On Covering Codes and Upper Bounds for the Dimension of Simple Games

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Abstract

Consider a situation with n agents or players where some of the players form a coalition with a certain collective objective. Simple games are used to model systems that can decide whether coalitions are successful (winning) or not (losing). A simple game can be viewed as a monotone boolean function. The dimension of a simple game is the smallest positive integer d such that the simple game can be expressed as the intersection of d threshold functions where each threshold function uses a threshold and n weights. Taylor and Zwicker have shown that d is bounded from above by the number of maximal losing coalitions. We present two new upper bounds both containing the Taylor/Zwicker-bound as a special case. The Taylor/Zwicker-bound imply an upper bound of $\binom{n}{\frac{n}{2}}$. We improve this upper bound significantly by showing constructively that d is bounded from above by the cardinality of any binary covering code with length n and covering radius 1. This result supplements a recent result where Olsen et al. showed how to construct simple games with dimension $|C|$ for any binary constant weight SECDED code C with length n. Our result represents a major step in the attempt to close the dimensionality gap for simple games.

1 Introduction

Consider a multi agent system where a coalition of agents is formed in order to solve a given task and where we have to predict if the coalition will succeed or not. We restrict our attention to cases obeying the natural monotonicity condition saying that the superset of any successful coalition will also succeed. In such a multi agent system we need some sort of system that can compute a prediction: "yes" or "no". The so-called *simple games* model such systems and simple games can also be viewed as monotone boolean functions or monotone hypergraphs. The agents in a simple game are referred to as players and successful and unsuccessful coalitions will be referred to as winning and losing coalitions respectively.

A *weighted game* is a special type of simple game where every player is assigned a weight and where a coalition is successful if and only if the total weight of the players in the coalition is meeting or exceeding a given quota. Any simple game can be implemented as the intersection of one or more weighted games and the *dimension* [14] of a simple game is the minimum number of weighted games we need to implement the simple game in this way. The dimension has a direct influence on the storage requirements and the efficiency for such a system. Real world voting systems can be seen as simple games and the dimension aspects of real world voting systems have been studied intensively within the field of *Computational Social Choice*.

In this paper, we consider the maximum dimension, d_n, that we can obtain for a simple game with n players. Taylor and Zwicker [14] have shown that $\binom{n}{\lfloor n/2 \rfloor}$ is an upper bound for d_n by demonstrating how to implement any simple game as the intersection of no more than $\binom{n}{\lfloor n/2 \rfloor}$ games (details will follow later). The main contribution of this paper is a constructive major improvement of the generic upper bound provided by Taylor and Zwicker that we present in the form of two new upper bounds both representing stronger versions of the upper bound presented by Taylor and Zwicker.

We apply a technique that – to the best of our knowledge – has not been used before to translate any simple game into the intersection of relatively few simple games. Recently, Olsen et al. [12] demonstrated a major improvement in the lower bound on d_n by using theory on *error correcting codes*. We use a significantly different and novel approach based on *covering codes* to obtain our upper bounds. The gap between the upper and lower bound has for some n gone from a factor n to $\sqrt{n}$ (roughly) through our improvement and to a factor $\ln n \sqrt{n}$ in general. We conclude by suggesting a direction that might lead to a further reduction of the dimensionality gap.

2 Related Work

Taylor and Zwicker [14] have constructed a sequence of games with dimension at least $2^{\frac{n}{2}-1}$ for $n = 2k$ with k odd. The dimension of the simple games presented by Taylor and Zwicker was later shown to be exactly $2^{\frac{n}{2}-1}$ [12]. Freixas and Puente [8] have shown how to construct another type of simple games with dimension $2^{\frac{n}{2}-1}$ for all even n. This lower bound of d_n was recently improved significantly by Olsen et al. [12] by establishing a connection to the theory on error correcting codes resulting in the following lower bound:

$$d_n \geq \frac{1}{n}\binom{n}{\lfloor \frac{n}{2} \rfloor} \in 2^{n-o(n)} \ . \tag{1}$$

Here it might be useful to consider the following identity for comparison with the previous lower bound:

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = (1 - o(1))\sqrt{\frac{2}{\pi n}} 2^n \ . \tag{2}$$

Kurz and Napel [10] also present a general approach for the determination of lower bounds for the dimension of a simple game.

A maximal losing coalition in a simple game is a losing coalition that has the property that adding any player will turn it into a winning coalition. Let L^M denote the collection of maximal losing coalitions. Taylor and Zwicker [14] demonstrate how to express any simple game as an intersection of at most $|L^M|$ weighted games implying an $|L^M|$-upper bound for d_n. Kurz and Napel [10] provide heuristic algorithms based on integer linear programming for constructing a representation of a given simple game as an intersection of weighted games.

As mentioned earlier, the dimension of real world voting systems has been the focus for several studies. The Amendment of the Canadian constitution [9] and the US federal legislative system [15] have dimension 2. The voting systems of the Legislative Council of Hong Kong [3] and the Council of the European Union under its Treaty of Nice rules [6] have a dimension of exactly 3. Kurz and Napel [10] have established that the dimension of the voting system of the Council of the European Union under its Treaty of Lisbon rules is between 7 and 13 368.

There are obviously alternative ways for representing simple games. The *codimension* [7] is the minimum number of weighted games it takes to represent a simple game as a union of weighted games. Considering arbitrary combinations of unions and intersections leads to the notion of *boolean dimension*, which is introduced and studied in [5].

2.1 Outline of the Paper

The next section introduces the notation and the formal definitions for simple games. We also give a brief introduction for readers not familiar with covering codes. The algorithm behind our first upper bound on d_n is then presented in two sections. The first of the sections demonstrates how the algorithm works and the second section contains the technical details and proofs including a formal statement of the upper bound in terms of a theorem. The second upper bound and our second theorem is then presented in a section and finally, we wrap the paper up in the conclusion.

3 Preliminaries

In this section, we introduce the concepts and definitions that we consider in this paper. We start by presenting formal definitions for simple games. After that we give a brief introduction to covering codes.

3.1 Simple Games

We now formally the define simple games:

Definition 1. *A simple game* $\Gamma = (N, W)$ *is a pair where* $N = \{1, \ldots, n\}$ *for some positive integer* n *and* $W \subseteq 2^N$ *is a collection of subsets of* N *such that:*

- $\emptyset \notin W$

- $N \in W$

- $S \subseteq T \subseteq N$ *and* $S \in W$ *implies* $T \in W$

The members of N are referred to as players and subsets of N are referred to as coalitions. A coalition is said to be winning if it is a member of W and otherwise it is said to be losing. The first condition says that the coalition with no players loses and the second condition ensures that the coalition containing all players wins. The third condition is the monotonicity condition that says that any superset of a winning coalition is also winning. The set of losing coalitions is denoted by $L = 2^N \setminus W$.

A coalition is a maximal losing coalition if it is losing and all of its supersets are winning. The collection of coalitions $L^M \subset 2^N$ contains all the maximal losing coalitions. The collection of minimal winning coalitions W^m is defined accordingly. A simple game Γ can be defined by either of the sets W, L, W^m or L^M.

The weighted games that form a proper subset of the simple games are defined as follows:

Definition 2. *A simple game* $\Gamma = (N, W)$ *is weighted if there exists a* quota $q \in \mathbb{R}_+$ *and* weights $w_1, w_2, \ldots, w_n \in \mathbb{R}_+$ *such that* $S \in W$ *if and only if* $\sum_{i \in S} w_i \geq q$. *In this case we use the notation* $\Gamma = [q; w_1, w_2, \ldots, w_n]$.

The intersection $\Gamma_1 \cap \Gamma_2$ of the games $\Gamma_1(N, W_1)$ and $\Gamma_2(N, W_2)$ is the simple game with players N and $W = W_1 \cap W_2$. As previously mentioned, Taylor and Zwicker [14] have shown that any simple game can be expressed as the intersection of $|L^M|$ weighted games: For any game Γ, we have $\Gamma = \cap_{T \in L^M} \Gamma_T$ where a coalition S wins in Γ_T if and only if $S \not\subseteq T$. A weighted representation of Γ_T using weights 0 and 1 is given as follows: the game has quota 1 and a player in $N \setminus T$ is assigned the weight 1 and all other players are assigned weight 0.

The dimension of a simple game can now be formally defined:

Definition 3. *The dimension* d *of a simple game* Γ *is the smallest positive integer such that* $\Gamma = \cap_{i=1}^d \Gamma_d$ *where the games* Γ_i, $i \in \{1, 2, \ldots, d\}$, *are weighted.*

In this paper, we let d_n denote the maximum dimension that we can observe for a simple game with n players.

A maximal losing coalition cannot contain another maximal losing coalition, so we can apply Sperner's Lemma [11] and get the following upper bound on L^M: $|L^M| \leq \binom{n}{\lfloor n/2 \rfloor}$. From the construction by Taylor and Zwicker, we conclude the following:

$$d_n \leq |L^M| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \ . \tag{3}$$

The main objective of this paper is to improve this upper bound.

We will illustrate the definitions by an example.

Example 1. *Let the simple game* $\Gamma(N, L^M)$ *be defined as follows:*

$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$L^M = \{\{1, 2, 3\}, \{3, 4, 5, 6\}\} \ .$$

The coalition $\{1, 2\}$ loses in Γ since $\{1, 2\} \subseteq \{1, 2, 3\}$. The coalition $\{1, 4\}$ wins since $\{1, 4\} \not\subseteq \{1, 2, 3\}$ and $\{1, 4\} \not\subseteq \{3, 4, 5, 6\}$.

If we use the construction by Taylor and Zwicker, we get this representation of Γ as the intersection of two weighted games:

$$\Gamma = [1; 0, 0, 0, 1, 1, 1, 1] \cap [1; 1, 1, 0, 0, 0, 0, 1] .$$

The dimension of Γ is 2 since Γ cannot be weighted. We can realize this using a proof by contradiction that illustrates a classical way of establishing lower bounds for the dimension:

Assume that Γ was weighted with quota q. The coalitions $\{1, 2\}$ and $\{4, 5\}$ are both losing so the total weight of the players in the coalitions must be strictly smaller than $2q$. The two coalitions can exchange players and both win after the exchange: $\{1, 4\}$ and $\{2, 5\}$. We now arrive at a contradiction since the total weight of the players must be at least $2q$.

We now turn our attention to covering codes.

3.2   Covering Codes

A binary *code* is technically a set of bit vectors. A bit vector $x = x_1 x_2 \ldots x_n \in \{0, 1\}^n$ can be viewed as the coalition $S_x = \{i \in N : x_i = 1\}$. We will use this perspective and see a binary code as a collection of coalitions in order to align the notation of binary codes and simple games. The Hamming distance between two bit vectors is the number of coordinates where the two bit vectors differ. Using the perspective just described, we can define the Hamming distance between two coalitions x and y as follows:

$$d(x, y) = |x \setminus y| + |y \setminus x| .$$

A binary *covering code* [4] of length n and *covering radius* 1 can consequently be perceived as a collection $C \subset 2^N$ of coalitions such that any coalition is within Hamming distance 0 or 1 from at least one member of C: $\forall x \in 2^N \exists c \in C : d(x, c) \leq 1$. As an example, covering codes have applications within data compression. In this paper, K_n denotes the minimum cardinality of a binary covering code of length n with covering radius 1

Example 2. *The following set represents a binary covering code with length 4 and covering radius 1:*
$$C = \{\{\}, \{4\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$$

As an example, the coalition $\{2, 4\}$ is covered by the coalition $\{4\}$ in C since the Hamming distance between these coalitions is 1.

It is not possible to cover all subsets of $\{1, 2, 3, 4\}$ with fewer coalitions – we cannot cover more than $3 \cdot 5 = 15$ coalitions with 3 coalitions and there are 16 coalitions in total so our example shows that $K_4 = 4$.

As we saw earlier, a coalition cannot cover more than $n + 1$ other coalitions including itself within radius 1 so we need at least $2^n/(n + 1)$ coalitions for a binary

covering code with covering radius 1. The well known Hamming codes [2] defined for $n = 2^m - 1$ are so called *perfect* codes that meet this lower bound. For $n = 2^m$, we have the slightly smaller value in the denominator: $K_n = 2^n/n$ [13]. In general, it is hard to establish exact values for K_n but it is not hard to prove the upper bound $K_n \leq (\ln(n+1)+1)2^n/(n+1)$ using a classical result[1] from Alon and Spencer [1] on computing dominating sets.

4   The First Upper Bound

From now on, any simple game will be defined using maximal losing coalitions. Given a simple game $\Gamma(N, L^M)$, we now present an algorithm producing a representation of Γ as an intersection of no more than K_n weighted games. In this section, we will show how the algorithm works step by step. Each step will contain a formal explanation but we will also illustrate how each step works through an example. The technical details including the proof of correctness will follow in the next section.

The key idea for the algorithm is the result of a simple observation expressed by the following lemma:

Lemma 1. *If $L^M = \cup_{i=1}^{p} L_i$ then*

$$\Gamma(N, L^M) = \cap_{i=1}^{p} \Gamma(N, L_i) \ .$$

Proof. Now assume that $x \subseteq N$ is losing in $\Gamma(N, L^M)$. There must be an $y \in L^M$ such that $x \subseteq y$. This means that x loses in any of the games $\Gamma(N, L_i)$ with $y \in L_i$. On the other hand, x will lose in $\Gamma(N, L^M)$ if x loses in $\cap_{i=1}^{p} \Gamma(N, L_i)$ since x must be a subset of at least one y that is a member of L^M.   □

The objective for our algorithm is to use the lemma and partition L^M into a small number of sets such that all the corresponding games are weighted.

The game that we use as an example is the following simple game with players $\{1, 2, 3, 4\}$: A coalition wins if and only if both of the players 1 and 2 are members of the coalition or both of the players 3 or 4 are members. As an example, the coalition $\{1, 2, 4\}$ wins since both of the players 1 and 2 have joined the coalition. On the other hand, the coalition $\{1, 3\}$ is losing – and in fact it is a maximal losing coalition since this coalition will turn into a winning coalition if any of the other players joint it. As a side remark, this game belongs to a class of simple games that has been studied in detail by Freixas and Puente [8].

We are now ready to describe how our algorithm works:

4.1   Input

The input to the algorithm is a simple game $\Gamma(N, L^M)$. Example:

$$N = \{1, 2, 3, 4\}$$

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[1]Consider the graph where we have a vertex for each coalition and an edge between two vertices if and only if the distance between the corresponding coalitions is 1. A covering code corresponds to a dominating set in this graph where all vertices have degree n. Alon and Spencer present a lower bound for the size of such a dominating set. This upper bound is probably well known within the coding theory community.

$$L^M = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$$

4.2 Step 1

Construct a collection of coalitions $C \subset 2^N$ such that any coalition in L^M is within Hamming distance 0 or 1 from at least one coalition in C: $\forall x \in L^M \exists c \in C : d(x,c) \leq 1$. Example:

$$C = \{\{4\}, \{1,2,3\}\}$$

4.3 Step 2

Let $\{L_c\}_{c \in C}$ be a partition of L^M such that all members of L_c have distance 0 or 1 to c: $\forall x \in L_c : d(x,c) \leq 1$. Example:

$$L_{\{4\}} = \{\{1,4\}, \{2,4\}\}$$

$$L_{\{1,2,3\}} = \{\{1,3\}, \{2,3\}\}$$

4.4 Step 3

For each $c \in C$, we now represent $\Gamma(N, L_c)$ as a weighted game $[q^c; w_1^c, w_2^c, \ldots, w_n^c]$. We prove that $\Gamma(N, L_c)$ is weighted for any $c \in C$ and provide the details on how to compute the weights and the quota in Lemma 2 below. Example:

$$\Gamma(N, L_{\{4\}}) = [2; 1, 1, 2, 0]$$

$$\Gamma(N, L_{\{1,2,3\}}) = [2; 1, 1, 0, 2]$$

4.5 Output

Finally, we can use Lemma 1 and express Γ as the intersection of the weighted games that we have constructed in Step 3: $\Gamma = \cap_{c \in C}[q^c; w_1^c, w_2^c, \ldots, w_n^c]$. Example:

$$\Gamma = [2; 1, 1, 2, 0] \cap [2; 1, 1, 0, 2]$$

This concludes the description of our algorithm.

In Step 1, we can actually use a binary covering code of length n with covering radius 1 for *any* simple game involving n players. As a consequence, any simple game can be implemented as the intersection of no more than K_n weighted games. This allows us to set up the following upper bounds on d_n using the facts on K_n from the previous section:

$$d_n \leq K_n = \frac{2^n}{n+1} \text{ for } n = 2^m - 1 \tag{4}$$

$$d_n \leq K_n = \frac{2^n}{n} \text{ for } n = 2^m \tag{5}$$

$$d_n \leq K_n \leq (\ln(n+1) + 1)\frac{2^n}{n+1} \text{ for all } n . \tag{6}$$

Table 1: This table displays lower and upper bounds for d_n combining our findings with the results from [13] and [12].

n	Lower bound	Upper bound	$\binom{n}{\lfloor n/2 \rfloor} - 1$
6	4	12	19
7	7	16	34
8	14	32	69
9	18	62	125
10	36	120	251
11	66	192	461
12	132	380	923
13	166	704	1715
14	325	1408	3431
15	585	2048	6434

In all three cases, the upper bounds are considerably smaller than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ which can be seen from (2). The first two upper bounds represent an improvement on roughly a factor $\sqrt{n}$ and the bounds are all $o(\binom{n}{\lfloor \frac{n}{2} \rfloor})$.

It is important to observe that it might be a bad idea to use a binary covering code as a "one size fits all"-solution since we do not exploit the structure of L^M for the specific game at hand if we follow this approach.

Ostergård and Kaikkonen [13] have listed some upper bounds for K_n that we also can use as upper bounds for d_n. Table 1 presents these upper bounds together with lower bounds from [12].

5   Technical Details for the First Upper Bound

We now take another look at our approach where we formally prove our first upper bound and state the bound as a theorem.

We have to ensure is that our algorithm is correct in the sense that it is able to express any input game as an intersection of weighted games. It is clearly possible to produce the collection C in Step 1 and to construct the partition of L^M in Step 2. The algorithm uses the decomposition approach suggested by Lemma 1 so we only have to check that all the games considered in Step 3 are weighted.

Lemma 2. $\Gamma(N, L_c)$ *is weighted for any* $c \in C$.

Proof. All the members of L_c are maximal losing coalitions so it is not possible to find two members of L_c such that one of them contains the other. This means there are three cases that we have to consider: 1) $\forall x \in L_c : x \subset c$, 2) $L_c = \{c\}$, or 3) $\forall x \in L_c : c \subset x$. We now show how to express $\Gamma(N, L_c)$ as a weighted game in all three cases.

Case 1: The set L_c consists of coalitions were exactly one element has been removed from c for each member of L_c. Let R denote the set of removed elements: $R = \cup_{x \in L_c}(c \setminus x)$. Let us consider a set S that is winning and is contained in c. For

any $x \in L_c$, we know that S is not contained in x so S must contain the element that has been removed from c to form x. In other words, S cannot win in $\Gamma(N, L_c)$ unless $S \setminus c \neq \emptyset$ or $R \subseteq S$. On the other hand, it is not hard see that S wins if $S \setminus c \neq \emptyset$ or $R \subseteq S$. This means that we can implement $\Gamma(N, L_c)$ as the weighted game with $q = |R|$ and weights as follows: $w_i = |R|$ for $i \notin c$, $w_i = 1$ for $i \in R$ and $w_i = 0$ for the remaining players.

As an example, we consider the game $\Gamma(N, L_c)$ with $N = \{1, 2, 3, 4, 5\}$, $c = \{1, 2, 3, 4\}$ and $L_c = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. For this game we have $R = \{2, 3, 4\}$ and $\Gamma(N, L_c) = [3; 0, 1, 1, 1, 3]$.

Case 2: In this case, we can use the weighted game with quota $q = 1$ where we assign the weight 0 to all players in c and the weight 1 to all other players.

Case 3: All the members of L_c are constructed by adding exactly one element to c. Let A denote the set of added elements: $A = \cup_{x \in L_c}(x \setminus c)$. If a coalition S wins and S does not contain any players in $c \cup A$ then S has to contain at least two players in A (otherwise S would lose). Conversely, S wins if S contains a player not in $c \cup A$ or at least two of the players in A. This implies that $\Gamma(N, L_c)$ can be expressed as a weighted game with quota $q = 2$ and the following weight distribution: $w_i = 2$ for $i \notin c \cup A$, $w_i = 1$ for $i \in A$ and $w_i = 0$ for the players in c.

An example for case 3: $\Gamma(N, L_c)$ with $N = \{1, 2, 3, 4, 5, 6, 7\}$, $c = \{1, 2, 3\}$ and $L_c = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}\}$. Here we have $A = \{4, 5, 6\}$ and $\Gamma(N, L_c) = [2; 0, 0, 0, 1, 1, 1, 2]$. □

We are now ready to formally state the main contribution of our paper:

Theorem 1. *Let $\Gamma(N, L^M)$ be a simple game and let $C \subset 2^N$ be a collection of coalitions such that $\forall x \in L^M \exists c \in C : d(x, c) \leq 1$. The dimension of $\Gamma(N, L^M)$ is bounded from above by $|C|$.*

Proof. We can use our algorithm to produce a representation of Γ as the intersection of $|C|$ weighted games. Lemma 1 and Lemma 2 guarantee that our algorithm is correct. □

It is important to note that the special case $C = L^M$ corresponds to the $|L^M|$-upper bound presented by Taylor and Zwicker [14].

If we have a binary covering code with covering radius 1 then we can use it as C in the theorem. We therefore have the following corollary:

Corollary 1.
$$d_n \leq K_n$$

It is important to stress that we only require C to "cover" the set L^M in the theorem above. We might be able to exploit the structure of L^M in order to achieve a better upper bound than in the corollary where the underlying collection covers all possible coalitions. As an example, we might use the fact that L^M is a Sperner family where no member contains another member of the family. This explains why we have chosen to express the bound $d_n \leq K_n$ as a corollary since the theorem is a stronger result.

6 The Second Upper Bound

In this section, we will once again use the key idea from Lemma 1 and prove another upper bound generalizing the $|L^M|$-upper bound presented by Taylor and Zwicker [14]. This upper bound is related to SECDED codes that are binary codes where any two of the members have pairwise distance at least 4.

Theorem 2. *Let $\Gamma(N, L^M)$ be a simple game. The dimension of $\Gamma(N, L^M)$ is bounded from above by $\frac{1}{2}(|L^M|+|C|)$ for some collection $C \subseteq L^M$ of maximal losing coalitions satisfying $\forall x, y \in C : d(x, y) \geq 4$.*

Proof. Let M be a maximal set of pairs $(x, y) \in L^M \times L^M$ such that $x \neq y$ and $d(x, y) \leq 3$ and such that an element in L^M occurs in no more than one pair. We claim that the game $\Gamma(N, \{x, y\})$ is weighted for any $(x, y) \in M$. We will prove it for the case $d(x, y) = 3$ and leave the only remaining case $d(x, y) = 2$ to the reader (there are no more cases since L^M is a Sperner family).

Without loss of generality, we assume that $x \setminus y$ contains two players and $y \setminus x$ contains one player. A coalition wins in the game if and only if: 1) the coalition contains at least one player in $N \setminus (x \cup y)$, or 2) the coalition contains one of the players in $x \setminus y$ and the player in $y \setminus x$. We implement the game $\Gamma(N, \{x, y\})$ as a weighted game with quota $q = 3$. The players in $N \setminus (x \cup y)$ get weight 3. The two players in $x \setminus y$ get weight 1 and the player in $y \setminus x$ gets weight 2. All the players in $x \cap y$ are assigned the weight 0.

Let us illustrate the construction with the example with $N = \{1, 2, 3, 4, 5, 6, 7\}$, $x = \{1, 2, 3, 4\}$ and $y = \{2, 3, 5\}$. The corresponding weighted game is $[3; 1, 0, 0, 1, 2, 3, 3]$.

Let C be the set of coalitions that have not been paired in M. All the coalitions in C have pairwise distance at least 4 since M is maximal. The pairs in M and the coalitions in C considered as single element sets constitute a partition of L^M where all the corresponding games are weighted. This partition consists of no more than $\frac{1}{2}(|L^M| - |C|) + |C|$ coalitions. $\square$

A corollary of the theorem is as follows:

Corollary 2. *The dimension of $\Gamma(N, L^M)$ is less than $|L^M|$ if L^M is not a SECDED code.*

7 Conclusion

We have presented two new upper bounds on the maximum dimension d_n for simple games with n players. The bounds are related to binary codes and they represent improvements of the $|L^M|$-upper bound presented by Taylor and Zwicker [14].

The recent development [12] for the lower bound of d_n can be illustrated as follows:

$$2^{\frac{n}{2}-1} \rightarrow \frac{1}{n}\binom{n}{\lfloor\frac{n}{2}\rfloor} = (1 - o(1))\sqrt{\frac{2}{\pi n}}\frac{2^n}{n} \leq d_n \ . \tag{7}$$

On the other hand, one of the upper bounds in our paper represents the following improvement with respect to the upper bound for $n = 2^m - 1$:

$$d_n \leq \frac{2^n}{n+1} \leftarrow (1 - o(1)) \sqrt{\frac{2}{\pi n}} 2^n = \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{8}$$

The dimensionality gap for the simple games is now considerably smaller and the upper bound is roughly within a factor $\sqrt{n}$ away from the lower bound for some values of n.

As previously mentioned, we only have to cover L^M with a binary covering code with radius 1 to obtain an upper bound on the dimension as expressed by Theorem 1. It is not known – at least to the authors of this paper – whether it is possible but it seems plausible to improve the upper bound from (8) by using the fact that L^M has a certain structure.

The key idea behind our upper bounds is to decompose L^M into a union of collections of maximal losing coalitions such that any of the simple games defined by the component collections are weighted. This can be done in many ways and it is highly likely that there are smarter decompositions than the ones presented in our paper. It is an open problem to find smarter decompositions.

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