part a) $E \cap \bigcap_{S \in D} S \neq \emptyset$, i.e., one or more elements in E belong to all feasible solutions.

Remark 1. *For a cost function be of type MAX, Theorem 10 does not hold.*

Proof: Consider the following example:

Example 3.

- $\mathcal{E} = \{x, y, z\}$ *with* $c(x) = 3$, $c(y) = 7$, $c(z) = 12$,
- $E = \{x, y\}$, $e_1 = x$, $e_2 = y$,
- $D = \{\, \{x, y\}, \{x, z\}, \{y, z\} \,\}$,
- *the cost function is of type MAX.*

$\{x, y\}$ is the optimal solution with cost 7, and two further feasible solutions exist, namely $\{x, z\}$ and $\{y, z\}$ with cost 12.
Obviously, $E \cap \bigcap_{S \in D} S = \emptyset$, i.e., **a)** does not hold.
Choose $\beta \in \mathbb{R}$ with $\beta \geq 5$ arbitrary. Increase the cost of e_1 by $\alpha_1 = \beta + 4$ and the cost of e_2 by $\alpha_2 = \beta$, and let $\vec{\alpha} = (\alpha_1, \alpha_2)$. Then all three feasible solutions have the same cost $\beta + 7$ and are optimal with respect to $\mathcal{P}_{c_{\vec{\alpha}, E}}$. As $\beta \geq 5$ can be chosen arbitrarily large, it follows that $u_{\mathcal{P}}(E) = +\infty$, i.e., **b)** holds.
By Theorem 2(c), it holds that $u_{\mathcal{P}}(e_1) = 9$, $u_{\mathcal{P}}(e_2) = 5$. Thus,

$$\max\, \{u_{\mathcal{P}}(e_1), u_{\mathcal{P}}(e_2)\} \;=\; 9 \;<\; +\infty.$$

Thus, **c)** does not hold. □

Theorem 11. *Let $\mathcal{P}$ be an instance and $E \in UTS_{\mathcal{P}}$. Then,*

$$E \subseteq \bigcap_{S^\star \in D^\star} S^\star \quad \Rightarrow \quad u_{\mathcal{P}}(E) > 0.$$

Proof: Let $E \subseteq \bigcap_{S^\star \in D^\star} S^\star$. It follows that $e_l \in \bigcap_{S^\star \in D^\star} S^\star$ for $l = 1, 2, \ldots, k$. By Theorem 3, it follows that $\max_{l=1}^{k}\{u_{\mathcal{P}}(e_l)\} > 0$, and by Theorem 9a) that $u_{\mathcal{P}}(E) > 0$. □

Finally, we present a criterion for the uniqueness of an optimal solution based on set upper tolerances.

Theorem 12. *Let $\mathcal{P}$ be an instance where the cost function is of type $\sum$ or $\prod$. Then the following statements are equivalent:*

a) *Only one optimal solution of $\mathcal{P}$ exists.*

b) $u_{\mathcal{P}}(E) > 0$ *for all* $E \in UTS_{\mathcal{P}}$.

Proof: By Eq. (3), the condition that $u_{\mathcal{P}}(e) > 0$ for all $e \in UTE_{\mathcal{P}}$ is equivalent to the condition that $u_{\mathcal{P}}(e) > 0$ for all $e \in \bigcup_{S^\star \in D^\star} S^\star$. By Theorem 3a), this is equivalent to $\bigcup_{S^\star \in D^\star} S^\star \subseteq \bigcap_{S^\star \in D^\star} S^\star$ and equivalent to $|D^\star| = 1$.
Thus, **a)** is equivalent to

c) $u_{\mathcal{P}}(e) > 0$ for all $e \in UTE_{\mathcal{P}}$.

Therefore, it is sufficient to show the equivalence of **b)** and **c)**.

" b) $\Rightarrow$ c)" Let $e \in UTE_{\mathcal{P}}$. By Eqs. (8), (10), it holds that $E := \{e\} \in UTS_{\mathcal{P}}$ and $u_{\mathcal{P}}(E) = u_{\mathcal{P}}(e)$. By **b)**, $u_{\mathcal{P}}(e) = u_{\mathcal{P}}(E) > 0$.

" c) $\Rightarrow$ b)" This follows from Theorem 9a). □

5. Set Lower Tolerances

This section is structured in the same way as Section 5. It introduces set lower tolerances, provides a definition of them, and derives bounds on the values that set lower tolerances can attain, and one exact value. Finally, we consider set lower tolerances in special cases, such as the presence of unique optimal solutions.
Let $\mathcal{P}$ be an instance, $S^\star$ an optimal solution of $\mathcal{P}$, and $E = \{e_1, e_2, \ldots, e_k\} \subseteq \mathcal{E} \setminus S^\star$. Extending the single lower tolerance, define the *set lower tolerance* $l_{S^\star}(E)$ of E with respect to $S^\star$ as the supremum of all those α such that the costs of all elements $e \in E$ are not increased, the cost function remains of type $\sum$, $\prod$, or MAX, the sum of all decreases equals α, and $S^\star$ remains optimal, provided

that the costs of all elements $\bar{e} \in \mathcal{E} \setminus E$ remain unchanged, i. e., the set lower tolerance of E is defined as follows:

$$\begin{aligned} l_{S^\star}(E) \quad := \quad & \sup \Big\{\alpha \in \mathbb{R} \;\Big|\; \exists\, \alpha_1, \alpha_2, \ldots, \alpha_k \text{ with } \alpha = \sum_{l=1}^{k} \alpha_l, \\ & \vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k),\ 0 \le \alpha_1 < \delta(e_1),\, 0 \le \alpha_2 < \delta(e_2), \\ & \ldots, 0 \le \alpha_k < \delta(e_k),\ S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{-\vec{\alpha},E}}\Big\}. \end{aligned} \tag{13}$$

By definition, it holds that:

$$l_{S^\star}(\{e\}) \;=\; l_{S^\star}(e). \tag{14}$$

It holds that $l_{S^\star}(E)$ is either an element of $\mathbb{R}_0^+$ or infinite. More precisely, it holds for all $E \subseteq \mathcal{E} \setminus S^\star$ that:

$$0 \le l_{S^\star}(E) \le \sum_{l=1}^{k} \delta(e_l),$$

For a cost function of type $\sum$ or MAX, it holds for all $E \subseteq \mathcal{E} \setminus S^\star$ with $l_{S^\star}(E) < +\infty$ that:

$$\begin{aligned} l_{S^\star}(E) \quad := \quad & \max \Big\{\alpha \in \mathbb{R} \;\Big|\; \exists\, \alpha_1, \alpha_2, \ldots, \alpha_k \text{ with } \alpha = \sum_{l=1}^{k} \alpha_l, \\ & \vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k),\ 0 \le \alpha_1 < \delta(e_1),\, 0 \le \alpha_2 < \delta(e_2), \\ & \ldots, 0 \le \alpha_k < \delta(e_k),\ S^\star \text{ is an optimal solution of } \mathcal{P}_{c_{-\vec{\alpha},E}}\Big\}. \end{aligned} \tag{15}$$

Similar to Theorem 7 it holds that the set lower tolerance does not depend on the particular optimal solution of $\mathcal{P}$: for all optimal solutions S^* such that $E \subseteq \mathcal{E} \setminus S^*$, it holds that the value of the set lower tolerance is the same. This is shown in the following theorem.

Theorem 13. *Let $\mathcal{P}$ be an instance. It holds that:*

$$\forall S_1, S_2 \in D^\star\ \forall E \subseteq \mathcal{E} \setminus (S_1 \cup S_2): \qquad l_{S_1}(E) = l_{S_2}(E).$$

Proof: Let $E = \{e_1, e_2, \ldots, e_k\}$ and $S_1, S_2 \in D^\star$ with $E \subseteq \mathcal{E} \setminus (S_1 \cup S_2)$. Because of $S_1, S_2 \in D^\star$, it holds that $f_c(S_1) = f_c(S_2)$. It follows that $f_{c_{-\vec{\alpha},E}}(S_1) = f_c(S_1) = f_c(S_2) = f_{c_{-\vec{\alpha},E}}(S_2)$ for all $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $0 \le \alpha_l < \delta(e_l)$ for $l = 1, 2, \ldots, k$. By definition, it follows that $l_{S_1}(E) = l_{S_2}(E)$. □

We refer to the set lower tolerance of E with respect to an optimal solution $S^\star$ as the set lower tolerance of E with respect to $\mathcal{P}$, and denote it by $l_{\mathcal{P}}(E)$. Let $LTS_{\mathcal{P}} := \{E \subseteq \mathcal{E} \;\big|\; \exists S^\star \in D^\star :\, E \subseteq \mathcal{E} \setminus S^\star\}$ be the set of subsets of $\mathcal{E}$ for which the set lower tolerance is defined with respect to $\mathcal{P}$. By definition, it holds that:

$$e \in LTE_{\mathcal{P}} \;\Leftrightarrow\; \{e\} \in LTS_{\mathcal{P}}, \tag{16}$$

$$E \in LTS_{\mathcal{P}} \;\Rightarrow\; \forall e \in E:\; e \in LTE_{\mathcal{P}}. \tag{17}$$

For lower tolerances we have to distinguish between cost functions of type $\sum$ or MAX in Theorem 14 on the one hand and of type $\prod$ in Theorem 15 on the other hand. For a cost function of type $\prod$ we should take into account that a cost decrease of any $e \in LTE_{\mathcal{P}}$ of more than $c(e)$ units changes the sign of the objective values of solutions that include e and a cost decrease of precisely $c(e)$ makes the costs of all these solutions 0. It is not unreasonable to exclude such cost decreases from our analysis of the cost function $\prod$. For cost functions of type $\sum$ or MAX negative cost coefficients are inconsequential.

Theorem 14. *Let $\mathcal{P}$ be an instance where the cost function is of type $\sum$ or MAX. Furthermore, let $E \in LTS_{\mathcal{P}}$. Then it holds that:*

$$E \subseteq \mathcal{E} \setminus \bigcup_{S \in D} S \quad \Rightarrow \quad l_{\mathcal{P}}(E) = +\infty.$$

Proof: Let $E = \{e_1, e_2, \ldots, e_k\}$ and $E \subseteq \mathcal{E} \setminus \bigcup_{S \in D} S$. Decreasing the cost of e_l by $\alpha_l \geq 0$ for $l = 1, 2, \ldots, k$ does not change the cost of any feasible solution, and optimal solutions remain optimal. Therefore, $l_{\mathcal{P}}(E) = +\infty$. □

Theorem 15. *Let $\mathcal{P}$ be an instance where the cost function is of type $\prod$. Furthermore, let $E = \{e_1, e_2, \ldots, e_k\} \in LTS_{\mathcal{P}}$. Then it holds that:*

$$E \subseteq \mathcal{E} \setminus \bigcup_{S \in D} S \quad \Leftrightarrow \quad l_{\mathcal{P}}(E) = \sum_{l=1}^{k} c(e_l)$$

Proof:

"⇒" Let $E \subseteq \mathcal{E} \setminus \bigcup_{S \in D} S$. Set $\alpha_l = c(e_l) - \epsilon$ for $l = 1, 2, \ldots, k$ where $\epsilon \in \mathbb{R}^+$ is chosen arbitrarily small, and $\alpha = \sum_{l=1}^{k} \alpha_l$. This vector leads to the supremum of all those α with the condition that the instance is still of type $\prod$ after the decreases. Furthermore, as $E \subseteq \mathcal{E} \setminus \bigcup_{S \in D} S$, the cost of any feasible solution is not changed, and optimal solutions remain optimal. Therefore, $l_{\mathcal{P}}(E) = \sum_{l=1}^{k} c(e_l)$.

"⇐" Let $l_{\mathcal{P}}(E) = \sum_{l=1}^{k} c(e_l)$. Assume that $E \nsubseteq \mathcal{E} \setminus \bigcup_{S \in D} S$. Then a $t \in \{1, 2, \ldots, k\}$ and $S \in D$ exist with $e_t \in S$. Let $S^\star$ be an optimal solution of $\mathcal{P}$ with $E \subseteq \mathcal{E} \setminus S^\star$. Choose $\epsilon \in \mathbb{R}^+$ with

$$\epsilon \quad < \quad \frac{f_c(S^\star) \cdot c(e_t)}{f_c(S)}. \tag{18}$$

By Eq. (13), $\alpha \in \mathbb{R}^+$ and $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ exist with $\alpha = \sum_{l=1}^{k} \alpha_l$, $\alpha > \sum_{l=1}^{k} c(e_l) - \epsilon$ and $0 \leq \alpha_l < c(e_l)$ for $l = 1, 2, \ldots, k$ such that $S^\star$ is an optimal solution of $\mathcal{P}_{c_{-\vec{\alpha},E}}$. It follows that:

$$\alpha_t \quad > \quad c(e_t) - \epsilon. \tag{19}$$

Hence,

$$\begin{aligned} f_{c-\vec{\alpha},E}(S^\star) &= f_c(S^\star) \\ &> \frac{f_c(S)\cdot\epsilon}{c(e_t)}, && \text{because of Eq. (18)} \\ &> \frac{f_c(S)\cdot(c(e_t)-\alpha_t)}{c(e_t)}, && \text{because of Eq. (19)} \\ &\geq f_{c-\vec{\alpha},E}(S). \end{aligned}$$

Thus, $S^\star$ is not an optimal solution of $\mathcal{P}_{c-\vec{\alpha},E}$, and we have a contradiction. Therefore, $E \subseteq \mathcal{E} \setminus \bigcup_{S\in D} S$. □

Similarly to Theorem 8, Theorem 16 shows that no optimal solution exists that does not contain any element of E, if the joint cost decrease α of all elements in a set exceeds the set lower tolerance.

Theorem 16. *Let $\mathcal{P}$ be an instance, $E = \{e_1, e_2, \ldots, e_k\} \in LTS_{\mathcal{P}}$ with $l_{\mathcal{P}}(E) \neq +\infty$. Furthermore, let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\alpha = \sum_{l=1}^{k} \alpha_l$ with $0 \leq \alpha_l < \delta(e_l)$ for $l = 1, 2, \ldots, k$ and $\alpha > l_{\mathcal{P}}(E)$. Then E overlaps with each optimal solution of $\mathcal{P}_{c-\vec{\alpha},E}$.*

Proof: Assume that $E \subseteq \mathcal{E} \setminus S$ for an optimal solution S of $\mathcal{P}_{c-\vec{\alpha},E}$. Because of Condition 1, S is a feasible solution of $\mathcal{P}$. As $E \in LTS_{\mathcal{P}}$ holds, an optimal solution $S^\star$ of $\mathcal{P}$ with $E \subseteq \mathcal{E} \setminus S^\star$ exists. It holds that $f_c(S^\star) \leq f_c(S)$. As $E \subseteq \mathcal{E} \setminus (S \cup S^\star)$, it follows that:

$$f_{c-\vec{\alpha},E}(S^\star) \;=\; f_c(S^\star) \;\leq\; f_c(S) \;=\; f_{c-\vec{\alpha},E}(S). \tag{20}$$

As $\alpha > l_{\mathcal{P}}(E) = l_{S^\star}(E)$ and because of Eq. (13), $S^\star$ is not an optimal solution of $\mathcal{P}_{c-\vec{\alpha},E}$. Because of Eq. (20), S is also not an optimal solution of $\mathcal{P}_{c-\vec{\alpha},E}$. Thus, we have a contradiction, and E overlaps with each optimal solution of $\mathcal{P}_{c-\vec{\alpha},E}$. □

In Theorem 17 we provide bounds to the values of the set lower tolerances. Interestingly, if the cost function is of type MAX, we can express the set lower tolerance value as the sum of the individual single lower tolerance values of the elements.

Theorem 17. *Let $\mathcal{P}$ be an instance and $E = \{e_1, e_2, \ldots, e_k\} \in LTS_{\mathcal{P}}$. Then the following inequalities hold:*

a) $\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} \;\leq\; l_{\mathcal{P}}(E) \;\leq\; \sum_{l=1}^{k} l_{\mathcal{P}}(e_l)$.

b) *If the cost function is of type MAX, $l_{\mathcal{P}}(E) \;=\; \sum_{l=1}^{k} l_{\mathcal{P}}(e_l)$.*

Proof: Let $S^\star$ be an optimal solution of $\mathcal{P}$ with $E \subseteq \mathcal{E} \setminus S^\star$.

a) We show the claimed inequalities.

- $\boldsymbol{\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} \leq l_{\mathcal{P}}(E)}$.

  Trivially, the inequality is true if $l_{\mathcal{P}}(E) = +\infty$. In the following let $l_{\mathcal{P}}(E) \neq +\infty$.

  Assume that $\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} > l_{\mathcal{P}}(E)$. Let $t \in \{1, 2, \ldots, k\}$ with $l_{\mathcal{P}}(e_t) = \max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\}$. By Eq. (13), for all $\alpha \in \mathbb{R}$ with $l_{\mathcal{P}}(E) < \alpha$, $\alpha = \sum_{l=1}^{k} \alpha_l$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $0 \leq \alpha_l < \delta(e_l)$ for $l = 1, 2, \ldots, k$, $S^\star$ is not an optimal solution of $\mathcal{P}_{c_{-\vec{\alpha},E}}$. In particular, this holds for the following definition where $l = 1, 2, \ldots, k$: $\alpha_l = \begin{cases} (l_{\mathcal{P}}(E) + l_{\mathcal{P}}(e_t))/2, & \text{if } l = t \\ 0, & \text{otherwise} \end{cases}$. With $(l_{\mathcal{P}}(E) + l_{\mathcal{P}}(e_t))/2 < l_{\mathcal{P}}(e_t)$ we receive a contradiction to Eq. (5).

- $\boldsymbol{l_{\mathcal{P}}(E) \leq \sum_{l=1}^{k} l_{\mathcal{P}}(e_l)}$.

  Trivially, the inequality is true if $\sum_{l=1}^{k} l_{\mathcal{P}}(e_l) = +\infty$. In the following let $\sum_{l=1}^{k} l_{\mathcal{P}}(e_l) \neq +\infty$.

  Assume that $l_{\mathcal{P}}(E) > \sum_{l=1}^{k} l_{\mathcal{P}}(e_l)$. Choose $\alpha \in \mathbb{R}$ with $\sum_{l=1}^{k} l_{\mathcal{P}}(e_l) < \alpha \leq l_{\mathcal{P}}(E)$, $\alpha = \sum_{l=1}^{k} \alpha_l$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $0 \leq \alpha_l < \delta(e_l)$ for $l = 1, 2, \ldots, k$ so that $S^\star$ is an optimal solution of $\mathcal{P}_{c_{-\vec{\alpha},E}}$. Then a $t \in \{1, 2, \ldots, k\}$ exists with $\alpha_t > l_{\mathcal{P}}(e_t)$. By Eq. (4), a feasible solution S exists with $f_{c_{-\alpha_t,e_t}}(S) < f_{c_{-\alpha_t,e_t}}(S^\star)$. It follows that $f_{c_{-\vec{\alpha},E}}(S) \leq f_{c_{-\alpha_t,e_t}}(S) < f_{c_{-\alpha_t,e_t}}(S^\star) = f_{c_{-\vec{\alpha},E}}(S^\star)$. Thus, $S^\star$ is not an optimal solution of $\mathcal{P}_{c_{-\vec{\alpha},E}}$, which is a contradiction.

b) By **a)**, it remains to be shown that $\sum_{l=1}^{k} l_{\mathcal{P}}(e_l) \leq l_{\mathcal{P}}(E)$.

Trivially, the inequality is true if $l_{\mathcal{P}}(E) = +\infty$. If $\sum_{l=1}^{k} l_{\mathcal{P}}(e_l) = +\infty$, then it also holds that $\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} = +\infty$. From a) it follows that $l_{\mathcal{P}}(E) = +\infty$, and the inequality is also true. In the following let both terms $l_{\mathcal{P}}(E)$ and $\sum_{l=1}^{k} l_{\mathcal{P}}(e_l)$ be not equal $+\infty$.

Let $\alpha \geq 0$ with $\alpha = \sum_{l=1}^{k} \alpha_l$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\alpha_l = l_{\mathcal{P}}(e_l)$ for $l = 1, 2, \ldots, k$. By Theorem 5c), $l_{\mathcal{P}}(e_l) = c(e_l) - f_c(\mathcal{P}) \neq +\infty$. Let S be an arbitrary feasible solution of $\mathcal{P}_{c_{-\vec{\alpha},E}}$.

Case 1: $E \subseteq \mathcal{E} \setminus S$.

Then, $f_{c_{-\vec{\alpha},E}}(S^\star) = f_c(S^\star) \leq f_c(S) = f_{c_{-\vec{\alpha},E}}(S)$.

Case 2: $E \nsubseteq \mathcal{E} \setminus S$.

Then an $l \in \{1, 2, \ldots, k\}$ exists with $e_l \in S$ and it holds that:

$$\begin{aligned} f_{c_{-\vec{\alpha},E}}(S^\star) &= f_c(S^\star) \\ &= f_c(\mathcal{P}) \\ &= c(e_l) - (c(e_l) - f_c(\mathcal{P})) \\ &= c(e_l) - \alpha_l \\ &\leq f_{c_{-\vec{\alpha},E}}(S). \end{aligned}$$

In both cases, $S^\star$ is an optimal solution of $\mathcal{P}_{c_{-\vec{\alpha},E}}$, and $\alpha = \sum_{l=1}^{k} l_{\mathcal{P}}(e_l)$ $\leq l_{\mathcal{P}}(E)$ because of Eq. (15). □

The following two theorems are generalizations of Theorem 4 and Theorem 6, respectively, and show the values of the set lower tolerances if elements in the considered set belong to no feasible solutions or to no optimal solutions, respectively.

Theorem 18. *Let $\mathcal{P}$ be an instance where the cost function is of type $\sum$. Furthermore, let $E = \{e_1, e_2, \ldots, e_k\} \in LTS_{\mathcal{P}}$. Then the following statements are equivalent:*

a) $E \nsubseteq \bigcup_{S \in D} S$.

b) $l_{\mathcal{P}}(E) = +\infty$.

c) $\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} \;=\; +\infty$.

Proof:

" a) ⇔ c)" The following statements are equivalent:

$$E \nsubseteq \bigcup_{S \in D} S,$$

$$\exists\, l \in \{1, 2, \ldots, k\} : \; e_l \in \mathcal{E} \setminus \bigcup_{S \in D} S,$$

$$\exists\, l \in \{1, 2, \ldots, k\} : \; l_{\mathcal{P}}(e_l) \;=\; +\infty, \quad \text{because of Theorem 4 and Eq. (17)}$$

$$\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} \;=\; +\infty.$$

" b) ⇒ c)" This follows from the second inequality of Theorem 17a), because if $\sum_{l=1}^{k} l_{\mathcal{P}}(e_l) = +\infty$, then it also holds that $\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} = +\infty$.

" c) ⇒ b)" This follows from the first inequality of Theorem 17a). □

In practice, one would use part c) of Theorem 18 to show part a), i.e., to show that E contains elements that do not belong to any feasible solution.

Clearly, Theorem 18 cannot hold for a cost function of type $\prod$, but it does also not hold for a cost function be of type MAX, as can be seen in the following remark.

Remark 2. *For a cost function be of type MAX, Theorem 18 does not hold.*

Proof: Consider the following example:

Example 4.

- $\mathcal{E} = \{w, x, y, z\}$ *with* $c(w) = c(x) = c(y) = c(z) = 1$*,*
- $E = \{y, z\}, e_1 = y,\ e_2 = z$*,*
- $D = \{\,\{w, x\},\ \{w, y\},\ \{x, z\}\,\}$*,*
- *the cost function is of type MAX.*

The three feasible solutions $\{w, x\}$, $\{w, y\}$ and $\{x, z\}$ are optimal with cost 1. Obviously, $E \subseteq \bigcup_{S \in D} S = \mathcal{E}$, i.e., **a)** does not hold.
By Theorem 5(c), it holds that $l_{\mathcal{P}}(e_1) = +\infty$, $l_{\mathcal{P}}(e_2) = +\infty$. Thus, $\max_{l=1}^{k} \{l_{\mathcal{P}}(e_l)\} = +\infty$, and **c)** holds.
By Theorem (17)(a), it follows that $l_{\mathcal{P}}(E) = +\infty$, i.e., **b)** holds.

Theorem 19. *Let $\mathcal{P}$ be an instance and $E \in LTS_{\mathcal{P}}$. Then,*

$$E \subseteq \mathcal{E} \setminus \bigcup_{S^\star \in D^\star} S^\star \quad \Rightarrow \quad l_{\mathcal{P}}(E) > 0.$$

Proof: Let $E \subseteq \mathcal{E} \setminus \bigcup_{S^\star \in D^\star} S^\star$. It follows that $e_l \in \mathcal{E} \setminus \bigcup_{S^\star \in D^\star} S^\star$ for $l = 1, 2, \ldots, k$. By Theorem 6, it follows that $\max_{l=1}^{k}\{l_{\mathcal{P}}(e_l)\} > 0$, and by Theorem 17a) that $l_{\mathcal{P}}(E) > 0$. □

Finally, we present a criterion for the uniqueness of an optimal solution based on set lower tolerances.

Theorem 20. *Let $\mathcal{P}$ be an instance where the cost function is of type $\sum$ or $\prod$. Then the following statements are equivalent:*

a) *Only one optimal solution of $\mathcal{P}$ exists.*

b) $l_{\mathcal{P}}(E) > 0$ *for all* $E \in LTS_{\mathcal{P}}$*.*

Proof: By Eq. (6), the condition that $l_{\mathcal{P}}(e) > 0$ for all $e \in LTE_{\mathcal{P}}$ is equivalent to the condition that $l_{\mathcal{P}}(e) > 0$ for all $e \in \mathcal{E} \setminus \bigcap_{S^\star \in D^\star} S^\star$. By Theorem 6a), this is equivalent to $\mathcal{E} \setminus \bigcap_{S^\star \in D^\star} S^\star \subseteq \mathcal{E} \setminus \bigcup_{S^\star \in D^\star} S^\star$ and equivalent to $|D^\star| = 1$.
Thus, **a)** is equivalent to

c) $l_{\mathcal{P}}(e) > 0$ for all $e \in LTE_{\mathcal{P}}$.

Therefore, it is sufficient to show the equivalence of **b)** and **c)**.

" b) $\Rightarrow$ c)" Let $e \in LTE_{\mathcal{P}}$. By Eqs. (14), (16), it holds that $E := \{e\} \in LTS_{\mathcal{P}}$ and $l_{\mathcal{P}}(E) = l_{\mathcal{P}}(e)$. By **b)**, $l_{\mathcal{P}}(e) = l_{\mathcal{P}}(E) > 0$.

" c) $\Rightarrow$ b)" This follows from the first inequality of Theorem 17a). □

6. Basic Computational Examples

In this section, we illustrate the main theoretical results in Sections 4 and 5, namely the computation of (bounds to) the values of set upper and lower tolerances, for the three types of cost functions by simple examples of each of them.

Example 5. *(Cost function of type $\sum$)*

- $\mathcal{E} = \{v, w, x, y, z\}$ *with* $c(v) = 2$, $c(w) = 4$, $c(x) = 8$, $c(y) = 7$, $c(z) = 9$,
- $D = \{\, \{v, w\}, \{v, x\}, \{y, z\}\,\}$.

$S_1 := \{v, w\}$ is the only optimal solution with cost 6, and two further feasible solutions exist, namely $S_2 := \{v, x\}$ with cost 10, and $S_3 := \{y, z\}$ with cost 16. Because of Theorem 2a), it holds that:

$$u_{\mathcal{P}}(v) = 10,\ u_{\mathcal{P}}(w) = 4,$$

and because of Theorem 5a), it holds that:

$$l_{\mathcal{P}}(x) = 4,\ l_{\mathcal{P}}(y) = 10,\ l_{\mathcal{P}}(z) = 10.$$

- Let $E = \{v, w\} \subseteq S_1$, $e_1 = v$, $e_2 = w$.
  It follows from Eq. (7) that:

  $$u_{\mathcal{P}}(E) \quad = \quad \sup_{\alpha_1,\alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

  with the constraint

  $$f_{c_{(\alpha_1,\alpha_2),E}}(S_1) \quad \leq \quad \min\,\{f_{c_{(\alpha_1,\alpha_2),E}}(S_2), f_{c_{(\alpha_1,\alpha_2),E}}(S_3)\},$$

  which is equivalent to

  $$u_{\mathcal{P}}(E) \quad = \quad \sup_{\alpha_1,\alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

with the constraints

$$\begin{aligned} 4 + \alpha_2 &\leq 8, \\ 6 + \alpha_1 + \alpha_2 &\leq 16. \end{aligned}$$

It follows that $\alpha_1 = 6$ and $\alpha_2 = 4$, and then (cf. Theorem 9a),b)):

$$\max\{u_{\mathcal{P}}(v), u_{\mathcal{P}}(w)\} \;=\; u_{\mathcal{P}}(E) \;=\; 10 \;<\; u_{\mathcal{P}}(v) + u_{\mathcal{P}}(w) \;=\; 14.$$

Thus, this is an example where the lower bound of Theorem 9a) is attained.

- Let $E = \{y, z\} \subseteq \mathcal{E} \setminus S_1$, $e_1 = y$, $e_2 = z$.
  It follows from Eq. (13) that:

$$l_{\mathcal{P}}(E) \;=\; \sup_{\alpha_1, \alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

with the constraint

$$f_{c_{(\alpha_1,\alpha_2),E}}(S_1) \;\leq\; \min\{f_{c_{(\alpha_1,\alpha_2),E}}(S_2), f_{c_{(\alpha_1,\alpha_2),E}}(S_3)\},$$

which is equivalent to

$$l_{\mathcal{P}}(E) \;=\; \sup_{\alpha_1, \alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

with the constraint

$$16 - \alpha_1 - \alpha_2 \;\geq\; 6.$$

It follows that $\alpha_1 + \alpha_2 = 10$, and then (cf. Theorem 17a)):

$$\max\{l_{\mathcal{P}}(y), l_{\mathcal{P}}(z)\} \;=\; l_{\mathcal{P}}(E) \;=\; 10 \;<\; l_{\mathcal{P}}(y) + l_{\mathcal{P}}(z) \;=\; 20.$$

Thus, this is an example where the lower bound of Theorem 17a) is attained.

Analogously, it can be shown for $E = \{x, y\}$:

$$\max\{l_{\mathcal{P}}(x), l_{\mathcal{P}}(y)\} \;=\; 10 \;<\; l_{\mathcal{P}}(E) \;=\; l_{\mathcal{P}}(x) + l_{\mathcal{P}}(y) \;=\; 14.$$

Thus, this is an example where the upper bound of Theorem 17a) is attained.

For $E = \{x, z\}$ it holds that:

$$\max\{l_{\mathcal{P}}(x), l_{\mathcal{P}}(z)\} \;=\; 10 \;<\; l_{\mathcal{P}}(E) \;=\; l_{\mathcal{P}}(x) + l_{\mathcal{P}}(z) \;=\; 14.$$

Thus, this is another example where the upper bound of Theorem 17a) is attained.

For $E = \{x, y, z\}$ it holds that:

$$\begin{aligned} \max\{l_{\mathcal{P}}(x), l_{\mathcal{P}}(y), l_{\mathcal{P}}(z)\} \;=\; 10 \;&<\; l_{\mathcal{P}}(E) \;=\; 14 \\ &<\; l_{\mathcal{P}}(x) + l_{\mathcal{P}}(y) + l_{\mathcal{P}}(y) \;=\; 24. \end{aligned}$$

Thus, this is an example where neither the lower bound nor the upper bound of Theorem 17a) is attained.

Example 6. *(Cost function of type $\prod$)*

- $\mathcal{E} = \{w, x, y, z\}$ *with* $c(w) = 2,\ c(x) = 4,\ c(y) = 1,\ c(z) = 36,$
- $D = \{\, \{w, x\},\ \{y, z\} \,\}.$

$S_1 := \{w, x\}$ is the only optimal solution with cost 8, and $S_2 := \{y, z\}$ the second feasible solution with cost 36. Because of Theorem 2b), it holds that:

$$u_{\mathcal{P}}(w) = 7,\ u_{\mathcal{P}}(x) = 14,$$

and because of Theorem 5b), it holds that:

$$l_{\mathcal{P}}(y) = 7/9,\ l_{\mathcal{P}}(z) = 28.$$

- Let $E = \{w, x\} \subseteq S_1,\ e_1 = w,\ e_2 = x.$

  From Eq. (7), it follows that:

  $$u_{\mathcal{P}}(E) \;=\; \sup_{\alpha_1, \alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

  with the constraint that S_1 is an optimal solution of $\mathcal{P}_{c_{(\alpha_1,\alpha_2),E}}$.

  This is equivalent to

  $$u_{\mathcal{P}}(E) \;=\; \sup_{\alpha_1, \alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

  with the constraint

  $$f_{c_{(\alpha_1,\alpha_2),E}}(S_1) \;\leq\; f_{c_{(\alpha_1,\alpha_2),E}}(S_2),$$

  and equivalent to

  $$u_{\mathcal{P}}(E) \;=\; \sup_{\alpha_1, \alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

  with the constraint

  $$(2 + \alpha_1) \cdot (4 + \alpha_2) \;\leq\; 36.$$

  As a sum of two positive numbers with fixed product is maximized, if and only if the difference of both numbers is maximized, it follows that $\alpha_1 = 0$, $\alpha_2 = 14$, and then (cf. Theorem 9a),b)):

  $$\max\{u_{\mathcal{P}}(w), u_{\mathcal{P}}(x)\} \;=\; u_{\mathcal{P}}(E) \;=\; 14 \;<\; u_{\mathcal{P}}(w) + u_{\mathcal{P}}(x) \;=\; 21.$$

  Thus, this is an example where the lower bound of Theorem 9a) is attained.

- Let $E = \{y, z\} \subseteq \mathcal{E} \setminus S_1$, $e_1 = y$, $e_2 = z$.

  It follows from Eq. (13) that:

$$l_{\mathcal{P}}(E) \quad = \quad \sup_{\alpha_1 \leq 1,\ \alpha_2 \leq 36} \{\alpha_1 + \alpha_2\}$$

  with the constraint

$$f_{c_{(\alpha_1,\alpha_2),E}}(S_1) \quad \leq \quad f_{c_{(\alpha_1,\alpha_2),E}}(S_2)$$

  which is equivalent to

$$l_{\mathcal{P}}(E) \quad = \quad \sup_{\alpha_1 \leq 1,\ \alpha_2 \leq 36} \{\alpha_1 + \alpha_2\}$$

  with the constraint

$$8 \quad \leq \quad (1 - \alpha_1) \cdot (36 - \alpha_2).$$

  The largest value of $\alpha = \alpha_1 + \alpha_2$ such that this inequality holds is achieved for $\alpha_1 = 0$, $\alpha_2 = 28$, and then (cf. Theorem 17a)):

$$\max \{l_{\mathcal{P}}(y), l_{\mathcal{P}}(z)\} \ = \ l_{\mathcal{P}}(E) \ = \ 28 \ < \ l_{\mathcal{P}}(w) + l_{\mathcal{P}}(x) \ = \ 28\frac{7}{9}.$$

  Thus, this is an example where the lower bound of Theorem 17a) is attained.

Example 7. *(Cost function of type MAX)*

- $\mathcal{E} = \{w, x, y, z\}$ *with* $c(w) = 3$, $c(x) = 5$, $c(y) = 4$, $c(z) = 8$,
- $D = \{\, \{w, x\},\ \{x, y\},\ \{w, z\} \,\}$.

$S_1 := \{w, x\}$ and $S_2 := \{x, y\}$ are optimal solutions with cost 5. The third feasible solution $S_3 := \{w, z\}$ has cost 8. Because of Theorem 2c), it holds that:

$$u_{\mathcal{P}}(w) = 2,\ u_{\mathcal{P}}(x) = 3,\ u_{\mathcal{P}}(y) = 1,$$

and because of Theorem 5c), it holds that:

$$l_{\mathcal{P}}(w) = +\infty,\ l_{\mathcal{P}}(y) = +\infty,\ l_{\mathcal{P}}(z) = 3.$$

- Let $E = \{w, x\} \subseteq S_1$, $e_1 = w$, $e_2 = x$.

  It follows from Eq. (7) that:

$$u_{\mathcal{P}}(E) \quad = \quad \sup_{\alpha_1, \alpha_2 \in \mathbb{R}_0^+} \{\alpha_1 + \alpha_2\}$$

  with the constraint

$$f_{c_{(\alpha_1,\alpha_2),E}}(S_1) \quad \leq \quad \min \{f_{c_{(\alpha_1,\alpha_2),E}}(S_2), f_{c_{(\alpha_1,\alpha_2),E}}(S_3)\},$$

which is equivalent to

$$u_{\mathcal{P}}(E) \quad = \sup_{\alpha_1,\alpha_2\in\mathbb{R}_0^+} \{\alpha_1+\alpha_2\}$$

with the constraints

$$\begin{aligned} 3+\alpha_1 &\leq 5+\alpha_2, \\ 5+\alpha_2 &\leq \max\{3+\alpha_1, 8\}. \end{aligned}$$

It follows that $\alpha_1 = d+2$, $\alpha_2 = d$ where d can be chosen arbitrarily large. Then it follows (cf. Theorem 9a),c)):

$$\max\{u_{\mathcal{P}}(w), u_{\mathcal{P}}(x)\} \;=\; 3 \;<\; u_{\mathcal{P}}(w)+u_{\mathcal{P}}(x) \;=\; 5 \;<\; u_{\mathcal{P}}(E) \;=\; \infty.$$

Thus, this is an example where the lower bound of Theorem 9c) is *not* attained.

- Let $E = \{x, y\} \subseteq S_2$, $e_1 = x$, $e_2 = y$.
  It follows from Eq. (7) that:

$$u_{\mathcal{P}}(E) \quad = \sup_{\alpha_1,\alpha_2\in\mathbb{R}_0^+} \{\alpha_1+\alpha_2\}$$

with the constraint

$$f_{c_{(\alpha_1,\alpha_2),E}}(S_2) \quad \leq \quad \min\{f_{c_{(\alpha_1,\alpha_2),E}}(S_1), f_{c_{(\alpha_1,\alpha_2),E}}(S_3)\},$$

which is equivalent to

$$u_{\mathcal{P}}(E) \quad = \sup_{\alpha_1,\alpha_2\in\mathbb{R}_0^+} \{\alpha_1+\alpha_2\}$$

with the constraints

$$\begin{aligned} 4+\alpha_2 &\leq 5+\alpha_1, \\ 5+\alpha_1 &\leq 8. \end{aligned}$$

It follows that $\alpha_1 = 3$, $\alpha_2 = 4$, and then (cf. Theorem 9a),c)):

$$\max\{u_{\mathcal{P}}(x), u_{\mathcal{P}}(y)\} \;=\; 3 \;<\; u_{\mathcal{P}}(x)+u_{\mathcal{P}}(y) \;=\; 4 \;<\; u_{\mathcal{P}}(E) \;=\; 7.$$

Thus, this is another example where the lower bound of Theorem 9c) is *not* attained.

- Let $E = \{y, z\} \subseteq \mathcal{E} \setminus S_1$, $e_1 = y$, $e_2 = z$.
  It holds (cf. Theorem 17b)):

$$\max\{l_{\mathcal{P}}(y), l_{\mathcal{P}}(z)\} \;=\; l_{\mathcal{P}}(E) \;=\; l_{\mathcal{P}}(y)+l_{\mathcal{P}}(z) \;=\; +\infty.$$

Thus, this is an example for the exact formula of Theorem 17b).

7. Computation and Discussion of Set Tolerances

In Section 3 we introduced numerical examples of two combinatorial minimization problems. We also considered joint cost changes in multiple elements. In this section we use set tolerances to assess the impact of these cost changes on the optimal solutions. Moreover, we discuss the limitations of set tolerances based on these examples.

Example 8. *(Continuation of Example 1)*
In Example 1 we solved a Linear Assignment Problem (LAP) instance with 3 *workers and* 3 *jobs.*
The question is which cost increase α *in the job-to-worker assignments (the elements in our problem) can be accommodated such that* S_1 *remains optimal. We consider the cost increases of the elements in the set* $E_1 = \{(v_1, v_1), (v_3, v_3)\}$ *and cost decreases of the elements in* $E_2 = \{(v_1, v_2), (v_3, v_2)\}$.

- *Let* $E_1 = \{(v_1, v_1), (v_3, v_3)\} \subseteq S_1$, $e_1 = (v_1, v_1)$, $e_2 = (v_3, v_3)$.
  According to Theorem 9a),b) and Example 1 it holds that:

  $$\begin{aligned} \max\{u_{\mathcal{P}}((v_1, v_1)), u_{\mathcal{P}}((v_3, v_3))\} = 13 &\leq u_{\mathcal{P}}(E_1) \\ &\leq u_{\mathcal{P}}((v_1, v_1)) + u_{\mathcal{P}}((v_3, v_3)) = 24. \end{aligned}$$

  Thus, for a value of $\alpha \leq 13$ *we know that there exist* α_1, α_2, $\alpha_1 + \alpha_2 = \alpha$ *such that* S_1 *remains optimal and for* $\alpha > 24$ *we know from Theorem 8 that no such* α_1, α_2 *exist.*

  In fact, it is easy to see that increasing the cost of $e_1 = (v_1, v_1)$ *by* $\alpha_1 = 6$ *and increasing the cost of* $e_2 = (v_3, v_3)$ *by* $\alpha_2 = 7$ *does not change the optimality of* S_1. *If we increase the cost any further than* $\alpha = 13$, *we obtain that the solution* $\{(v_1, v_3), (v_2, v_2), (v_3, v_1)\}$ *with cost* 13 *becomes the optimal solution, as it does not contain* (v_1, v_1) *and it does not contain* (v_3, v_3).
  It follows that:

  $$u_{\mathcal{P}}((v_1, v_1), (v_3, v_3)) = 13.$$

  We have (cf. Example 1 and Theorem 9a),b)):

  $$\max\{u_{\mathcal{P}}((v_1, v_1)), u_{\mathcal{P}}((v_3, v_3))\} = u_{\mathcal{P}}(E_1) = 13$$

  In this case the set upper tolerance equals the lower bound. An interpretation of its value is that workers 1 *and* 3 *could increase their joint rewards by* 13 *units without changing the optimal assignment. One corresponding distribution is an increase* $\alpha_1 = 6$ *for the first worker and* $\alpha_2 = 7$ *for the second one.*

- *Let* $E_2 = \{(v_1, v_2), (v_3, v_2)\} \subseteq \mathcal{E} \setminus S_1$*,* $e_1 = (v_1, v_2)$*,* $e_2 = (v_3, v_2)$*.*

  According to Theorem 17a),b) it holds that:

$$\begin{aligned} \max\{l_{\mathcal{P}}((v_1, v_2)), l_{\mathcal{P}}((v_3, v_2))\} = 15 &\leq l_{\mathcal{P}}(E_2) \\ &\leq l_{\mathcal{P}}((v_1, v_2)) + l_{\mathcal{P}}((v_3, v_2)) = 26. \end{aligned}$$

  As $\{(v_1, v_2), (v_2, v_1), (v_3, v_3)\}$ *with cost* 11 *is the best solution containing* (v_1, v_2)*, as* $\{(v_1, v_1), (v_2, v_3), (v_3, v_2)\}$ *with cost* 15 *is the best solution containing* (v_3, v_2) *and as no feasible solution exists containing both* (v_1, v_2) *and* (v_3, v_2)*, it follows that:*

$$l_{\mathcal{P}}((v_1, v_2), (v_3, v_2)) = 11 + 15 = 26 = l_{\mathcal{P}}((v_1, v_2)) + l_{\mathcal{P}}((v_3, v_2)).$$

  In this case the upper bound in Theorem 17a) is attained.

  The interpretation could be that if we can reduce the costs of the assignments of workers 1 *and* 3 *to job* 2 *by* 26 *units in total, there exists a distribution* $\alpha = \alpha_1 + \alpha_2$*, namely* $\alpha_1 = 11, \alpha_2 = 15$ *such that* S_1 *remains an optimal solution, which means that such a cost reduction has no impact on the total costs of performing the three jobs.*

Similarly we compute the set upper and lower tolerances of the ABTSP instance from Example 2, which has a bottleneck cost function (MAX).

Example 9. *(Continuation of Example 2)*
We consider the cost increases of the elements in the sets $E_1 = \{(v_1, v_3), (v_4, v_1)\}$ *and cost decreases of the elements in* $E_2 = \{(v_1, v_4), (v_4, v_3)\}$*.*

- *Let* $E_1 = \{(v_1, v_3), (v_4, v_1)\} \subseteq S_2$*,* $e_1 = (v_1, v_3)$*,* $e_2 = (v_4, v_1)$*.*

  Theorem 9c)) provides a lower bound to the set upper tolerance of E_1 *in case of a bottleneck cost function, namely (cf. Example 2):*

$$u_{\mathcal{P}}((v_1, v_3)) + u_{\mathcal{P}}((v_4, v_1)) \ = \ 7 \ \leq \ u_{\mathcal{P}}(E_1).$$

  We now know that a joint cost increase in (v_1, v_3) *and* (v_4, v_1) *of 7 units can be distributed over these elements while keeping* S_2 *optimal.*

  In fact, the true value of $u_{\mathcal{P}}(E_1)$ *is strictly larger in this example. As* $\{(v_1, v_4, v_3, v_2, v_1)\}$ *with cost* 11 *is the best solution that does not contain* (v_1, v_3) *and* (v_4, v_1)*, it follows that:*

$$u_{\mathcal{P}}(E_1) \ = \ (11 - 2) + (11 - 8) \ = \ 12.$$

- *Let* $E_2 = \{(v_1, v_4), (v_4, v_3)\} \subseteq \mathcal{E} \setminus S_1$*,* $e_1 = (v_1, v_4)$*,* $e_2 = (v_4, v_3)$*.*

  We have (cf. Theorem 17b) and Example 2):

$$l_{\mathcal{P}}((E_2)) \ = \ l_{\mathcal{P}}((v_1, v_4)) + l_{\mathcal{P}}((v_4, v_3)) \ = \ +\infty.$$

The examples highlight two challenges concerning set tolerances. The first challenge is to establish exact computations of the set upper and lower tolerance values. One way to do this is by showing that the bounds in Theorem 9 and 17 are sharp for specific problems. The second challenge is for which vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ with $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$ it holds that the solution remains optimal. Such a result would be useful for the purpose of sensitivity analysis for multiple parameter changes. For instance, in Example 8 we know that the solution remains optimal for joint cost increases (α_1, α_2) in $e_1 = (v_1, v_1)$, $e_2 = (v_3, v_3)$ such that $\alpha_1 = 6$ and $\alpha_2 = 7$, but also for any $\vec{\alpha}$ such that $\alpha_1 \leq 6$ and $\alpha_2 \leq 7$. However, increases of $\alpha_1 = 9$ and $\alpha_2 = 4$ also keep the solution optimal, but not increases of $\alpha_1 = 12$ and $\alpha_2 = 1$.

8. Conclusions and Future Work

This paper introduces the concept of set tolerances. In fact, since individual elements form sets of cardinality one, the theory on set tolerances can be regarded as a generalization of the theory of single tolerances. Set tolerances allow us to compute the largest change in the costs of sets of elements such that the current solution remains optimal. The set upper and lower tolerances represent the largest cost increase and decrease, respectively, α such that the current optimal solutions remain optimal for a set of elements E. This paper provides bounds to the value of the set upper and lower tolerances and one exact formula for the value of the set lower tolerance for a cost function of type MAX.
After the formal introduction of set tolerances of this work, we suggest the following research questions for set tolerances to different combinatorial minimization problems such as the MSTP, LAP, TSP, and ABTSP:

- How can (upper and lower) set tolerances be exactly computed? Can set upper and lower tolerances be computed in terms of single upper and lower tolerances?
- What is the complexity of this computation?
- How do the computation and its complexity depend on the cardinality of the set?
- Are the upper and lower bounds from this paper sharp for specific problems?
- How can cost changes smaller than or equal to the set tolerances be distributed over the elements in the set while keeping the current optimal solution optimal?

If such effective computations are found, it is natural to create set tolerance based algorithms for $\mathcal{NP}$-hard combinatorial optimization problems. For instance, such algorithms for the TSP can be based on the computation of set tolerances for the MSTP (see [19] for the application of single tolerances) and the LAP (see [9, 14, 34] for the application of single tolerances).

References

[1] Balas, E., Saltzman, M.J.: An Algorithm for the Three-Index Assignment Problem. Oper. Res. 39(1) (1991) 150-161.

[2] Bekker, H., Braad, E.P., Goldengorin, B.: Selecting the Roots of a Small System of Polynomial Equations by Tolerance Based Matching. in: Proc. 4th WEA 2005, LNCS 3503, pp. 610-613.

[3] Burkard, R.E., Dell'Amico, M., Martello, S.: Assignment Problems. SIAM, Philadelphia, 2009.

[4] Burkard, R.E., Sandholzer, W.: Efficiently Solvable Special Cases of Bottleneck Travelling Salesman Problems. Discrete Applied Mathematics 32(1) (1991) 61-76.

[5] Chin, F., Houck, D.: Algorithms for Updating Minimal Spanning Trees. J. Comput. System Sci. 16(3) (1978) 333-344.

[6] Dong, C., Jäger, G., Richter, D., Molitor, P.: Effective Tour Searching for TSP by Contraction of Pseudo Backbone Edges. in: Proc. 5th AAIM 2009, LNCS 5564, pp. 175-187.

[7] Gal, T.: Sensitivity Analysis, Parametric Programming, and Related Topics: Degeneracy, Multicriteria Decision Making, Redundancy. de Gruyter W., Berlin and New York (1995).

[8] Gal, T., Greenberg, H.J. (Eds.): Advances in Sensitivity Analysis and Parametric Programming. Internat. Ser. Oper. Res. Management Sci. 6. Kluwer Academic Publishers, Boston (1997).

[9] Germs, R., Goldengorin, B., Turkensteen, M.: Lower Tolerance-Based Branch and Bound Algorithms for the ATSP. Comput. Oper. Res. 39(2) (2012) 291-298.

[10] Ghosh, D., Goldengorin, B., Gutin, G., Jäger, G.: Tolerance-based Algorithms for the Traveling Salesman Problem. Chapter 5 of Mathematical Programming and Game Theory for Decision Making, 47-59. World Scientific, New Jersey (2008).

[11] Goldengorin, B., Jäger, G., Molitor, P.: Some Basics on Tolerances. in: Proc. 2nd AAIM 2006, LNCS 4041, pp. 194-206.

[12] Goldengorin, B., Jäger, G., Molitor, P.: Tolerances Applied in Combinatorial Optimization. J. Comput. Sci. 2(9) (2006) 716-734.

[13] Goldengorin, B., Jäger, G., Molitor, P.: Tolerance Based Contract-or-Patch Heuristic for the Asymmetric TSP. in: Proc. 3rd CAAN 2006, LNCS 4235, pp. 86-97.

[14] Goldengorin, B., Sierksma, G., Turkensteen, M.: Tolerance Based Algorithms for the ATSP. in: Proc. 30th WG 2004, LNCS 3353, pp. 222-234.

[15] Greenberg, H.J.: An Annotated Bibliography for Post-Solution Analysis in Mixed Integer and Combinatorial Optimization. Advances in Computational and Stochastic Optimization, Logic Programming, and Heuristic Search. Kluwer Academic Publishers, 97-148 (1998).

[16] Gusfield, D.: A Note on Arc Tolerances in Sparse Minimum-Path and Network Flow Problems. Networks 13(2) (1983) 191-196.

[17] Gutin, G., Goldengorin, B., Huang, J.: Worst Case Analysis of Max-Regret, Greedy and Other Heuristics for Multidimensional Assignment and Traveling Salesman Problems. J. Heuristics 14 (2008) 169-181.

[18] Hall, N.G., Posner, M.E.: Sensitivity Analysis for Scheduling Problems. J. Scheduling 7(1) (2004) 49-83.

[19] Helsgaun, K.: An Effective Implementation of the Lin-Kernighan Traveling Salesman Heuristic. European J. Oper. Res. 126(1) (2000) 106-130.

[20] Jäger, G., Dong, C., Goldengorin, B., Molitor, P., Richter, D: A Backbone Based TSP Heuristic for Large Instances. J. Heuristics 20(1) (2014) 107-124.

[21] Jäger, G., Goldengorin, B., Pardalos, P.M.: The Theory of Set Tolerances. in: Proc. 8th LION 2014, LNCS 8426, pp. 362-367.

[22] Jäger, G., Molitor, P.: Algorithms and Experimental Study for the Traveling Salesman Problem of Second Order. in: Proc. 2nd COCOA 2008, LNCS 5165, pp. 211-224.

[23] Libura, M.: Sensitivity Analysis for Minimum Hamiltonian Path and Traveling Salesman Problems. Discrete Appl. Math. 30(2-3) (1991) 197-211.

[24] Libura, M.: A Note on Robustness Tolerances for Combinatorial Optimization Problems. Inf. Process. Lett. 110(16) (2010) 725-729.

[25] Murty, K.G.: An Algorithm for Ranking all the Assignments in Order of Increasing Cost. Oper. Res. 16 (1968) 682-687.

[26] Van der Poort, E.S., Libura, M., Sierksma, G., van der Veen, J.A.A.: Solving the k-Best Traveling Salesman Problem. Comput. Oper. Res. 26 (1999) 409-425.

[27] Ramaswamy, R., Orlin, J.B., Chakravarti, N.: Sensitivity Analysis for Shortest Path Problems and Maximum Capacity Path Problems in Undirected Graphs. Math. Program., Ser. A 102(2) (2005) 355-369.

[28] Reinfeld, N.V., Vogel, W.R.: Mathematical Programming. Prentice-Hall, Englewood Cliffs, N.J. (1958).

[29] Richter, D., Goldengorin, B., Jäger, G., Molitor, P.: Improving the Efficiency of Helsgaun's Lin-Kernighan Heuristic for the Symmetric TSP. in: Proc. 4th CAAN 2007, LNCS 4852, pp. 99-111.

[30] Shier, D.R., Witzgall, C.: Arc Tolerances in Shortest Path and Network Flow Problems. Networks 10(4) (1980) 277-291.

[31] Sotskov, Y.N.: The Stability of the Approximate Boolean Minimization of a Linear Form. USSR Comput. Math. Math. Phys. 33(5) (1993) 699-707.

[32] Sotskov, Y.N., Leontev, V.K., Gordeev, E.N.: Some Concepts of Stability Analysis in Combinatorial Optimization. Discrete Appl. Math. 58(2) (1995) 169-190.

[33] Tarjan, R.E.: Sensitivity Analysis of Minimum Spanning Trees and Shortest Path Trees. Inform. Process. Lett. 14(1) (1982) 30-33.

[34] Turkensteen, M., Ghosh, D., Goldengorin, B., Sierksma, G.: Tolerance-Based Branch and Bound Algorithms for the ATSP. European J. Oper. Res. 189(3) (2007) 775-788.

[35] Van der Veen, J.A.A.: An $O(n)$ Algorithm to Solve the Bottleneck Traveling Salesman Problem Restricted to Ordered Product Matrices. Discrete Applied Mathematics 47(1) (1993) 57-75

[36] Volgenant, A.: An Addendum on Sensitivity Analysis of the Optimal Assignment. European J. Oper. Res. 169(1) (2006) 338-339.