Selfdecomposable Fields

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Abstract

In the present paper we study selfdecomposability of random fields, as defined directly rather than in terms of finite-dimensional distributions. The main tools in our analysis are the master Lévy measure and the associated Lévy-Itô representation. We give the dilation criterion for selfdecomposability analogous to the classical one. Next, we give necessary and sufficient conditions (in terms of the kernel functions) for a Volterra field driven by a Lévy basis to be selfdecomposable. In this context we also study the so-called Urbanik classes of random fields. We follow this with the study of existence and selfdecomposability of integrated Volterra fields. Finally, we introduce infinitely divisible field-valued Lévy processes, give the Lévy-Itô representation associated with them and study stochastic integration with respect to such processes. We provide examples in the form of Lévy semistationary processes with a Gamma kernel and Ornstein-Uhlenbeck processes.

Keywords: selfdecomposability of random fields, Urbanik classes of random fields, random fields, Volterra fields

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1 Introduction

Of the many interesting classes of infinitely divisible distributions (cf. for example Bondesson (1992)) that of selfdecomposable laws – the SD class – has a foremost position. Originally this class was defined as the family of limit distributions of normalized partial sums. Paul Lévy was the first to study this family in depth. In particular he determined the form of, what is now known as the Lévy measure, of a selfdecomposable distribution. In the early literature the class was also referred to as Lévy's probability measures. For a long time these laws held a rather anonymous position. Even in Michel Loève's detailed and beautifully written biographical account of Lévy's life and contributions to Probability Theory (Loève (1973)) the concept of selfdecomposability is not mentioned. And in volume II of Feller's "An Introduction to Probability Theory and Its Applications" it is treated only very briefly (Section XVII.8), as a 'special topic' under the name of class L. Lévy himself described his work on selfdecomposability in his monumental monograph *Theorie de l'addition des variables aléatoires* (1937).

The more recent prominence of selfdecomposability came from the realisation, due to Wolfe (1982), that any SD distribution can be represented as that of a stochastic integral with respect to a Lévy process, the integrand being the negative exponential; or, otherwise put, the limit law of the solution to a linear stochastic differential equation driven by a Lévy process L is selfdecomposable provided the Lévy measure of the Lévy seed of L satisfies a log moment condition; and all selfdecomposable laws are representable in this way. In turn this gave rise to the concept of Lévy-driven OU processes, continuous time Markov processes whose marginals are SD. For an account of the developments in regard to stochastic integral representations of classes of ID laws in the period 1982–2010 see Jurek (2011).

Looking at this from a modelling point of view, suppose that subject matter knowledge and empirical data indicate that the phenomenon under study might be described as a continuous time stationary process. The simplest type of such a process is a Markov process and for a model to be consistent with this the one-dimensional marginal of the process must be SD, an assumption that may be supported by the available knowledge.

In general, a stochastic process has traditionally been said to be SD if all its finite-dimensional distributions are SD. However, in Barndorff-Nielsen et al. (2006a) it was proposed to define selfdecomposability of a stochastic process directly, saying that a process $X = (X_t)_{t \in \mathbb{R}}$ is SD if for all $q \in (0, 1)$ it can be represented in law as the sum of qX plus an independent stochastic process $V^{(q)}$. It is this approach we take in the present paper where we more generally study selfdecomposable stochastic fields.

In Mathematical Finance, in Turbulence and in other fields, OU processes, and the extended concept of supOU processes, have had an important role as models for stochastic volatility, see for instance Barndorff-Nielsen and Stelzer (2013). We intend here to develop the similar approach for stochastic fields, with the aim of incorporating such fields in models of Ambit Stochastics type, in particular as regards applications to turbulence studies. Note that in Turbulence, and many other fields of Physics, stochastic volatility is referred to as intermittency.

A central object in this context is the master Lévy measure of a stochastic field, a concept introduced in Maruyama (1970), see also Barndorff-Nielsen et al. (2006a), and recently brought on a more analytically tractable footing by Rosiński (2007a,b, 2008, 2013). Importantly, there Rosiński also discusses an associated Lévy-Itô representation. In the following we will build substantially on the results and propositions presented in Rosiński (2007a,b, 2008, 2013).

With the master Lévy measure and the associated Lévy-Itô representation in hand it is in particular possible to characterize volatility/intermittency fields generated from SD fields in much the same way that OU processes are engendered from SD random variables.

The present paper is organized as follows. Section 2 provides background material on ambit fields, Volterra fields, Lévy bases and integration with respect to Lévy bases. In Section 3, based on the recent work of Rosiński (2007a,b, 2008, 2013), we introduce the concept of the master Lévy measure of ID fields and present the associated Lévy-Itô representation. Finally, we give the dilation criterion for selfdecomposability of ID fields. Section 4 is devoted to the study of selfdecomposability of Volterra fields. In particular, we study the master Lévy measure of Volterra fields and give conditions on the kernel of a Volterra field that ensure inheritance of the SD property of the background driving noise to the resulting Volterra field. We close Section 4 with the converse result, that is we give conditions under which the Volterra field is SD if and only if the background driving noise is SD. All of the results mentioned above hold if we exchange SD class with the Urbanik class $\mathbb{L}_m$. In Section 5 we study the existence and selfdecomposability of integrated Volterra fields. Section 6 is devoted to ID field-valued Lévy processes. We give a Lévy-Itô representation of such processes and study integration with respect to such processes. We close the section with the study of Volterra and OU type field-valued processes and their selfdecomposability. Section 7 concludes.

2 Background

In the present section we give the definition of ambit fields and we recall basic results related to Lévy bases and stochastic integration with respect to Lévy bases. Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space

2.1 Ambit fields, Volterra fields and LSS processes

Ambit fields are random fields describing the dynamics in a stochastically developing field, for instance a turbulent wind field. A key characteristic of the modelling framework of ambit stochastics, which distinguishes this from other approaches is that beyond the most basic kind of random input it also specifically incorporates additional, often drastically changing, inputs referred to as volatility or intermittency. Another distinguishing feature is the presence of ambit sets that delineate which part of space-time may influence the value of the field at any given point in space-time.

In terms of mathematical formulae, in its original form an ambit field is specified by

$$\begin{aligned} Y(t,x) = \mu &+ \int_{A(t,x)} g(t,s,x,\xi)\sigma(s,\xi)\, L(dsd\xi) \\ &+ \int_{D(t,x)} q(t,s,x,\xi)\chi(s,\xi)\, dsd\xi \end{aligned}$$

where t denotes time while x gives the position in d-dimensional Euclidean space. Further, $A(t,x)$ and $D(t,x)$ are subsets of $\mathbb{R} \times \mathbb{R}^d$, termed ambit sets, g and q are deterministic weight functions, σ and χ are stochastic fields representing the volatility or intermittency. Finally, L denotes a Lévy basis (i.e. an independently scattered and infinitely divisible random measure). For aspects of the theory and applications of Ambit processes and fields see Barndorff-Nielsen et al. (2014b, 2011, 2014a,c, 2005); Barndorff-Nielsen and Schmiegel (2007); Chong and Klüppelberg (2013); Hedevang and Schmiegel (2014); Podolskij (2014) and Pakkanen (2014).

A *Lévy semistationary process* (LSS) is a stochastic process $(Y_t)_{t\in\mathbb{R}}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ which is described by the following dynamics

$$Y_t = \theta + \int_{-\infty}^{t} g(t-s)\sigma_s\, dL_s + \int_{-\infty}^{t} q(t-s)a_s\, ds, \qquad t \in \mathbb{R},$$

where $\theta \in \mathbb{R}$, L is a Lévy process with triplet (γ, b, ν), g and q are deterministic functions such that $g(x) = q(x) = 0$ for $x \leq 0$, and σ and a are adapted càdlàg processes. When L is a two-sided Brownian motion Y is called *Brownian semistationary process* (BSS). Observe that an LSS process is a null-space Ambit field. For further references to theory and applications of Lévy semistationary processes, see for instance Veraart and Veraart (2012); Benth et al. (2014); Brockwell et al. (2013).

2.2 Lévy bases

Denoting by $\mathrm{ID}(\mathbb{R}^n)$ the space of infinitely divisible (ID for short) distributions on $\mathbb{R}^n$, we recall that any $\mu \in \mathrm{ID}(\mathbb{R}^n)$ has a Lévy-Khintchine representation given by

$$\log \widehat{\mu}(\theta) = i\langle\theta, \gamma\rangle - \tfrac{1}{2}\langle\theta, B\theta\rangle + \int_{\mathbb{R}^n} \left[e^{i\langle\theta,x\rangle} - 1 - i\langle\tau_n(x), \theta\rangle\right] \nu(dx), \qquad \theta \in \mathbb{R}^n,$$

where $\widehat{\mu}$ is the characteristic function of the law of μ, $\gamma \in \mathbb{R}^n$, B is a symmetric non-negative definite matrix on $\mathbb{R}^{n\times n}$ and ν is a Lévy measure, i.e. $\nu(\{0^n\}) = 0$, with 0^n denoting the origin in $\mathbb{R}^n$, and $\int_{\mathbb{R}^n} 1 \wedge |x|^2 \nu(dx) < \infty$. Here, we assume that the truncation function τ_n is given by $\tau_n(x_1, \dots, x_n) = (\frac{x_i}{1\vee|x_i|})_{i=1}^n$, $(x_1, \dots, x_n) \in \mathbb{R}^n$. By $\mathrm{SD}(\mathbb{R}^n)$, we mean the subset of $\mathrm{ID}(\mathbb{R}^n)$ of selfdecomposable (SD) distributions on $\mathbb{R}^n$. More precisely, $\mu \in \mathrm{ID}(\mathbb{R}^n)$ belongs to $\mathrm{SD}(\mathbb{R}^n)$ if and only if for any $q > 1$ there exists $\mu_q \in \mathrm{ID}(\mathbb{R}^n)$ such that

$$\widehat{\mu}(\theta) = \widehat{\mu}(q^{-1}\theta)\widehat{\mu}_q(\theta) \qquad \text{for any } \theta \in \mathbb{R}^n.$$

Let $\mathcal{S}$ be a non-empty set and $\mathcal{R}$ a δ-ring of subsets of $\mathcal{S}$ having the property that exists an increasing sequence $\{S_n\} \subset \mathcal{S}$ with $\bigcup_n S_n = \mathcal{S}$. A real-valued stochastic field $L = \{L(A) : A \in \mathcal{R}\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called *independently scattered random measure* (i.s.r.m. for short), if for every sequence $\{A_n\}_{n\geq 1}$ of disjoint sets in $\mathcal{R}$, the random variables $(L(A_n))_{n\geq 1}$ are independent, and if $\bigcup_{n\geq 1} A_n$ belongs to $\mathcal{R}$, then we also have

$$L\Big(\bigcup_{n\geq 1} A_n\Big) = \sum_{n\geq 1} L(A_n) \qquad \text{a.s.},$$

where the series is assumed to converge almost surely. When the law of $L(A)$ belongs to $\mathrm{ID}(\mathbb{R})$ for any $A \in \mathcal{R}$, L is called a *Lévy basis*. Any Lévy basis admits a Lévy-Khintchine representation:

$$\mathrm{C}\{\theta \ddagger L(A)\} = \int_A \psi(\theta, s)c(ds), \qquad \theta \in \mathbb{R}, A \in \mathcal{R},$$

where $\mathrm{C}\{\theta \ddagger X\}$ denotes the cumulant function of a random variable X and

$$\psi(\theta, s) := \gamma(s)\theta - \tfrac{1}{2}b^2(s)\theta^2 + \int_{\mathbb{R}} [e^{i\theta x} - 1 - i\theta\tau_1(x)]\rho(s, dx), \qquad \theta \in \mathbb{R}, s \in \mathcal{S}. \quad (2.1)$$

The functions γ, b and $\rho(\,\cdot\,, dx)$ are measurable with $b \geq 0$ and $\rho(s, \,\cdot\,)$ is a Lévy measure for every $s \in \mathcal{S}$. The measure c is defined on $\mathcal{B}_{\mathcal{S}} := \sigma(\mathcal{R})$ and is called the *control measure* of L. We will refer to $(\gamma(s), b(s), \rho(s, dx), c(ds))$ as the *characteristic quadruplet* of L. If L has characteristic quadruplet, $(\gamma(s), b(s), \rho(s, dx), c(ds))$, the associated family of random variables $(L'(s))_{s\in\mathcal{S}}$ such that $L'(s)$ is ID and has characteristic triplet $(\gamma(s), b(s), \rho(s, dx))$ is called *Lévy seeds.* When $b \equiv 0$, we say that L is *Poissonian.* If γ, b and ρ do not depend on s we say that L is *factorizable.* Moreover, L will be called *homogeneous* if it is factorizable and c is proportional to the Lebesgue measure.

If we put $\mathcal{R} = \mathcal{B}_b(\mathbb{R}^k)$ the bounded Borel sets and add the extra condition $L(\{x\}) = 0$ a.s. for all $x \in \mathbb{R}^k$, L has a Lévy-Itô decomposition: We have that almost surely

$$L(A) = \int_A \gamma(s)c(ds) + W(A) + \int_A \int_{|x|>1} xN(dxds) + \int_A \int_{|x|\leq 1} x\widetilde{N}(dxds), \quad A \in \mathcal{R},$$

where W is a centered Gaussian Lévy basis with $\mathbb{E}(W(A)W(B)) = \int_{A\cap B} b(s)c(ds)$ for all $A, B \in \mathcal{R}$, $\widetilde{N}$ and N are compensated and non-compensated Poisson random measures on $\mathbb{R}^k \times \mathbb{R}$ with intensity $\rho(s, dx)c(ds)$, respectively. Additionally, W and N are independent. See Pedersen (2003) for more details.

2.3 Stochastic integration with respect to Lévy bases

In the following, we present a short review of Rajput and Rosiński (1989) concerning to the existence of stochastic integrals of the form $\int_{\mathcal{S}} f(s)L(ds)$, where $f\colon \mathcal{S} \to \mathbb{R}$ is a measurable function and L a Lévy basis with characteristic quadruplet $(\gamma(s), b(s), \rho(s, dx), c(ds))$.

Let $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ be the space of real-valued random variables endowed with convergence in probability. Consider ϑ, the space of simple functions on $(\mathcal{S}, \mathcal{R})$, i.e. $f \in \vartheta$ if and only if f can be written as

$$f(s) = \sum_{i=1}^{k} a_i \mathbb{1}_{A_i}(s), \qquad s \in \mathcal{S},$$

where $A_i \in \mathcal{R}$ and $a_i \in \mathbb{R}$ for $i = 1, \ldots, k$. Given $f \in \vartheta$, define the linear operator $m\colon \vartheta \to \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ by

$$m(f) := \sum_{i=1}^{k} a_i L(A_i). \tag{2.2}$$

In stochastic integration theory, commonly one is looking for a linear extension of operators of the form (2.2) to a suitable space, let's say I_m, such that $m(f)$ can be approximated by simple integrals of elements of ϑ. More precisely, if m can be extended to I_m and ϑ is dense in this set, we say that f is L-integrable or $f \in I_m$ and we define its stochastic integral with respect to L as

$$\int_{\mathcal{S}} f(s)L(ds) := \mathbb{P}\text{-}\lim_{n\to\infty} m(f_n), \tag{2.3}$$

provided that $f_n \in \vartheta$ and $f_n \to f$ c-a.e.

In Rajput and Rosiński (1989), it has been shown that the simple integral (2.2) can be extended to the so-called Musielak-Orlicz space:

$$I_m = \{f\colon (\mathcal{S},\mathcal{B}_\mathcal{S}) \to (\mathbb{R},\mathcal{B}(\mathbb{R})) : \int_\mathcal{S} \Phi_0(|f(s)|,s)c(ds) < \infty\},$$

where

$$\Phi_p(r,s) := \sup_{|c|\le 1} H(cr,s) + b^2(s)r^2 + \int_\mathbb{R} [|xr|^p \mathbb{1}_{\{|xr|>1\}} + |xr|^2 \mathbb{1}_{\{|xr|\le 1\}}]\rho(s,dx), \quad (2.4)$$

with $p \ge 0, r \in \mathbb{R}, s \in \mathcal{S}$ and

$$H(r,s) := \big|\gamma(s)r + \int_\mathbb{R} [\tau_1(xr) - r\tau_1(x)]\rho(s,dx)\big|, \qquad r \in \mathbb{R}, s \in \mathcal{S}. \quad (2.5)$$

For a comprehensive introduction to Musielak-Orlicz spaces, we refer to Rao and Ren (1994).

When $f \in I_m$, $\int_\mathcal{S} f(s)L(ds)$ is ID and

$$\mathrm{C}\left\{\theta \ddagger \int_\mathcal{S} f(s)L(ds)\right\} = \int_\mathcal{S} \psi(f(s)\theta,s)c(ds), \qquad \theta \in \mathbb{R}, \quad (2.6)$$

with ψ as in (2.1).

Fix $p \ge 0$ such that $\mathbb{E}(|L(A)|^p) < \infty$ for all $A \in \mathcal{R}$ and define

$$\mathcal{L}_{\Phi_p} := \{f\colon (\mathcal{S},\mathcal{B}_\mathcal{S}) \to (\mathbb{R},\mathcal{B}(\mathbb{R})) : \int_\mathcal{S} \Phi_p(|f(s)|,s)c(ds) < \infty\}. \quad (2.7)$$

$\mathcal{L}_{\Phi_p}$ is the space of L-integrable functions having finite p-moment. When $p = 0$, $\mathcal{L}_{\Phi_0} = I_m$, i.e. $\mathcal{L}_{\Phi_0}$ is the space of L-integrable functions. Furthermore, $\mathcal{L}_{\Phi_p}$ endowed with the Luxemburg norm

$$\|f\|_{\Phi_p} := \inf\{a > 0 : \int_\mathcal{S} \Phi_p(a^{-1}|f(s)|,s)c(ds) \le 1\}, \quad (2.8)$$

is a separable Banach space. Observe that $f \in \mathcal{L}_{\Phi_p}$ if and only if $\|f\|_{\Phi_p} < \infty$.

Recall that $\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$ is the space of random variables endowed with the convergence in probability. The following properties of $\int_\mathcal{S} f(s)L(ds)$ will be useful for the rest of the paper, see Rajput and Rosiński (1989) for proofs:

1. The mapping $(f \in \mathcal{L}_{\Phi_p}) \longmapsto \big(\int_\mathcal{S} f(s)L(ds) \in \mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})\big)$ is continuous, i.e. if $f_n \to 0$ in $\mathcal{L}_{\Phi_p}$, then $\int_\mathcal{S} f_n(s)L(ds) \to 0$ in $\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$;
2. If L is symmetric or centered, then for any $p \ge 0$ the mapping $(f \in \mathcal{L}_{\Phi_p}) \longmapsto \big(\int_\mathcal{S} f(s)L(ds) \in \mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})\big)$ is an isomorphism between $\mathcal{L}_{\Phi_p}$ and $\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$.

3 Some Lévy theory of ID fields

In this part we introduce the notions of infinite divisible and selfdecomposable fields as well as some basic properties of such fields.

3.1 Infinite divisibility and selfdecomposability of stochastic fields

Let U be a non-empty index set and $X = (X_u)_{u\in U}$ be a real-valued stochastic field defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that X is *infinitely divisible*, writing $\mathcal{L}(X) \in \mathrm{ID}(\mathbb{R}^U)$, with $\mathcal{L}(X)$ denoting the law of the field X, if for any $n \in \mathbb{N}$ there are $X^{i,n} = (X_u^{i,n})_{u\in U}$, $i = 1, \ldots, n$, independent and identically distributed stochastic fields such that

$$X \stackrel{d}{=} X^{1,n} + X^{2,n} + \cdots + X^{n,n}.$$

In the same way, we say that X is *selfdecomposable*, writing $\mathcal{L}(X) \in \mathrm{SD}(\mathbb{R}^U)$, if for any $q > 1$ there exists X', an independent copy of X, and V^q a random field independent of X', such that

$$X \stackrel{d}{=} q^{-1}X' + V^{(q)}.$$

Observe that when U is finite, the definition of infinite divisibility and selfdecomposability coincide with the usual concepts of ID and SD random vectors. We denote by $\mathbb{L}_m(\mathbb{R}^U)$, for $m = 0, 1, \ldots$, the m-th *Urbanik class*, i.e. $\mathcal{L}(X) \in \mathbb{L}_m(\mathbb{R}^U)$ if and only if X is SD and the field $\mathcal{L}(V^{(q)}) \in \mathbb{L}_{m-1}(\mathbb{R}^U)$. Here $\mathbb{L}_0(\mathbb{R}^U) = \mathrm{SD}(\mathbb{R}^U)$.

3.2 The master Lévy measure of an ID field

In this subsection we extend the family of Lévy measures associated to an ID field to a measure in the space of paths, what we refer to as the master Lévy measure. Such a measure was originally introduced in Maruyama (1970). Later on, it was studied in depth by Rosiński (2007a,b, 2008, 2013), who also established the associated Lévy-Itô representation. For completeness of our discussion of this result, in the appendix of this paper we present a detailed proof.

Let us introduce some notation. For a given non-empty set U, denote by $\widehat{U}$ the collection of all finite subsets of U. For any $\widehat{u} \in \widehat{U}$, we write $\mathbb{R}^{\widehat{u}} := \Pi_{u\in\widehat{u}}\mathbb{R}$, i.e. $\mathbb{R}^{\widehat{u}}$ is $\#\widehat{u}$-dimensional Euclidean space with $\#\widehat{u}$ denoting the cardinality of $\widehat{u}$. Furthermore, $0^{\widehat{u}}$ denotes the origin in $\mathbb{R}^{\widehat{u}}$ and $X_{\widehat{u}} := \pi_{\widehat{u}}(X) = (X_u)_{u\in\widehat{u}}$. Here $\pi_{\widehat{u}}\colon \mathbb{R}^U \to \mathbb{R}^{\widehat{u}}$ is the natural projection of $\mathbb{R}^U$ into $\mathbb{R}^{\widehat{u}}$. For any $\widehat{u}, \widehat{v} \in \widehat{U}$, with $\widehat{u} \subset \widehat{v}$, $\pi_{\widehat{v}\widehat{u}}$ denotes the natural projection of $\mathbb{R}^{\widehat{v}}$ into $\mathbb{R}^{\widehat{u}}$.

As an extension of (Barndorff-Nielsen et al., 2006a, Theorems 3.4, 3.6 and 3.7), we have that $\mathcal{L}(X) \in \mathrm{ID}(\mathbb{R}^U)$ if and only if $\mathcal{L}(X_{\widehat{u}}) \in \mathrm{ID}(\mathbb{R}^{\widehat{u}})$ for any $\widehat{u} \in \widehat{U}$. An analogous statement applies for selfdecomposability and for the Urbanik classes. Moreover, the field X has associated a consistent system of characteristic triplets in the sense of the following proposition:

Proposition 3.1. *Let $X = (X_u)_{u\in U}$ be an ID field. For any $\widehat{u} \in \widehat{U}$, let $(\gamma_{\widehat{u}}, B_{\widehat{u}}, \nu_{\widehat{u}})$ be the characteristic triplet of $\mathcal{L}(X_{\widehat{u}})$. Then, there are unique functions $B\colon U \times U \to \mathbb{R}$ and $\Gamma\colon U \to \mathbb{R}$, such that $\gamma_{\widehat{u}} = \pi_{\widehat{u}}(\Gamma)$ and $B_{\widehat{u}} = (B(u,v))_{u,v\in\widehat{u}}$. In addition, we have*

$$\nu_{\widehat{u}} = \nu_{\widehat{v}} \circ \pi_{\widehat{v}\widehat{u}}^{-1}, \qquad \text{on } \mathcal{B}(\mathbb{R}^{\widehat{u}} \setminus 0^{\widehat{u}}), \text{ for any } \widehat{u}, \widehat{v} \in \widehat{U} \text{ and } \widehat{u} \subset \widehat{v}. \tag{3.1}$$

Reciprocally, given functions B and Γ as before and a collection of Lévy measures satisfying (3.1)*, there exists a unique (in law) field X having characteristic triplets $(\gamma_{\widehat{u}}, B_{\widehat{u}}, \nu_{\widehat{u}})$.*

Remark 3.2. Observe that (3.1) only holds on $\mathcal{B}(\mathbb{R}^{\widehat{u}} \setminus 0^{\widehat{u}})$. Indeed, since in general $\nu_{\widehat{v}} \circ \pi^{-1}_{\widehat{v}\widehat{u}}(\{0^{\widehat{u}}\}) \neq 0$, $\nu_{\widehat{v}} \circ \pi^{-1}_{\widehat{v}\widehat{u}}$ could have an atom in the origin of $\mathbb{R}^{\widehat{u}}$, consequently $\nu_{\widehat{u}}$ and $\nu_{\widehat{v}} \circ \pi^{-1}_{\widehat{v}\widehat{u}}$ coincide only outside of a neighborhood of zero.

From Proposition 3.1 and Remark 3.2 , we have that $(\nu_{\widehat{u}})_{\widehat{u}\in\widehat{U}}$ does not form a projective system of measures, so in general it is not possible to extend $(\nu_{\widehat{u}})_{\widehat{u}\in\widehat{U}}$ to a unique measure on $\mathcal{B}(\mathbb{R})^U$, the cylindrical σ-algebra of U, by standard arguments. But even when it is possible, such measure could not in general be σ-finite, mainly because $\{0^U\} \notin \mathcal{B}(\mathbb{R})^U$ when U is uncountable. This was already pointed out in Rosiński (2007a,b, 2008, 2013)

From now on, we will assume that U is uncountable. The countable case is well known. In view of the pointed out before, we introduce the concept of a measure that does not charge zero.

Definition 3.3. *Let U be an arbitrary index set. A measure ν on $\mathcal{B}(\mathbb{R})^U$, the cylindrical σ-algebra of $\mathbb{R}^U$,* does not charge zero *if there exists $U_0 \subset U$ countable, such that*

$$\nu(\pi^{-1}_{U_0}(0^{U_0})) = 0. \tag{3.2}$$

With all the notation above, we are now ready to present one of the main results that is going to be fundamental for the rest of the paper:

Theorem 3.4 (Rosiński 2013)**.** *Let $X = (X_u)_{u\in U}$ be an ID field with $(\gamma_{\widehat{u}}, B_{\widehat{u}}, \nu_{\widehat{u}})$ being its system of characteristic triplets. Then there are $B\colon U\times U \to \mathbb{R}$ and $\Gamma\colon U \to \mathbb{R}$ unique functions, such that $\gamma_{\widehat{u}} = \pi_{\widehat{u}}(\Gamma)$ and $B_{\widehat{u}} = (B(u,v))_{u,v\in\widehat{u}}$. Additionally there is a measure on $(\mathbb{R}^U, \mathcal{B}(\mathbb{R})^U)$ such that*

$$\nu_{\widehat{u}}(A) = \nu \circ \pi^{-1}_{\widehat{u}}(A), \qquad A \in \mathcal{B}(\mathbb{R}^{\widehat{u}} \setminus 0^{\widehat{u}}), \widehat{u} \in \widehat{U}, \tag{3.3}$$

and

$$\int_{\mathbb{R}^U} 1 \wedge |\pi_u(x)|^2 \nu(dx) < \infty, \qquad u \in U. \tag{3.4}$$

If ν does not charge zero, then ν is the unique measure that doesn't charge zero for which (3.3) *holds.*

A proof of this theorem is presented in the appendix.

Remark 3.5. In general, the measure ν in Theorem 3.4 is not unique. It is mainly because ν may not be σ-finite. This issue has been already pointed out in Rosiński (2007a,b, 2008, 2013). At this point the concept of a measure in the cylindrical σ-field that does not charge zero plays an important role. However, the uniqueness can be obtained without condition (3.2).

From the preceding proposition, for a given measure ν satisfying (3.3) and (3.4) we can construct a consistent system of Lévy measures by putting $\nu_{\widehat{u}}(\,\cdot\,) = \nu[\pi^{-1}_{\widehat{u}}(\,\cdot\, \setminus 0^{\widehat{u}})]$. If in addition, we consider functions $B\colon U \times U \to \mathbb{R}$ and $\Gamma\colon U \to \mathbb{R}$, such that for any $\widehat{u} \in \widehat{U}$, $\Gamma_{\widehat{u}} := \pi^{-1}_{\widehat{u}}(\Gamma) \in \mathbb{R}^{\widehat{u}}$ and $B_{\widehat{u}} = (B(u,v))_{u,v\in\widehat{u}}$ is non-negative definite, then there is a unique (in law) ID field X having characteristic triplets $(\Gamma_{\widehat{u}}, B_{\widehat{u}}, \nu_{\widehat{u}})$. This remark induces naturally the following definition:

Definition 3.6. *Let* $(\Gamma_{\widehat{u}}, B_{\widehat{u}}, \nu_{\widehat{u}})_{\widehat{u}\in\widehat{U}}$ *be the system of characteristic triplets associated to the ID field* $X = (X_u)_{u\in U}$*. A measure* ν *on* $\mathcal{B}(\mathbb{R})^U$*, the cylindrical* σ*-algebra of* $\mathbb{R}^U$*, is said to be* a pseudo master Lévy measure *of* X *if* (3.3) *and* (3.4) *hold. If in addition,* ν *does not charge zero, we say that* ν *is* the master Lévy measure *of* X*. In this case, we refer to* (Γ, B, ν) *as the* characteristic triplet *of* X*. When we write that* X *has characteristic triplet* (Γ, B, ν)*, we are going to assume that* ν *does not charge zero.*

The following Lévy-Itô representation follows easily from Theorem 3.4 and, like Theorem 3.4, it was introduced in Rosiński (2007a,b, 2008, 2013).

Proposition 3.7 (Rosiński 2013)**.** *Let* $X = (X_u)_{u\in U}$ *be an infinitely divisible field with characteristic triplet* (Γ, B, ν)*. Then the field*

$$\widetilde{X}_u := \pi_u(\Gamma) + W_u + \int_{\mathbb{R}^U} \pi_u(x)[N(dx) - \mathbb{1}_{\{|\pi_u(x)|\le 1\}}\nu(dx)], \qquad u \in U,$$

is well defined and it is a version of X*. Here* W *is a centered Gaussian process with* $Cov(X_{\widehat{u}}) = (B(u,v))_{u,v\in\widehat{u}}$*, for any* $\widehat{u} \in \widehat{U}$*. Further,* N *is a Poisson random measure with intensity* ν *and it is independent of* W*.*

3.3 Criterion for selfdecomposability of ID fields

As in the classical theory, the selfdecomposability of an ID field can be characterized via dilations.

Proposition 3.8. *Let* $X = (X_u)_{u\in U}$ *be an infinitely divisible field with characteristic triplet* (Γ, B, ν)*. Then* X *is selfdecomposable if and only if for any* $q > 1$

$$\nu(qA) \le \nu(A), \qquad A \in \mathcal{B}(\mathbb{R})^U. \tag{3.5}$$

Proof. Let $(\Gamma_{\widehat{u}}, B_{\widehat{u}}, \nu_{\widehat{u}})_{\widehat{u}\in\widehat{U}}$ be the system of characteristic triplets associated to X. If (3.5) holds, then from (3.3), we have that the same expression holds for $\nu_{\widehat{u}}$, which means that $\mathcal{L}(X_{\widehat{u}}) \in \mathrm{ID}(\mathbb{R}^{\widehat{u}})$ for any $\widehat{u} \in \widehat{U}$, proving thus that X is SD.

Now suppose that X is SD, i.e. $\mathcal{L}(X_{\widehat{u}}) \in \mathrm{ID}(\mathbb{R}^{\widehat{u}})$ for any $\widehat{u} \in \widehat{U}$. Let us observe that in general $qA \notin \mathcal{B}(\mathbb{R})^U$, for instance if $q = 0$, $qA = \{0^U\} \notin \mathcal{B}(\mathbb{R})^U$ if U is uncountable. Thus, we firstly verify that for any $q > 0$ we have $qA \in \mathcal{B}(\mathbb{R})^U$. Define

$$\mathcal{A}_q := \left\{A \in \mathcal{B}(\mathbb{R})^U : qA \in \mathcal{B}(\mathbb{R})^U\right\}.$$

Observe that $\mathcal{C}$, the set of the cylinders in $\mathcal{B}(\mathbb{R})^U$, belongs to $\mathcal{A}_q$. Moreover, $\mathcal{A}_q$ is a σ-algebra. Indeed, obviously $\mathbb{R}^U \in \mathcal{A}_q$. Due to

$$q(A \cap B) = qA \cap qB;$$
$$q(A \cup B) = qA \cup qB,$$

it follows easily that $\mathcal{A}_q$ is closed under complements and countable unions. This shows that $qA \in \mathcal{B}(\mathbb{R})^U$ for any $q > 0$ and $A \in \mathcal{B}(\mathbb{R})^U$. To prove (3.5), fix $q > 1$ and define

$$\mathcal{A}_q^\nu := \left\{A \in \mathcal{B}(\mathbb{R})^U : \nu(qA) \le \nu(A)\right\}.$$

Thanks to Lemma A.3 (see the appendix) and the Monotone Class Theorem, we only need to check that $\mathcal{B}_0 \cup \{\mathbb{R}^U\} \subset \mathcal{A}_q^\nu$, where $\mathcal{B}_0$ is as in A.3. In view that ν is the master Lévy measure of X, we have that ν does not charge zero, or equivalently, it is σ-finite, so without loss of generality we can assume that ν is finite. Clearly $\mathbb{R}^U \in \mathcal{A}_q^\nu$. Moreover, by consistency and equation (A.1) we see that for any $A_0 \in \mathcal{B}_0$

$$\begin{aligned}
\nu(qA_0) &= \nu[q\pi_{\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}})] \\
&\leq \nu[\pi_{\widehat{u}}^{-1}(qA \setminus 0^{\widehat{u}})] \\
&= \nu_{\widehat{u}}(qA) \\
&\leq \nu_{\widehat{u}}(A) \\
&= \nu(\pi_{\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}})) \\
&= \nu(A_0).
\end{aligned}$$

provided that $A_0 = \pi_{\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}})$ for some $\widehat{u} \in \widehat{U}$ and $A \in \mathcal{B}(\mathbb{R}^{\widehat{u}})$. The inequality above follows from the selfdecomposability of the finite-dimensional distributions, i.e. the system of finite-dimensional Lévy measures $\{\nu_F\}_{\widehat{u} \in \widehat{U}}$ fulfills (3.5) on $\mathcal{B}(\mathbb{R}^{\widehat{u}})$ for any $\widehat{u} \in \widehat{U}$. Therefore $\mathcal{B}_0 \cup \{\mathbb{R}^U\} \subset \mathcal{A}_c^\nu$. This completes the proof. □

Remark 3.9. From the proof of the previous proposition, if there is a pseudo master Lévy measure (i.e. it may not fulfill (3.2)) of X satisfying (3.5), then the process X is selfdecomposable. However, the reverse may not be true in general.

4 Selfdecomposability of Volterra fields

In this section we study the selfdecomposability of ID Volterra fields induced by a Lévy basis. In particular, we show that under some conditions on the kernel, the selfdecomposability of the field is equivalent to the selfdecomposability of the Lévy basis.

Let $(L_t)_{t \in \mathbb{R}}$ be a $\mathbb{R}^n$-valued two-sided Lévy process and f a measurable function. It is well known that the mapping $\mathcal{L}(L_1) \mapsto \mathcal{L}(\int_{\mathbb{R}} f(s) dL_s)$ is not in general one-to-one (e.g. Barndorff-Nielsen et al. (2008)). Note that $\mathcal{L}(\int_{\mathbb{R}} f(s) dL_s)$ corresponds to the marginal distribution of the stationary process $X_u := \int_{\mathbb{R}} f(u-s) dL_s$. There are several important classes of infinitely divisible distributions that can be characterized using such mapping. Perhaps the most important example corresponds to the class SD($\mathbb{R}^n$) of selfdecomposable distributions on $\mathbb{R}^n$. In this case $\mathcal{L}(L_1) \mapsto \mathcal{L}(\int_0^\infty e^{-s} dL_s)$ creates a bijection between the class of ID distributions on $\mathbb{R}^n$ whose Lévy measure has log-moment outside of zero and the class SD($\mathbb{R}^n$). Moreover, $\mathcal{L}(\int_0^\infty e^{-s} dL_s)$ is the marginal distribution of a stationary OU process driven by L. Observe that in this case $\mathcal{L}(\int_0^\infty e^{-s} dL_s) \in$ SD($\mathbb{R}^n$) even if $\mathcal{L}(L_1) \notin$ SD($\mathbb{R}^n$). Nevertheless, as it has been shown in (Barndorff-Nielsen et al., 2006a, Theorem 3.4), this is not true for SD fields, e.g. the OU process is SD if and only if L is SD as well. We generalize such result for ID Volterra fields.

4.1 The master Lévy measure of an ID Volterra field

In this part we investigate the master Lévy measure of certain class of infinitely divisible fields which can be expressed in terms of stochastic integrals, namely Volterra fields.

For the rest of the section $p \geq 0$ is such that $\mathbb{E}(|L(A)|^p) < \infty$ for all $A \in \mathcal{R}$. Recall that the stochastic integral $\int_{\mathcal{S}} f(s)L(ds)$ is well defined and has finite p-moment if and only if $f \in \mathcal{L}_{\Phi_p}$. Here $\mathcal{L}_{\Phi_0}$ is the space of L-integrable functions. Let L be a Poissonian Lévy basis on $(\mathcal{S}, \mathcal{R})$ with quadruplet $(\gamma(s), 0, \rho(s, dx), c(ds))$. An ID *Volterra field* driven by L is a field

$$X_u := \int_{\mathcal{S}} f(u, s)L(ds), \qquad u \in U, \tag{4.1}$$

where U is a separable space and $f\colon U \times \mathcal{S} \to \mathbb{R}$ a measurable function such that $f(u, \cdot\,) \in \mathcal{L}_{\Phi_0}$ for all $u \in U$. Note that the expression in (4.1) is also called spectral representation of an infinitely divisible process. The next proposition describes the master Lévy measure of X. Recall that a function $g\colon U \to \mathbb{R}$ is lower (upper) continuous if $\liminf_{u\to u_0} g(u) \geq g(u_0)$ ($\limsup_{u\to u_0} g(u) \leq g(u_0)$) for any $u_0 \in U$.

Proposition 4.1. *Let X be as in* (4.1) *with L a Poissonian Lévy basis with quadruplet $(\gamma(s), 0, \rho(s, dx), c(ds))$. Define*

$$\nu := \eta \circ g^{-1}, \tag{4.2}$$

where $g\colon \mathbb{R} \times \mathcal{S} \to \mathbb{R}^U$ is the function defined as $g_u(x, s) = xf(u, s)$, $u \in U$ and $\eta(dxds) = \rho(s, dx)c(ds)$. Suppose that $f(\,\cdot\,, s)$ is non-identically zero and lower or upper continuous for c-almost all $s \in \mathcal{S}$. Then ν as in (4.2) *is the master Lévy measure of X.*

For the proof of this proposition we need the next result:

Proposition 4.2. *Let X be as in* (4.1) *with L a Poissonian Lévy basis. Then, for any $\widehat{u} \in \widehat{U}$, $X_{\widehat{u}}$ has characteristic triplet $(\Gamma_{\widehat{u}}, 0, \nu_{\widehat{u}})$ with*

$$\Gamma_{\widehat{u}} = \int_{\mathcal{S}} \Big\{\gamma(s)\pi_{\widehat{u}}(f(\,\cdot\,, s)) + \int_{\mathbb{R}} [\tau_{\#\widehat{u}}[\pi_{\widehat{u}}(xf(\,\cdot\,, s))] - \pi_{\widehat{u}}(f(\,\cdot\,, s))\tau_1(x)]\rho(s, dx)\Big\}c(ds),$$

and

$$\nu_{\widehat{u}} = \nu \circ \pi_{\widehat{u}}^{-1}, \qquad \text{on } \mathcal{B}(\mathbb{R}^{\widehat{u}} \setminus 0^{\widehat{u}}), \tag{4.3}$$

with ν given by (4.2).

Proof. Let us start by noting that since the mapping $(x, s) \mapsto g_u(x, s)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}_{\mathcal{S}}/\mathcal{B}(\mathbb{R})$-measurable for all $u \in U$ we have that ν is well defined. Now, let $\widehat{u} \in \widehat{U}$ and observe that for all $\theta \in \mathbb{R}^{\widehat{u}}$

$$\langle X_{\widehat{u}}, \theta\rangle = \int_{\mathcal{S}} \langle \pi_{\widehat{u}}[f(\,\cdot\,, s)], \theta\rangle L(ds).$$

Thus, from (2.6), the cumulant function of $X_{\widehat{u}}$ satisfies

$$\mathrm{C}\{\theta \ddagger X_{\widehat{u}}\} = \int_{\mathcal{S}} \psi\left(\langle \pi_{\widehat{u}}[f(\,\cdot\,, s)], \theta\rangle, s\right) c(ds),$$

with ψ as in (2.1). But for c-almost all $s \in \mathcal{S}$

$$\begin{aligned}
&\psi(\langle \pi_{\widehat{u}}[f(\,\cdot\,,s)],\theta\rangle,s)\\
&\quad= i\langle \gamma(s)\pi_{\widehat{u}}[f(\,\cdot\,,s)],\theta\rangle + \int_{\mathbb{R}} \left[e^{ix\langle \pi_{\widehat{u}}[f(\cdot,s)],\theta\rangle} - 1 - i\langle \pi_{\widehat{u}}[f(\,\cdot\,,s)],\theta\rangle \tau_1(x)\right] \rho(s,dx)\\
&\quad= i\Big\langle \gamma(s)\pi_{\widehat{u}}[f(\,\cdot\,,s)] + \int_{\mathbb{R}} \{\tau_{\#\widehat{u}}[\pi_{\widehat{u}}(xf(\,\cdot\,,s))] - \pi_{\widehat{u}}(f(\,\cdot\,,s))\tau_1(x)\}\rho(s,dx),\theta\Big\rangle\\
&\qquad+ \int_{\mathbb{R}} \left\{e^{i\langle \pi_{\widehat{u}}[g.(x,s)],\theta\rangle} - 1 - i\langle \tau_{\#\widehat{u}}[\pi_{\widehat{u}}(g.(x,s))],\theta\rangle\right\} \rho(s,dx).
\end{aligned}$$

Integrating the previous equation with respect to c and invoking the uniqueness of the triplet, the result follows. □

Proof of Proposition 4.1. From Theorem 3.4 and the previous proposition we only need to check that ν does not charge zero. Let U_0 be a dense set in U, then from the definition of ν

$$\begin{aligned}
\nu(\pi_{U_0}^{-1}(0^{U_0})) &= \eta(\{(s,x) : xf(u,s) = 0 \ \forall\ u \in U_0\})\\
&= \eta(\{(s,x) : x = 0 \text{ or } f(u_0,s) = 0 \ \forall\ u \in U_0\})\\
&= \lim_{n\to\infty} \eta(\{(s,x) : f(u_0,s) = 0 \ \forall\ u \in U_0, |x| > 1/n\})\\
&= \lim_{n\to\infty} \int_{\{s\,:\,f(u,s)=0\,\forall\,u\in U_0\}} \rho(s,\{|x| > 1/n\})c(ds) = 0,
\end{aligned}$$

because if $f(u,s) = 0 \,\forall\, u \in U_0$, by the lower\upper continuity we have that $f(u,s) = 0$ for all $u \in U$, which is contradictory. □

Remark 4.3. Observe that equation (4.3) holds for any ID Volterra field. This means that the measure ν defined by (4.2) is always a pseudo master Lévy measure of an ID Volterra field. However, it is not clear that in general such a measure does not charge zero for general index set U.

Remark 4.4. Note that ν can be viewed as a general Υ^0-transformation (see for instance Barndorff-Nielsen et al. (2013c)). Indeed, let $(E,\mathcal{B}_E)$ be a measurable space. For any fixed measurable function $g\colon \mathcal{S}\times\mathbb{R}^d \to E$ and c a σ-finite measure on $\mathcal{S}$, define the functional

$$\Upsilon^0_{g,c}(\rho)(A) := \int_{\mathcal{S}}\int_{\mathbb{R}^d} \mathbb{1}_{g^{-1}(A)}(x,s)\rho(s,dx)c(ds), \qquad A \in \mathcal{B}_E,$$

where $(\rho(s,dx))_{s\in\mathcal{S}}$ is a measurable collection of measures. Notice that $\Upsilon^0_{g,c}(\rho) = \eta\circ g^{-1}$, with $\eta(dxds) = \rho(s,dx)c(ds)$. In particular, if $\mathcal{S} = \mathbb{R}^+$, $g(x,s) = xs$ and ρ does not depend on s, $\Upsilon^0_{g,c}$ coincides with the usual Υ^0-transformation of ρ via c. In this case, it is well known that such transformation is generally not one-to-one. More generally, if $g(x,s) = T(s)x$, where T is a measurable collection of linear mappings on $\mathbb{R}^d$, we have that $\Upsilon^0_{g,c}(\rho)$ is the Lévy measure of $\Upsilon_T(\mu)$, the probability measure with cumulant

$$\mathrm{C}\{\theta \ddagger \Upsilon_T(\mu)\} = \int_{\mathcal{S}} \mathrm{C}\{T(s)\theta \ddagger L'\}c(ds), \qquad L' \sim \mu \text{ and } \mu \in \mathrm{ID}(\mathbb{R}^d).$$

with L' the Lévy seed of a factorizable Lévy bases. Hence, $\Upsilon^0_{g,c}(\rho)$ can be viewed as the Lévy measure of the probability measure $\Upsilon_f(\mu) \in \mathrm{ID}(\mathbb{R}^U)$, with μ being the distribution of L in $\mathrm{ID}(\mathbb{R}^{\mathcal{R}})$, characterized by

$$\mathrm{C}\{\theta \ddagger \Upsilon_f(\mu) \circ \pi_{\widehat{u}}^{-1}\} = \int_{\mathcal{S}} \mathrm{C}\{\langle \pi_{\widehat{u}}[f(\,\cdot\,,s)]\,,\theta\rangle \ddagger \mu(s)\} c(ds) \qquad \text{for any } \widehat{u} \in \widehat{U}, \theta \in \mathbb{R}^{\widehat{u}},$$

and with $\mu(s)$ the distribution of the Lévy seed $L'(s)$ for all $s \in \mathcal{S}$. See Barndorff-Nielsen et al. (2008) and Barndorff-Nielsen et al. (2013c) for an extensive discussion on Υ^0-transformations and generalizations.

4.2 Inherited selfdecomposability from the Lévy basis

Typically the selfdecomposability of stochastic integrals can be obtained by assuming that the integrator is also selfdecomposable. In the case of random fields such result is also true. In this part we verify this property for ID Volterra fields by using the characterization provided in Proposition 3.8. The converse of such property will be discussed in the next subsection.

Proposition 4.5. *Let X be as in* (4.1) *with $L = \{L(A) : A \in \mathcal{R}\}$ a Lévy basis. Suppose that $\mathcal{L}(L) \in \mathbb{L}_m(\mathbb{R}^{\mathcal{R}})$, then the law of X belongs to $\mathbb{L}_m(\mathbb{R}^U)$.*

Proof. We will only check the case $m = 0$, the general case follows by induction. Fix $q > 1$ and suppose that $\mathcal{L}(L) \in \mathbb{L}_0(\mathbb{R}^{\mathcal{R}})$. Firstly, let us observe that in this case for any non-negative measurable function $h\colon \mathcal{S} \times \mathbb{R} \to \mathbb{R}$, we have

$$\int_{\mathcal{S}} \int_{\mathbb{R}} h(q^{-1}x, s)\rho(s, dx)c(ds) \leq \int_{\mathcal{S}} \int_{\mathbb{R}} h(x, s)\rho(s, dx)c(ds). \tag{4.4}$$

Indeed, since $\mathcal{L}(L) \in \mathbb{L}_0(\mathbb{R}^{\mathcal{R}})$, it follows that $\mathcal{L}(L(A)) \in \mathbb{L}_0(\mathbb{R})$ for all $A \in \mathcal{R}$. Therefore, $\nu_A(qB) \leq \nu_A(B)$ for any $A \in \mathcal{R}$ and $B \in \mathcal{B}(\mathbb{R})$, where $\nu_A(\,\cdot\,)$ is the Lévy measure of $L(A)$. But in view of $\nu_A(B) = \int_{\mathcal{S}} \int_{\mathbb{R}^d} \mathbb{1}_{A\times B}(x, s)\rho(s, dx)c(ds)$, (4.4) holds for every function of the form $\mathbb{1}_{A\times B}$ with $A \in \mathcal{R}$ and $B \in \mathcal{B}(\mathbb{R})$. The general case follows by the Functional Monotone Class Theorem.

Thanks to Proposition 3.8 and Remark 3.9, in order to show that $\mathcal{L}(X) \in \mathbb{L}_m(\mathbb{R}^U)$ it is enough to check that (3.5) holds for some pseudo master Lévy measure of X (its existence is guaranteed by Theorem 3.4). Let ν be as in Proposition 4.1, then ν is a pseudo master Lévy measure of X and

$$\begin{aligned} \nu(qA) &= \int_{\mathcal{S}} \int_{\mathbb{R}} \mathbb{1}_{qA}[g(x, s)]\rho(s, dx)c(ds) \\ &= \int_{\mathcal{S}} \int_{\mathbb{R}} \mathbb{1}_{A}[q^{-1}xf(\,\cdot\,, s)]\rho(s, dx)c(ds) \\ &= \int_{\mathcal{S}} \int_{\mathbb{R}} \mathbb{1}_{A}[g(q^{-1}x, s)]\rho(s, dx)c(ds) \\ &\leq \int_{\mathcal{S}} \int_{\mathbb{R}} \mathbb{1}_{A}[g(x, s)]\rho(s, dx)c(ds) \\ &= \nu(A), \qquad A \in \mathcal{B}(\mathbb{R})^U, \end{aligned}$$

where we used (4.4). Thus, $\mathcal{L}(X) \in \mathbb{L}_0(\mathbb{R}^U)$. $\square$

4.3 Identification problem for ID Volterra fields and selfdecomposability

As we showed in the previous subsection, in general, if the Lévy basis is SD, the associated Volterra process is also SD. In this part we give sufficient conditions for which the converse holds. Based on Sauri (2014), we show that such conditions can be checked easily for the class of stationary Volterra ID fields.

Let X be as in (4.1) with L a Poissonian Lévy basis with characteristic quadruplet $(\gamma(s), 0, \rho(s, dx), c(ds))$. Recall that $p \geq 0$ is such that $\mathbb{E}(|L(A)|^p) < \infty$ for all $A \in \mathcal{R}$. Define $S_{\Phi_p}(f) := \overline{\text{span}}\{f(u, \cdot)\}_{u\in U}$ in $\mathcal{L}_{\Phi_p}$ and $S_p(X) := \overline{\text{span}}\{X_u\}_{u\in U}$ in $\mathcal{L}^p(\Omega)$. In order to present the main theorem of this section we need to introduce the following condition:

Condition 4.6. For any $A \in \mathcal{R}$, we have $\mathbb{1}_A \in S_{\Phi_p}(f)$ (or equivalently $S_{\Phi_p}(f) = \mathcal{L}_{\Phi_p}$).

Remark 4.7. Note that when $p = 2$ (i.e. L is square-integrable) and L is centered and homogeneous, Condition 4.6 is equivalent to $S_{\Phi_2}(f) = \mathcal{L}^2(\mathbb{R}, ds)$.

Theorem 4.8. *Let X be as in (4.1) with $L = \{L(A) : A \in \mathcal{R}\}$ a Lévy basis with characteristic quadruplet $(\gamma(s), b(s), \rho(s, dx), c(ds))$. Suppose that $\mathcal{L}(L) \in \mathbb{L}_m(\mathbb{R}^{\mathcal{R}})$, then the law of X belongs to $\mathbb{L}_m(\mathbb{R}^U)$. Conversely, assume that Condition 4.6 holds. Then $\mathcal{L}(X) \in \mathbb{L}_m(\mathbb{R}^U)$ implies that $\mathcal{L}(L) \in \mathbb{L}_m(\mathbb{R}^{\mathcal{R}})$.*

For the proof of this theorem we need some auxiliary results:

Lemma 4.9. *Suppose that $X \sim \mu$ with $\mu \in \mathbb{L}_m(\mathbb{R}^d)$. Then for any linear transformation $T\colon \mathbb{R}^d \to \mathbb{R}^k$, the law of $T(X)$ is in $\mathbb{L}_m(\mathbb{R}^k)$.*

Proof. The proof is straightforward thus omitted. □

Proposition 4.10. *Assume that $\mathbb{1}_A \in S_{\Phi_p}(f)$ for some $A \in \mathcal{R}$. Then $L(A) \in S_p(X)$. Conversely, if L is symmetric (or centered) and $L(A) \in S_p(X)$ for some $A \in \mathcal{R}$, then $\mathbb{1}_A \in S_{\Phi_p}(f)$.*

Proof. The result follows from the continuity of the mapping

$$(f \in \mathcal{L}_{\Phi_p}) \longmapsto \Big(\int_S f(s)L(ds) \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})\Big)$$

and the fact that when L is symmetric or centered such mapping is in fact an isomorphism. □

Proof Theorem 4.8. We will only check the case $m = 0$, the general case follows by induction. The first part was already proved in Proposition 4.5. Suppose that $\mathcal{L}(X) \in \mathbb{L}_0(\mathbb{R}^U)$ and Condition 4.6 holds. Let $A_1, \ldots, A_k \in \mathcal{R}$. From Condition 4.6 and Proposition 4.10, for any $j = 1, \ldots k$ there are, $\theta_j^n := (\theta_i^j)_{i=1}^n \in \mathbb{R}^n$ and $\widehat{u}_j^n := (u_i^j)_{i=1}^n \subset U^n$ with $n \in \mathbb{N}$, such that $\langle \theta_j^n, X_{\widehat{u}_j^n} \rangle \xrightarrow{\mathcal{L}^p(\Omega)} L(A_j)$ for any $j = 1, \ldots k$. Putting $u^n := \bigcup_{j=1}^k \widehat{u}_j^n$ we have that there exists $M(\theta)$, a $k \times \#u^n$ matrix only depending on θ_j^n for $j = 1, \ldots k$, such that

$$M(\theta)X_{u^n} = (\langle \theta_j^n, X_{\widehat{u}_j^n} \rangle)_{j=1}^k \xrightarrow{\mathcal{L}^p(\Omega)} (L(A_j))_{j=1}^k \qquad \text{as } n \to \infty. \tag{4.5}$$

From Lemma (4.9) and the fact that $\mathcal{L}(X) \in \mathbb{L}_0(\mathbb{R}^U)$, we have $\mathcal{L}[M(\theta)X_{u^n}] \in \mathbb{L}_0(\mathbb{R}^k)$ for any $n \in \mathbb{N}$. The closedness of $\mathbb{L}_0(\mathbb{R}^k)$ under weak limits guarantees that the weak limit of $\mathcal{L}[M(\theta)X_{u^n}]$ belongs to $\mathbb{L}_0(\mathbb{R}^k)$, or in other words $\mathcal{L}((L(A_j))_{j=1}^k) \in \mathbb{L}_0(\mathbb{R}^k)$, the selfdecomposability of L. □

Remark 4.11. In view of Proposition 3.8 and equation (4.5), Condition 4.6 allows us to determine the Lévy basis through the process X by linear approximations, i.e. under this assumption for some $p \geq 0$

$$\overline{\text{span}}\{X_u\}_{u\in U} = \overline{\text{span}}\{L(A)\}_{A\in\mathcal{R}} \qquad \text{in } \mathcal{L}^p(\Omega). \tag{4.6}$$

Thus, this can be considered as an identification condition also discussed in more depth in Sauri (2014).

Due to Sauri (2014), in the stationary case, Condition 4.6 can be easily checked as the following theorem shows:

Theorem 4.12. *Let L be an homogeneous Lévy basis on $\mathcal{B}_b(\mathbb{R}^d)$ and $g \in \mathcal{L}^1(\mathbb{R}^d, ds) \cap \mathcal{L}_{\Phi_0}$ having non-vanishing Fourier transform. Then the law of the ID Volterra field*

$$X_u := \int_{\mathbb{R}^d} g(u-s)L(ds), \qquad u \in \mathbb{R}^d,$$

belongs to $\mathbb{L}_m(\mathbb{R}^{\mathbb{R}^d})$ if and only if $\mathcal{L}(L) \in \mathbb{L}_m(\mathbb{R}^{\mathcal{B}_b(\mathbb{R}^d)})$.

Proof. From Theorem 13 in Sauri (2014), we have that in this case $S_{\Phi_0}(g) = \mathcal{L}_{\Phi_0}$, which implies that Condition 4.6 is fulfilled. The result follows from this and the previous theorem. □

Example 4.13 (Ornstein-Uhlenbeck processes)**.** Suppose that L is a Lévy process with characteristic triplet (γ, b, ρ). Let

$$f(u,s) = \varphi_0(u-s) := e^{-(u-s)}\mathbb{1}_{\{s\leq u\}}, \qquad u, s \in \mathbb{R}.$$

The resulting ID Volterra field is the classic OU process driven by L. It is well known that such processes are well defined if and only if $\int_{|x|>1}\log(|x|)\rho(dx) < \infty$. Moreover, in this case $\mathcal{L}(X_u) \in \mathbb{L}_0(\mathbb{R})$ for all $u \in \mathbb{R}$ and it is uniquely determined by L and vice versa. But, since $\widehat{\varphi_0}$, the Fourier transform of φ_0, never vanishes, we conclude that an OU process is SD if and only if the background Lévy process is SD as well, just as in (Barndorff-Nielsen et al., 2006a, Theorem 3.4).

Example 4.14 (LSS process with a Gamma kernel)**.** Take L to be a Lévy process with characteristic triplet (γ, b, ρ). Let $\alpha > -1$ and consider

$$f(u,s) = \varphi_\alpha(u-s) := e^{-(u-s)}(u-s)^\alpha\mathbb{1}_{\{s\leq u\}}, \qquad u, s \in \mathbb{R}. \tag{4.7}$$

It has been shown in Basse-O'Connor (2014), that $f(u, \cdot\,) \in \mathcal{L}_{\Phi_0}$ for every (equivalently for some) $u \in \mathbb{R}$ if and only if the following two conditions are satisfied:

1. $\int_{|x|>1}\log(|x|)\rho(dx) < \infty$,

2. One of the following conditions holds:
   (a) $\alpha > -1/2$;
   (b) $\alpha = -1/2$, $b = 0$ and $\int_{|x|\leq 1} |x|^2 |\log(|x|)| \rho(dx) < \infty$;
   (c) $\alpha \in (-1, -1/2)$, $b = 0$ and $\int_{|x|\leq 1} |x|^{-1/\alpha} \rho(dx) < \infty$.

Moreover, according to Pedersen and Sauri (2014), for any $-1 < \alpha < 0$, $\mathcal{L}(X_u) \in \mathbb{L}_0(\mathbb{R})$ for all $u \in \mathbb{R}$. However, since the Fourier transform of φ_α is given by

$$\widehat{\varphi_\alpha}(\xi) = \frac{\Gamma(\alpha+1)}{\sqrt{2\pi}}(1 - i\xi)^{-\alpha-1}, \qquad \xi \in \mathbb{R},$$

the law of the Lévy semistationary process $X_u = \int_{-\infty}^{u} e^{-(u-s)}(u-s)^\alpha dL_s$ is in $\mathbb{L}_0(\mathbb{R}^U)$ if and only if L is selfdecomposable.

Example 4.15 (Fractional Lévy motions)**.** Suppose that L is a centered and square-integrable Lévy process with characteristic triplet (γ, b, ρ). For $\alpha \in (0, 1/2)$ consider

$$f(u,s) = (u-s)_+^\alpha - (-s)_+^\alpha, \qquad u, s \in \mathbb{R},$$

where $(x)_+$ denotes the positive part of x. In Cohen and Maejima (2011), it was shown that $\{f(u, \cdot)\}_{u\in U}$ is total in $\mathcal{L}^2(ds)$, which according to Remark 4.7, implies Condition 4.6. Furthermore, the authors also noted that in general the marginal distribution of the Volterra process induced by this function is not selfdecomposable unless L is SD, as Theorem 4.8 shows.

5 Integrated ID Volterra fields

In this section we are interested in the random variable

$$\mu(X; A) = \int_A X_u \mu(du), \qquad A \in \mathcal{B}_b(\mu), \tag{5.1}$$

where X is an ID Volterra field, μ a σ-finite measure and $\mathcal{B}_b(\mu) := \{A : \mu(A) < \infty\}$. We will consider the following associated field

$$X^\mu = (\mu(X; A))_{A\in\mathcal{B}_b(\mu)}, \tag{5.2}$$

We start by giving sufficient conditions for $\mu(X; A)$ to exists.

5.1 Existence of $\mu(X; A)$

In this part we present sufficient conditions for which $\mu(X; A)$ as in (5.1) exists. To do this we use the Stochastic Fubini Theorem presented in Barndorff-Nielsen and Basse-O'Connor (2011).

Let $(U, \mathcal{B}(U), \mu)$ be a measurable space, where U is a Polish space, i.e. a complete and separable metric space, $\mathcal{B}(U)$ its Borel σ-algebra and μ a σ-finite measure. Note that defining $\mu(X; A)$ involves two issues. Firstly, we must to verify that the process

X has at least a measurable modification with respect to $\mathcal{F} \otimes \mathcal{B}(U)$. The second consists in providing sufficient conditions which guarantee that $X \in \mathcal{L}^1(U, \mathcal{B}(U), \mu)$. In particular, for ID Volterra fields, it would be desirable to relate such conditions directly to the kernel. Let X be as in (4.1). In this case, it has been shown in Barndorff-Nielsen and Basse-O'Connor (2011) that X always admits a measurable modification. Furthermore, the following Stochastic Fubini Theorem for Lévy bases provides sufficient conditions for $X \in \mathcal{L}^1(U, \mathcal{B}(U), \mu)$.

Theorem 5.1 (Stochastic Fubini Theorem (Barndorff-Nielsen and Basse-O'Connor, 2011, Theorem 3.1))**.** *Let L be a centered Lévy basis with characteristic triplet $(\gamma(s), b(s), \rho(s, dx), c(ds))$. Consider $f\colon U \times \mathcal{S} \to \mathbb{R}$ be a $\mathcal{B}(U)/\mathcal{B}_{\mathcal{S}}$-measurable function such that $f(u, \cdot\,) \in \mathcal{L}_{\Phi_0}$ for all $u \in U$. Assume that for $A \in \mathcal{B}(U)$*

$$\int_A \|f(u, \cdot\,)\|_{\Phi_1} \mu(du) < \infty. \tag{5.3}$$

where $\|\cdot\|_{\Phi_1}$ is as in (2.8)*. Then $f(\,\cdot\,, s) \in \mathcal{L}^1(U, \mathcal{B}(U), \mu)$ for c-almost every $s \in \mathcal{S}$ and the mapping $s \longmapsto \int_A f(u, s)\mu(du)$ belongs to $\mathcal{L}_{\Phi_1}$. In this case, all the integrals below exist and almost surely*

$$\int_A \left[\int_{\mathcal{S}} f(u, s) L(ds)\right] \mu(du) = \int_{\mathcal{S}} \left[\int_A f(u, s)\mu(du)\right] L(ds).$$

Moreover, if μ is finite, (5.3) *is equivalent to*

$$\int_A \int_{\mathcal{S}} \Big[f^2(u, s)b^2(s) + \int_{\mathbb{R}} |xf(u, s)| \wedge |xf(u, s)|^2\Big] c(ds)\mu(du) < \infty. \tag{5.4}$$

In spirit of the previous theorem, for the rest of this section L will be assumed to be centered.

Remark 5.2. Note that in the stationary case, i.e. when L is homogeneous and $f(u, s) = g(u - s)$, with $g \in \mathcal{L}_{\Phi_0}$, (5.3) holds if and only if $\mu(A) < \infty$ and $g \in \mathcal{L}_{\Phi_1}$. Indeed, this follows from the fact that in this case

$$\|f(u, \cdot\,)\|_{\Phi_1} = \|g\|_{\Phi_1} \qquad \text{for all } u \in U. \tag{5.5}$$

Using the previous theorem, it is easy to check the validity of the next proposition:

Proposition 5.3. *Assume that* (5.3) *holds. Then the random variable $\mu(X; A)$ in* (5.1) *is well defined and it is infinitely divisible.*

5.2 Selfdecomposability of X^μ

In this part we study the selfdecomposability of the fields X^μ defined in (5.2).

Let $(U, \mathcal{B}(U), \mu)$ be a measurable space as in the previous subsection. From Theorem 5.1, if (5.3) holds for all $A \in \mathcal{B}_b(\mu)$, the random field $X^\mu = (\mu(X; A))_{A \in \mathcal{B}_b(\mu)}$ is well defined and it admits the following representation

$$\mu(X; A) = \int_{\mathcal{S}} \mu_f(A, s) L(ds), \qquad A \in \mathcal{B}_b(\mu),$$

where

$$\mu_f(A,s) := \int_A f(u,s)\mu(du), \qquad A \in \mathcal{B}_b(\mu), s \in \mathcal{S}.$$

In addition, X^μ is an ID field with system of characteristic triplets given as in Proposition (4.2). However, since the indexing set of the field is not separable in general, we can only argue that the measure given in (4.2) is a pseudo master Lévy measure of X^μ.

Due to Theorem 4.8, if $\overline{\text{span}}(\{\mu_f(A,\cdot)\}_{A\in\mathcal{B}_b(\mu)}) = \mathcal{L}_{\Phi_1}$, we have that the law of Y_μ is in $\mathbb{L}_m(\mathbb{R}^{\mathcal{B}_b(\mu)})$ if and only if $\mathcal{L}(L) \in \mathbb{L}_m(\mathbb{R}^{\mathcal{R}})$. Here a natural question appears, is the selfdecomposability of X (or $\overline{\text{span}}(\{f(u,\cdot)\}_{u\in U}) = \mathcal{L}_{\Phi_1}$) necessary and sufficient for the one on X^μ?. In the stationary case the answer is affirmative as the following theorem shows:

Theorem 5.4. *Let L be an homogeneous centered Lévy basis on $\mathcal{B}_b(\mathbb{R}^d)$ and $g \in \mathcal{L}^1(\mathbb{R}^d, ds) \cap \mathcal{L}_{\Phi_1}$ having non-vanishing Fourier transform. Assume that μ is a finite measure such that $\mu \sim Leb^d$. Then, the law of the integrated process*

$$\mu(X;A) = \int_A X_u \mu(du), \qquad A \in \mathcal{B}_b(U),$$

where

$$X_u := \int_{\mathbb{R}^d} g(u-s)L(ds), \qquad u \in \mathbb{R}^d,$$

belongs to $\mathbb{L}_m(\mathbb{R}^{\mathcal{B}_b(\mu)})$ if and only if $\mathcal{L}(X) \in \mathbb{L}_m(\mathbb{R}^{\mathbb{R}^d})$ or equivalently $\mathcal{L}(L) \in \mathbb{L}_m(\mathbb{R}^{\mathcal{R}})$.

Proof. From the discussion above, we only need to check that

$$\overline{\text{span}}(\{\mu_f(A,\cdot)\}_{A\in\mathcal{B}_b(\mu)}) = \mathcal{L}_{\Phi_1}.$$

Suppose the opposite, this is (thanks to the Hahn-Banach Theorem), there exists h a non-zero measurable function in the dual of $\mathcal{L}_{\Phi_1}$ such that (see Rao and Ren (1994) or Corollary 3 in Sauri (2014))

$$\int_{\mathbb{R}^d} \mu_f(A,s)h(s)ds = 0, \qquad \text{for all } A \in \mathcal{B}_b(\mathbb{R}^d). \tag{5.6}$$

Since μ is finite, $\mathcal{B}_b(\mu) = \mathcal{B}(\mathbb{R}^d)$. Moreover, due to (5.3) and $g \in \mathcal{L}^1(\mathbb{R}^d, ds) \cap \mathcal{L}_{\Phi_1}$, we have that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \mu_f(A,s)h(s)\,ds \\ &= \int_{\mathbb{R}^d}\int_A g(u-s)h(s)\,\mu(du)ds \\ &= \int_A\int_{\mathbb{R}^d} g(u-s)h(s)\,ds\mu(du) \qquad \text{for all } A \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^d} g(u-s)h(s)ds = 0 \qquad \text{for } \mu\text{-almost all } u \in U.$$

But $\mu \sim Leb^d$, consequently the previous equation holds for almost all $u \in U$. Therefore, from the proof of Theorem 13 in Sauri (2014), we obtain that $h = 0$, a contradiction. □

Remark 5.5. Note that in the non-stationary case, we are able to show that equation (5.6) implies that $\mu(\{u \in U : \int_{\mathbb{R}^d} f(u,s)h(s)c(ds) = 0\}) = 0$. However, in general it is not possible to verify from this that

$$\int_{\mathbb{R}^d} f(u,s)h(s)c(ds) = 0 \qquad \text{for all } u \in U,$$

which under Condition 4.6 occurs if and only if $h = 0$.

Example 5.6 (Ornstein-Uhlenbeck processes)**.** Let L be a centered Lévy process with characteristic triplet (γ, b, ρ), f as in Example 4.13 and μ being a σ-finite measure on $\mathbb{R}$. Then (5.3) holds for every $A \in \mathcal{B}_b(\mu)$. Indeed, from Remark 5.2 we only need to verify that $f \in \mathcal{L}_{\Phi_1}$. This occurs (see Section 1) if and only if

$$\int_{\mathbb{R}} \int_0^\infty \left[|xe^{-s}| \wedge |xe^{-s}|^2 \right] ds\rho(dx) < \infty.$$

We have that

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty \left[|xe^{-s}| \wedge |xe^{-s}|^2 \right] ds\rho(dx) \\
&\quad = \frac{1}{2} \int_{|x|\leq 1} |x|^2 \rho(dx) + \int_{|x|>1} \int_0^{\log(|x|)} |xe^{-s}| ds\rho(dx) \\
&\qquad + \int_{|x|>1} \int_{\log(|x|)}^\infty |xe^{-s}|^2 ds\rho(dx) \\
&\quad = \frac{1}{2} \int_{\mathbb{R}} 1 \wedge |x|^2 \rho(dx) + \int_{|x|>1} [\frac{|x|^2 - 1}{|x|}] \rho(dx) < \infty,
\end{aligned}$$

due to the fact that $\int_{|x|>1} |x|\rho(dx) < \infty$ (because L has first moment). Therefore, the integrated process

$$X^\mu(A) = \int_A X_u \mu(du), \qquad A \in \mathcal{B}_b(\mu),$$

is well defined. In particular, if $A = [0,t]$ and $\mu(du) = du$, we get that

$$\begin{aligned}
X_t^\mu &:= X^\mu([0,t]) \\
&= \int_0^t X_u \, du \\
&= L_t - (X_t - X_0), \qquad t \geq 0,
\end{aligned}$$

the Langevin equation. Since $(X_t - X_0)$ is independent of L_t, $(X_t^\mu)_{t\geq 0}$ is SD if and only if X is SD or L is SD. Note that this result is true in general for any Lévy process for which $\int_{|x|>1} \log(|x|)\rho(dx) < \infty$, which means that the condition on L in Theorem 5.4 is sufficient but not necessary.

Example 5.7 (LSS process with a Gamma kernel)**.** Let L and μ be as in the previous example. Consider f as in Example 4.14. We want to check that (5.3) holds for any μ-bounded set. To do this, we observe that $\varphi_\alpha \in \mathcal{L}_{\Phi_0}$ if and only if $\varphi_\alpha \in \mathcal{L}_{\Phi_1}$, where φ_α is as in (4.7). Obviously if $\varphi_\alpha \in \mathcal{L}_{\Phi_1}$ we have that $\varphi_\alpha \in \mathcal{L}_{\Phi_0}$, so suppose that $\varphi_\alpha \in \mathcal{L}_{\Phi_0}$. Then, from Example 4.14 necessarily the following two conditions are satisfied:

1. $\int_{|x|>1} \log(|x|)\rho(dx) < \infty$,
2. One of the following conditions holds:
   (a) $\alpha > -1/2$;
   (b) $\alpha = -1/2$, $b = 0$ and $\int_{|x|\le 1} |x|^2 |\log(|x|)|\rho(dx) < \infty$;
   (c) $\alpha \in (-1, -1/2)$, $b = 0$ and $\int_{|x|\le 1} |x|^{-1/\alpha}\rho(dx) < \infty$.

Since $|\varphi_\alpha| \le c_1\phi_\alpha$, where

$$\phi_\alpha(s) := \begin{cases} s^\alpha \mathbb{1}_{\{0<s\le 1\}} + e^{-s}\mathbb{1}_{\{s>1\}} & \text{for } -1<\alpha<0; \\ e^{-s}\mathbb{1}_{\{s\ge 0\}} & \text{for } \alpha \ge 0, \end{cases}$$

we only need to check that in this case $\phi_\alpha \in \mathcal{L}_{\Phi_1}$. If $\alpha > 0$, from the previous example we obtain immediately that $\phi_\alpha \in \mathcal{L}_{\Phi_1}$. Assume that $\alpha \in (-1, 0)$. Obviously $b^2 \int_0^\infty \phi_\alpha^2(s)ds < \infty$, so it suffices to show that $\int_{\mathbb{R}} \int_0^\infty \left[|x\phi_\alpha(s)| \wedge |x\phi_\alpha(s)|^2\right] ds\rho(dx) < \infty$. From the previous example

$$\int_{\mathbb{R}} \int_1^\infty \left[|x\phi_\alpha(s)| \wedge |x\phi_\alpha(s)|^2\right] ds\rho(dx) \le \int_{\mathbb{R}} \int_0^\infty \left[|xe^{-s}| \wedge |xe^{-s}|^2\right] ds\rho(dx) < \infty.$$

Moreover

$$\begin{aligned} &\int_{\mathbb{R}} \int_0^1 \left[|x\phi_\alpha(s)| \wedge |x\phi_\alpha(s)|^2\right] ds\rho(dx) \\ &= \frac{1}{\alpha+1} \int_{|x|>1} |x|\rho(dx) + \int_{|x|\le 1} \int_0^{|x|^{-\frac{1}{\alpha}}} |xs^\alpha| ds\rho(dx) \\ &\quad + \int_{|x|\le 1} \int_{|x|^{-\frac{1}{\alpha}}}^1 |xs^\alpha|^2 ds\rho(dx) \\ &= \frac{1}{\alpha+1} \int_{\mathbb{R}} |x|^{-\frac{1}{\alpha}} \wedge |x|\rho(dx) + \frac{1}{2\alpha+1} \int_{|x|\le 1} (|x|^2 - |x|^{-\frac{1}{\alpha}})\rho(dx) < \infty, \end{aligned}$$

due to the conditions 1. and 2. Thus $\phi_\alpha \in \mathcal{L}_{\Phi_1}$.

All above implies that in this case, the integrated process X^μ is well defined for any $\alpha > -1$. Now, for $\beta > -1$ consider the finite measure

$$\mu(du) := \varphi_\beta(u)\mathbb{1}_{\{u\ge 0\}}du,$$

and consider the following integrated process

$$\begin{aligned} X_t^\mu &:= \int_0^\infty X_{t-u}\mu(du) \\ &= \int_{-\infty}^t \varphi_\beta(t-u)X_u du, \qquad t \in \mathbb{R}. \end{aligned}$$

From what we have shown above, we see that X_t^μ is well defined and for each $t \in \mathbb{R}$ almost surely

$$X_t^\mu = \int_{-\infty}^t e^{-(t-s)} \int_s^t (t-u)^\beta (u-s)^\alpha du dL_s.$$

Since

$$\int_s^t (t-u)^\beta (u-s)^\alpha du = k_{\alpha,\beta}(t-s)^{\beta+\alpha+1}, \qquad t > s,$$

where $k_{\alpha,\beta} := \int_0^1 x^\alpha (1-x)^\beta dx < \infty$, we have that for $-1 < \alpha < 0$, $\beta = -\alpha - 1$ and $t \in \mathbb{R}$, almost surely

$$X_t^\mu = k_\alpha \int_{-\infty}^t e^{-(t-s)} dL_s,$$

with $k_\alpha = k_{\alpha,-\alpha-1}$, i.e. X^μ is an OU process. Here we see immediately that Theorem 5.4 holds. Let us remark, that the technique of Gamma convolutions has been first used in Barndorff-Nielsen et al. (2013b) and it has also been applied in Sauri (2014).

6 ID field-valued processes

In this section we build Lévy processes whose realizations are ID fields. We propose a way to define stochastic integrals with respect to such processes and in particular we show that any SD field can be expressed as a stochastic integral with respect to an element of this class of Lévy processes.

6.1 ID field-valued Lévy processes

In this part we construct a process which has independent and stationary increments taking values in the space of random fields. Thus, in analogy with Lévy processes in $\mathbb{R}^d$, these will be called ID field-valued Lévy processes.

Let $X = (X_u)_{u \in U}$ be an ID field with characteristic triplet (Γ, B, ν) and suppose that ν does not charge zero. For any $\widehat{u} \in \widehat{U}$ the law of $X_{\widehat{u}}$ belongs to $\mathrm{ID}(\mathbb{R}^{\widehat{u}})$ and has characteristic triplet $(\pi_{\widehat{u}}(\Gamma), B_{\widehat{u}}, \nu \circ \pi_{\widehat{u}}^{-1})$ with $B_{\widehat{u}} = (B(u,v))_{u,v \in \widehat{u}}$. Consider $L_{\widehat{u}}^X$ to be a two-sided Lévy process in $\mathbb{R}^{\widehat{u}}$, such that $L_{\widehat{u}}^X(1) \overset{d}{=} X_{\widehat{u}}$. The cumulant function of $L_{\widehat{u}}^X$ is given by

$$\begin{aligned} \mathrm{C}\{\theta \ddagger L_{\widehat{u}}^X(t)\} &= |t| \mathrm{C}\{\theta \ddagger X_{\widehat{u}}\} \qquad (6.1) \\ &= i\langle \theta, \Gamma_{\widehat{u}}^t \rangle - \tfrac{1}{2}\langle \theta, B_{\widehat{u}}^t \theta \rangle + \int_{\mathbb{R}^{\widehat{u}}} \left[e^{i\langle \theta, x \rangle} - 1 - i\langle \tau_{\#\widehat{u}}(x), \theta \rangle \right] \nu_{\widehat{u}}^t(dx), \end{aligned}$$

where $\theta \in \mathbb{R}^{\widehat{u}}, t \in \mathbb{R}$, and

$$\begin{aligned} \Gamma_{\widehat{u}}^t &= |t| \pi_{\widehat{u}}(\Gamma); \\ B_{\widehat{u}}^t &= |t| B_{\widehat{u}}; \\ \nu_{\widehat{u}}^t &= |t| \nu \circ \pi_{\widehat{u}}^{-1}. \end{aligned}$$

Since the system $(\pi_{\widehat{u}}(\Gamma), B_{\widehat{u}}, \nu \circ \pi_{\widehat{u}}^{-1})$ is consistent, we have that for any $t \in \mathbb{R}$ the system $(\Gamma_{\widehat{u}}^t, B_{\widehat{u}}^t, \nu_{\widehat{u}}^t)$ is consistent as well, thus from Theorem 3.4 there exists a unique (in law) ID field $L^X(t)$ with characteristic triplet $(|t|\Gamma, |t|B, |t|\nu)$. The process $L^X = (L^X(t))_{t \in \mathbb{R}}$ will be called *ID field-valued Lévy process*. The motivation of this name is given in the Proposition 6.1 below. Before we present such result, we need to introduce some notation. For any non empty index set U, let

$$(\mathbb{R}^U)' := \{y \in \mathbb{R}^U : y(u) = 0 \text{ for all but finitely many } u \in U\}. \tag{6.2}$$

Define the pseudo bilinear form $\langle \cdot\,, \cdot \rangle_{\mathbb{R}^U} : \mathbb{R}^U \times (\mathbb{R}^U)' \to \mathbb{R}$ as

$$\langle x, y\rangle_{\mathbb{R}^U} := \sum_{u\in U} x(u)y(u).$$

Note that $\langle \cdot\,, \cdot \rangle_{\mathbb{R}^U}$ is well defined. In particular, if U is countable, $\langle \cdot\,, \cdot \rangle_{\mathbb{R}^U}$ is the restriction of the inner product in l^2 to $(\mathbb{R}^U)'$.

Proposition 6.1. *The process $L^X = (L^X(t))_{t\in\mathbb{R}}$ has independent and stationary increments and the process*

$$\widetilde{L}^X_{\cdot}(t) := \pi_{\cdot}(t\Gamma) + W_{\cdot}(t) + \int_{\mathbb{R}^U} \pi_{\cdot}(x) \left[N(dx, ds) - \mathbb{1}_{\{|\pi_{\cdot}(x)|\le 1\}}\widetilde{\nu}(dx, ds)\right], \quad t \in \mathbb{R}, \tag{6.3}$$

is a version of L^X. Here $W_{\cdot}(t)$ is the Gaussian process with covariance matrix $|t|(B(u, v))_{u,v\in U}$ and $N(dx, ds)$ is a Poisson measure independent of $W_{\cdot}(t)$ with intensity $\widetilde{\nu}(dx, ds) = \nu(dx)ds$. Moreover, we have that $\lim_{t\to s}\langle L^X(t) - L^X(s), y\rangle_{\mathbb{R}^U}$ exists almost surely and

$$\mathbb{P}\text{-}\lim_{t\to s}\langle L^X(t) - L^X(s), y\rangle_{\mathbb{R}^U} = 0, \qquad \textit{for all } y \in (\mathbb{R}^U)'. \tag{6.4}$$

Proof. By construction for any $\widehat{u} \in \widehat{U}$, $L^X_{\widehat{u}}(\cdot)$ is a Lévy process in law in $\mathbb{R}^{\widehat{u}}$. Therefore $L^X_{\widehat{u}}(\cdot)$ has independent and stationary increments for any $\widehat{u} \in \widehat{U}$, so L^X does as well. Since $L^X_{\widehat{u}}(\cdot)$ has independent and stationary increments and $\widetilde{L}^X_{\cdot}(1) \overset{d}{=} X \overset{d}{=} L^X_{\cdot}(1)$, we have that L^X and $\widetilde{L}^X$ have the same law. Finally, since for any $y \in (\mathbb{R}^U)'$, there exists $\widehat{u} \in \widehat{U}$ such that $y(u) = 0$ for all $u \in \widehat{u}^c$, we deduce

$$\begin{aligned}\langle L^X(t) - L^X(s), y\rangle_{\mathbb{R}^U} &= \sum_{u\in\widehat{u}}(L^X_u(t) - L^X_u(s))y(u)\\ &= \langle L^X_{\widehat{u}}(t) - L^X_{\widehat{u}}(s), y_{\widehat{u}}\rangle_{\mathbb{R}^{\widehat{u}}},\end{aligned}$$

where $\langle \cdot\,, \cdot \rangle_{\mathbb{R}^{\widehat{u}}}$ denotes the inner product in $\mathbb{R}^{\widehat{u}}$. This implies necessarily that the limit $\lim_{t\to s}\langle L^X(t) - L^X(s), y\rangle_{\mathbb{R}^U}$ exists and that (6.4) holds. □

Remark 6.2. Observe that in general, the concept of càdlàg paths cannot be defined for L^X. It is because $\mathbb{R}^U$ is not in general metric and the topologies that can be defined in such space are not tractable. Therefore, the object $\lim_{t\to s} L^X(t)$ may not be well defined. However, if $U \subset \mathbb{R}^d$ and for any $t \in \mathbb{R}$, $L^X(t) \in \mathcal{L}^2(\mathbb{R})$, (6.4) is equivalent to continuity in probability under the $\mathcal{L}^2$-norm. It is an open problem to verify that if $L^X(t) \in D(U, \mathbb{R})$ (the Skorohod space) for every t and (6.4) holds, then L^X is càdlàg under the norm in $D(U, \mathbb{R})$.

6.2 Integration with respect to ID field-valued Lévy processes

In this part, using as a starting point the stochastic integration of deterministic functions with respect to Lévy bases on $\mathbb{R}^d$ (see Rajput and Rosiński (1989) or Section 2.3 and for the $\mathbb{R}^d$-valued case see Barndorff-Nielsen and Stelzer (2011)), we define the stochastic integral of an operator from $\mathbb{R}^U$ into itself with respect to an ID field-valued Lévy process.

Let $X = (X_u)_{u\in U}$ be an ID field with characteristic triplet (Γ, B, ν) and suppose that ν does not charge zero. Consider L^X to be the Lévy process ID field-valued constructed in the previous subsection. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function integrable with respect to $L^X_{\widehat{u}}$, i.e.

$$\int_{\mathbb{R}} \left| f(s)\Gamma_{\widehat{u}} + \int_{\mathbb{R}^{\widehat{u}}} (\tau_{\#\widehat{u}}(f(s)x) - f(s)\tau_{\#\widehat{u}}(x))\nu_{\widehat{u}}(dx) \right| ds < \infty;$$

$$\int_{\mathbb{R}^{\widehat{u}}} f^2(s)dsB_{\widehat{u}} < \infty;$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}^{\widehat{u}}} (1 \wedge |f(s)x|^2)\nu_{\widehat{u}}(dx)ds < \infty.$$

For the sake of brevity, we introduce the notation

$$I_{\widehat{u}}(f \ddagger X) := \int_{\mathbb{R}^{\widehat{u}}} f(s)dL^X_{\widehat{u}}(s), \qquad \widehat{u} \in \widehat{U}.$$

Then $I_{\widehat{u}}(f \ddagger X)$ is ID with characteristic triplet $(\Gamma^{I(f\ddagger X)}_{\widehat{u}}, B^{I(f\ddagger X)}_{\widehat{u}}, \nu^{I(f\ddagger X)}_{\widehat{u}})$ given by

$$\begin{aligned} \Gamma^{I(f\ddagger X)}_{\widehat{u}} &= \int_{\mathbb{R}} \Big[f(s)\Gamma_{\widehat{u}} + \int_{\mathbb{R}^{\widehat{u}}} (\tau_{\#\widehat{u}}(f(s)x) - f(s)\tau_{\#\widehat{u}}(x))\nu_{\widehat{u}}(dx) \Big] ds, \qquad (6.5) \\ B^{I(f\ddagger X)}_{\widehat{u}} &= \int_{\mathbb{R}^{\widehat{u}}} f^2(s)dsB_{\widehat{u}}, \\ \nu^{I(f\ddagger X)}_{\widehat{u}}(A) &= \int_{\mathbb{R}}\int_{\mathbb{R}^{\widehat{u}}} \mathbb{1}_A(f(s)x)\nu_{\widehat{u}}(dx)ds, \qquad A \in \mathcal{B}(\mathbb{R}^{\widehat{u}}). \end{aligned}$$

This procedure generates a system of characteristic triplets $(\Gamma^{I(f\ddagger X)}_{\widehat{u}}, B^{I(f\ddagger X)}_{\widehat{u}}, \nu^{I(f\ddagger X)}_{\widehat{u}})$ which, as the next proposition shows, is consistent.

Proposition 6.3. *The system of characteristic triplets* $(\Gamma^{I(f\ddagger X)}_{\widehat{u}}, B^{I(f\ddagger X)}_{\widehat{u}}, \nu^{I(f\ddagger X)}_{\widehat{u}})$ *is consistent.*

Proof. Let $\widehat{v}, \widehat{u} \in \widehat{U}$ with $\widehat{v} \subset \widehat{u}$. From Proposition 3.1 and (6.5), we only need to check that

$$\nu^{I(f\ddagger X)}_{\widehat{v}} = \nu^{I(f\ddagger X)}_{\widehat{u}} \circ \pi^{-1}_{\widehat{v}\widehat{u}} \qquad \text{on } \mathcal{B}(\mathbb{R}^{\widehat{v}} \setminus 0^{\widehat{v}}).$$

From (6.5), it follows that for any $A \in \mathcal{B}(\mathbb{R}^{\widehat{v}} \setminus 0^{\widehat{v}})$

$$\begin{aligned} \nu^{I(f\ddagger X)}_{\widehat{u}} \circ \pi^{-1}_{\widehat{v}\widehat{u}}(A) &= \int_{\mathbb{R}}\int_{\mathbb{R}^{\widehat{u}}} \mathbb{1}_A(f(s)\pi_{\widehat{v}\widehat{u}}(x))\nu_{\widehat{u}}(dx)ds \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^{\widehat{v}}} \mathbb{1}_A(f(s)x)\nu_{\widehat{v}}(dx)ds \\ &= \nu^{I(f\ddagger X)}_{\widehat{v}}(A), \end{aligned}$$

which is enough. ∎

These results mean that we can lift the system of finite-dimensional triplets $(\Gamma_{\widehat{u}}^{I(f\ddagger X)}, B_{\widehat{u}}^{I(f\ddagger X)}, \nu_{\widehat{u}}^{I(f\ddagger X)})$ to a triplet $(\Gamma^{I(f\ddagger X)}, B^{I(f\ddagger X)}, \nu^{I(f\ddagger X)})$ of an ID field, let's say $I(f \ddagger X)$. In this case, we define the *stochastic integral of f with respect to L^X* to be the ID field given by

$$\int_{\mathbb{R}} f(s)dL^X(s) \coloneqq I(f \ddagger X).$$

Note that, from (6.5),

$$\nu^{I(f\ddagger X)}(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^U} \mathbb{1}_A(f(s)x)\nu(dx)ds, \qquad A \in \mathcal{B}(\mathbb{R}^U).$$

Thus, if ν does not charge zero, $\nu^{I(f\ddagger X)}$ does not either. Hence in this case, $\nu^{I(f\ddagger X)}$ is the master Lévy measure of $I(f \ddagger X)$. Moreover, a modification of $I(f \ddagger X)$ can be obtained by the Lévy-Itô representation for ID fields.

Proposition 6.4. *Let $X = (X_u)_{u\in U}$ be an infinitely divisible field with characteristic triplet (Γ, B, ν) such that ν does not charge zero and let L^X be the ID field-valued Lévy process induced by X. For a given $\widehat{u} \in \widehat{U}$ denote by $\mathcal{L}_{\Phi_0}^{\widehat{u}}$ the Musiela-Orlicz space induced by the triplet of $L_{\widehat{u}}^X$ (see Section 2.3). Consider $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \in \mathcal{L}_{\Phi_0}^{\widehat{u}}$ for every $\widehat{u} \in \widehat{U}$. Then the process*

$$\begin{aligned}\widetilde{I}_u(f \ddagger X) \coloneqq& \int_{\mathbb{R}} f(s)ds\pi_u(\Gamma) + \int_{\mathbb{R}} f(s)W_u(ds)\\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)x\left[N_u(dx, ds) - \mathbb{1}_{\{|f(s)x|\leq 1\}}\nu_u(dx)ds\right],\end{aligned}$$

is a modification of $I(f \ddagger X)$. Here N_u has compensator $\nu_u(dx)ds$.

Proof. By the Lévy-Itô decomposition, almost surely for $t > s$

$$\begin{aligned}L_u^X(t) - L_u^X(s) =& (t - s)\pi_u(\Gamma) + (W_u(t) - W_u(s))\\ &+ \int_s^t \int_{\mathbb{R}^U} x\left[N_u(dx, ds) - \mathbb{1}_{\{|f(s)x|\leq 1\}}\nu_u(dx)ds\right].\end{aligned}$$

Therefore, for any $u \in U$, almost surely $I_u(F \ddagger X) = \int_{\mathbb{R}} f(s)dL_u^X(s) = \widetilde{I}_u(F \ddagger X)$, as required. □

Remark 6.5. The procedure above allows to extend the class of integrands to linear operators as follows: Let $(f_u)_{u\in U}$ be a family of measurable functions. For every $\widehat{u} \in \widehat{U}$, take $F_{\widehat{u}}\colon \mathbb{R} \to \mathbb{M}_{\widehat{u}}(\mathbb{R})$ with $F_{\widehat{u}}(\,\cdot\,) = \operatorname{diag}(f_{\widehat{u}}(\,\cdot\,))$ and $\mathbb{M}_{\widehat{u}}(\mathbb{R})$ denotes the set of $\#\widehat{u} \times \#\widehat{u}$ matrices with real entries. The integral of $F_{\widehat{u}}$ with respect to $L_{\widehat{u}}^X$ (if it exists) can be considered to have a consistent system of characteristic triplets $(\Gamma_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)}, B_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)}, \nu_{\widehat{u}}^{I(F_{\widehat{u}})\ddagger X})$. Moreover, the collection $(F_{\widehat{u}})_{\widehat{u}\in\widehat{U}}$ can be lifted to an indexed linear operator F from $\mathbb{R}^U$ into itself. Therefore, the integral of F with respect to L^X, denoted by $I(F \ddagger X)$, is the ID field with system of characteristic triplets given by $(\Gamma_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)}, B_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)}, \nu_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)})$. However, it is not clear how to extend this procedure to more general linear operators, as the consistency of the system $(\Gamma_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)}, B_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)}, \nu_{\widehat{u}}^{I(F_{\widehat{u}}\ddagger X)})$ may fail in the case of non-diagonal operators.

6.3 Volterra and OU type field-valued processes and selfdecomposability

Following the steps of the previous subsection, in this part we define Volterra type ID field-valued processes, focusing on the OU case.

Let $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a measurable function. Suppose that for each $t\in\mathbb{R}$, $F(t,\,\cdot\,)$ is integrable with respect to L^X as in the previous subsection. Then the process $Y(t) := I[F(t,\,\cdot\,) \ddagger X]$ is well defined as an ID field-valued process and we will refer to it as a *Volterra type ID field-valued process*.

A simple yet important example of the functions $F(t,\,\cdot\,)$ is the one that gives rise to the *Ornstein-Uhlenbeck* ID field-valued process. Namely

$$F(t,s) = \mathbb{1}_{(-\infty,t]}(s)e^{-(t-s)}.$$

It is well known that F is $L^X_{\widehat{u}}$-integrable if and only if $\int_{|x|>1}\log(|x|)\nu_{\widehat{u}}(dx)<\infty$ or equivalently $\int_{|\pi_{\widehat{u}}(x)|>1}\log(|\pi_{\widehat{u}}(x)|)\nu(dx)<\infty$. In this setting, the process

$$Y(t) = \int_{-\infty}^{t} e^{-(t-s)}dL^X(s), \tag{6.6}$$

is well defined, provided that $\int_{|\pi_{\widehat{u}}(x)|>1}\log(|\pi_{\widehat{u}}(x)|)\nu(dx)<\infty$ for any $\widehat{u}\in\widehat{U}$. The field-valued process Y is ID and stationary and will be called *field-valued Ornstein-Uhlenbeck process*. The following proposition generalizes the classical result concerning the marginal distributions of OU processes driven by a Lévy process.

Proposition 6.6. *Let $X=(X_u)_{u\in U}$ be an infinitely divisible field with characteristic triplet (Γ,B,ν) such that $\displaystyle\int_{|\pi_{\widehat{u}}(x)|>1}\log(|\pi_{\widehat{u}}(x)|)\nu(dx)<\infty$ for every $\widehat{u}\in\widehat{U}$. Take Y to be as in* (6.6)*. Then, for every $t\in\mathbb{R}$, the field $Y(t)$ is SD and $Y(t)\overset{d}{=}\int_0^\infty e^{-s}dL^X(s)$. Reciprocally, for a given SD field Y, there exists a unique in law ID field-valued Lévy process L^Y, such that $Y\overset{d}{=}\int_0^\infty e^{-s}dL^Y(s)$.*

Proof. Let $\widehat{u}\in\widehat{U}$. Then $Y_{\widehat{u}}(t)\overset{d}{=}\int_{-\infty}^t e^{-(t-s)}dL^X_{\widehat{u}}(s)\overset{d}{=}\int_0^\infty e^{-s}dL^X_{\widehat{u}}(s)$. It is well known that the law of $\int_0^\infty e^{-s}dL^X_{\widehat{u}}(s)$ belongs to $\mathrm{SD}(\mathbb{R}^{\widehat{u}})$. Consequently, the ID field $Y(t)$ is selfdecomposable. Reciprocally, let Y be a selfdecomposable field, then $\mathcal{L}(Y_{\widehat{u}})\in\mathrm{SD}(\mathbb{R}^{\widehat{u}})$, thus there exists a unique (in law) Lévy process $L^Y_{\widehat{u}}$ such that $Y_{\widehat{u}}\overset{d}{=}\int_0^\infty e^{-s}dL^Y_{\widehat{u}}(s)$. Put $\widetilde{Y}_{\widehat{u}}(t):=\int_{-\infty}^t e^{-(t-s)}dL^Y_{\widehat{u}}(s)$, $t\in\mathbb{R}$, then $\widetilde{Y}_{\widehat{u}}$ is stationary and its marginal distributions are equal to $\mathcal{L}(Y_{\widehat{u}})$. By the Langevin equation

$$L^Y_{\widehat{u}}(1) = \widetilde{Y}_{\widehat{u}}(1)-\widetilde{Y}_{\widehat{u}}(1)+\int_0^1\widetilde{Y}_{\widehat{u}}(s)ds, \qquad \widehat{u}\in\widehat{U},$$

i.e. $L^Y_{\widehat{u}}(1)$ is a functional of $\widetilde{Y}_{\widehat{u}}$. Due to the consistency of the characteristic triplets of $\widetilde{Y}_{\widehat{u}}$ we have that the triplets of $L^Y_{\widehat{u}}(1)$ are consistent as well, thus there exists a unique (in law) ID field-valued Lévy process L^Y, whose finite-dimensional distributions correspond to those of $L^Y_{\widehat{u}}$. This concludes the proof. □

7 Conclusion

The purpose of this paper has been to study selfdecomposability of random fields, as defined directly rather than in terms of finite-dimensional distributions. Applications of the results to modelling within the framework of Ambit Stochastics will be discussed elsewhere. The exposition we present is based on the concept of master Lévy measures of which we give a thorough discussion, building on the recent work of Rosiński (2007a,b, 2008, 2013).

A Appendix

In this appendix we present a proof of Theorem 3.4. We want to emphasize that we construct such a proof based on the remarks given by Jan Rosiński in Rosiński (2007a, 2008, 2013).

Let us start with the next lemma.

Lemma A.1. *Let $(\mathcal{X}, \mathcal{B}, \mu)$ be a measure space and $\mathcal{L}^1(\mathcal{X}, \mathcal{B}, \mu)$ the Lebesgue space of real-valued integrable functions. We have that μ is σ-finite if and only if there is $f \in \mathcal{L}^1(\mathcal{X}, \mathcal{B}, \mu)$ such that f is strictly positive.*

Proof. The proof is straightforward, thus omitted. □

Lemma A.2. *Let ν be a measure on $\mathcal{B}(\mathbb{R})^U$ satisfying (3.4). Then ν does not charge zero if and only if ν is σ-finite and for all $A \in \mathcal{B}(\mathbb{R})^U$ there exists $U_A \subset U$ countable, such that*

$$\nu(A) = \nu(A \setminus \pi_{U_A}^{-1}(0^{U_A})). \tag{A.1}$$

Proof. Assume that Equations (3.2) and (3.4) hold. Then, for any $A \in \mathcal{B}(\mathbb{R})^U$, $\nu(A \cap \pi_{U_0}^{-1}(0^{U_0})) = 0$, thus

$$\nu(A) = \nu(A \cap \pi_{U_0}^{-1}(0^{U_0})) + \nu(A \setminus \pi_{U_0}^{-1}(0^{U_0})) = \nu(A \setminus \pi_{U_0}^{-1}(0^{U_0})),$$

proving thus (A.1). On the other hand, due to (3.4), for any $u \in U_0$

$$\int_{A_0} 1 \wedge |\pi_u(x)|^2 \nu(dx) < \infty,$$

with $A_0 := \mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0})$. Using that $1 \wedge |\pi_u(\,\cdot\,)|^2$ is strictly positive on A_0 and the previous lemma, we have that ν restricted to A_0 is σ-finite, i.e. there exists $\{S_n'\}_{n\geq 1}$, such that $S_n' \uparrow \mathbb{R}^U$ and $\nu(S_n' \cap A_0) < \infty$ for all $n \in \mathbb{N}$. Putting $S_n = (S_n' \cap A_0) \cup \pi_{U_0}^{-1}(0^{U_0})$, we see that $S_n \uparrow \mathbb{R}^U$ and thanks to (3.2)

$$\nu(S_n) \leq \nu(S_n' \cap A_0) < \infty,$$

i.e. ν is σ-finite. Conversely, assume that ν is σ-finite and (A.1) holds, then without loss of generality we may and do assume that ν is finite. Thanks to (A.1), we have that there exist $U_0 \subset U$ countable, such that

$$\nu(\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0})) = \nu(\mathbb{R}^U),$$

which implies (4.1). □

Lemma A.3. *The collection* $\mathcal{B}_0 := \bigcup_{\widehat{u}\in\widehat{U}} \pi_{\widehat{u}}^{-1}[\mathcal{B}(\mathbb{R}^{\widehat{u}}\setminus 0^{\widehat{u}})]$ *is a ring for which* $\mathcal{B}(\mathbb{R})^U = \sigma(\mathcal{B}_0)$. *Let* ν *and* $\widetilde{\nu}$ *be two* σ*-finite measures defined on* $\mathcal{B}(\mathbb{R})^U$ *which coincide in* $\mathcal{B}_0$. *If* ν *and* $\widetilde{\nu}$ *do not charge zero, then* $\nu \equiv \widetilde{\nu}$.

Proof. Obviously $\mathcal{B}_0$ is a ring and $\mathcal{B}_0 \subset \mathcal{B}(\mathbb{R})^U$. Therefore, $\mathcal{B}(\mathbb{R})^U = \sigma(\mathcal{B}_0)$ if $\pi_{\widehat{u}}^{-1}(A) \in \sigma(\mathcal{B}_0)$ for any $A \in \mathcal{B}(\mathbb{R}^{\widehat{u}})$ and $\widehat{u} \in \widehat{U}$. Fix $\widehat{u} \in \widehat{U}$ and take an arbitrary $A \in \mathcal{B}(\mathbb{R}^{\widehat{u}})$. Now, if $0^{\widehat{u}} \notin A$, $\pi_{\widehat{u}}^{-1}(A) \in \pi_{\widehat{u}}^{-1}[\mathcal{B}(\mathbb{R}^{\widehat{u}} \setminus 0^{\widehat{u}})]$, i.e. $\pi_{\widehat{u}}^{-1}(A) \in \sigma(\mathcal{B}_0)$. In counterpart, if $0^{\widehat{u}} \in A$, $0^{\widehat{u}} \notin A^c$, thus as before $\pi_{\widehat{u}}^{-1}(A^c) \in \sigma(\mathcal{B}_0)$, which implies necessary that $\pi_{\widehat{u}}^{-1}(A) = [\pi_{\widehat{u}}^{-1}(A^c)]^c \in \sigma(\mathcal{B}_0)$.

On the other hand, let ν and $\widetilde{\nu}$ be two σ-finite measures coinciding on $\mathcal{B}_0$. By the σ-finiteness, we may and do assume that ν and $\widetilde{\nu}$ are finite measures. Invoking the Monotone Class Theorem, we deduce that ν and $\widetilde{\nu}$ coincide on $\mathcal{SR}(\mathcal{B}_0)$ the σ-ring generated by $\mathcal{B}_0$. Suppose now that ν and $\widetilde{\nu}$ do not charge zero. Then, there are $U_0^1, U_0^2 \subset U$ countable such that $\nu(\mathbb{R}^U \setminus \pi_{U_0^1}^{-1}(0^{U_0^1})) = \widetilde{\nu}(\mathbb{R}^U \setminus \pi_{U_0^2}^{-1}(0^{U_0^2})) = 0$. Putting $U_0 = U_0^1 \cup U_0^2$, we get $\nu(\pi_{U_0}^{-1}(0^{U_0})) = \widetilde{\nu}(\pi_{U_0}^{-1}(0^{U_0})) = 0$. This means that in order to complete the proof we only need to check that $\nu(\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0})) = \widetilde{\nu}(\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0}))$, because in this case ν and $\widetilde{\nu}$ will coincide on $\mathcal{B}_0 \cup \{\mathbb{R}^U\}$, which implies, by the first part of the lemma and once you apply the Monotone Class Theorem, that ν and $\widetilde{\nu}$ coincide on $\mathcal{SR}(\mathcal{B}_0 \cup \{\mathbb{R}^U\}) = \sigma(\mathcal{B}_0) = \mathcal{B}(\mathbb{R})^U$.

Let us verify that $\nu(\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0})) = \widetilde{\nu}(\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0}))$. To do this, we show that $\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0}) \in \mathcal{SR}(\mathcal{B}_0)$. Assume that U_0 is finite. In view of $\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0}) = \pi_{U_0}^{-1}(\mathbb{R}^{U_0} \setminus 0^{U_0})$ we see that $\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0}) \in \mathcal{B}_0 \subset \mathcal{SR}(\mathcal{B}_0)$. Suppose now that U_0 has infinitely many elements, lets say $(u_n)_{n\in\mathbb{N}} \subset U$. Define $U_0^n := (u_i)_{i=1}^n$, then $U_0^n \in \widehat{U}$, $U_0^n \uparrow U_0$ and $\pi_{U_0^n}^{-1}(0^{U_0^n}) \downarrow \pi_{U_0}^{-1}(0^{U_0})$. Consequently $\pi_{U_0^n}^{-1}(\mathbb{R}^{U_0^n} \setminus 0^{U_0^n}) \in \mathcal{B}_0 \subset \mathcal{SR}(\mathcal{B}_0)$ for all $n \in \mathbb{N}$ and

$$\mathbb{R}^U \setminus \pi_{U_0}^{-1}(0^{U_0}) = \bigcup_{n\geq 1} \pi_{U_0^n}^{-1}(\mathbb{R}^{U_0^n} \setminus 0^{U_0^n}) \in \mathcal{SR}(\mathcal{B}_0),$$

which completes the proof. □

Lemma A.4. *Suppose that for all* $n \in \mathbb{N}$ *we have a non-empty compact set* $C_n \subseteq \mathbb{R}^n$ *such that* $(x_1, \ldots, x_{n+1}) \in C_{n+1}$ *implies that* $(x_1, \ldots, x_n) \in C_n$. *Then there is* $y_\infty = (y_1, y_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ *such that* $(y_1, \ldots, y_n) \in C_n$ *for all* $n \in \mathbb{N}$.

Proof. By the continuity of projections, for any $m, n \in \mathbb{N}$, $m > n$ the set $\pi_{mn}(C_m)$ is non-empty and compact. It is easy to check that $C_n = \pi_{nn}(C_n) \supset \pi_{n+1\,n}(C_{n+1}) \supset \ldots$, for any $n \in \mathbb{N}$. Since an intersection of a decreasing sequence of non-empty compact sets is itself non-empty, there is an y_1 such that $y_1 \in \bigcap_{n\geq 1} \pi_{n1}(C_n)$.

Consider $C_n(y_1) = \pi_{n-1}[\pi_{n1}^{-1}(\{y_1\}) \cap C_n]$ for $n \geq 2$. By construction, the family $\{C_n(y_1)\}_{n\geq 2}$ satisfies the assumptions of this Lemma. Repeating the argument that lead to the choice of y_1, we can find y_2 such that for any $n > 2$, $(y_1, y_2) \in \pi_{n2}(C_n)$. Now, by induction, we obtain $y = (y_1, y_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ such that $\pi_n(y) \in C_n$ for each $n \in \mathbb{N}$, as required. □

Now we are ready to show a proof of Theorem 3.4:

Proof of Theorem 3.4. Firstly we prove the uniqueness. From Proposition 3.1, the functions Γ and B are unique, so we only need to check that if there is a measure $\widetilde{\nu}$ that does not charge zero and satisfies (3.3), then $\nu \equiv \widetilde{\nu}$. If ν and $\widetilde{\nu}$ are two measures satisfying (3.3) that do not charge zero, then (3.4) holds for ν and $\widetilde{\nu}$ and they coincide on $\mathcal{B}_0$. By Lemma A.2 ν and $\widetilde{\nu}$ are σ-finite. The uniqueness follows from Lemma A.3.

Now we proceed to prove the existence. We divide the proof in four steps to make it easier to read. As a starting point, we define a measure on $\mathcal{B}_0$, and then we extend it to the σ-algebra generated by $\mathcal{B}_0$, which in virtue of Lemma A.3 coincides with $\mathcal{B}(\mathbb{R})^U$.

Step 1: Defining the pre-measure

Recall that if $A_0 \in \mathcal{B}_0$, then there is a $\widehat{u} \in \widehat{U}$ and $A \in \mathcal{B}(\mathbb{R}^{\widehat{u}})$, such that $A_0 = \pi_{\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}})$. Let us define the set function $\nu_0 \colon \mathcal{B}_0 \to [0, \infty]$, by

$$\nu_0(A_0) = \nu_{\widehat{u}}(A), \qquad A_0 \in \mathcal{B}_0, \tag{A.2}$$

provided that $A_0 = \pi_{\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}})$. Here $\nu_{\widehat{u}}$ is the Lévy measure of $X_{\widehat{u}}$. We claim that ν is well defined. Indeed, suppose that there are $\widehat{u}, \widehat{v} \in \widehat{U}$ and $A^1 \in \mathcal{B}(\mathbb{R}^{\widehat{u}})$, $A^2 \in \mathcal{B}(\mathbb{R}^{\widehat{v}})$ such that $A_0 = \pi_{\widehat{u}}^{-1}(A^1 \setminus 0^{\widehat{u}}) = \pi_{\widehat{v}}^{-1}(A^2 \setminus 0^{\widehat{v}})$. Then, $A_0 = \pi_{\widehat{u}}^{-1}(A^1 \setminus 0^{\widehat{u}}) = \pi_{\widehat{w}}^{-1}[\pi_{\widehat{w}\widehat{v}}^{-1}(A^2 \setminus 0^{\widehat{v}})]$ where $\widehat{w} = \widehat{u} \cup \widehat{v} \in \widehat{U}$. Thus, by Proposition 3.1

$$\begin{aligned} \nu_{\widehat{u}}(A^1) &= \nu_0(A_0) \\ &= \nu_{\widehat{w}}(\pi_{\widehat{w}\widehat{v}}^{-1}(A^2 \setminus 0^{\widehat{v}})) \\ &= \nu_{\widehat{v}}(A^2), \end{aligned}$$

where we used that $\pi_{\widehat{w}\widehat{v}}^{-1}(A^2 \setminus 0^{\widehat{v}}) \in \mathcal{B}(\mathbb{R}^{\widehat{w}} \setminus 0^{\widehat{w}})$. This implies that ν is well defined.

Step 2: ν_0 is finitely additive

In this step we show that ν_0 is finitely additive. Let $A_0, B_0 \in \mathcal{B}_0$ with $A_0 \cap B_0 = \emptyset$. There exist $\widehat{u}, \widehat{v} \in \widehat{U}$, $A \in \mathcal{B}(\mathbb{R}^{\widehat{u}})$ and $B \in \mathcal{B}(\mathbb{R}^{\widehat{v}})$ such that $A_0 = \pi_{\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}})$ and $B_0 = \pi_{\widehat{v}}^{-1}(B \setminus 0^{\widehat{v}})$. Put $\widehat{w} = \widehat{u} \cup \widehat{v} \in \widehat{U}$. Again, by Proposition 3.1

$$\begin{aligned} \nu_0(A_0 \cup B_0) &= \nu_0\{\pi_{\widehat{w}}^{-1}[\pi_{\widehat{w}\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}}) \cup \pi_{\widehat{w}\widehat{v}}^{-1}(B \setminus 0^{\widehat{v}})]\} \\ &= \nu_{\widehat{w}}[\pi_{\widehat{w}\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}}) \cup \pi_{\widehat{w}\widehat{v}}^{-1}(B \setminus 0^{\widehat{v}})] \\ &= \nu_{\widehat{w}}[\pi_{\widehat{w}\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}})] + \nu_{\widehat{w}}[\pi_{\widehat{w}\widehat{v}}^{-1}(B \setminus 0^{\widehat{v}})] \\ &= \nu_0(A_0) + \nu_0(B_0), \end{aligned}$$

thanks to $\pi_{\widehat{w}\widehat{u}}^{-1}(A \setminus 0^{\widehat{u}}) \cap \pi_{\widehat{w}\widehat{v}}^{-1}(B \setminus 0^{\widehat{v}}) = \emptyset$. This also implies that $\nu_0(\emptyset) = 0$.

Step 3: Continuity at the empty set

Recall that a set function μ defined on $\mathcal{R}$ a ring of sets which is finitely additive is σ-additive if and only if it is continuous at the empty set, i.e. if $A_n \downarrow \emptyset$ with $A_n \in \mathcal{R}$ then $\mu(A_n) \to 0$ or equivalently if $(A_n)_{n \geq 1}$ is a decreasing sequence on $\mathcal{R}$ with $\inf_{n \geq 1} \mu(A_n) > 0$ then $\bigcap_{n \geq 1} A_n \neq \emptyset$.

Now we prove that ν_0 as in (A.2) is continuous at the empty set. Let $(A_n^0)_{n\geq 1} \subset \mathcal{B}_0$, then there are $\widehat{u}_n \in \widehat{U}$ and $A_n \in \mathcal{B}(\mathbb{R}^{\widehat{u}_n})$, such that $A_n^0 = \pi_{\widehat{u}_n}^{-1}(A_n \setminus 0^{\widehat{u}_n})$ for any $n \in \mathbb{N}$. Without loss of generality we may assume that $\widehat{u}_n \subset \widehat{u}_{n+1}$, otherwise put $\widehat{u}'_n = \bigcup_{i=}^n \widehat{u}_n$ and use that $\pi_{\widehat{u}_n}^{-1}(A_n \setminus 0^{\widehat{u}_n}) = \pi_{\widehat{u}'_n}^{-1}(\pi_{\widehat{u}'_n \widehat{u}_n}^{-1}(A_n \setminus 0^{\widehat{u}_n}))$.

Consider $(A_n^0)_{n\geq 1}$ be a decreasing sequence with $\inf_{n\geq 1} \nu_0(A_n^0) > 0$. We want to show that $\bigcap_{n\in\mathbb{N}} A_n^0 \neq \emptyset$. The condition with the infimum is equivalent to saying that there is an $\epsilon > 0$ such that for all $n \in \mathbb{N}$, $\nu_{\widehat{u}_n}(A_n) > \epsilon$. Considering that $\nu_{\widehat{u}_n}$ is a Lévy measure for any $\widehat{u}_n \in \widehat{U}$, we have that there is a compact set $K_n \subseteq A_n \backslash 0^{\widehat{u}_n} \subset \mathbb{R}^{\widehat{u}_n} \backslash 0^{\widehat{u}_n}$ such that

$$\nu_{\widehat{u}_n}[(A_n \setminus 0^{\widehat{u}_n}) \setminus K_n] = \nu_{\widehat{u}_n}(A_n \setminus K_n) < \frac{\epsilon}{2^{n+1}}, \qquad n \in \mathbb{N}. \tag{A.3}$$

Let $C_n = \bigcap_{k=1}^n \pi_{\widehat{u}_n \widehat{u}_k}^{-1}(K_k)$. We see that C_n is compact on $\mathbb{R}^{\widehat{u}_n}$ with $C_n \subset K_n \subset A_n \setminus 0^{\widehat{u}_n}$. Further, C_n is non-empty for all $n \in \mathbb{N}$. Indeed, from (A.3) and the fact that $0^{\widehat{u}_n} \notin C_n$, we have

$$\epsilon < \nu_{\widehat{u}_n}(A_n \setminus C_n) + \nu_{\widehat{u}_n}(C_n) < \nu_{\widehat{u}_n}(C_n) + \tfrac{\epsilon}{2},$$

or in other words $\nu_{\widehat{u}_n}(C_n) > \frac{\epsilon}{2}$. By construction $\bigcap_{n\geq 1} \pi_{\widehat{u}_n}^{-1}(C_n) \subset \bigcap_{n\geq 1} A_n^0$, meaning that in order to show that $\bigcap_{n\in\mathbb{N}} A_n^0 \neq \emptyset$ we only need to check that $\bigcap_{n\geq 1} \pi_{\widehat{u}_n}^{-1}(C_n) \neq \emptyset$. Further, defining $\widehat{u}_\infty := \bigcup_{n\geq 1} \widehat{u}_n$ and putting $C_n^\infty := \pi_{\widehat{u}_\infty \widehat{u}_n}^{-1}(C_n)$ with $n \in \mathbb{N}$, we get $\bigcap_{n\geq 1} \pi_{\widehat{u}_n}^{-1}(C_n) = \pi_{\widehat{u}_\infty}^{-1}(\bigcap_{n\geq 1} C_n^\infty)$. Therefore, it suffices to prove that $\bigcap_{n\geq 1} C_n^\infty$ is non-empty. Note that if $\widehat{u}_\infty$ is a finite set, then $(C_n^\infty)_{n\geq 1}$ is a collection of non-empty compact sets on $\mathbb{R}^{\widehat{u}_\infty}$, implying trivially that $\bigcap_{n\geq 1} C_n^\infty \neq \emptyset$, so we only consider the case when $\widehat{u}_\infty$ has infinitely many elements. Since $\widehat{u}_n \subset \widehat{u}_{n+1}$, we can assume that there is $(u_n)_{n\in\mathbb{N}} \subset U$ such that $\widehat{u}_n = (u_i)_{i=1}^n$. Note that in this case $\pi_{\widehat{u}_{n+1}\widehat{u}_n}(C_{n+1}) \subset C_n$ due to $\pi_{\widehat{u}_{n+1}\widehat{u}_k}^{-1}(K_k) = \pi_{\widehat{u}_{n+1}\widehat{u}_n}^{-1}(\pi_{\widehat{u}_n\widehat{u}_k}^{-1}(K_k))$. Hence, from Lemma A.4 there is $x \in \mathbb{R}^{\widehat{u}_\infty}$ such that $\pi_{\widehat{u}_\infty \widehat{u}_n}(x) \in C_n$ for all $n \in \mathbb{N}$ concluding thus that $\bigcap_{n\geq 1} C_n^\infty \neq \emptyset$.

Step 4: Extending ν_0

At this point we have so far that ν_0 is a σ-additive measure on the ring $\mathcal{B}_0$. By the Carathéodory Extension Theorem, it follows that there is an extension of ν_0, lets say ν, to $\sigma(\mathcal{B}_0) = \mathcal{B}(\mathbb{R})^U$, such that $\nu \mid_{\mathcal{B}_0} = \nu_0$. This step concludes the proof. □

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