Incremental Similarity and Turbulence

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Abstract

This paper discusses the mathematical representation of an empirically observed phenomenon, referred to as Incremental Similarity. We discuss this feature from the viewpoint of stochastic processes and present a variety of non-trivial examples, including those that are of relevance for turbulence modelling.

1 Introduction

This paper discusses an empirically observed feature that we refer to as *Incremental Similarity.* Its most remarkable manifestation is in turbulence but it is also found, in a less prononunced form, in finance.

In high frequency recordings of velocities of homogeneous, isotropic and stationary turbulence the non-Gaussian and skewed distributions of velocity increments from different experiments and different lags are essentially identical provided they have the same variance. Note however that this identification is effectuated lag by lag and not by a simple common transformation.

The observation of incremental similarity for turbulent velocity time series adds a new type of universality to the statistical stylised features of turbulent flows. In the time domain, the term universality traditionally refers to universal scaling properties of structure functions defined as moments of velocity increments. These scaling laws are only realised in the limit of very large Reynolds numbers, i.e. strongly turbulent flows. Even for large Reynolds numbers only approximate scaling is observed and only for a certain range of time scales, the so-called inertial range, which covers merely part of the dynamically active scales.

The new stylised feature of incremental similarity points towards a completely different type of universality. Incremental similarity characterises the distributions of velocity increments at all dynamically active scales. Furthermore, our empirical analysis shows that incremental similarity is not restricted to any high Reynolds number limit. And finally, incremental similarity provides a relatively simple mapping that directly connects measurements from different experiments, different in Reynolds numbers and different in boundary conditions.

The phenomenon of incremental similarity was first noted in the paper [4] which revealed this trait through a detailed analysis of the recordings of the main component of the velocity vector from three different types of experiments, with Reynold's numbers 80, 190, 17000. The collapsibility of the incremental distributions was of extraordinary degree. This type of analysis has since been extended to a much larger class of experiments, as reported in [5], fully confirming the original observation. However, up till now a mathematical representation of the phenomenon has not been given. Such a representation is proposed in this paper.

As stated here the property of incremental (or distributional) similarity is of nonparametric character but, as demonstrated in [4], the laws of the velocity differences can be fitted by the normal inverse Gaussian distribution, denoted NIG, to a high degree of precision. This distribution was introduced in [1] and has since found a multitude of applications in a variety of fields. Recently the fact that NIG describes the velocity increments to high precision has found a theoretical counterpart in studies of Birnir concerning a stochastic version of the Navier-Stokes equations, see [7, 8, 9].

The empirical evidence for incremental similarity is summarised in Section 2. Section 3 presents a mathematical definition of incremental similarity - or IS - and an extended concept - extIS - and a variety of examples of stochastic processes and fields that meet the definition of incremental similarity exactly are presented there. For the relation to the statistical theory of turbulence it is crucial to have examples where the processes considered are stationary, and herein lies the main difficulty. Section 4 briefly adresses the question of modelling the timewise behaviour of the main component of the velocity vector in homogeneous turbulence by stochastic processes embodying the main universal features of this type of dynamics, including that of incremental similarity.

2 Incremental similarity: empirical evidence

The empirical verification of incremental similarity, presented in [4] and [5] and briefly outlined here, is based on the analysis of 17 experimental data sets (a wake-flow experiments, a free-jet experiment, 13 helium jet experiments, one wind tunnel experiment, one data set from the atmospheric boundary layer) and on one data set from a direct numerical simulation (DNS) of the Navier-Stokes equation. The Reynolds numbers covered by these data range from 80 up to 20000. The empirical data consist of stationary time series of recordings of the main component of the turbulent velocity vector measured at a fixed position in space. The DNS data are spatially homogeneous at a fixed point in time. All data are standardised for the velocities to have a unit variance.

Let $v_t^{(i)}$ denote the velocity signal at time t belonging to the data set (i). We denote by

$$u_s^{(i)} = v_{t+s}^{(i)} - v_t^{(i)}$$

the velocity increment at time scale s. Here we skip reference to t since we are only dealing with stationary time series. (For the DNS data t denotes the spatial position.)

A key observation related to the densities of velocity increments is depicted in Figure 1. Each graph corresponds to the densitiy of velocity increments at a certain time scale s with s increasing from top left to bottom right. These densities evolve from heavy tails at small time scales towards a more Gaussian shape at the large time scales. This evolution across scales is well known in the literature and sometimes called aggregational Gaussianity. Figure 1 also shows, as solid lines, the approximation of these densities within the class of normal inverse Gaussian distributions. The normal inverse Gaussian distributions fit the empirical densities equally well for all time lags and all amplitudes.

Figure 1 provides one example for the evolution across scales, similar results are observed for the other data sets we analysed. The larger the Reynolds number the more the heavyness of the tails. And, different data sets show different distributions at the same time scale. But the key question that is of interest here is: Do different experiments show the same distribution of increments just at different time scales? In other words: Is the evolution across scales exemplified in Figure 1 universal in the sense that the distributions of $u_s^{(i)}$ and $u_{s'}^{(j)}$ are the same if s' is properly chosen given s? Obviously, an affirmative answer requires that the variances at these time scales are the same

$$\mathrm{Var}\left(u_s^{(i)}\right) = \mathrm{Var}\left(u_{s'}^{(j)}\right).$$

Figure 2 shows the corresponding densities of velocity increments for 12 fixed values of the variance. Each plot corresponds to a different value, increasing from top left to bottom right. The densities within each plot correspond to different experiments and different time lags, but the variances are the same. We clearly observe the collapse of the densities during the whole evolution across scales. In this sense the evolution across scales is universal.

The shape of the distributions in Figure 2 is, to a good approximation, a universal function of the variance. For normal distributions, this is trivial. But here we clearly have distributions that are not normal.

3 Incremental similarity: mathematical considerations

As theoretical counterpart to the incremental similarity features discussed in Sections 1 and 2 we introduce the following definition

Definition *Incremental similarity* Let X and Z be two stochastic processes on $\mathbb{R}$. Then X and Z are said to be incrementally similar, or IS, provided that for any $t \in \mathbb{R}$ and any $u > 0$ there exists a $t' \in \mathbb{R}$ and a positive number u' such that the law of $Z(t'+u') - Z(t')$ is the same as that of $X(t+u) - X(t)$.

More generally, if $\mathcal{X}$ and $\mathcal{Z}$ are two classes of stochastic process on $\mathbb{R}$ then $\mathcal{X}$ and $\mathcal{Z}$ are said to have the IS property if all pairs X and Z such that $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$ are of IS type. For brevity we will then say that $(\mathcal{X}, \mathcal{Z})$ is IS and that $\mathcal{X}$ is IS if that is the case of $(\mathcal{X}, \mathcal{X})$. $\square$

There are many trivial examples of increment similarity. For instance, any continuous Gaussian process is incrementally similar to Brownian motion. Another example is where X and Z are stationary processes on $\mathbb{R}_+$ with Z equal in law to a proportional timechange of X, that is $Z_t = X_{ct}$ for some $c > 0$. Various non-obvious examples may be based on the following concept of extended incremental similarity or extIS.

Definition *Extended incremental similarity* Let R be a class of positive, continuous and decreasing functions r on $[0, \infty)$. Further, let $\mathcal{X} = \left\{X^{[r]} : r \in R\right\}$ be a parametrised family of stationary processses $X^{[r]}$ on $\mathbb{R}$ with the property that for any pair of time points $(t, t+u)$ the joint law of $X_t^{[r]}$ and $X_{t+u}^{[r]}$ is fully determined by $r(u)$; this in particular implies that the same holds for the law of the increment $X_{t+u}^{[r]} - X_t^{[r]}$. We furthermore assume that the joint law of $X_t^{[r]}$ and $X_{t+u}^{[r]}$ is the same as the joint law of $X_t^{[\tilde{r}]}$ and $X_{t+\tilde{u}}^{[\tilde{r}]}$ provided $r(u) = \tilde{r}(\tilde{u})$ whatever r and $\tilde{r}$ in R. We then say that, relative to R, $\mathcal{X}$ has the property of extended incremental similarity or that $\mathcal{X}$ is extIS. $\square$

If $\mathcal{X}$ is extIS then it is in particular IS. As a another direct consequence of the definition of extIS we have

Proposition Let $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$ denote independent extIS families relative to the same index class R. If F is a real (measurable) function on $\mathbb{R}^n$ and if $\mathcal{Y} = \left\{Y^{[r]} : r \in R\right\}$ is the class of processes given by

$$Y_t^{[r]} = F\left(X_{1t}^{[r]}, ..., X_{nt}^{[r]}\right)$$

where $X_j^{[r]} \in \mathcal{X}_j$, $j = 1, ..., n$, then the family $\mathcal{Y}$ is extIS, and hence IS. In fact, the same conclusion holds if F is random, provided it is independent of $(\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n)$. $\square$

Applications of this result will be discussed in Section 4.

We proceed to present some classes of stationary processes on $\mathbb{R}$ having the extIS property

Let $\mathcal{U}$ be the class of stationary Gaussian processes on $\mathbb{R}$ of mean 0 and variance 1 such that for any member X of $\mathcal{U}$ the autocorrelation function r of X is positive, continuous and strictly decreasing to 0. Then, as is easily seen, $\mathcal{U}$ is extIS.

The concept of trawl processes, introduced in [2], offers a range of extIS classes. As discussed in [3], the simplest type of trawl processes X on $\mathbb{R}$ are of the form

$$X_t = L(A_t) \tag{1}$$

where L is a homogeneous Lévy basis on $\mathbb{R}^2$ and $A_t = A + (t, 0)$ for a Borel set A in $\mathbb{R}^2$ with positive Lebesgue measure, points in $\mathbb{R}^2$ being denoted by (t, x). In this case, since for any $\phi, \psi \in \mathbb{R}$ we have

$$\begin{aligned} \phi X_t + i\psi X_{t+u} &= \phi L(A_t \backslash A_{t+u}) + (\phi + \psi) L(A_t \cap A_{t+u}) + \psi L(A_{t+u} \backslash A_t) \\ &= \phi L(A \backslash A_u) + (\phi + \psi) L(A \cap A_u) + \psi L(A_u \backslash A) \end{aligned}$$

the cumulant function of (X_t, X_{t+u}) is given by

$$\begin{aligned} \mathrm{C}\{\phi, \psi \ddagger (X_t, X_{t+u})\} &= |A \backslash A_u| \, \mathrm{C}\{\phi \ddagger L'\} + |A \cap A_u| \, \mathrm{C}\{\phi + \psi \ddagger'\} + |A_u \backslash A| \, \mathrm{C}\{\phi \ddagger L'\} \\ &= |A| \left[\mathrm{C}\{\phi \ddagger L'\} + \mathrm{C}\{\psi \ddagger L'\} + r(u) \left(\mathrm{C}\{\phi \ddagger L'\} + \mathrm{C}\{\psi \ddagger L'\} - \mathrm{C}\{\phi + \psi \ddagger'\}\right)\right] \end{aligned}$$

where L' denotes the Lévy seed of L, $|\cdot|$ indicates Lebesgue measure and

$$r(u) = \frac{|A \cap A_u|}{|A|}. \tag{2}$$

Now consider the class $\mathcal{A}$ of Borel sets A such that (2) is positive for all real u with r continuous and strictly decreasing on $\mathbb{R}_+$ and tending to 0 as $u \longrightarrow \infty$. Let R be the corresponding class of functions r. Furthermore, for any $c > 0$ let $\mathcal{A}_c$ be the subclass of $\mathcal{A}$ given by $\mathcal{A}_c = \{A \in \mathcal{A} : |A| = c\}$. If $\mathcal{X}$ is the class of trawl processes corresponding to a given seed L' and a given $\mathcal{A}_c$ then $\mathcal{X}$ is extIS and hence IS.

We note that

$$\mathrm{C}\{\phi \ddagger X_{t+u} - X_t\} = |A| (1 - r(u)) (\mathrm{C}\{\phi \ddagger L'\} + \mathrm{C}\{-\phi \ddagger L'\}).$$

In case L' is square integrable then r is the autocorrelation function of the process Y. In general we will refer to r as the *autodependence* function of Y. By suitable choice of A the autodependence function can be selected to show short, middle or long term dependence.

With reference to Section 2 we note that if the law of L' is normal inverse Gaussian and symmetric then Y_t and all increments of Y are also normal inverse Gaussian distributed.

Next, suppose that X_t is a stationary process of the form

$$X_t = \int_{-\infty}^{t} g\left(t-s\right) L\left(\mathrm{d}s\right) \tag{3}$$

where L is a symmetric α-stable Lévy process on $\mathbb{R}$ and the kernel function g satisfies $g\left(s\right)=0$ for $s<0$ and

$$I_\alpha\left(g\right) = \int_0^\infty g\left(s\right)^\alpha \mathrm{d}s < \infty. \tag{4}$$

for some $\alpha \in (0,2)$. Then the integral (3) exists and, since the Lévy seed L' of L has cumulant function

$$\mathrm{C}\{\phi \ddagger X\} = -\gamma \left|\phi\right|^\alpha, \tag{5}$$

we find, for $u>0$, that

$$\mathrm{C}\{\phi,\psi \ddagger X_t, X_{t+u}\} = -\gamma\left|\phi\right|^\alpha \int_0^u g\left(s\right)^\alpha \mathrm{d}s - \gamma \int_{-\infty}^{0} \left|\phi g\left(u-s\right) - \psi g\left(-s\right)\right|^\alpha \mathrm{d}s. \tag{6}$$

Now, let $\mathcal{G}$ be the class of kernel functions g such that g is continuous and strictly decreasing to 0 and let $\mathcal{X}$ be the corresponding class of stochastic processes (3). Suppose that g and h are both members of $\mathcal{G}$ and consider the analogue of (6), i.e.

$$\mathrm{C}\{\phi,\psi \ddagger Z_t, Z_{t+u}\} = -\gamma\left|\phi\right|^\alpha \int_0^u h\left(s\right)^\alpha \mathrm{d}s - \gamma \int_{-\infty}^{0} \left|\phi h\left(u-s\right) - \psi h\left(-s\right)\right|^\alpha \mathrm{d}s \tag{7}$$

where Z denotes the element of $\mathcal{X}$ corresponding to h. Only in quite exceptional cases will it be possible for every $u>0$ to find a $v>0$ such that $\mathrm{C}\{\phi,\psi\ddagger Z_t, Z_{t+v}\} = \mathrm{C}\{\phi,\psi\ddagger X_t, X_{t+u}\}$. In other words, in the present setting interesting examples of extIS do not exist.

On the other hand,

$$\mathrm{C}\{\phi \ddagger X_{t+u} - X_t\} = -\gamma\left|\phi\right|^\alpha \hat{g}\left(u;\alpha\right) \tag{8}$$

where

$$\hat{g}\left(u;\alpha\right) = \int_{-\infty}^{u} \left|g\left(u-s\right) - g\left(-s\right)\right|^\alpha \mathrm{d}s \tag{9}$$

and therefore there are subclasses of $\mathcal{X}$ that are IS. Specifically, for a fixed $\alpha \in (0,2)$, consider the subclass $\mathcal{G}_\alpha$ of $\mathcal{G}$ of kernels g such that $I_\alpha\left(g\right)$ does not depend on g. Then the processes X in $\mathcal{X}$ have the same one-dimensional marginal distribution, and $\mathcal{X}$ is IS. In fact, for any $g,h \in \mathcal{G}_\alpha$ and any $u>0$ there exists a $v>0$ such that the condition $\left|\hat{h}\left(v;\alpha\right)\right| = \left|\hat{g}\left(u;\alpha\right)\right|$ is met.

From the viewpoint of applications the question now is whether in principle it is possible, given a class of processes that are known to be of IS type (or suspected to be so), to determine the transformation that effectuates the collapsibility of the laws of increments from the different members of the class.

Empirically, given that high frequency and extensive datasets are available from each of the processes in question and that stationarity of the series is a realistic assumption, the most immediate way is to first standardise the series to have the same marginal variance and then estimate the variances of the increments for a suitable range of lags, as was done for the turbulence data discussed in Section 2. However, in case the data are suspected to come from a fractional regime where second order moments may not in principle exist, such as the α-stable setting considered above, one may resort to other methods of lag transformation, for instance resorting to estimation of fractional moments.

4 Models of $\mathcal{BSS}/\mathcal{LSS}$ type

In [6] the concept of Brownian semistationary processes - or $\mathcal{BSS}$ - processes was introduced. These are stationary processes of the form

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\,\sigma_s \mathrm{d}B_s + \int_{-\infty}^{t} q(t-s)\,a_s \mathrm{d}s \tag{10}$$

where B is Brownian motion, σ and a are stationary processes and the kernels g and q are deterministic functions. The particular setting

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\,\sigma_s \mathrm{d}B_s + \int_{-\infty}^{t} q(t-s)\,\sigma_s^2 \mathrm{d}s \tag{11}$$

is of particular interest. (It may be seen as a stationary analogue of the BNS model studied in financial econometrics.) The $\mathcal{LSS}$ class is obtained by substitution of B in (10) by a general Lévy process L.

The primary aim of the definitions of $\mathcal{BSS}$ and $\mathcal{LSS}$ was to model the timewise behaviour of the main velocity component in a homogeneous turbulent flow, but the same kind of processes have found many applications elsewhere, see for instance [3], [12] and references given there.

$\mathcal{BSS}$ processes of type (11) have been demonstrated to be capable of modelling classical stylised features of turbulence very accurately, cf. [11]. And a recent study [10] of the Helium data discussed in Section 2, based on exponential $\mathcal{LSS}$ processes, has revealed a new type of universality for the energy dissipation.

However, the $\mathcal{BSS}$ structure does not have the property of extended increment similarity and it is therefore natural to ask whether there is a modification of that structure having dynamic behaviour closely similar to $\mathcal{BSS}$ but also exhibiting extIS.

To this end we now introduce the following variant of $\mathcal{BSS}$. Let X be a stationary Gaussian process such that the one-dimensional marginals of X have mean 0 and variance 1. Let σ be a stationary, cadlag and square integrable process with autocorrelation function ρ, and let

$$Y_t = \mu + \sigma_t X_t + \beta\sigma_t^2 \tag{12}$$

for some constant $\beta \in \mathbb{R}$. We shall refer to this type as a $\mathcal{BSS}'$ process. Colloquially speaking, the alternative view amounts to moving the volatility process σ in (11) outside the integration signs.

Now, let $\mathcal{Y}$ be the class of processes (12) obtained by letting the autocorrelation function r of X vary over the set R for which r is positive, continous and decreasing on $\mathbb{R}_+$ and taking σ^2 to vary over a family Σ which in itself is extIS with the same index set R and such that σ is paired with X so as to have the same index r. Then, in view of the Proposition presented in Section 3, the class $\mathcal{Y}$ is extIS.

For instance, Σ could be chosen to be an extIS class of processes σ^2 for which the $\log\sigma^2$ are trawl process $L(A+(t,0))$, as discussed in Section 3 In particular, L may be taken to be an NIG basis, as in [10].

As a further example, suppose that Z is an independent copy of X and let $\sigma_t^2 = |Z_t|^{1/2}$. In this case the autovariance function of the corresponding process Y of (12) is given by

$$\mathrm{E}\left\{(Y_t - Y_0)^2\right\} = 2\left[\mathrm{E}\{\sigma_0\}^2(1-r(t)) + \mathrm{V}\{\sigma_0\}\,r(t)\,(1-\rho(t)) + 2\beta^2\mathrm{V}\left\{\sigma_0^2\right\}(1-\varrho(t))\right]$$

where ϱ is the autocorrelation function of σ^2, with

$$\mathrm{E}\left\{(Y_t - Y_0)^2\right\} = 2[(3+96\beta^2) - r(t) - 2r(t)^3 - 72\beta^2 r(t)^2 - 24\beta^2 r(t)^4$$

which is a monotonely decreasing function of $r(t)$ enabling direct lag identification.

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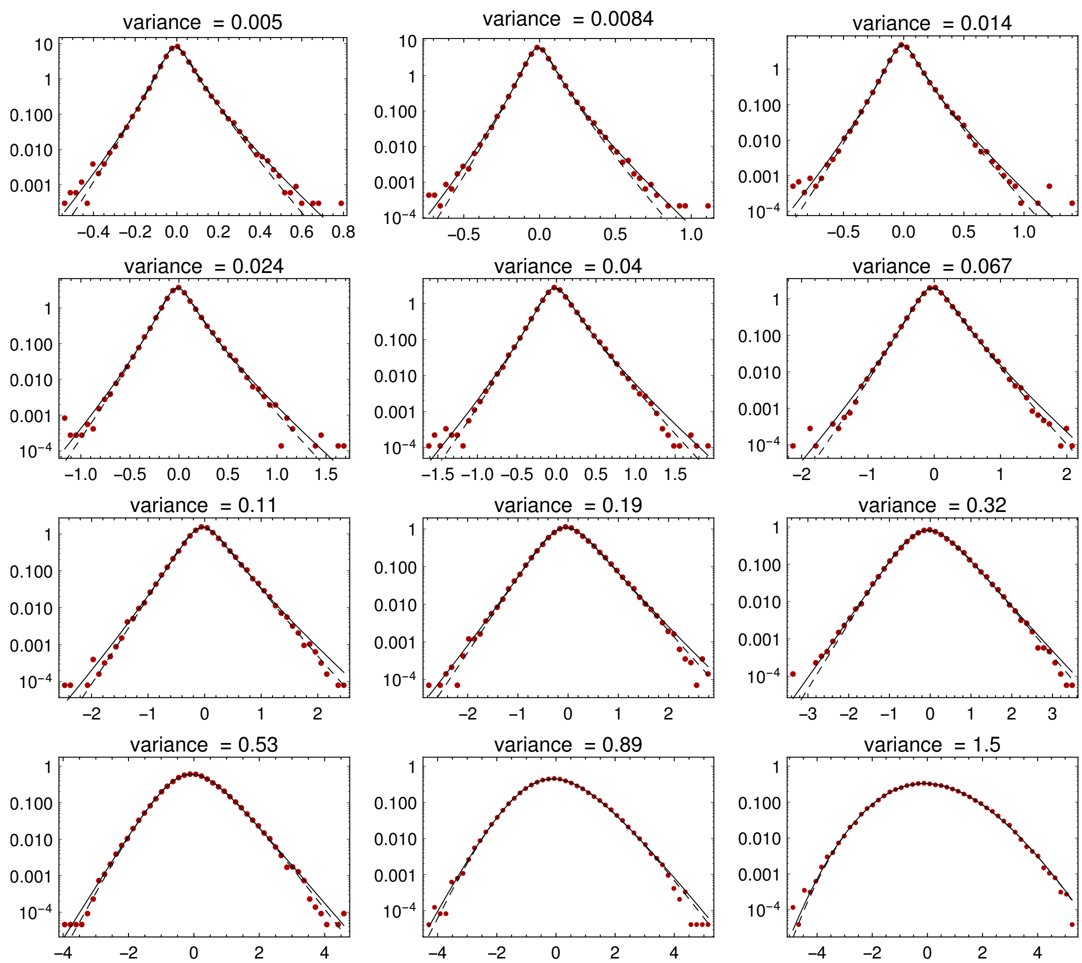

Figure 1: Logarithmic representation of probability density functions. Red dots: empirical pdf of the wind tunnel dataset. Solid black: pdf of the corresponding estimated normal inverse Gaussian distribution. Dashed black: pdf of the normal inverse Gaussian distribution fitted from all sixteen data sets pooled into a single one.

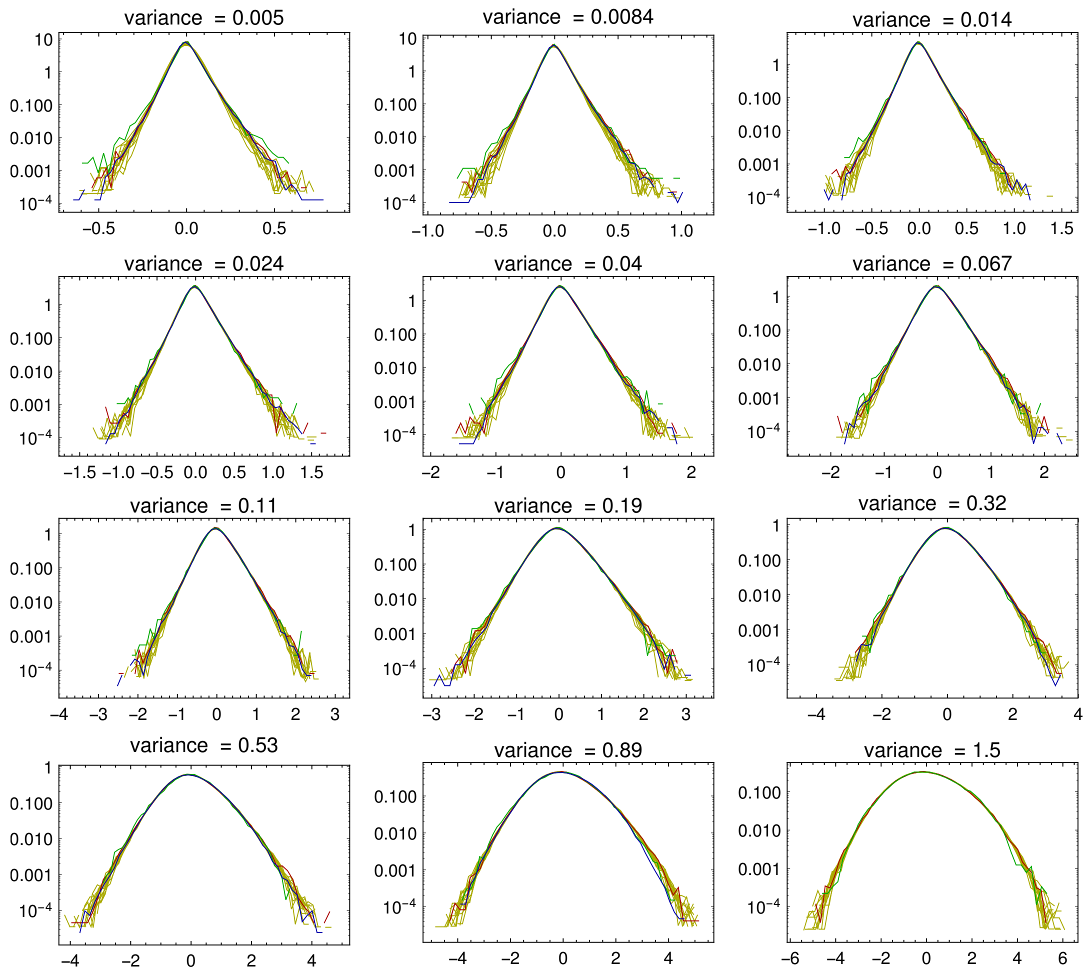

Figure 2: Logarithm representation of the empirical probability density function of the sixteen datasets, after pairing of variances. The thirteen Helium jet data sets are displayed using the same color (yellow).The wind tunnel dataset is displayed in red and the data set from the atmospheric boundary layer is displayed in green. The DNS dataset (blue) is absent in the case of the largest variance.