ON THE UNIQUENESS OF VORTEX EQUATIONS AND ITS GEOMETRIC APPLICATIONS

QIONGLING LI

Abstract. We study the uniqueness of a vortex equation involving an entire function on the complex plane. As geometric applications, we show that there is a unique harmonic map $u : \mathbb{C} \to \mathbb{H}^2$ satisfying $\partial u \neq 0$ with prescribed polynomial Hopf differential; there is a unique affine spherical immersion $u : \mathbb{C} \to \mathbb{R}^3$ with prescribed polynomial Pick differential. We also show that the uniqueness fails for non-polynomial entire functions with finite zeros.

1. Introduction

Let $\phi = \phi(z)$ be a nonzero entire function on $\mathbb{C}$ and $k \geq 2$ be an integer. We consider the following vortex equation on the complex plane

$$\triangle w = e^w - |\phi|^2 e^{-(k-1)w}, \tag{1}$$

where $\triangle = 4\partial_z \partial_{\bar{z}}$. In the case $k = 2$, it is induced from the harmonic map equation from $\mathbb{C}$ to $\mathbb{H}^2$, which is extensively studied, see [?, ?, ?, ?, ?]. In the case $k = 3$, it is Wang's equation for affine spherical immersions, see [?, ?, ?]. For general k, such equations are first considered in [?] and later generalized in [?].

There is a geometric interpretation for Equation (??). Let $\sigma = e^w |dz|^2$ be a conformal metric on $\mathbb{C}$. Equation (??) becomes

$$2K_\sigma = -1 + |\phi|^2_\sigma,$$

where $K_\sigma = -\frac{1}{2\sigma} \triangle \log \sigma$ is the curvature of σ. Roughly speaking, twice of the curvature of σ differs from the curvature of the hyperbolic metric by the norm of ϕdz^k with respect to the metric σ.

The first main theorem we show is

Theorem 1.1. *Let ϕ be a nonzero polynomial function on $\mathbb{C}$, there is a unique C^∞ solution w on $\mathbb{C}$ to Equation (??).*

As an application of the case $k = 3$, the above theorem gives a negative answer to the question asked by Dumas and Wolf on page 1765 in [?], which is one of the original motivations of our paper.

Question 1.2. *(Dumas-Wolf) Does there exist an affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with polynomial Pick differential whose Blaschke metric has positive curvature at some point?*

For a nonzero entire function ϕ, we first show the following proposition.

Proposition 1.3. *Let ϕ be a nonzero entire function on $\mathbb{C}$, there is a unique C^∞ solution w on $\mathbb{C}$ to Equation (**??**) such that $e^w|dz|^2$ is complete.*

So Theorem **??** shows a stronger uniqueness result when ϕ is a polynomial. The proof of Proposition **??** is adapted on previous work of [**?**, **?**, **?**, **?**, **?**] and uses techniques in [**?**, **?**, **?**] of the case $k = 3$ where they show the completeness of the Blaschke metric implies that it is negatively curved.

We then study the case where ϕ is a non-polynomial entire function with finite zeros. Any such entire function is of the form $\phi(z) = P(z)e^{Q(z)}$, where $P(z)$ is a polynomial and $Q(z)$ is any nonconstant entire function. In this case, ϕ has an essential singularity at infinity. For such ϕ, the uniqueness of the solution fails. The second main theorem is

Theorem 1.4. *Suppose ϕ is a non-polynomial entire function with finite zeros, then Equation (**??**) has at least two C^∞ solutions $w_1 > w_2$ where $e^{w_1}|dz|^2$ is a complete metric on $\mathbb{C}$ while $e^{w_2}|dz|^2$ is an incomplete metric on $\mathbb{C}$.*

Remark 1.5. *It would be interesting to know what happens when ϕ has infinite zeros, for example, $\sin \pi z$.*

1.1. **Geometric Applications.**

1.1.1. *Harmonic maps with prescribed Hopf differential.* Consider a harmonic map $u : \mathbb{C} \to \mathbb{H}^2$. Suppose ∂u does not vanish. Writing $w = \frac{1}{2}\log|\partial u|^2$ and let q be the Hopf differential of u. Then u being harmonic is equivalent to q being holomorphic and that the following equation holds on $\mathbb{C}$:

$$\triangle w = e^{2w} - |q|^2 e^{-2w}.$$

As an application of Theorem **??** and **??**, we have

Theorem 1.6. *Suppose q is a nonzero entire function on $\mathbb{C}$ with finite zeros, then*
(1) if q is a polynomial, there exists a unique harmonic map $u : \mathbb{C} \to \mathbb{H}^2$ satisfying $\partial u \neq 0$ with its Hopf differential qdz^2 up to an isometry of $\mathbb{H}^2$. Moreover, it is an orientation-preserving harmonic embedding.
(2) if q is not a polynomial, there are at least two harmonic maps $u_1, u_2 : \mathbb{C} \to \mathbb{H}^2$ satisfying $\partial u_i \neq 0$ not related by an isometry of $\mathbb{H}^2$ with the given qdz^2 as their Hopf differential.

1.1.2. *Affine spherical immersions with prescribed Pick differential.* Consider a locallystrictly convex immersed hypersurface $f : M \to \mathbb{R}^3$. The affine differential geometry associates to such a locally convex immersed surface a special transverse vector field ξ, the affine normal. Being a hyperbolic affine spherical immersion means the affine normal of each image point $f(p)$ meets at a point (the center), which lies in a convex side of the hypersurface.

Relative to the affine normal, there are two important objects on M: (1) the second fundamental form induces a Riemannian metric h, called the Blaschke metric; (2) a holomorphic cubic differential Udz^3, called the Pick differential, measuring the difference between the induced connection of the immersion and the Levi-Civita connection with respect to h.

In the case when (M, h) is conformal to $(\mathbb{C}, |dz|^2)$, we reparametrize the affine spherical immersion as $f : \mathbb{C} \to \mathbb{R}^3$. Following Wang [**?**] and Simon-Wang [**?**], f being the affine spherical immersion is equivalent to U being holomorphic and the Blaschke metric $e^w|dz|$ satisfies

$$\triangle w = 2e^w - 4|U|^2 e^{-2w}.$$

As an application of Theorem **??** and **??**, we have

Theorem 1.7. *Suppose U is a nonzero entire function on $\mathbb{C}$ with finite zeros, then*
(1) if U is a polynomial, there is a unique affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with its Pick differential as Udz^3 up to a projective transformation of $\mathbb{R}^3$. Moreover, it is complete, properly embedded and its Blaschke metric is negatively curved.
(2) if U is not a polynomial, there are at least two affine spherical immersions $f_1, f_2 : \mathbb{C} \to \mathbb{R}^3$ not related by a projective transformation of $\mathbb{R}^3$ with the given Udz^3 as their Pick differential.

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2. Main tools

We collect here the main tools for complete manifolds used in the paper.

Omori-Yau Maximum Principle. ([**?**, **?**]) Suppose (M, h) is a complete manifold with Ricci curvature bounded from below. Then for a C^2 function $u : M \to \mathbb{R}$ bounded from above, there exists a $m_0 \in \mathbb{N}$ and a family of points $\{x_m\} \in M$ such that for each $m \geq m_0$,

$$u(x_m) \geq \sup u - \frac{1}{m}, \quad |\nabla_h u(x_m)| \leq \frac{1}{m}, \quad \triangle_h u(x_m) \leq \frac{1}{m},$$

where ∇_h, $\triangle_h$ are the gradient and the Laplacian with respect to the background metric h respectively.

Cheng-Yau Maximum Principle. ([?]) Suppose (M, h) is a complete manifold with Ricci curvature bounded from below. Let u be a C^2-function defined on M such that $\triangle_h u \geq f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a function. Suppose there is a continuous positive function $g(t) : [a, \infty) \to \mathbb{R}_+$ such that
(i) g is non-decreasing;
(ii) $\liminf_{t\to\infty} \frac{f(t)}{g(t)} > 0$;
(iii) $\int_a^\infty (\int_b^t g(\tau)d\tau)^{-\frac{1}{2}} dt < \infty$, for some $b \geq a$,
then the function u is bounded from above. Moreover, if f is lower semi-continuous, $f(\sup u) \leq 0$.

Supersolution and subsolution method. Let $F(x, u)$ be a C^∞ function defined on $M \times \mathbb{R}$ where (M, h) is a complete Riemannian manifold. Furthermore, assume that

$$\frac{\partial F}{\partial u} > 0 \qquad \text{for all} \quad (x, u) \in M \times \mathbb{R}.$$

Lemma 2.1. *(see [?]) If there exists $w_+, w_- \in C^0(M) \cap H^1_{loc}(M)$ such that $w_- \leq w_+$ and*

$$\triangle w_+ \leq F(x, w_+), \qquad \triangle w_- \geq F(x, w_-)$$

are satisfied weakly (in the sense of distribution), then there is a C^∞ solution w to the equation

$$\triangle w = F(x, w),$$

such that $w_- \leq w \leq w_+$.

In our setting $F(x, u) = e^u - |\phi|^2 e^{-(k-1)u}$, satisfying $\frac{\partial F}{\partial u} > 0$.

3. Complete solutions

In this section, given an entire function ϕ on $\mathbb{C}$, we show the existence and uniqueness of a solution w to

$$\triangle w = e^w - |\phi|^2 e^{-(k-1)w}$$

such that the metric $e^w|dz|^2$ is complete.

Theorem 3.1. *Let ϕ be a nonzero entire function on $\mathbb{C}$, there is a unique C^∞ solution w to Equation (**??**) on $\mathbb{C}$ such that the metric $e^w|dz|^2$ is complete.*

Remark 3.2. *Under the substitution by $w' := cw + \log d$, the above theorem can also be applied to the equation*

$$c \cdot \triangle w = de^{cw} - \frac{1}{d^{k-1}}|\phi|^2 e^{-c(k-1)w},$$

for any $c, d > 0$. So through the paper, whenever referring to the above theorem, we use it with different constants without further explanation.

We prove the above theorem in the end of this section.

To show the existence of solutions on $\mathbb{C}$, we start with the existence of solutions on a domain (U, h), which lifts to the unit disk $\mathbb{D}$ with the Poincaré metric. The proof of the following proposition is easily adapted from the proof of Proposition 4 in [**?**].

Proposition 3.3. *Let ϕdz^k be a holomorphic k-differential on (U, h), then Equation (**??**) has a unique C^∞ solution w on U such that $e^w|dz|^2$ is a complete metric on U and $|\phi|^2 e^{-kw} \leq 1$. Moreover, w satisfies $h \leq e^w|dz|^2$.*

The following existence theorem mainly follows from [**?**], where they work with the case $k = 2$. We include the proof here.

Proposition 3.4. *(Existence) Let ϕ be a nonzero entire function on $\mathbb{C}$, there is a C^∞ solution w to Equation (**??**) on $\mathbb{C}$ such that $e^w|dz|^2$ is complete.*

Proof. Without loss of generality, we assume $\phi(0) \neq 0$. Choose R_1 such that the zeros of ϕ are outside the ball B_{R_1}. Choose $R_2 < R_1$. Let u_1 be the unique solution on B_{R_1} and u_2 be the unique solution in the complement of the closed disk $\overline{B}_{R_2}$ as described in Proposition **??**.

We first construct a supersolution w_+ as follows:
on $\{|z| < R_2\}$, let $w_+ = u_1$;
on $\{R_2 \leq |z| \leq R_1\}$, let $w_+ = \min\{u_1, u_2\}$;
on $\{|z| > R_1\}$, let $w_+ = u_2$.
By Proposition **??**, since $e^{u_1}|dz|^2$ dominates the hyperbolic metric on B_{R_1}, the metric $e^{u_1}|dz|^2$ blows up at boundary of B_{R_1}. Then at the neighbourhood of ∂B_{R_1}, $w_+ = u_2$. Similarly, since $e^{u_2}|dz|^2$ dominates the hyperbolic metric on the complement of B_{R_2}, the metric $e^{u_2}|dz|^2$ blows up at boundary of B_{R_2}. Then at the neighbourhood of ∂B_{R_2}, $w_+ = u_1$. Hence w_+ is a continuous function. Since u_1, u_2 are both supersolutions on the annulus, then the minimum of u_1 and u_2 is a supersolution in the weak sense. Therefore $w_+ \in C^0(M) \cap H^1_{loc}(M)$ is a supersolution.

Secondly, we construct a subsolution w_-. Choose R_1', R_2' such that $R_2 < R_2' < R_1' < R_1$. On the annulus $\{R_2' \leq |z| \leq R_1'\}$, u_2 has a maximum and $\frac{2}{k}\log|\phi|$ has a minimum using that the zeros of ϕ are all outside B_{R_1}. Then there exists a constant $c > 0$ such that

$$\frac{2}{k}\log|\phi| \geq u_2(z) - c$$

holds on the annulus $\{R_2' \leq |z| \leq R_1'\}$.

We construct the subsolution w_- as follows:
onn $|z| < R_1'$, let $w_- = \frac{2}{k}\log|\phi|$;
onn $|z| \geq R_2'$, let $w_- = \max\{\frac{2}{k}\log|\phi|, u_2(z) - c\}$. Our choice of c assures w_- is continuous. Since $\frac{2}{k}\log|\phi|, u_2 - c$ are both subsolutions on $\{|z| \geq R_2'\}$, then the maximum of $\frac{2}{k}\log|\phi|$ and $u_2 - c$ is a subsolution in the weak sense. Therefore $w_- \in C^0(M) \cap H^1_{loc}(M)$ is a subsolution.

By Proposition **??** $e^{u_i} \geq |\phi|^{\frac{2}{k}}$, for $i = 1, 2$. It is clear that $w_- \leq w_+$. Applying Lemma **??**, there is a C^∞ solution w satisfying $w_- \leq w \leq w_+$. Given a path α diverging to ∞ on $\mathbb{C}$, then

$$\int_\alpha e^{\frac{1}{2}w}|dz| \geq \int_\alpha e^{\frac{1}{2}w_-}|dz| = e^{-\frac{1}{2}c}\int_{\alpha\cap B^c_{R'_1}} e^{\frac{1}{2}u_2}|dz| = \infty,$$

by the completeness of $e^{u_2}|dz|^2$. Hence the metric $e^w|dz|^2$ is complete. □

Now we show a weaker uniqueness result under two conditions that the metric $e^w|dz|^2$ is complete and $|\phi|^2 e^{-kw} \leq 1$. The method is similar to the proof in [**?**], [**?**].

Lemma 3.5. *Let ϕ be an entire function on $\mathbb{C}$. If there is a C^∞ solution w to Equation (**??**) on $\mathbb{C}$ such that $e^w|dz|^2$ is complete and $|\phi|^2e^{-kw} \leq 1$. Then for any another C^∞ solution w_1, either $w_1 < w$ on $\mathbb{C}$ or $w_1 \equiv w$ on $\mathbb{C}$.*

Proof. We first have

$$\triangle(w_1 - w) = (e^{w_1} - e^w) - |\phi|^2(e^{-(k-1)w_1} - e^{-(k-1)w}).$$

Denote $\eta = w_1 - w$, using the background metric $g = e^w|dz|^2$, we rewrite the above equation as

$$\triangle_g\eta = (e^\eta - 1) - |\phi|^2e^{-kw}(e^{-(k-1)\eta} - 1) \geq (e^\eta - 1) - e^{-(k-1)\eta},$$

where $\triangle_g = \frac{1}{e^w}\triangle$ and the inequality uses $|\phi|^2e^{-kw} \leq 1$.

Since the metric $g = e^w|dz|^2$ is complete and the curvature $K_g = -\frac{1}{2}\triangle_g w \geq -\frac{1}{2} + \frac{1}{2}|\phi|^2e^{-kw} \geq -\frac{1}{2}$, we can apply the Cheng-Yau maximum principle. Choosing $f(t) = e^t - 1 - e^{-(k-1)t}$ and $g(t) = e^t$, one can check (i) g is increasing; (ii) $\liminf_{t\to\infty}\frac{f(t)}{g(t)} = 1$; (iii) $\int_0^\infty(\int_0^t g(\tau)d\tau)^{-\frac{1}{2}}dt < \infty$. Hence we obtain that η is bounded from above.

By the Omori-Yau maximum principle, since η is bounded from above, there exists a sequence of points $\{x_m\}$ such that there exists m_0 such that for $m \geq m_0$, $\triangle_g\eta(x_m) < \frac{1}{m}$, $\eta(x_m) \geq \sup\eta - \frac{1}{m}$.

Then at point x_m,

$$\frac{1}{m} \geq (e^{\sup\eta - \frac{1}{m}} - 1) - |\phi|^2e^{-kw}(x_m)(e^{-(k-1)(\sup\eta - \frac{1}{m})} - 1)$$

Suppose $\sup\eta > 0$ and then $\sup\eta > \frac{1}{M}$ for some $M > 0$. Then for $m > \max\{m_0, M\}$, the term $-|\phi|^2e^{-kw}(x_m)(e^{-(k-1)(\sup\eta-\frac{1}{m})} - 1)$ is positive. Therefore

$$\frac{1}{m} \geq e^{\sup\eta - \frac{1}{m}} - 1.$$

As $m \to \infty$,

$$0 \geq e^{\sup\eta} - 1.$$

Therefore, $\sup\eta \leq 0$, contradiction.

So $\sup\eta \leq 0$ and then $w_1 \leq w$.

Consider the equation for η

$$\triangle_g \eta = (e^{\eta} - 1) - |\phi|^2 e^{-kw}(e^{-(k-1)\eta} - 1)$$

Applying the strong maximum principle to the equation for η. Since $\eta \leq 0$, then $\eta < 0$ on $\mathbb{C}$ or $\eta \equiv 0$. The lemma follows. $\square$

We are ready to show Theorem **??**.

Proof. (of Theorem **??**) It suffices to show the uniqueness. Using Lemma **??**, it is enough to show that $|\phi|^2 e^{-kw} \leq 1$. Consider the function $h = \frac{|\phi|^2}{e^{kw}}$ and we calculate the equation for $\tau = \log(h+1)$. Calculate $h_z = (\phi_z - kw_z\phi)\bar{\phi}e^{-kw}$, $h_{\bar{z}} = (\bar{\phi}_{\bar{z}} - k\bar{w}_{\bar{z}}\phi)\phi e^{-kw}$, and $h_{z\bar{z}} = -kw_{z\bar{z}}h + |\phi_z - kw_z\phi|^2 e^{-kw}$.

Then

$$\begin{aligned}
\triangle \tau &= 4\frac{h_{z\bar{z}}(h+1) - h_z h_{\bar{z}}}{(h+1)^2} \\
&= -4\frac{khw_{z\bar{z}}}{h+1} + 4\frac{|\phi_z - kw_z\phi|^2 e^{-kw}}{(h+1)^2} \\
&\geq -4\frac{khw_{z\bar{z}}}{h+1} = -kh(e^w - |\phi|^2 e^{-(k-1)w})e^{-\tau} \\
&= -kh(1 - |\phi|^2 e^{-kw})e^{-\tau}e^w \\
&= -k(e^{\tau} - 1)(2 - e^{\tau})e^{-\tau}e^w \\
&= k(e^{\tau} - 3 + 2e^{-\tau})e^w.
\end{aligned}$$

Let $g = e^w|dz|^2$ be the new background metric on $\mathbb{C}$, the above equation is

$$\triangle_g \tau \geq k(e^{\tau} - 3 + 2e^{-\tau}).$$

Since the metric $e^w|dz|^2$ is complete and the curvature $K_g \geq -\frac{1}{2}$, we can apply the Cheng-Yau maximum principle. By choosing $f(t) = k(e^t - 3 + 2e^{-t})$ and $g(t) = e^t$, one can check
(i) g is increasing; (ii) $\liminf_{t\to\infty} \frac{f(t)}{g(t)} = 1$; (iii) $\int_0^{\infty}(\int_0^t g(\tau)d\tau)^{-\frac{1}{2}}dt < \infty$.
We then have that τ is bounded from above and moreover $f(\sup\tau) \leq 0$. Hence $e^{\sup\tau} \leq 2$ and then $h = \frac{|\phi|^2}{e^{kw}} \leq 1$. $\square$

Proposition 3.6. *If ϕ is not a constant, then the unique solution w to Equation (**??**) such that $e^w|dz|^2$ is complete satisfies $|\phi|^2 e^{-kw} < 1$.*

Proof. Consider the function $\sigma = \log(\frac{|\phi|^2}{e^{kw}})$ which is well-defined away from the zeros of ϕ. We have the equation for σ

$$\triangle_g \sigma = k(|\phi|^2 e^{-kw} - 1) = k(e^{\sigma} - 1).$$

Since the right hand side is monotone increasing with respect to σ, one can apply the strong maximum principle to this equation. Since $\sigma \leq 0$, then either $\sigma < 0$ on $\mathbb{C}$ or $\sigma \equiv 0$. $\square$

4. Discussion on the uniqueness

In this section, we consider the uniqueness separately for the case ϕ is a polynomial and the case ϕ is not a polynomial.

Firslty, we restrict ourselves to work with polynomials and show the uniqueness of solutions to Equation (**??**) without any requirement.

Theorem 4.1. *If ϕ be a polynomial function on $\mathbb{C}$, then there is a unique solution to Equation (**??**) on $\mathbb{C}$.*

Proof. The existence is shown in the previous section. It suffices to show the uniqueness. We first show that w has a lower bound. Since ϕ is a polynomial, there exists a large enough R such that outside the ball B_R, $|\phi|^2 \geq 1$. Outside B_R, we have the inequality

$$\triangle w \leq e^w - e^{-(k-1)w}.$$

Denote $B = \max_{B_R}(-w)$ and define the function $f(t)$ on $\mathbb{R}$ as: for $t \leq B, f = -e^{-t}$; for $t > B, f = e^{(k-1)t} - e^{-t}$. Therefore we have

$$\triangle(-w) \geq f(-w).$$

Choose $g(t) = e^{(k-1)t}$, one can check
(i) g is increasing; (ii) $\liminf_{t\to\infty} \frac{f(t)}{g(t)} = 1$; (iii) $\int_0^\infty (\int_0^t g(\tau)d\tau)^{-\frac{1}{2}} dt < \infty$.
Since the background metric is complete and has curvature 0, we can apply the Cheng-Yau maximum principle. Hence we have $\sup -w < \infty$ and w is bounded from below. Then the metric $e^w|dz|^2$ is complete. By Theorem **??**, the solution is unique. □

Next, we study the case when ϕ is a non-polynomial entire function with finite zeros. In this case we show that Equation (**??**) has at least two distinct solutions.

Theorem 4.2. *Let ϕ be an entire function on $\mathbb{C}$ with finite zeros. If ϕ is not a polynomial, then Equation (**??**) has at least two solutions $w_1 > w_2$ where $e^{w_1}|dz|^2$ defines a complete metric and $e^{w_2}|dz|^2$ defines an incomplete metric. Moreover, $|\phi|^2 e^{-kw_1} < 1$ and $|\phi|^2 e^{-kw_2} \leq 1$.*

The main idea is that we construct a solution using another pair of a supersolution and a subsolution in the following Lemma **??** if ϕ is with finite zeros. When the differential ϕ is not a polynomial, the solution constructed here differs from the one constructed in Theorem **??**.

Lemma 4.3. *Let ϕ be a nonzero entire function on $\mathbb{C}$ with finite zeros, there is a C^∞ solution w to Equation (**??**) on $\mathbb{C}$ such that there are constants $R > 0, a > 1$ such that for $|z| \geq R$,*

$$|\phi|^{\frac{2}{k}} \leq e^w \leq a|\phi|^{\frac{2}{k}}.$$

Proof. (of Lemma **??**) If ϕ has no zero, then $w = \frac{2}{k}\log|\phi|$ is a solution.
If ϕ has finite zeros, we can construct a supersolution and a subsolution. Since ϕ has finite zeros, choose R_2 such that the ball $B_{R_2} = \{|z| < R_2\}$

contains all zeros of ϕ. Choose $R_2 < R_2' < R_1' < R_1$. Let u_1 be the unique solution on the domain B_{R_1} defined in Proposition **??**.

On the annulus $\{R_2' \leq |z| \leq R_1'\}$, u_1 has a maximum and $\frac{2}{k}\log|\phi|$ has a minimum using the fact that the zeros of ϕ are all inside B_{R_2}. Then there exists a constant $c > 0$ such that

$$\frac{2}{k}\log|\phi| \geq u_1(z) - c$$

holds on the annulus $\{R_2' \leq |z| \leq R_1'\}$.

We first construct the supersolution w_+ as follows:
on $\{|z| < R_2'\}$, let $w_+ = u_1$;
on the annulus $\{R_2' \leq |z| \leq R_1\}$, let $w_+ = \min\{u_1, \frac{2}{k}\log|\phi| + c\}$;
outside B_{R_1}, let $w_+ = \frac{2}{k}\log|\phi| + c$.
On the annulus $\{R_2' \leq |z| \leq R_1'\}$, $u_1 \leq \frac{2}{k}\log|\phi| + c$. Therefore, in the neighbourhood of $\partial B_{R_2'}$, $w_+ = u_1$ is continuous. Since the metric $e^{u_1}|dz|^2$ dominates the hyperbolic metric on B_{R_1}, u_1 blows up on ∂B_{R_1}. Then in the neighbourhood of ∂B_{R_1}, $w_+ = \frac{2}{k}\log|\phi| + c$ is continuous. Both u_1 and $\frac{2}{k}\log|\phi| + c$ are supersolutions, so their minimum is still a supersolution in the weak sense. Therefore $w_+ \in C^0(\mathbb{C}) \cap H^1_{loc}(\mathbb{C})$ is again a supersolution.

Secondly, we construct the subsolution w_- as follows:
on $\{|z| < R_1'\}$, let $w_- = \max\{u_1 - c, \frac{2}{k}\log|\phi|\}$;
outside $B_{R_1'}$, let $w_- = \frac{2}{k}\log|\phi|$.
On the annulus $\{R_2' \leq |z| \leq R_1'\}$, $u_1 - c \leq \frac{2}{k}\log|\phi|$. Then in the neighbourhood of $\partial B_{R_1'}$, $w_- = \frac{2}{k}\log|\phi|$ is continuous. Both $u_1 - c$ and $\frac{2}{k}\log|\phi|$ are subsolutions, so their maximum is still a subsolution in the weak sense. Therefore $w_- \in C^0(\mathbb{C}) \cap H^1_{loc}(\mathbb{C})$ is again a subsolution.

One can check that $w_- \leq w_+$ as follows:
on $B_{R_2'}$, $w_- = \max\{u_1 - c, \frac{2}{k}\log|\phi|\} \leq u_1 = w_+$, using $\frac{|\phi|^{\frac{2}{k}}}{e^{u_1}} < 1$;
on the annulus $\{R_2' \leq |z| \leq R_1'\}$,
$w_- = \max\{u_1 - c, \frac{2}{k}\log|\phi|\} \leq w_+ = \min\{u_1, \frac{2}{k}\log|\phi| + c\}$;
on the annulus $\{R_1' \leq |z| \leq R_1\}$,
$w_- = \frac{2}{k}\log|\phi| \leq w_+ = \min\{u_1, \frac{2}{k}\log|\phi| + c\}$, using $\frac{|\phi|^{\frac{2}{k}}}{e^{u_1}} < 1$;
outside B_{R_1}, let $w_- = \frac{2}{k}\log|\phi| \leq w_+ = \frac{2}{k}\log|\phi| + c$.

Applying Lemma **??**, there is a C^∞ solution w satisfying $w_- \leq w \leq w_+$. Note when $|z| > R_1'$, we have $\frac{2}{k}\log|\phi| = w_- \leq w \leq w_+ = \frac{2}{k}\log|\phi| + c$. □

The following lemma follows from Lemma 9.6 in [**?**]. For readers' convenience, we include the argument here.

Lemma 4.4. *Let ϕ be a nonzero entire function on $\mathbb{C}$ with finite zeros, the solution w in Lemma* **??** *defines a complete metric $e^w|dz|^2$ if and only if ϕ is a polynomial.*

Proof. If ϕ is a polynomial, for a large enough R, $\phi(z) > 1$ for all $|z| > R$. Then the metric $e^w|dz|^2 > |\phi|^{\frac{2}{k}}|dz|^2 \geq |dz|^2$ for $|z| > \max\{R, R_1\}$. Hence the metric $e^w|dz|^2$ is complete.

Suppose the metric $e^w|dz|^2$ is complete. Then for every path C which diverges to infinity, we have

$$\int_C |\phi(z)|^{\frac{1}{k}}|dz| = \infty. \tag{2}$$

Since $\phi(z)$ has finite zeros, it is of the form $P(z)e^{G(z)}$, where $P(z)$ is a polynomial and $G(z)$ is an entire function. Then there exists a positive integer N such that $|\phi(z)|^{\frac{1}{k}} \leq O(|z^N e^{G(z)}|)$. Define the entire function $F(z)$ as

$$F(z) = \int z^N e^{G(z)} dz, \quad F(z) = 0.$$

So $F(z)$ vanishes at 0 of order $N+1$. There exists a single-value branch $\zeta(z) = [F(z)]^{\frac{1}{N+1}}$ in a neighborhood of $z = 0$ satisfying $\zeta'(z) \neq 0$. Hence we have an inverse $z(\zeta)$ in a neighborhood of $\zeta = 0$.

We claim the inverse map can extend to the whole ζ-plane. Suppose it is not the case, there would be a point ζ_0 satisfying $|\zeta_0| \leq R$ over which $z(\zeta)$ cannot be extended. Consider the path C as image of the segment from 0 to ζ_0 under the map $z(\zeta)$,

$$\int_C |z^N e^{G(z)}||dz| = \int_C |F'(z)||dz| = R^{N+1}.$$

By Equation (**??**), the path C does not diverge to infinity. Then along this path there exists a sequence $z_n \to z_0$ such that $\zeta(z_n) \to \zeta_0$. But since $F'(z_0)$ is nonzero, we can extend $z(\zeta)$ over ζ_0. We thus have the inverse map $z(\zeta)$ is defined on the whole ζ-plane.

Thus $\zeta(z)$ is entire and invertible, hence $\zeta(z) = Az$ for $z \neq 0$. So $F(z) = (z/A)^{N+1}$. And then $G(z)$ is a constant. Therefore ϕ is a polynomial. □

Proof. (of Theorem **??**) By Lemma **??**, the solution w constructed in Lemma **??** defines a complete metric if and only if ϕ is polynomial. Then, if ϕ is not a polynomial, the solution constructed in Lemma **??** defines an incomplete metric $e^w|dz|^2$. By Theorem **??**, there is also another solution w_1 such that $e^{w_1}|dz|^2$ is complete. Hence $w \neq w_1$. By Lemma **??**, $w < w_1$.

It is clear that $|\phi|^2 e^{-kw} \leq 1$ and by Proposition **??**, $|\phi|^2 e^{-kw_1} < 1$. □

We also prove the following property of a solution to Equation (**??**).

Proposition 4.5. *For any nonzero entire function ϕ on $\mathbb{C}$, there is no solution w to Equation (**??**) satisfies $|\phi|^2 e^{-kw} \leq \delta$, for some positive constant $\delta < 1$.*

Proof. Suppose there exists a solution w such that $|\phi|^2 e^{-kw} \leq \delta < 1$, then the solution w satisfying

$$\triangle w = e^w(1 - |\phi|^2 e^{-kw}) \geq (1 - \delta)e^w.$$

Applying the Cheng-Yau maximum principle, we obtain that w is bounded from above. Therefore $|\phi|^2 \leq \delta e^{kw} < \infty$. Hence ϕ is bounded. By Liouville theorem, ϕ is a constant function $c \neq 0$. But in this case we can show that $|c|^2 e^{-kw} \equiv 1$. Consider the equation for $-w$,

$$\triangle(-w) = |c|^2 e^{-(k-1)w} - e^w.$$

Applying the Cheng-Yau maximum principle by choosing $f(t) = |c|^2 e^{(k-1)t} - e^{-t}$ and $g(t) = e^{(k-1)t}$, we obtain that $-w$ is bounded from above. Therefore $e^w|dz|^2$ is complete. We also see the constant function $\frac{2}{k}\log|c|^2$ is a solution. Using the uniqueness of complete solutions in Lemma **??**, we obtain that $|c|^2 e^{-kw}$ is identically 1, contradiction. □

5. Harmonic maps

We consider harmonic maps $u : \mathbb{C} \to (\mathbb{H}^2, h)$. Call $|\partial u|^2$ the ∂-energy density. Following Schoen-Yau [**?**], when $\partial u \neq 0$, the following equation holds

$$\triangle \log|\partial u|^2 = 2(|\partial u|^2 - |\overline{\partial} u|^2). \tag{3}$$

Writing $w = \frac{1}{2}\log|\partial u|^2$. Let q be the quadratic differential defined by $q(z) = (u^*h)^{2,0}$, called the Hopf differential. The harmonicity of u is equivalent to the Hopf differential q being holomorphic and that Equation (**??**) relates (w, q) as follows

$$\triangle w = e^{2w} - |q|^2 e^{-2w}. \tag{4}$$

The Jacobian J of the harmonic map is $J = |\partial u|^2 - |\overline{\partial} u|^2 = e^{2w} - |\phi|^2 e^{-2w}$.

Theorem 5.1. *Given a nonconstant polynomial function q on $\mathbb{C}$, there exists a unique harmonic map $u : \mathbb{C} \to \mathbb{H}^2$ satisfying $\partial u \neq 0$ with its Hopf differential qdz^2 up to an isometry of $\mathbb{H}^2$. Moreover, it is an orientation-preserving harmonic embedding.*

Proof. The existence is standard as shown in [**?**]. Given such a q, by Theorem **??**, there exists a solution w to Equation (**??**) such that $e^{2w}|dz|^2$ is complete. We then construct a harmonic map giving back such data through constructing space-like constant mean curvature immersions in the Minkowski 3-space. The Minkowski 3-space $\mathbb{M}^{2,1}$ is $\mathbb{R}^2 \times \mathbb{R}^1$ endowed with the metric $ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2$. Applying Milnor's method [**?**], we develop a constant mean curvature immersion from $\mathbb{C}$ to $\mathbb{M}^{2,1}$ using the pair (w, q) as follows: define h_{ij} such that

$$h_{11} + h_{22} = e^{2w}, \quad h_{11} - h_{22} = 2\Re q, \quad h_{12} = h_{21} = -\Im q;$$

and define the first and second fundamental form I and II as

$$I = e^{2w}|dz|^2, \quad II = h_{ij}dx^i dx^j.$$

Then the form I and II satisfy the integrability condition of the Fundamental Theorem of Differential Geometry using the equation of q and w. Since the domain $\mathbb{C}$ is simply connected, we can develop a constant mean curvature space-like immersion from $\mathbb{C}$ to $\mathbb{M}^{2,1}$. Then its Gauss map is a harmonic map from $\mathbb{C}$ to $\mathbb{H}^2$ with the Hopf differential is the given q and the ∂-energy density is the given $e^{2w}|dz|^2$, hence $\partial u \neq 0$.

By [?], since the space-like immersion f is complete with respect to the induced Riemannian metric $e^{2w}|dz|^2$, then f is an entire graph, meaning that the natural projection $\Pi : f(\Omega) \to \mathbb{R}^2$ is onto. Because the immersion f is entire, by Theorem 4.8 in [?], the Gauss map of the immersion is a harmonic diffeomorphism onto its image. Therefore there exists an orientation-preserving harmonic embedding with the prescribed Hopf differential q.

Next, we show the uniqueness. From Theorem **??**, the unique solution w to Equation (**??**) satisfies $e^{-4w}|\phi|^2 < 1$. So the Jacobian $J(u) = e^{2w} - |\phi|^2 e^{-2w} > 0$. Hence any harmonic map $u : \mathbb{C} \to \mathbb{H}^2$ satisfying $\partial u \neq 0$ with a polynomial Hopf differential q is an orientation-preserving local diffeomorphism.

Suppose there are two harmonic maps $u_1, u_2 : \mathbb{C} \to \mathbb{H}^2$ satisfying $\partial u_i \neq 0$ inducing the same polynomial Hopf differential q. Then they are both orientation-preserving local diffeomorphisms. Therefore, there exist two domains $\Omega_1, \Omega_2 \in \mathbb{H}^2$ such that $u_2^{-1} : \Omega_2 \to \mathbb{C}$ is well-defined and $u_1 \circ u_2^{-1} : \Omega_2 \to \Omega_1$ is a diffeomorphism. By Theorem **??**, the ∂-energy densities for u_1, u_2 are the same. Then the pullback metrics of both maps u_1, u_2 is the same, written as $\phi dz^2 + (e^{2w} + |\phi|^2 e^{-2w})|dz|^2 + \bar{\phi} d\bar{z}^2$. Hence the map $u_1 \circ u_2^{-1} : \Omega_2 \to \Omega_1$ is an isometry. Since the hyperbolic metric on $\mathbb{H}^2$ is analytic, $u_1 \circ u_2^{-1}$ is a restriction of an isometry τ of $\mathbb{H}^2$. Therefore u_1 and $\tau \circ u_2$ coincide on a domain $u_2(\Omega)$. Then by the unique continuation theorem of harmonic maps shown in [?], u_1 and $\tau \circ u_2$ are identical. The uniqueness follows. $\square$

Remark 5.2. *When q is a polynomial of degree k, the image of the unique harmonic map is an ideal (k+2)-polygon in $\mathbb{H}^2$, as shown in* [?, ?].

Next, we study the case when q is not a polynomial.

Theorem 5.3. *Given a non-polynomial entire function q with finite zeros, there are at least two harmonic maps $u_1, u_2 : \mathbb{C} \to \mathbb{H}^2$ satisfying $\partial u_i \neq 0$ with the given qdz^2 as their Hopf differential not related by an isometry of $\mathbb{H}^2$, where u_1 is an orientation-preserving harmonic embedding.*

Proof. Applying Theorem **??**, we have two distinct solutions w_1, w_2 to Equation (**??**) for the same holomorphic quadratic differential qdz^2. Then by the argument in the proof of Theorem **??**, both pairs (w_1, q) and (w_2, q) develop parabolic constant mean curvature immersions from $\mathbb{C}$ into $\mathbb{M}^{2,1}$ whose Gauss maps are harmonic maps u_1, u_2 from $\mathbb{C}$ to $\mathbb{H}^2$ giving back these two given pairs $(w_1, q), (w_2, q)$ respectively. As in the proof in Theorem **??**, the harmonic map u_1 is an orientation-preserving harmonic embedding. $\square$

6. Hyperbolic affine spherical immersion

For a non-compact simply connected 2-manifold M, consider a locally strictly convex immersed hypersurface $f : M \to \mathbb{R}^3$. Affine differential geometry associates to such a locally convex hypersurface a transversal vector field ξ, called the affine normal. Being an affine spherical immersion means the affine normal meets at a point (the center). By applying a translation, we can move the center of the affine sphere to the origin and write $\xi(p) = -Hp$, for all $p \in f(M) \subset \mathbb{R}^3$ for some constant $H \in \mathbb{R}$, the affine curvature. In the case when H is negative, call the affine spherical immersion hyperbolic. After renormalization, we obtain a hyperbolic affine spherical immersion with center 0 and of affine curvature -1.

Decomposing the standard connection D of $\mathbb{R}^3$ into tangent direction of $f(M)$ and affine normal components:

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \quad \forall X, Y \in T_{f(p)} f(M).$$

The second fundamental form h of the image $f(M)$ relative to the affine normal ξ can define a Riemannian metric h on M, the Blaschke metric. This induces a complex structure on M. Also, the decomposition defines an induced connection ∇ on TM. Let ∇^h be the Levi-Civita connection of the Blaschke metric h and the Pick form $A(X,Y,Z) = h((\nabla - \nabla^h)_X Y, Z)$ is a 3-tensor, which uniquely determines a cubic differential $U = U(z)dz^3$ such that $\Re U = A$, the Pick differential. We focus on the case where (M, h) is conformal to $(\mathbb{C}, |dz|^2)$. In this case, we can reparametrize the hyperbolic affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with the Blaschke metric is $e^w|dz|^2$.

Write the Blaschke metric $h = e^w|dz|^2$ and $U = U(z)dz^3$. Denoting $f_z, f_{\bar{z}}$ for $f_*(\frac{\partial}{\partial z}), f_*(\frac{\partial}{\partial \bar{z}})$ respectively. Following Wang [**?**] and Simon-Wang [**?**], the immersion f being an affine spherical immersion implies that the frame field $(f, f_z, f_{\bar{z}})$ satisfying a linear first-order system of PDEs

$$\frac{\partial}{\partial z}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & w_z & Ue^{-w} \\ \frac{1}{2}e^w & 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix},$$

$$\frac{\partial}{\partial \bar{z}}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2}e^w & 0 & 0 \\ 0 & \bar{U}e^{-w} & w_{\bar{z}} \end{pmatrix}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}}. \end{pmatrix}$$

Hence, the integrability for the above system (i.e. $f_{z\bar{z}} = f_{\bar{z}z}$) is equivalent to U is holomorphic and

$$\triangle w = 2e^w - 4|U|^2 e^{-2w}. \tag{5}$$

The curvature of the Blaschke metric $h = e^w|dz|^2$ is

$$k_h = -\frac{1}{2}\triangle_h w = -1 + 2|U|^2 e^{-3w}.$$

Conversely, given a function $w : \mathbb{C} \to \mathbb{R}$ and a holomorphic cubic differential $U = U(z)dz^3$ satisfying Equation (**??**), we can integrate the linear first-order system (**??**) and develop an affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ since the domain $\mathbb{C}$ is simply connected. And f has Pick differential $U = U(z)dz^3$ and Blaschke metric $e^w|dz|^2$.

In conclusion, any pair (w, U) satisfying U is holomorphic and Equation (**??**) is in one-to-one correspondence with an affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with the Blaschke metric $e^w|dz|^2$ and the Pick differential Udz^3 up to a projective transformation of $\mathbb{R}^3$.

Proposition 6.1. *([**?**, **?**]) If the Blaschke metric of an affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ is complete, then f is a proper embedding, and its image is asymptotic to the boundary of an open convex set.*

We then drop the condition on the completeness of the Blaschke metric.

Theorem 6.2. *If U is a polynomial function, there is a unique affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with its Pick differential as Udz^3 up to a projective transformation of $\mathbb{R}^3$. Moreover, it is complete, properly embedded and its Blaschke metric is negatively curved.*

Proof. Applying Theorem **??**, the polynomial U uniquely determines the pair (w, U) satisfying Equation (**??**) Then by the discussion before Prop **??**, there is a unique affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with its Pick differential Udz^3 up to a projective transformation of $\mathbb{R}^3$. Moreover since the Blaschke metric $e^w|dz|^2$ is complete, by Proposition **??**, f is a proper embedding. The curvature of Blaschke metric is $k_h = -1 + 2|U|^2e^{-3w}$, by $2|U|^2e^{-3w} < 1$, we obtain that the curvature of Blaschke metric $k_h < 0$. □

Theorem **??** gives a negative answer to the following question.

Question 6.3. *([**?**]) Does there exist an affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with polynomial Pick differential whose Blaschke metric has positive curvature at some point?*

Remark 6.4. *When U is a polynomial of degree k, the image of the properly embedded affine sphere is asymptotic to a cone generated by a $(k+3)$-polygon in $\mathbb{R}^2$, as shown in [**?**].*

Next, we study the case when U is not a polynomial.

Theorem 6.5. *Given a non-polynomial entire function U with finite zeros, then there are at least two affine spherical immersions $f_1, f_2 : \mathbb{C} \to \mathbb{R}^3$ with the given Udz^3 as their Pick differential, where*
(1) f_1 is properly embedded. The Blaschke metric is complete and negatively curved; while
(2) f_2 is not properly embedded. The Blaschke metric is incomplete and nonpositively curved.

Proof. Applying Theorem **??**, we obtain two solutions (w_1, w_2) to Equation where $e^{w_1}|dz|^2$ is complete while $e^{w_2}|dz|^2$ is incomplete. Then the two pairs (w_1, Udz^3) and (w_2, Udz^3) develop two affine spherical immersions f_1, f_2 :

$\mathbb{C} \to \mathbb{R}^3$ respectively. Since $e^{w_1}|dz|^2$ is complete, by Proposition **??**, f_1 is a proper embedding. By Theorem **??**, w_1 satisfies $2|U|^2 e^{-3w_1} < 1$, implying the curvature $k_1 = -1 + 2|U|^2 e^{-3w_1} < 0$; w_2 satisfies $2|U|^2 e^{-3w_2} \leq 1$, implying the curvature $k_2 = -1 + 2|U|^2 e^{-3w_2} \leq 0$. □

Centre for Quantum Geometry of Moduli Spaces (QGM), Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark

Department of Mathematics, California Institute of Technology, 1200 East California Boulevard, Pasadena, US

E-mail address: qiongling.li@gmail.com