Facets for Node-Capacitated Multicut Polytopes from Path-Block Cycles with Two Common Nodes☆

Michael M. Sørensen

Department of Economics and Business Economics, Aarhus University, Fuglesangs Allé 4, DK-8210 Aarhus V, Denmark

Abstract

A path-block cycle is a graph that consists of several cycles that all intersect in a common subset of nodes. The associated path-block-cycle inequalities are valid, and sometimes facet-defining, inequalities for polytopes in connection with graph partitioning problems and corresponding multicut problems. Special cases of the inequalities were introduced by De Souza & Laurent (1995) and shown to be facet-defining for the equicut polytope. Generalizations of these inequalities were shown by Ferreira *et al.* (1996) to be valid for node-capacitated graph partitioning polytopes on general graphs.

This paper considers the special case of the inequalities, where all cycles intersect in two nodes, and establishes conditions under which these inequalities induce facets of node-capacitated multicut polytopes and bisection cut polytopes. These polytopes are associated with simple versions of the node-capacitated graph partitioning and bisection problems, where all node weights are assumed to be 1.

Keywords: graph bisection, graph partitioning, polyhedral combinatorics
2000 MSC: 90C27, 90C57

1. Introduction

The node-capacitated graph partitioning problem, described below in detail, is an NP-hard combinatorial optimization problem. We work towards a polyhedral approach to solve this problem. In this approach, one starts out by formulating the problem as an integer linear program (ILP). Subsequently, one derives strong valid — preferably facet-defining — inequalities that can be used as cutting planes to tighten the continuous relaxation of the ILP. The purpose of this paper is to present one such type of inequalities for the problem and

☆Accepted for publication in *Discrete Optimization*, March 2017.
DOI: 10.1016/j.disopt.2017.03.001
Email address: mim@econ.au.dk (Michael M. Sørensen)

to determine its facet-defining properties. We refer the reader to [1] and the book [12] for more information on this approach.

A path-block cycle (PBC) is a graph that consists of several cycles that all intersect in a common subset of nodes. De Souza & Laurent [5] introduced a special case of PBC inequalities and established the conditions under which these inequalities define facets of the equicut polytope. Ferreira *et al.* [7] considered the general form of PBC inequalities and gave conditions for the inequalities to be valid for node-capacitated multicut polytopes.

Except for the case in [5], little is known about the facet-defining properties of PBC inequalities for various versions of node-capacitated multicut polytopes. In this paper, we characterize the facet-defining PBC inequalities that can be obtained from PBCs where all cycles intersect in exactly two nodes.

The structure of the paper is as follows. Below, we define the node-capacitated graph partitioning problem and consider the associated polytope and a few other polytopes that are of interest here. This also involves a short literature review. Section 2 considers the general form of the PBC inequalities and introduces some necessary conditions for their facet-defining properties. Sections 3 and 4 deal with the PBC inequalities that are obtained from PBCs with two common nodes and establish their facetness properties. Section 3 considers the bisection cut and node-capacitated multicut polytopes that are associated with the PBC graphs, while Section 4 considers the polytopes on complete graphs. Finally, Section 5 contains a few concluding remarks.

1.1. The node-capacitated graph partitioning problem and notation

Given a graph $G = (V, E)$ and a partition $\{S_1, \ldots, S_m\}$ of V, where $m \geq 2$ and $S_i \neq \emptyset$, for $i = 1, \ldots, m$, the corresponding multicut is the subset of edges $\delta_E(S_1, \ldots, S_m) = \{uv \in E : u \in S_i,\ v \in S_j;\ i \neq j\}$. We refer to the node subsets S_i of the partition of V as the *shores* of the multicut. Given $b \in \mathbb{Z}_+$, we call a multicut *feasible* if $|S_i| \leq b$, for $i = 1, \ldots, m$. Note that the number m of shores of a feasible multicut is not fixed, but can be any integer between $\lceil |V|/b \rceil$ and $|V|$. Given weights $w_e \in \mathbb{R}$, for all $e \in E$, the node-capacitated graph partitioning problem associated with graph G is to determine a feasible multicut in G that minimizes (or maximizes) the sum of the weights of the edges in the multicut. This problem is known to be NP-hard in the strong sense; see A.2.2 in [9]. We will assume in the following that all multicuts considered are feasible, unless otherwise stated.

Now, let us introduce some shorthand notation that will be used in connection with the complete graph $K_n = (V_n, E_n)$ on n nodes. We will often skip explicit references to the edge set E_n. For example, cuts or multicuts in E_n are denoted by $\delta(\cdot)$ instead of $\delta_{E_n}(\cdot)$. Accordingly, $M = \delta(S_1, \ldots, S_m) \subset E_n$ identifies a multicut M with shores $S_1, \ldots, S_m$. For a subset of nodes $U \subseteq V_n$, $E_n(U)$ is the set of edges with both end nodes in U. When we deal with vectors $a, y \in \mathbb{R}^{E_n}$, we may write $a^T y$ instead of $\sum_{e \in E_n} a_e y_e$, and, for a subset of edges $F \subseteq E_n$, we let $y(F)$ denote $\sum_{e \in F} y_e$.

The following concepts will be used quite often. Consider a graph $G = (V, E)$ and suppose that $P(G)$ is an associated multicut polytope. Let $\sum_{e \in E} a_e y_e \geq a_0$

be a valid inequality for $P(G)$. The set $F_a = \{y \in P(G) : \sum_{e\in E} a_e y_e = a_0\}$ is the *face* of the polytope that is defined by this inequality. A *facet* of the polytope is a face that is properly contained in the polytope and that is maximal in the sense that it is not contained in any other face. The *incidence vector* of a multicut $M \subset E$ is the vector $\gamma \in \{0,1\}^E$ with entries $\gamma_e = 1$, for all $e \in M$, and $\gamma_e = 0$, for all $e \in E \setminus M$. An incidence vector γ that belongs to the face F_a, i.e., $\sum_{e\in E} a_e \gamma_e = a_0$, is a *root* of the inequality that defines F_a, and we say that M belongs to the *root set* of the inequality.

1.2. The node-capacitated multicut polytope

We denote the *node-capacitated multicut polytope* associated with graph G by $\mathrm{NCMC}_b(G)$. We will often consider this polytope when G is a complete graph. In this case, $\mathrm{NCMC}_b(K_n)$ is the convex hull of the vectors $y \in \{0,1\}^{E_n}$ that satisfy the inequalities

$$y_{ij} + y_{ik} - y_{jk} \geq 0 \qquad (jk \in E_n;\ i \in V_n \setminus \{j,k\}) \tag{1}$$
$$y(\delta(i)) \geq n - b \qquad (i \in V_n), \tag{2}$$

where $\delta(i)$ is the set of edges in E_n that are incident on node i. For a subgraph $G = (V,E) \subset K_n$ with $V = V_n$, the polytope $\mathrm{NCMC}_b(G)$ is obtained from $\mathrm{NCMC}_b(K_n)$ by projecting out the variables that belong to the missing edges; see [4, 7] for further details. $\mathrm{NCMC}_b(G)$ has full dimension equal to $|E|$; see [7, Proposition 4.1 and Corollary 4.2]. This means that the facets of $\mathrm{NCMC}_b(G)$ have dimension $|E|-1$, and all facet-defining inequalities are unique up to scalar multiplication.

$\mathrm{NCMC}_b(K_n)$ is the complement of the simple graph partitioning polytope $\mathrm{SGP}_b(K_n)$ studied in [13, 14]. Consider vectors $x, y \in \{0,1\}^{E_n}$ such that $x_e = 1 - y_e$, for all $e \in E_n$. Then

$$x \in \mathrm{SGP}_b(K_n) \quad \Leftrightarrow \quad y \in \mathrm{NCMC}_b(K_n).$$

The implication of this is that facets of one polytope can be transformed to facets of the other one by complementing the variables.

Several classes of valid and facet-defining inequalites for $\mathrm{NCMC}_b(K_n)$ are known (see again [13, 14]). For example, the above triangle inequalities (1), star inequalities (2), and the upper-bound constraints $y_e \leq 1$ define facets of $\mathrm{NCMC}_b(K_n)$ under very mild conditions. Of particular interest here are the following *cycle inequalities* due to [6, 7]. We will assume throughout that all cycles are proper in the sense that each node in a cycle belongs to exactly two edges.

Proposition 1. *Let $C \subset E_n$ be the edge set of a cycle such that $|C| \geq b+1$. Then the cycle inequality*

$$y(C) \geq 2 \tag{3}$$

is valid for $\mathrm{NCMC}_b(K_n)$*. It defines a facet of* $\mathrm{NCMC}_b(K_n)$ *if and only if* $|C| = b+1$.

In the following, we will also consider special cases of the *multistar inequalities* [14]. For disjoint node sets S, T, let $\delta(S, T) \subset E_n$ be the set of edges with one end node in S and the other one in T, and let $E_n(S)$ be the set of edges with both end nodes in S.

Proposition 2. *Let $d \in \{1, \ldots, b-1\}$ be an integer, and let $S, T \subset V_n$ be disjoint such that $|S| \geq 2$ and $|T| = (b-1)|S| - d$. Then the multistar inequality*

$$\sum_{e \in \delta(S,T)} (1 - y_e) + (2 - d) \sum_{e \in E_n(S)} (1 - y_e) \leq |T| \tag{4}$$

defines a facet of $\mathrm{NCMC}_b(K_n)$*, for* $b \geq 3$.

Figure 1 shows an example of a multistar with $|S| = 2$, which satisfies the conditions of this proposition.

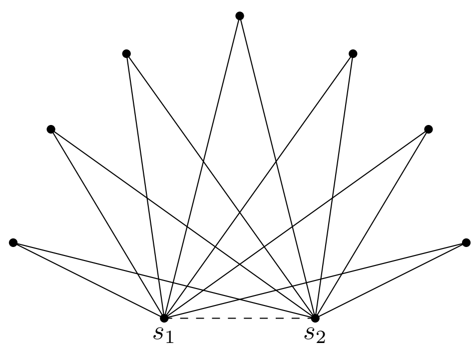

Figure 1: A multistar where $b = 7$, $d = 5$, $|S| = 2$ ($S = \{s_1, s_2\}$), and $|T| = 7$.

1.3. The bisection cut and equicut polytopes

We now consider two polytopes that are related to $\mathrm{NCMC}_b(G)$. Armbruster *et al.* [2] study the following graph bisection problem. Consider a graph $G = (V, E)$ with node weights $\varphi_v \in \mathbb{Z}_+$, for all $v \in V$, and edge weights $w_e \in \mathbb{R}$, for all $e \in E$, and let $F \in \mathbb{Z}_+$ be a node-weight capacity such that $F \geq \lceil \varphi(V)/2 \rceil$, where $\varphi(U) = \sum_{v \in U} \varphi_v$, for $U \subseteq V$. The minimum bisection problem is to determine a partition $\{S_1, S_2\}$ of V such that $\varphi(S_i) \leq F$, for $i = 1, 2$, and such that the sum of the weights of all edges in the bisection cut $\delta_E(S_1, S_2)$ is minimal.

We will consider the *simple* version of this problem, where all node weights are 1 and $F = b$ bounds the number of nodes in each shore; see also [10]. Then, every bisection cut is a multicut with two shores. We denote the corresponding *bisection cut polytope* by $\mathrm{BC}_b(G)$.

It is clear that $\mathrm{BC}_b(G) \subset \mathrm{NCMC}_b(G)$, so that all inequalities valid for $\mathrm{NCMC}_b(G)$ are also valid for $\mathrm{BC}_b(G)$. Furthermore, $\mathrm{BC}_b(G)$ is full dimensional when $n \leq 2b - 2$ (see again [7, Proposition 4.1 and Corollary 4.2]) so that any facet-defining inequality for full-dimensional $\mathrm{BC}_b(G)$ also defines a facet of $\mathrm{NCMC}_b(G)$, provided that the inequality is valid for $\mathrm{NCMC}_b(G)$.

A cut $\delta_E(S_1, S_2)$ is called an equicut if $||S_1| - |S_2|| \in \{0, 1\}$. The *equicut polytope* $\mathrm{EC}(G)$ on G is the convex hull of the incidence vectors of all equicuts in E. This polytope is studied in [3, 4, 5]. The following results on the dimension of the polytope are established in [3, lemmas 3.4–3.6]: $\mathrm{EC}(G)$ is full dimensional if and only if G has an odd number of nodes and G is not a complete graph. For the complete graph, the dimension of $\mathrm{EC}(K_n)$ is equal to $|E_n|-1$ when n is odd and equal to $|E_n| - n$ when n is even. Fortunately, it is possible to obtain the facets of these polytopes from each case to the other one [3], so that it suffices to consider only the case when $n = 2p + 1$ is odd. In this case, $\mathrm{EC}(K_{2p+1})$ is contained in the hyperplane that is defined by $y(E_{2p+1}) = p(p + 1)$.

Note that $\mathrm{BC}_{p+1}(G)$ coincides with $\mathrm{EC}(G)$ for all graphs G on $2p+1$ nodes. Otherwise, $\mathrm{EC}(G) \subset \mathrm{BC}_b(G)$. From the discussion above, it immediately follows that a facet-defining inequality for full-dimensional $\mathrm{EC}(G)$ also defines a facet of $\mathrm{BC}_b(G)$ and $\mathrm{NCMC}_b(G)$, provided that the inequality is valid for the polytopes. In the case of the equicut polytope on the complete graph K_{2p+1}, it is also possible to obtain facets of $\mathrm{NCMC}_b(K_{2p+1})$ under a certain condition, as shown in the next proposition.

Proposition 3. *Suppose the inequality $a^T y \geq a_0$ defines a facet of* $\mathrm{EC}(K_{2p+1})$ *and that it is valid for* $\mathrm{NCMC}_{p+1}(K_{2p+1})$. *If there exists a multicut whose incidence vector $\gamma \in$* $\mathrm{NCMC}_{p+1}(K_{2p+1})$ *is a root of the inequality such that $\gamma(E_{2p+1}) \neq p(p + 1)$, then $a^T y \geq a_0$ defines a facet of* $\mathrm{NCMC}_{p+1}(K_{2p+1})$.

Proof. Let $n = 2p + 1$, let F be the face of $\mathrm{NCMC}_b(K_n)$ that is defined by $a^T y \geq a_0$, and let F' be the corresponding facet of $\mathrm{EC}(K_n)$. Then, F contains F' and γ. As $F' \subset \{y \in \mathbb{R}^{E_n} : y(E_n) = p(p + 1)\}$ and $\gamma(E_n) \neq p(p + 1)$, $\dim(F) = \dim(F') + 1$. □

Proposition 3 can be used to prove that the PBC inequalities of De Souza & Laurent define facets of the node-capacitated multicut polytope $\mathrm{NCMC}_{p+1}(K_n)$, for $n \geq 2p + 1$. However, we will not give details here.

2. PBC inequalities

In this section, we present the general form of the PBC inequalities which were shown in [7] to be valid for $\mathrm{NCMC}_b(G)$. Subsequently, we present some structural properties which these inequalities must satisfy in order to induce facets of $\mathrm{NCMC}_b(K_n)$.

2.1. General PBC inequalities

As mentioned in the introduction, a general PBC is a graph that is obtained as the union of $r \geq 2$ cycles that share a common subset of nodes. Let $C^j = (V^j, E^j)$, for $j = 1, \dots, r$, be cycles such that $V^i \cap V^j = I = \{i_1, \dots, i_t\}$, for all distinct $i, j \in \{1, \dots, r\}$. We assume that $|I| \geq 2$ and that the nodes of I occur

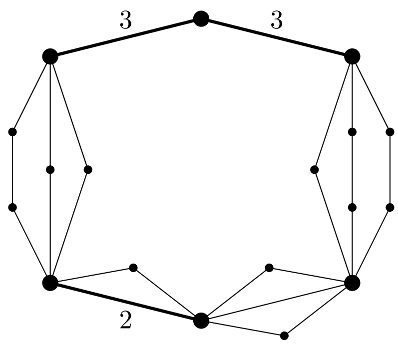

Figure 2: A PBC from 3 cycles with 6 common nodes (shown as the bigger dots). Thick lines represent edges shared by several cycles, the number of which appears next to each of these lines.

in the same sequence in all cycles. Then, the graph $G = (V, E) = \bigcup_{j=1}^{r} C^j$ is a PBC. Figure 2 shows an example.

The edge set of each cycle E^j decomposes into $t = |I|$ paths P_{kj}, $k = 1, \dots, t$, starting at i_k followed by a possibly empty node set Q_{kj}, and terminating at i_{k+1} (taking $t + 1 \equiv 1$). Following [7], we denote by $Q^1, Q^2, \dots, Q^m$, for $1 \leq m \leq rt$, the m distinct node sets Q_{kj} of largest cardinalities such that $|Q^i| \geq |Q^{i+1}|$. Ferreira *et al.* established the following result.

Proposition 4. *Let G' be any graph containing the path-block cycle $G = (V, E)$ as defined above. For each $e \in E$, let a_e be the number of occurrences of edge e in the r cycles. Then, the path-block-cycle inequality*

$$\sum_{e \in E} a_e y_e \geq 2r \tag{5}$$

is valid for $\text{NCMC}_b(G')$ *if and only if* $|V| - \sum_{j=1}^{r-1} |Q^j| \geq b + 1$.

A cycle is always cut in an even number of edges, and validity of inequality (5) follows from the fact that $\bar{V} = V \setminus \bigcup_{j=1}^{r-1} Q^j$ contains more than b nodes and, therefore, cannot be contained in a single shore. Otherwise, if $\bar{V}$ consisted of b or fewer nodes, a feasible multicut might cut fewer than r cycles.

Throughout the remainder of this paper, we will assume that $I \subset V$ is proper such that there exist nonempty node sets Q_{kj}.

2.2. Some properties of facet-inducing PBCs

PBC inequalities come in many different forms. Therefore, it will be useful to identify some common structural properties of the inequalities that induce facets of node-capacitated multicut polytopes. Here, we will concentrate on the node-capacitated multicut polytopes $\text{NCMC}_b(K_n)$ that are associated with complete graphs.

We begin by presenting an operation on PBCs that we refer to as *triangular reduction*.

Definition 1. Let $C = (V_C, E_C) \subset K_n$ be a cycle. Let $v \in V_C$ be any node, and let $u, w \in V_C$ be the unique neighbor nodes of v in C. The triangular reduction $\nabla_C(v)$ of C results in cycle $C' = (V'_C, E'_C)$, where $V'_C = V_C \setminus \{v\}$ and $E'_C = \{uw\} \cup E_C \setminus \{uv, vw\}$.

Let $G = (V, E) \subset K_n$ be a PBC, and let $v \in V$ be any node.

- If $v \notin I$, the triangular reduction $\nabla_G(v)$ of G is the PBC that is obtained by using cycle $C' = \nabla_C(v)$ instead of cycle C in G.
- If $v \in I$ and $|I| \geq 3$, a triangular reduction $\nabla_G(v)$ of G is the PBC that is obtained by using any $r-1$ cycles $C'_j = \nabla_{C^j}(v)$ instead of cycles C^j in G.

Suppose that $G = (V, E)$ is a PBC and consider any $v \in V \setminus I$. The PBC $G' = (V', E') = \nabla_G(v)$ has $V' = V \setminus \{v\}$ and $E' = \{uw\} \cup E \setminus \{uv, vw\}$. This means that inequality (5) associated with graph G is obtained as the sum of the triangle inequality $y_{uv} + y_{vw} - y_{uw} \geq 0$ and the PBC inequality associated with G', so that any valid PBC inequality that can be obtained by triangular reduction dominates the original PBC inequality.

When triangular reduction is applied to a node $v \in I$, the effect is that v is removed from I, but not from V, and a similar argument applies, except that $r-1$ (possibly identical) triangle inequalities are involved. We will only need to use this kind of triangular reductions when node v and its neighbors u, w all belong to I. Figure 3 shows examples of both kinds of triangular reductions.

The following proposition states three properties that must be satisfied by facet-defining PBC inequalities. Here, $\delta_E(i)$ is the set of edges in E that are incident on node i.

Proposition 5. *Let $G = (V, E) \subset K_n$ be a PBC. If inequality (5) having G as a support induces a facet of* $\mathrm{NCMC}_b(K_n)$*, then*

$$|Q^j| = |Q^r| \geq 1 \qquad (j = 1, \ldots, r-1), \tag{6}$$

$$|V| - \sum_{j=1}^{r-1} \left|Q^j\right| = b + 1, \quad \textit{and} \tag{7}$$

$$|\delta_E(i)| \geq 3 \qquad (i \in I). \tag{8}$$

Proof. Suppose that $|Q^r| = 0$. Then $I = V \setminus \bigcup_{j=1}^{r-1} Q^j$, and $|I| \geq b+1$ by validity. This implies that the PBC is obtained from r cycles, each of length at least $b+1$, so that inequality (5) is obtained as a sum of valid cycle inequalities. Hence, $\left|Q^j\right| \geq |Q^r| \geq 1$, for $j < r$.

In all remaining cases, where one of these conditions is not satisfied, we will show that a dominating inequality can be obtained by reducing the left-hand-side terms of the inequality. Consider again (6) and suppose that $\left|Q^i\right| > |Q^r|$. Take any $v \in Q^i$ and eliminate it from Q^i by triangular reduction. This removes

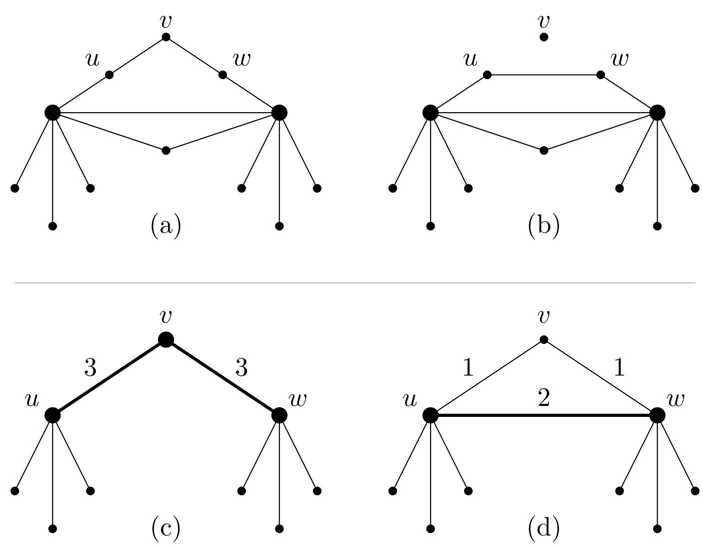

Figure 3: Triangular reductions applied to node v. (a): $v \in V \setminus I$ before reduction and (b): $v \notin V$ after reduction. (c): $\{u, v, w\} \subseteq I$ before reduction and (d): $\{u, w\} \subseteq I$ and $v \in V \setminus I$ after reduction.

v from Q^i and V so that $|V| - \sum_{j=1}^{r-1} |Q^j| \geq b + 1$ still holds true. Therefore, the resulting PBC inequality is still valid for $\mathrm{NCMC}_b(K_n)$, and it dominates the original one.

Now, suppose that (7) does not hold true such that $|V| - \sum_{j=1}^{r-1} |Q^j| \geq b + 2$. Taking any $v \in Q^r$ and eliminating it from the PBC by triangular reduction, we obtain a dominating PBC inequality.

Finally, consider (8) and suppose that, for some $v \in I$, $\delta_E(v) = \{uv, vw\}$ has cardinality 2. This means that $u, v, w \in I$, and if we apply triangular reduction to G on node v, the resulting PBC inequality is still valid for $\mathrm{NCMC}_b(K_n)$, and its left-hand side will be strengthened. □

3. PBCs with two common nodes

In this section and the next one, we consider PBCs that are associated with cycles that intersect in two common nodes. We start out by considering a special case of the PBC inequalities that corresponds to multistar inequalities. Then we present the structure of the PBCs that will be considered in the remainder of this paper, introduce some notation, and establish a few structural properties of the PBCs. Subsequently, we distinguish between PBCs G that consist of at most $2b$ nodes and more nodes than that and establish the facet defining properties of the inequalities for $\mathrm{BC}_b(G)$ and $\mathrm{NCMC}_b(G)$.

3.1. PBCs and multistars

Consider the multistar inequalities of Proposition 2 and suppose that $|S| = 2$. In this case, inequality (4) takes the form

$$\sum_{e\in\delta(S,T)} y_e + (2-d)y_{i_1 i_2} \geq |T| + 2 - d \qquad (S = \{i_1, i_2\}). \tag{9}$$

In particular, when parameter d equals 1 or 2, the support of inequality (9) is a PBC, obtained either as the union of $|T|/2$ cycles of length 4 (when $d = 2$) or as the union of $(|T| - 1)/2$ cycles of length 4 and a cycle of length 3 (when $d = 1$). In both cases, inequality (9) coincides with a PBC inequality (5).

By modifying the proof in [14] with the use of bisection cuts and equicuts only, we can establish the following result for the bisection cut polytope. However, we omit the proof here for the sake of brevity.

Proposition 6. *Let $d \in \{1, \ldots, b-1\}$ be an integer, and let $K_n = (V_n, E_n)$ be a complete graph such that $V_n = S \cup T$, where S, T are disjoint and satisfy $|S| = 2$ and $|T| = 2(b-1) - d$. Then inequality (9) defines a facet of $\mathrm{BC}_b(K_n)$, for $b \geq 3$.*

We remark that when $d = 1$ in Proposition 6, $n = |S| + |T| = 2b - 1$ so that $\mathrm{BC}_b(K_n) = \mathrm{EC}(K_n)$.

3.2. Structural properties and notation

The following definition describes the structure of the PBCs we will consider in the remainder of this paper. It respects all conditions of Proposition 5 when $|I| = 2$ in order to ensure that the resulting PBC inequalities may induce facets of $\mathrm{NCMC}_b(K_n)$ or $\mathrm{BC}_b(K_n)$.

Definition 2. Let $G = (V, E)$ be a PBC obtained from r cycles with two common nodes $\{i_1, i_2\} = I$. Each cycle decomposes into two paths that connect nodes i_1 and i_2. Let P^j, Q^j, for $j = 1, \ldots, r$, be the sets of inner nodes on these paths. If G satisfies

$$|Q^j| = \ell := |Q^r| \geq |P^j| \geq |P^{j+1}| \qquad (j = 1, \ldots, r-1), \tag{10}$$

$$|I| + \sum_{j=1}^{r} |P^j| + \ell = b + 1, \tag{11}$$

$$|P^1| \geq 1, \quad \text{and} \quad \ell \geq 2, \tag{12}$$

we denote G by $PBC_b(\lambda^1, \ldots, \lambda^r)$, where $\lambda^j = |P^j|$, for all j.

Conditions (10) and (12) imply (6), condition (11) then implies (7), and (8) is ensured by $|I| = 2$. The requirement $|P^1| \geq 1$ excludes the possibility that all cycles use edge $i_1 i_2$ and have length $b + 1$, in which case the PBC inequality is obtained as a sum of valid cycle inequalities (3). We have added the assumption that $\ell \geq 2$ for convenience, and because propositions 2 and 6 cover the case

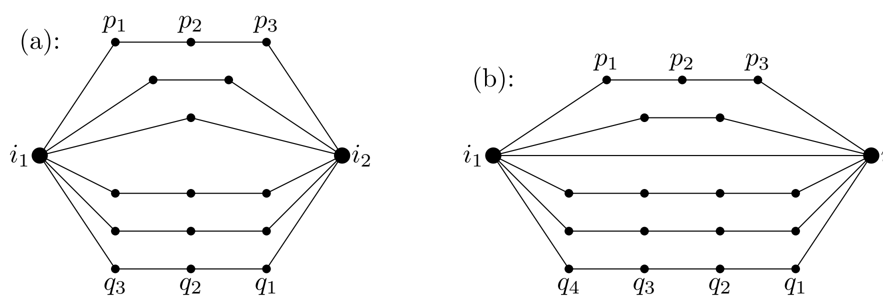

Figure 4: Two PBCs. (a): $PBC_{10}(3, 2, 1)$ which has $\ell = 3$. (b): $PBC_{10}(3, 2, 0)$ which has $\ell = 4$.

where $\ell = 1$. Figure 4 shows two examples of PBCs that conform to this definition.

We will now introduce some notation. In the following, we let $P = \bigcup_{j=1}^{r} P^j$ and $Q = \bigcup_{j=1}^{r} Q^j$ so that $V = I \cup P \cup Q$. Then, (11) is equivalent to $b = |P| + \ell + 1$, showing that $P \cup I$ does not admit an entire Q^j in the same shore. Later on, we will use r' to denote the maximal integer $j \in \{1, \dots, r\}$ for which $|P^j| \geq 1$.

We consider the node sets P^j, Q^j as sequences of nodes

$$P^j = \left(p_1^j, \dots, p_{\lambda^j}^j\right) = (p_1, \dots, p_\lambda)^j \quad \text{and}$$
$$Q^j = \left(q_1^j, \dots, q_\ell^j\right) = (q_1, \dots, q_\ell)^j.$$

Then, $(i_1, p_1, \dots, p_\lambda, i_2, q_1, \dots, q_\ell)^j$ identifies all nodes of the cycle on nodes $I \cup P^j \cup Q^j$. Accordingly, we may refer to nodes i_1, i_2 as members of the sequences, so that $i_1 \equiv p_0^j, q_{\ell+1}^j$ and $i_2 \equiv p_{\lambda^j+1}^j, q_0^j$. Usually, this notation becomes quite heavy and clumpsy. Therefore, whenever it is clear from the context which nodes we are dealing with, we will skip the subscripts and/or superscripts. We will even abuse this notation and write $p-1$ and $p+1$ instead of p_{i-1}^j and p_{i+1}^j, respectively, for any $p = p_i \in P^j$. The same applies to any $q \in Q^j$. Furthermore, we will also consider the sequences above as paths in E, e.g., $(i_1, p_1, \dots, p_\lambda)^j \subset E$.

Next, we give a characterization of the number of nodes and cycles that are involved in the PBCs that we consider here.

Lemma 7. *Let $G = (V, E)$ be a PBC that conforms to Definition 2, and let $k \in \mathbb{Z}$ such that $k \leq |P|$. Then $|V| = 2b - k$ if and only if*

$$(r-1)\ell = b - 1 - k \quad \textit{and} \quad (r-2)\ell = |P| - k. \tag{13}$$

Furthermore, if $|V| = 2b - k$, $r \geq 3$ unless $k = |P|$, and $|V| \leq 2b - 1$ if $r = 2$.

PROOF. By Definition 2, we have $|V| = 2+|P|+r\ell$, and (11) gives $|P| = b-\ell-1$. Then equation $2b - k = |V|$ resolves into $(r-1)\ell = b - 1 - k$. Using (11) again, we obtain that $(r-2)\ell = |P| - k$.

It follows directly from the last part of (13) that $r \geq 3$ when $k \neq |P|$. Finally, when $r = 2$, $|V| \leq 2b - 1$ because $k = |P| \geq 1$. □

Note that $|V| \leq 2b$ when $k \geq 0$ in Lemma 7, and $|V| > 2b$ when $k < 0$.

The next lemmas consider some properties of multicuts that belong to the root set of inequality (5).

Lemma 8. *Let $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ be a PBC that conforms to Definition 2. Let $M \subset E$ be a multicut that belongs to the root set of inequality (5) associated with G, and consider sequences $(i_1, p_1, \ldots, p_\lambda, i_2)^j$ and $(i_2, q_1, \ldots, q_\ell, i_1)^j$, for $j = 1, \ldots, r$.*

- *If nodes i_1, i_2 belong to two distinct shores, then M cuts each sequence exactly once.*
- *If nodes i_1, i_2 belong to the same shore, then M cuts exactly r distinct sequences twice.*

PROOF. Suppose that i_1, i_2 belong to different shores of M. Then M cuts each sequence at least once, and because there are $2r$ sequences, M cuts each of them only once. Next, suppose that i_1, i_2 belong to the same shore S of M. Then, each sequence which is not fully contained in S is cut at least twice by M. By validity, M cuts at least $2r$ edges, and since the cut edges belong to sequences that are not fully contained in S, it follows that exactly r sequences are cut twice. □

Lemma 9. *Let $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ be a PBC that conforms to Definition 2. Consider a multicut $M = \delta_E(S_1, \ldots, S_m)$, belonging to the root set of inequality (5) associated with G, such that $i_1 \in S_j$ and $i_2 \in S_k$, $j \neq k$. Then, $|V| \leq 2b$ and $m = 2$.*

PROOF. Suppose that $m \geq 3$. Note that this is the case if $|V| > 2b$. According to Lemma 8, M cuts each of the sequences $(i_1, p_1, \ldots, p_\lambda, i_2)^i$ and $(i_2, q_1, \ldots, q_\ell, i_1)^i$ exactly once. However, when M has three or more shores, not all sequences are fully contained in $S_j \cup S_k$, and any such sequence is cut twice or more. This contradicts that M belongs to the root set of (5). □

Lemma 10. *Let $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ be a PBC that conforms to Definition 2 such that $|V| > 2b$. Let $M = \delta_E(S_1, \ldots, S_m)$ be a multicut that belongs to the root set of inequality (5) associated with G. Then, $\{i_1, i_2\} \subset S_k$, for some $k \in \{1, \ldots, m\}$.*

PROOF. When $|V| > 2b$, $m \geq 3$. According to Lemma 9, nodes i_1, i_2 do not belong to distinct shores, and the statement follows. □

3.3. PBCs with no more than 2b nodes

Let $G = (V, E)$ be a PBC obtained from r cycles with two common nodes. We focus here on such PBCs with $|V| \leq 2b$ and show that the corresponding PBC inequalities define facets of $\mathrm{NCMC}_b(G)$ and $\mathrm{BC}_b(G)$. These inequalities have coefficients $a_e = 1$, for all $e \in E \setminus \{i_1 i_2\}$, and $a_{i_1 i_2} \in \{1, \ldots, r-1\}$ if $i_1 i_2 \in E$.

Proposition 11. *Let $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ be a PBC that conforms to Definition 2. If $|V| \leq 2b$, inequality (5) corresponding to G defines a facet of $\mathrm{NCMC}_b(G)$, for $b \geq 4$. If $|V| \leq 2b - 1$, the corresponding inequality defines a facet of $\mathrm{BC}_b(G)$, for $b \geq 4$.*

Proof. We first consider the case where $|V| < 2b$ and prove the statement for $\mathrm{BC}_b(G)$. Recall from section 1.3 that $\mathrm{BC}_b(G)$ is full dimensional in this case. This follows immediately when $|V| \leq 2b-2$. When $|V| = 2b-1$, $\mathrm{BC}_b(G)$ equals $\mathrm{EC}(G)$ and is full dimensional.

Let $F_a = \{y \in \mathrm{BC}_b(G) : \sum_{e \in E} a_e y_e = 2r\}$ be the face that is defined by inequality (5) associated with G. There exists a facet-defining inequality $\sum_{e \in E} \pi_e y_e \geq \pi_0$ for $\mathrm{BC}_b(G)$ such that

$$F_a \subseteq F_\pi = \left\{ y \in \mathrm{BC}_b(G) : \sum_{e \in E} \pi_e y_e = \pi_0 \right\}.$$

The proof proceeds by showing that, for some $\alpha \in \mathbb{R}_+$, $\pi_e = \alpha a_e$, for all $e \in E$. This, in turn, implies that $\pi_0 = \alpha \sum_{e \in E} a_e y_e = \alpha 2r$, for $y \in F_a$, so that $F_a = F_\pi$.

Claim 1: $\pi_{q-1,q} = \pi_{q,q+1} = \alpha^j$, for all $q \in Q^j$.
For $q \in Q^j$, let $A_q = \delta_E(S_1, S_2)$ and (if $q \neq q_\ell$) $A_q^+ = \delta_E(S_1^+, S_2^+)$ be bisection cuts in G with shores

$$S_1 = I \cup P \cup Q^j \setminus \{q\}, \quad S_2 = \{q\} \cup (Q \setminus Q^j),$$
$$S_1^+ = S_1 \setminus \{q+1\}, \quad S_2^+ = S_2 \cup \{q+1\}.$$

Note that $|S_1| = b$ and $|S_2| = |V| - b$ such that all shores contain at most b nodes. Throughout the proof, we leave it to the reader to check that all bisection cuts and multicuts employed belong to the root set of inequality (5). However, Figure 5 shows a few examples of the bisection cuts that are used in this proof as they are applied to $PBC_{10}(3, 2, 1)$.

Consider first node $q_2 \in Q^j$, and compare bisection cuts A_{q_2} and $A_{q_1}^+$. It follows that $\pi_{q_1 q_2} = \pi_{i_2 q_1} := \alpha^j$. Next, for all $q = q_2, \ldots, q_\ell$, compare A_{q-1} to A_{q-1}^+. Then, $\pi_{q,q+1} = \pi_{q-1,q} = \alpha^j$.

Claim 2: $\pi_{p-1,p} = \pi_{p,p+1} = \beta^j$, for all $p \in P^j$.
Let $p = p_i \in P^j$ and consider the bisection cut $B_p = \delta_E(S_1, S_2)$ in G with shores

$$S_1 = (p_i, \ldots, p_\lambda)^j \cup (P \setminus P^j) \cup (i_2, q_1, \ldots, q_{i-1})^2 \cup Q^1,$$
$$S_2 = (i_1, p_1, \ldots, p_{i-1})^j \cup (q_i, \ldots, q_\ell)^2 \cup (Q \setminus (Q^1 \cup Q^2)).$$

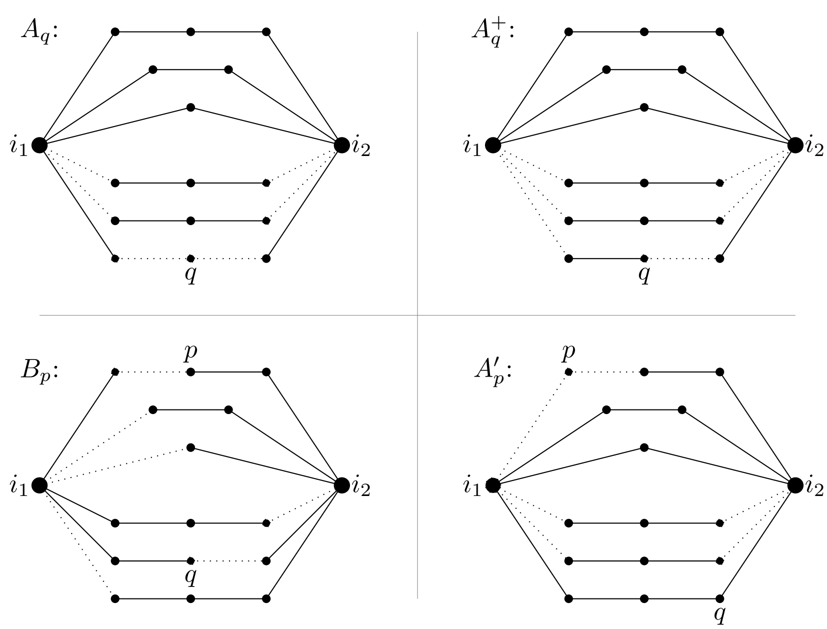

Figure 5: Examples of the bisection cuts that are used in the proof of Proposition 11. Dotted lines are cut edges.

Then, S_1 contains nodes $I \cup P \cup Q^1$, except for node i_1 and $i-1$ nodes from P^j, but in exchange for $i-1$ nodes from Q^2 so that $|S_1| = b$.

For each $p = p_1, \ldots, p_\lambda$, compare the bisection cuts B_p and B_{p+1} to see that $\pi_{p,p+1} + \pi_{q,q+1} = \pi_{p-1,p} + \pi_{q-1,q}$, where $q = q_i \in Q^2$. As $\pi_{q-1,q} = \pi_{q,q+1}$, cf. Claim 1, it follows that $\pi_{p-1,p} = \pi_{p,p+1} = \beta^j$.

Now, recall from the comments after Definition 2 that r' is the maximal integer in $\{1, \ldots, r\}$ for which $|P^j| \geq 1$.

Claim 3: $\alpha^j = \beta^k = \alpha$, for $j = 1, \ldots, r$, $k = 1, \ldots, r'$.
Let $q = q_1 \in Q^j$ and $p = p_1 \in P^k$. Compare the bisection cut A_q defined in Claim 1 to the bisection cut A'_p with shores $S_1 = I \cup Q^j \cup P \setminus \{p\}$ and $S_2 = \{p\} \cup Q \setminus Q^j$. Then, $\pi_{i_2 q} + \pi_{q,q+1} = \pi_{i_1 p} + \pi_{p,p+1}$ so that $2\alpha^j = 2\beta^k$. Combining different indices j, k, it follows that $\alpha^j = \beta^k = \alpha$, for all j and k.

The above claims show that $\pi_e = \alpha$, for all $e \in E \setminus \{i_1 i_2\}$. When $i_1 i_2 \notin E$, this completes the current part of the proof. Otherwise, we must consider the next claim.

Claim 4: $\pi_{i_1 i_2} = \alpha(r - r')$.
Let $q = q_\ell \in Q^1$ and $p = p_1 \in P^1$. Compare the bisection cut A_q defined in Claim 1 to the bisection cut B_p defined in Claim 2. Denoting $p(j) = p_1^j$ and

$q(j) = q_\ell^j$, it then follows that

$$\pi_{i_1 i_2} + \sum_{j=1}^{r'} \pi_{i_1 p(j)} = \pi_{q-1,q} + \sum_{j=2}^{r} \pi_{i_1 q(j)}$$

so that $\pi_{i_1 i_2} + \alpha r' = \alpha r$. This shows that $\pi_{i_1 i_2} = \alpha(r - r')$.

Now, consider the case where $|V| = 2b$. This part of the proof follows from arguments that are very similar to the ones above. Therefore, we will only explain the modifications to the above proof that are necessary to establish the result for this case. In accordance with Lemma 7, we can assume here that $r \geq 3$.

Consider the bisection cuts A_q^+ of Claim 1. They will be infeasible here, as the shores S_2^+ would contain $b+1$ nodes in this case. Therefore, we modify these bisection cuts as follows. Let $k \in \{1, \dots, r\} \setminus \{j\}$, and use instead the multicuts with three shores:

$$S_2^+ = \{q, q+1\} \cup (Q \setminus (Q^j \cup Q^k)), \quad S_3^+ = Q^k,$$

where S_1^+ remains the same. These multicuts cut the same edges in E as the original bisection cuts.

All other bisection cuts used in this proof have two shores with exactly b nodes in this case and, therefore, can be used without any changes. Considering F_a and F_π as faces of $\text{NCMC}_b(G)$, the proof follows from the same arguments as above. □

3.4. PBCs with more than 2b nodes

Consider a PBC $G = (V, E)$ with $|V| > 2b$. In this case G contains no feasible bisection cuts, and the PBC inequality is not facet defining for $\text{NCMC}_b(G)$. An implication of Lemma 10 is that inequality (5) can be strengthened by decreasing coefficient $a_{i_1 i_2}$. Recall again, from the comments after Definition 2, that r' is the number of cycles in the PBC with nonempty P^j, i.e., $|P^j| \geq 1$, for $j = 1, \dots, r'$. A PBC that is obtained from r cycles has $r - r'$ cycles that contain edge $i_1 i_2$, and $a_{i_1 i_2} = r - r'$ in (5). When the PBC contains more than $2b$ nodes, this coefficient can be reduced to $\bar{r} - r'$, where $\bar{r} < r$.

Let us take a look at an example of this. Consider the PBCs $PBC_6(2,1)$ and $PBC_6(2,1,0)$ as shown in Figure 6. Both PBCs contain fewer than $2b = 12$ nodes (9 and 11 nodes, respectively), and the associated coefficients $a_{i_1 i_2}$ in (5) are 0 and 1. Now, consider $G = PBC_6(2,1,0,0)$ which is also shown in Figure 6. G contains 13 nodes, and the corresponding inequality (5) has $a_{i_1 i_2} = 2$. However, this coefficient can be reduced to $a_{i_1 i_2} = 1$. This is so, because no more than $\bar{r} = 3$ of subsets Q^j, $j = 1, \dots, 4$, can be fully contained within two shores together with all nodes in $I \cup P$.

Lemma 12. *Let $G = (V, E) = PBC_b(\lambda^1, \dots, \lambda^r)$ with $|V| > 2b$, and let $\delta_E(S_1, \dots, S_m)$ be a feasible multicut in G. Suppose there are two distinct*

$h, i \in \{1, \ldots, m\}$ *such that* $I \cup P \subset S_h \cup S_i$, *and that* $Q^j \subset S_h \cup S_i$, *for* k *distinct* $j \in \{1, \ldots, m\}$. *Then,* $k \leq \bar{r}$, *where*

$$\bar{r} = \left\lfloor \frac{b-1}{\ell} \right\rfloor + 1. \tag{14}$$

PROOF. From (11) and (10) we have $|I| + |P| + \ell = b + 1$ and $|Q^j| = \ell$, for $j = 1, \ldots, r$. Requiring $b + 1 + (k-1)\ell \leq 2b$, because $|S_h \cup S_i| \leq 2b$, we get $k \leq (b-1)/\ell + 1$. $\bar{r}$ in (14) equals the largest integer k that satisfies this condition. □

We remark that $\bar{r} \geq 2$ because (11) implies $b - 1 = |P| + \ell$.

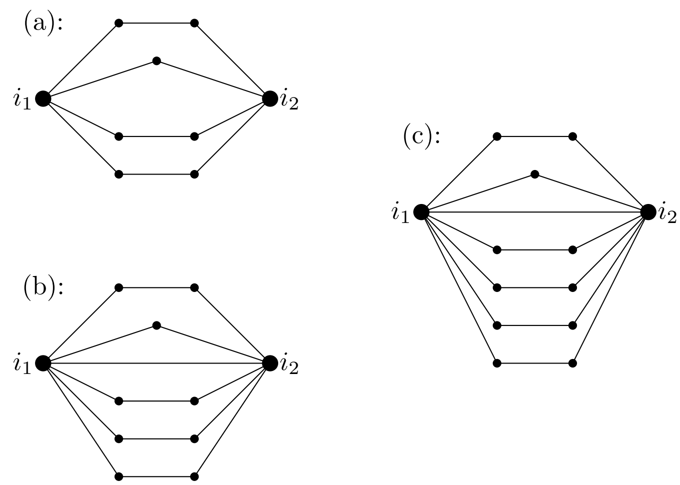

Figure 6: $PBC_6(\cdot)$ graphs. (a): $PBC_6(2,1)$. (b): $PBC_6(2,1,0)$. (c): $PBC_6(2,1,0,0)$.

We are now ready to present the *strengthened PBC inequalities.*

Proposition 13. *Let* $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ *be a PBC that conforms to Definition 2 such that* $|V| > 2b$. *Let* G' *be the graph that is obtained from* G *by adding edge* $i_1 i_2$, *if it is not already contained in* E, *and let* $F = E \setminus \{i_1 i_2\}$. *Then, the strengthened PBC inequality*

$$\sum_{e \in F} y_e + (\bar{r} - r') y_{i_1 i_2} \geq 2r \tag{15}$$

is valid for $\mathrm{NCMC}_b(G')$.

PROOF. Let $M \subset E \cup \{i_1 i_2\}$ be a feasible multicut in G' with shores $S_1, \ldots, S_m$, $m \geq 3$. If nodes i_1, i_2 are contained in the same shore, edge $i_1 i_2$ is not cut and validity follows via (5). So we will assume that $i_1 \in S_1$ and $i_2 \in S_2$. Consider sequences $(i_1, p_1 \ldots, p_\lambda, i_2)^j$ and $(i_2, q_1, \ldots, q_\ell, i_1)^j$. Every sequence that is not fully contained in $S_1 \cup S_2$ is cut at least twice, and every sequence that is fully contained in $S_1 \cup S_2$ is cut at least once. Suppose that $S_1 \cup S_2$ contains fully ρ sequences through P and η sequences through Q, and let $\phi = |M \cap F|$ be the number of edges in F that are cut. It then follows that $\phi \geq 2(r' - \rho) + 2(r - \eta) + \rho + \eta$, so that $\phi \geq 2r' + 2r - \rho - \eta$. Now, $\rho \leq r'$ and, by Lemma 12, $\eta \leq \bar{r}$, so that $\phi \geq 2r + r' - \bar{r}$. Since edge $i_1 i_2$ is also cut, we get $\phi + \bar{r} - r' \geq 2r$. □

We note that $\bar{r} - r'$ may be negative. As an example, consider $G = PBC_8(1, 1, 1)$ as shown in Figure 7. G contains $17 > 2b$ nodes, and $\bar{r} - r' = 2 - 3 = -1$.

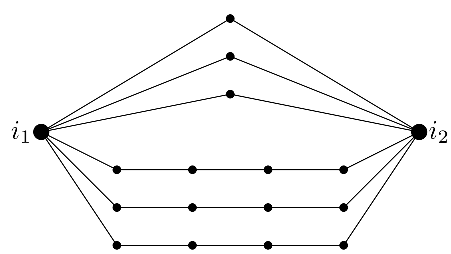

Figure 7: $PBC_8(1, 1, 1)$.

The strengthened PBC inequalities define facets of $\mathrm{NCMC}_b(G')$.

Proposition 14. *Let $G' = (V, E')$ be the graph that is obtained from a PBC with $|V| > 2b$ as in Proposition 13. Then, the corresponding inequality (15) defines a facet of* $\mathrm{NCMC}_b(G')$.

Proof. Let F_a be the face of $\mathrm{NCMC}_b(G')$ that is defined by (15), and let F_π be a facet of $\mathrm{NCMC}_b(G')$ such that $F_a \subseteq F_\pi$. Then, it suffices to show that $\pi_e = \alpha$, for all $e \in E' \setminus \{i_1 i_2\}$, and $\pi_{i_1 i_2} = \alpha(\bar{r} - r')$ in order to establish that $F_a = F_\pi$.

We will re-use all parts of the proof of Proposition 11, and in order to do that, we must make some simplifying assumptions. Specifically, for each multicut $M = \delta_{E'}(S_1, \ldots, S_m)$ in G', we fix the shores $S_3, \ldots, S_m$ and work only with shores S_1 and S_2. In doing this, we denote $Q = \bigcup_{j=1}^{\bar{r}} Q^j$ and assume that $S_1 \cup S_2 = I \cup P \cup Q$,

$$\bigcup_{k=3}^{m} S_k = \bigcup_{j=\bar{r}+1}^{r} Q^j, \quad \text{and} \quad \forall j \in \{\bar{r}+1, \ldots, r\} \; \exists k \in \{3, \ldots, m\} \; Q^j \subseteq S_k.$$

Note that by assuming $Q^j \subseteq S_k$, we ensure that, for each $j > \bar{r}$, M cuts exactly two edges, an edge incident on each of nodes i_1 and i_2. Also note that this fixing of shores can be done without loss of generality, since it is possible to assign different numbers to the indices of subsets Q^j and shores S_k.

Now, we can use the same arguments as in the proof of Proposition 11 to establish that $\pi_e = \alpha$, for all $e \in E' \setminus \{i_1 i_2\}$. It also follows that $\pi_{i_1 i_2} = (\bar{r} - r')\alpha$ because there are exactly $\bar{r}$ subsets Q^j in $S_1 \cup S_2$. □

A consequence of Proposition 14 is that it is possible to extend an existing PBC to obtain new facet-defining inequalities. Suppose that PBC $G = PBC_b(\lambda^1, \ldots, \lambda^k)$ satisfies the conditions in Definition 2 so that the associated inequality (5) or (15) defines a facet of $\mathrm{NCMC}_b(G)$. Then, the extended PBC $G' = PBC_b(\lambda^1, \ldots, \lambda^k, 0)$ has an associated inequality that defines a facet of $\mathrm{NCMC}_b(G')$.

4. PBC facets and complete graphs

The previous section considers the facet-defining properties of the PBC inequalites for the polytopes that are associated with the supports of the inequalities. In this section, we establish corresponding results for the polytopes on complete graphs that contain the supports as subgraphs.

One good reason to study node-capacitated multicut polytopes associated with complete graphs is that it is often trivial to obtain facets of the polytopes on subgraphs of the complete graph. All that is required is that the subgraph contains the support of a facet-defining inequality for the polytope on the complete graph. The following proposition is similar to Proposition 4.3 in [7] and Remark 5.2 in [4].

Proposition 15. *Suppose that inequality* $\sum_{e \in E} a_e y_e \geq a_0$ *defines a facet of* $\mathrm{NCMC}_b(K_n)$ *(resp.* $\mathrm{BC}_b(K_n)$*). Let* $G' = (V', E')$ *be a subgraph of* K_n *such that* $E \subseteq E'$*. Then the inequality* $\sum_{e \in E} a_e y_e \geq a_0$ *defines a facet of* $\mathrm{NCMC}_b(G')$ *(resp.* $\mathrm{BC}_b(G')$*).*

The following two lemmas will be useful in the proof of the next theorem. They are similar to lemmas 4.1 and 4.2 in [11], and therefore we state them without proof. In fact, Lemma 17 is a special case of Lemma 3.1 in [3].

Lemma 16. *Let* $\pi^T y \geq \pi_0$ *be a valid inequality for* $\mathrm{NCMC}_b(K_n)$ *and consider distinct nodes* $u, v \in V_n$*. Then,* $\pi_{uv} = 0$ *if the multicuts with the following shores all belong to the root set of the inequality:*

- $S_1, S_2, \ldots, S_m$*, where* $u \in S_1$*,* $v \in S_2$*,* $1 < |S_1| < b$*, and* $1 < |S_2| < b$*.*
- $S'_1, S'_2, \ldots, S'_m$*, where* $S'_1 = \{v\} \cup S_1 \setminus \{u\}$*,* $S'_2 = \{u\} \cup S_2 \setminus \{v\}$*, and* $S'_k = S_k$ *for* $k = 3, \ldots, m$*.*
- $\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_m$*, where* $\hat{S}_1 = S_1 \setminus \{u\}$*,* $\hat{S}_2 = \{u\} \cup S_2$*, and* $\hat{S}_k = S_k$ *for* $k = 3, \ldots, m$*.*
- $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_m$*, where* $\bar{S}_1 = \{v\} \cup S_1$*,* $\bar{S}_2 = S_2 \setminus \{v\}$*, and* $\bar{S}_k = S_k$ *for* $k = 3, \ldots, m$*.*

Lemma 17. *Let* $\pi^T y \geq \pi_0$ *be a valid inequality for* $\mathrm{NCMC}_b(K_n)$ *and consider distinct nodes* $u, v, w \in V_n$*. Then,* $\pi_{uv} = \pi_{vw}$ *if the multicuts with the following shores all belong to the root set of the inequality:*

- $S_1, S_2, \ldots, S_m$*, where* $u, v \in S_1$*,* $w \in S_2$*, and* $|S_2| < b$*.*

- $S_1', S_2', \ldots, S_m'$, *where* $S_1' = S_1 \setminus \{v\}$, $S_2' = \{v\} \cup S_2$, *and* $S_k' = S_k$ *for* $k = 3, \ldots, m$.
- $\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_m$, *where* $\hat{S}_1 = \{w\} \cup S_1 \setminus \{u\}$, $\hat{S}_2 = \{u\} \cup S_2 \setminus \{w\}$, *and* $\hat{S}_k = S_k$ *for* $k = 3, \ldots, m$.
- $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_m$, *where* $\bar{S}_1 = \hat{S}_1 \setminus \{v\}$, $\bar{S}_2 = \{v\} \cup \hat{S}_2$, *and* $\bar{S}_k = S_k$ *for* $k = 3, \ldots, m$.

These lemmas also apply to the bisection cut polytope $\mathrm{BC}_b(K_n)$, provided that there exist bisection cuts with shores S_1, S_2 that comply with the conditions of the lemmas. In particular, Lemma 16 only applies if $|S_1| < b$ and $|S_2| < b$, which is only the case when $n \leq 2b - 2$.

4.1. Facets of the bisection cut and equicut polytopes

We are now ready to establish that the PBC inequalities induce facets of the bisection cut polytopes on complete graphs. We present the result in two theorems. The first theorem treats the cases where $|V| \leq 2b - 2$ and $\mathrm{BC}_b(K_n)$ is full dimensional; the second theorem considers the case where $|V| = 2b - 1$ and $\mathrm{BC}_b(K_n)$ coincides with the equicut polytope and is not full dimensional.

Theorem 18. *Let* $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ *be a PBC that conforms to Definition 2 such that* $n = |V| \leq 2b - 2$, *and let* $K_n = (V, E_n)$ *be the complete graph on n nodes. Then inequality (5) associated with G defines a facet of* $\mathrm{BC}_b(K_n)$, *for* $b \geq 4$.

Proof. Let $F_a = \{y \in \mathrm{BC}_b(K_n) : a^T y = 2r\}$ be the face of $\mathrm{BC}_b(K_n)$ that is defined by (5), and let F_π be a facet of $\mathrm{BC}_b(K_n)$ defined by inequality $\pi^T y \geq \pi_0$ such that $F_a \subseteq F_\pi$. As $\mathrm{BC}_b(K_n)$ has full dimension $|E_n|$, it suffices to show that $\pi_e = \alpha a_e$, for some $\alpha \in \mathbb{R}_+$ and all $e \in E_n$. We will show that $\pi_e = 0$ for all $e \in E_n \setminus E$. Then, the rest of the proof follows from the same arguments as in the proof of Proposition 11.

In order to establish the proof, we rely on using lemmas 16 and 17. The main difficulty, then, is to construct bisection cuts with shores S_1, S_2 that fulfill the conditions of the lemmas. The construction of such bisection cuts is carefully explained, while the detailed applications of the lemmas, in terms of obtaining the other bisection cuts (e.g., S_1', S_2') that are used in the lemmas, will be clear from the context. Figure 8 shows examples of the construction of the bisection cuts in $PBC_{10}(3, 2, 1)$.

Claim 1: $\pi_{qq'} = 0$, for $qq' \in E_n(Q) \setminus E$.
We consider two cases; one in which q, q' belong to two distinct subsets, and one in which they belong to the same subset. In both cases, we will use Lemma 16, and the shores S_1, S_2 will be constructed such that $q \in S_1$, $q' \in S_2$, and $|S_1| = b - 1$.

Case 1: $q = q_i \in Q^j$ and $q' = q_h \in Q^k$, $j \neq k$.
Each node q, q' can be reached from i_1 and i_2 via two different paths in E on nodes in Q. Consider the following sequences of nodes in Q^j that will lead

to node q: $\bar{Q}_1^j = (q_\ell, q_{\ell-1}, \ldots, q_i)^j$ and $\bar{Q}_2^j = (q_1, \ldots, q_i)^j$. Sequence $\bar{Q}_1^j$ leads directly from i_1 to q, while $\bar{Q}_2^j$ leads from i_2 to q. Similar sequences $\bar{Q}_1^k$ and $\bar{Q}_2^k$ lead to q' in Q^k.

Suppose that there are sequences $\bar{Q}^j \in \left\{\bar{Q}_1^j, \bar{Q}_2^j\right\}$ and $\bar{Q}^k \in \left\{\bar{Q}_1^k, \bar{Q}_2^k\right\}$ such that $|\bar{Q}^j| + |\bar{Q}^k| = \rho < \ell$. Then, we can construct S_1 such that it contains nodes $I \cup P$ and nodes $\bar{Q}^j \cup \bar{Q}^k \setminus \{q'\}$ with a number of nodes equal to $\sigma = 1 + \rho + |P|$. We add $b - 1 - \sigma$ further nodes to S_1 to obtain $|S_1| = b - 1$. These nodes are added from the opposite end of Q^j than those in the sequence $\bar{Q}^j$. See Fig. 8(a) where such nodes are not needed.

Now, suppose that there are no sequences such that $|\bar{Q}^j| + |\bar{Q}^k| = \rho < \ell$. In this case, q and q' reside close to the middle of the sequences Q^j and Q^k, because there are exactly 2ℓ nodes in $Q^j \cup Q^k$ and it will be possible to reach q and q' from i_1 and i_2 via sequences that contain no more than $\ell + 1$ nodes. It will then be possible to reach q and q' from i_1 through sequences $\bar{Q}_1^j$ and $\bar{Q}_1^k$ that contain no more than $\ell + 2$ nodes. In this case, we construct S_1 so that it contains node i_1 and nodes $\bar{Q}_1^j \cup \bar{Q}_1^k \setminus \{q'\}$ with a number of nodes equal to $\sigma' \in \{\ell, \ell+1, \ell+2\}$. If $\sigma' < b - 1$, we also add $b - 1 - \sigma'$ nodes of P to S_1 in the following way. Let s be the first index in $1, \ldots, r'$ such that $\lambda^1 + \cdots + \lambda^s \geq b - 1 - \sigma'$, and let $\bar{P} = \bigcup_{j=1}^{s-1} P^j \cup (p_1, \ldots, p_t)^s$ such that $|\bar{P}| = b - 1 - \sigma'$, where $(p_1, \ldots, p_t)^s \subseteq (p_1, \ldots, p_\lambda)^s$. See Fig. 8(b).

Case 2: $q = q_i \in Q^k$ and $q' = q_j \in Q^k$, $j > i + 1$.
Here, we construct shore S_1 in such a way that it contains all nodes $I \cup P$, some nodes of Q^k, and possibly some nodes of Q^l, where $l \neq k$. We include nodes $(q_1, \ldots, q_i)^k \cup (q_{j+1}, \ldots, q_\ell)^k$ of Q^k in S_1, which together with $I \cup P$ amount to $\sigma \leq b - 1$ nodes. Let $h = b - 1 - \sigma$ and also include nodes $(q_1, \ldots, q_h)^l$ of Q^l such that we obtain $|S_1| = b - 1$. See Fig. 8(c).

In both cases, the bisection cuts with shores S_1 and $S_2 = V \setminus S_1$ belong to the root set of the PBC inequality, and they satisfy the conditions of Lemma 16. It therefore follows that $\pi_{qq'} = 0$.

Claim 2: $\pi_{pp'} = 0$, for $pp' \in E_n(P) \setminus E$.
As in Claim 1, we consider two cases; one in which p, p' belong to distinct subsets, and one in which they belong to the same subset. Lemma 16 is used in both cases, and we will construct bisection cuts with shores S_1, S_2 such that $p \in S_1$, $p' \in S_2$, $|S_1| \leq b - 1$, and $|S_2| \leq b - 1$.

Case 1: $p = p_i \in P^j$ and $p' = p_h \in P^k$, $j \neq k$.
We first include all nodes $\{i_2\} \cup Q^1 \cup P \setminus \left(P^j \cup P^k\right)$ in S_1. Next, we include nodes $(p_i, \ldots, p_\lambda)^j$ of P^j and nodes $(p_{h+1}, \ldots, p_\lambda)^k$ of P^k. This amounts to σ nodes in total, where $\sigma \leq b - 1$. If $\sigma < b - 1$, we include further nodes in S_1 from sequences $(q_1, q_2, \ldots)^l$ of Q^l, where $l = 2, 3$, until either $|S_1| = b - 1$ or $S_1 \supset Q$. See Fig. 8(d). It may happen that all nodes of Q are included in S_1 when $r = 2$, and in that case we have $S_2 = V \setminus S_1 \subset P^j \cup P^k \cup \{i_1\}$.

Case 2: $p = p_i \in P^k$ and $p' = p_j \in P^k$, $j > i + 1$.
Here, we first include in S_1 all nodes $I \cup Q^1 \cup P \setminus P^k$. Next, we include nodes

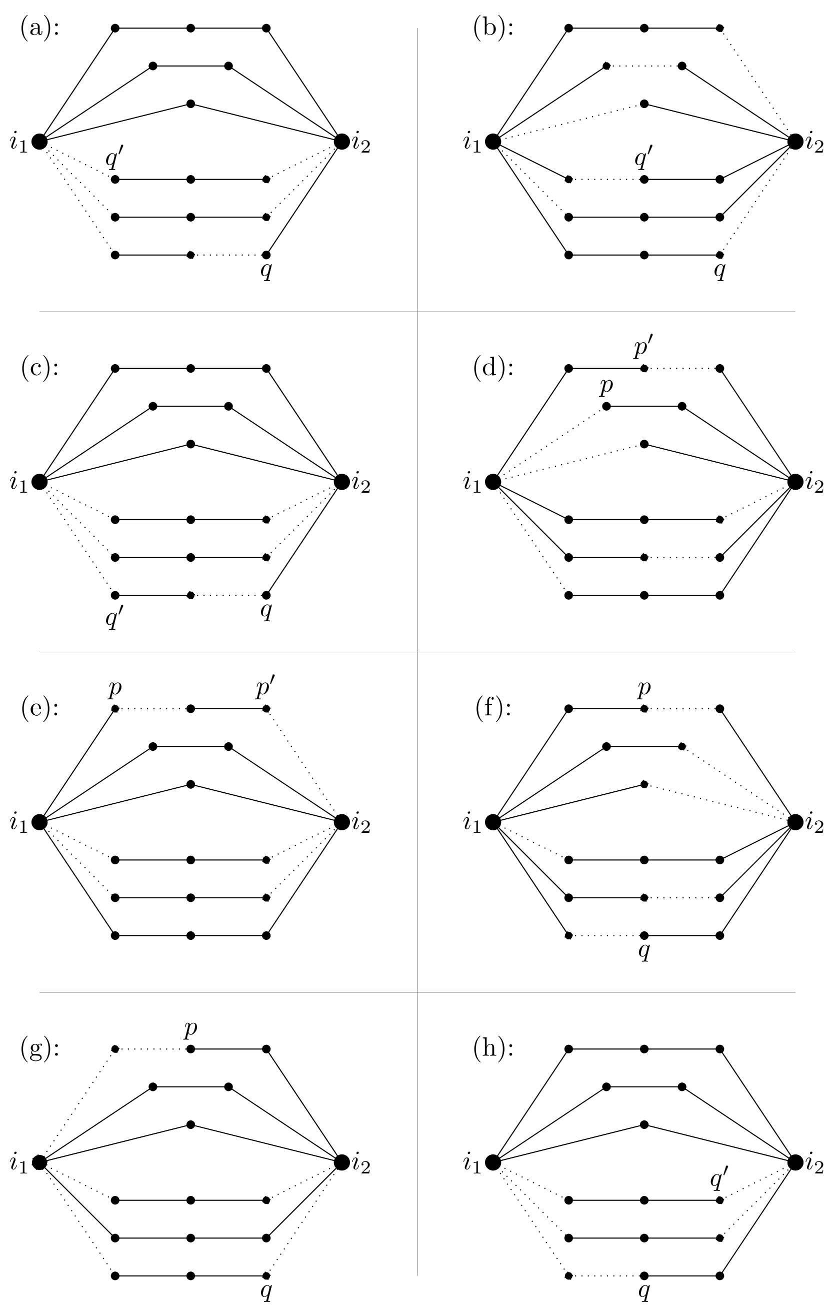

Figure 8: Examples of bisection cuts used in the proof of Theorem 18.

$(p_1,\ldots,p_i)^k$ and $(p_{j+1},\ldots,p_\lambda)^k$ of P^k and, if $h=j-i-2>0$, the first h nodes in the sequence $(q_1,q_2,\ldots)^2$ of Q^2. Then, $|S_1|=b-1$. See Fig. 8(e).

In both cases, the bisection cuts with shores S_1 and $S_2=V\setminus S_1$ comply with Lemma 16, and it follows that $\pi_{pp'}=0$.

Claim 3: $\pi_{pq}=0$, for $pq\in\delta(P,Q)$.
Assume that $p=p_i\in P^j$, $q=q_h\in Q^k$, where j,k need not be distinct. Once again, we use Lemma 16, and we construct a bisection cut with shores S_1,S_2 such that $p\in S_1$, $q\in S_2$, and $|S_1|=b-1$.

Include in S_1 nodes i_1 and $P\setminus P^j$ together with the sequence $(p_1,\ldots,p_i)^j$ of P^j and the sequence $(q_{h+1},\ldots,q_\ell)^k$ of Q^k. Then S_1 contains $\sigma\le b-1$ nodes, and if $\sigma<b-1$, also include in S_1 nodes from the sequence $(q_\ell,q_{\ell-1},\ldots)^l$ of Q^l, $l\ne k$, until $|S_1|=b-1$. See Fig. 8(f). Setting $S_2=V\setminus S_1$, the claim then follows via Lemma 16.

Claim 4: $\pi_e=0$, for $e\in\delta(\{i_1,i_2\},P\cup Q)\setminus E$.
We consider two cases; one in which $e\in\delta(\{i_1,i_2\},P)$, and one in which $e\in\delta(\{i_1,i_2\},Q)$. We use Lemma 17 to prove the claim. In the first case, we show that, for appropriate $p\in P$ and some $q\in Q$, $\pi_{i_1p}=\pi_{pq}=0$. It then follows by similar arguments and via symmetry that $\pi_{i_2p}=0$. In the other case, we show that $\pi_{i_1q}=\pi_{qq'}=0$, for appropriate $q,q'\in Q$. Once again, it then follows via symmetry that $\pi_{i_2q}=0$.

Case 1: $e\in\delta(\{i_1,i_2\},P)\setminus E$.
Assume that $p=p_i\in P^j$, $i>1$. Let $\bar P^j=(p_i,\ldots,p_\lambda)^j$ and $\bar Q^1=(q_1,\ldots,q_{i-1})^1$, and let $q=q^1_{i-1}$ be the last node in the sequence $\bar Q^1$. Construct shores S_1,S_2 such that

$$S_1=I\cup\bar P^j\cup\left(P\setminus P^j\right)\cup\bar Q^1\setminus\{q\}\cup Q^2$$

and $S_2=V\setminus S_1$. See Fig. 8(g). Then, $i_1,p\in S_1$, $q\in S_2$, and $|S_1|=b$, and the corresponding bisection cut satisfies the conditions of Lemma 17. It then follows that $\pi_{i_1p}=\pi_{pq}=0$, by Claim 3.

Case 2: $e\in\delta(\{i_1,i_2\},Q)\setminus E$.
Let $q=q_i\in Q^j$, $i<\ell$, and let $\bar Q^j=(q_1,\ldots,q_i)^j$. For some $k\in\{1,\ldots,r\}$, $k\ne j$, let $\bar Q^k=(q_1,\ldots,q_{\ell-i})^k$, and let $q'=q^k_{\ell-i}$ be the last node in this sequence. Let $S_1=I\cup P\cup\bar Q^j\cup\bar Q^k\setminus\{q'\}$ and $S_2=V\setminus S_1$. See Fig. 8(h). Then, $i_1,q\in S_1$, $q'\in S_2$, and $|S_1|=b$, and the bisection cut complies with Lemma 17. As a consequence, we obtain via Claim 1 that $\pi_{i_1q}=\pi_{qq'}=0$. This completes the proof. □

Next, we consider the facet-defining properties of PBC inequalities for the equicut polytope $\mathrm{EC}(K_n)$ with $n=2b-1$.

Theorem 19. *Let $G=(V,E)=PBC_b(\lambda^1,\ldots,\lambda^r)$ be a PBC that conforms to Definition 2 such that $n=|V|=2b-1$, and let $K_n=(V,E_n)$ be the complete graph on n nodes. Then inequality (5) associated with G defines a facet of $\mathrm{EC}(K_n)$ if and only if the PBC has $|P|\ge 2$.*

Proof. For the sake of brevity, we will only give a sketch of the proof and omit specific details. Recall that the equicut polytope $\mathrm{EC}(K_n)$ has dimension $|E_n|-1$ and is contained in the hyperplane defined by the equation $y(E_n) = b(b-1)$. Let F_a be the face of $\mathrm{EC}(K_n)$ that is defined by inequality (5).

In order to prove the only-if part of the theorem, we assume that $|P| = 1$. Then, $|P| = k = 1$ in (13) implies that $r = 2$ and $\ell = b-2$. This means that G is a path-block cycle of the form $PBC_b(1, 0)$. Let $q = q_i \in Q^1$ with $i = \lfloor(b+1)/2\rfloor$ and $q' = q_j \in Q^2$ with $j = \lfloor b/2\rfloor$, and let $\gamma \in \mathrm{EC}(K_n)$ be the incidence vector of an equicut. We can show that if q, q' are contained in the same shore as node i_2, then $\gamma \notin F_a$. Figure 9 illustrates this situation for the case where $b = 5$. Hence, it follows that $F_a \subset \{y \in \mathrm{EC}(K_n) : y_{i_2q} + y_{i_2q'} + y_{qq'} = 2\}$, and F_a has dimension strictly less than $|E_n| - 2$.

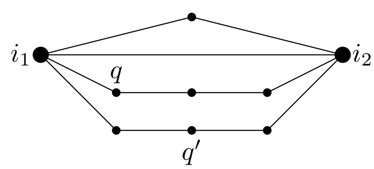

Figure 9: $PBC_5(1, 0)$.

In the proof of the if part of the theorem, we assume that $|P| \geq 2$ and consider a facet-inducing inequality $\pi^T y \geq \pi_0$ for $\mathrm{EC}(K_n)$ whose facet contains F_a. Then it suffices to show that $\pi^T y \geq \pi_0$ can be obtained as a linear combination of inequality (5) and the equation $y(E_n) = b(b-1)$. That is,

$$\left(\pi^T y \geq \pi_0\right) = \xi\left(a^T y \geq 2r\right) + \psi\left(y(E_n) = b(b-1)\right),$$

for $\xi \in \mathbb{R}_+$ and $\psi \in \mathbb{R}$.

When $|P| = 2$, the last part of (13) gives $(r-2)\ell = 1$ such that $r = 3$ and $\ell = 1$. This case is covered in Proposition 6 by the multistar inequality obtained from $PBC_4(1, 1, 0)$. Otherwise, using Lemma 17, it can be established that, for some $\beta \in \mathbb{R}$, $\pi_e = \beta$ for all $e \in E_n \setminus E$. This is not trivial, but the arguments are similar to the ones presented above; the proof is available in the supplement to this article. Subsequently, arguments that are almost identical to those in the proof of Proposition 11 will establish that, for some $\alpha \in \mathbb{R}_+$, $\pi_e = \alpha a_e$ for all $e \in E$. Given these findings, the result follows from $\xi = \alpha - \beta$ and $\psi = \beta$. □

4.2. Facets of the node-capacitated multicut polytope

Here we consider the facet-defining properties of the PBC inequalities for $\mathrm{NCMC}_b(K_n)$ where multicuts with more than two shores are feasible. Accordingly, we are not restricted to consider PBCs $G = (V, E)$ where $|V| \leq 2b$; rather, as in Section 3.4, we also treat the cases where $|V| > 2b$.

We will even allow the complete graph $K_n = (V_n, E_n)$ to be defined on a larger node set such that $V \subset V_n$. The following proposition, which is similar to Theorem 1 in [14], comes in handy in these situations.

Proposition 20. *Let $K_k = (V_k, E_k)$ be a complete subgraph of K_n, and suppose the inequality*

$$\sum_{e \in E_k} a_e y_e \geq a_0 \tag{16}$$

defines a facet of $\mathrm{NCMC}_b(K_k)$. *Let $V_k^1 \subseteq V_k$, such that*

- *for each $v \in V_k^1$, there exists a multicut in E_k, belonging to the root set of (16), that has a singleton shore $S = \{v\}$.*

Let $V_k^2 \subset V_k$, such that

- *for each $v \in V_k^2$, there exists a multicut in E_k, belonging to the root set of (16), that has a shore $S = \{v\} \cup S'$, where $|S| \leq b - 1$ and $S' \subseteq V_k^1$.*

Then inequality (16) defines a facet of $\mathrm{NCMC}_b(K_n)$ *if $V_k^1 \cup V_k^2 = V_k$.*

The next lemma considers the situation where the PBC consists of more than $2b$ nodes and $|V| = 2b + (r - \bar{r})\ell$ nodes, where $\bar{r}$ is defined in (14).

Lemma 21. *Let $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ be a PBC that conforms to Definition 2 such that $|V| = 2b + (r - \bar{r})\ell > 2b$. Consider a multicut $M \subset E_n$ with shores $S_1, \ldots, S_m$, belonging to the root set of inequality (15) associated with G, such that $i_1 \in S_j$ and $i_2 \in S_k$, $j \neq k$. Then $|S_j| = |S_k| = b$.*

Proof. When $|V| = 2b + (r - \bar{r})\ell$, $2 + |P| + \bar{r}\ell = 2b$ so that $S_j \cup S_k$ has capacity to contain exactly all nodes of P and $\bar{r}$ distinct subsets Q^i together with nodes i_1, i_2. It then follows from arguments that are similar to those in the proofs of lemmas 8 and 9 that $|S_j| = |S_k| = b$ when M belongs to the root set of (15). □

Finally, we consider the conditions under which inequalities (5) and (15) induce facets of the node-capacitated multicut polytopes $\mathrm{NCMC}_b(K_n)$.

Theorem 22. *Let $G = (V, E) = PBC_b(\lambda^1, \ldots, \lambda^r)$ be a PBC that conforms to Definition 2, and let $K_n = (V_n, E_n)$ be the complete graph on n nodes such that $V \subseteq V_n$.*

- *Inequality (5) associated with G defines a facet of* $\mathrm{NCMC}_b(K_n)$, *for $b \geq 4$, if and only if $|V| \leq 2b - 1$.*

- *Inequality (15) associated with G defines a facet of* $\mathrm{NCMC}_b(K_n)$, *for $b \geq 4$, if and only if $2b < |V| \neq 2b + (r - \bar{r})\ell$.*

Proof. We first prove the only-if parts of the theorem. Let $M \subset E_n$ be a multicut with shores $S_1, \ldots, S_m$ and let $\gamma \in \{0,1\}^{E_n}$ be its incidence vector. We first consider inequality (5). Suppose that $|V| \geq 2b$ and that M belongs to the root set of (5). It then follows from Lemma 9 that $|V| = 2b$ and $m = 2$ so that $|S_1| = |S_2| = b$ when nodes i_1, i_2 belong to distinct shores. On the other hand, when i_1, i_2 belong to the same shore S_j, both nodes are adjacent in M to all nodes in $V_n \setminus S_j$. For $k = 1, 2$, let $E^k = \delta(\{i_k\}, V_n \setminus I) \subset E_n$.

Then, irrespective of whether i_1, i_2 belong to the same shore or distinct shores, it follows that $\gamma(E^1) = \gamma(E^2)$. This means that the face of inequality (5) is contained in the hyperplane $\{y \in \text{NCMC}_b(K_n) : y(E^1) - y(E^2) = 0\}$ and, therefore, has dimension strictly less than $|E_n| - 1$.

Next, we consider inequality (15). Suppose that $|V| = 2b + (r - \bar{r})\ell$ and that M belongs to the root set of (15). Then, Lemma 21 states that $|S_i| = |S_j| = b$ when nodes i_1, i_2 belong to distinct shores S_i, S_j. As above, when i_1, i_2 belong to the same shore S_j, they are both adjacent to the same nodes $V_n \setminus S_j$ in M. Therefore, it follows that inequality (15) is contained in the same hyperplane as inequality (5) above and does not define a facet of $\text{NCMC}_b(K_n)$.

We now prove the if part of the theorem and assume first that $V_n = V$. Theorem 18 covers all cases where $|V| \leq 2b - 2$. So suppose that $|V| = 2b - 1$. Then, Theorem 19 states that inequality (5) defines a facet of $\text{EC}(K_n)$ if and only if $|P| \geq 2$. We will use Proposition 3 to show that the PBC inequality defines a facet of $\text{NCMC}_b(K_n)$ in this case. Consider multicut M with three shores: $S_1 = I \cup P$, $S_2 = Q^1$, and $S_3 = Q \setminus Q^1$. For $p = b - 1$, M cuts more than $p(p+1)$ edges in E_n, and the result follows.

In the case where $|V| = 2b-1$ and $|P| = 1$, we saw in the proof of Theorem 19 that there exist nodes $q \in Q^1$ and $q' \in Q^2$ such that the face of (5) is contained in the hyperplane defined by $y_{i_2 q} + y_{i_2 q'} + y_{qq'} = 2$. This is not true, when we consider the node-capacitated multicut polytope, because these nodes can belong to three distinct shores. We can prove that inequality (5) is also facet defining for $\text{NCMC}_b(K_n)$ in this case. However, this proof is quite tedious and we prefer not to show it here; instead it is available in the supplement.

Now, suppose that $|V| > 2b$. By fixing shores in the same way as in the proof of Proposition 14, the proof follows from the same arguments that are used when $|V| \leq 2b - 1$. This completes the proof that the PBC inequalities induce facets of $\text{NCMC}_b(K_n)$ when $V_n = V$.

Next, assume that $V \subset V_n$. We will use Proposition 20 to complete the proof for this case. Let $K_k = (V_k, E_k)$ be the complete graph on nodes $V_k = V$. We will show that all nodes of V_k belong to one of the subsets V_k^1, V_k^2 of Proposition 20. In order to do this, we will make modifications to some of the multicuts that are used in the proofs of propositions 11 and 14. It is important to note that here we consider these multicuts as multicuts in the complete graph instead of the support graph of the inequality, i.e., we will assume that $M \subset E_k$ rather than $M \subset E$.

First, let $q \in Q^j$, for any $j \in \{1, \ldots, r\}$, and consider multicut $A_q \subset E_k$ with shores $I \cup P \cup Q^j \setminus \{q\}$ and $\{q\} \cup (Q \setminus Q^j)$ (and additional 'fixed' shores when $k > 2b$). We split the second shore into two shores, $\{q\}$ and $Q \setminus Q^j$. The resulting multicut cuts the same edges in E as A_q and, therefore, also belongs to the root set of the inequality. This implies that $Q^j \subset V_k^1$, for $j = 1, \ldots, r$.

Next, let $p \in P^i$, for any $i \in \{1, \ldots, r'\}$, let $j \in \{1, \ldots, r\}$, and consider multicut A'_p with shores $I \cup Q^j \cup P \setminus \{p\}$ and $\{p\} \cup Q \setminus Q^j$ (and more shores, when $k > 2b$). Again, we split the second shore into two shores, $\{p\}$ and $Q \setminus Q^j$, to obtain a modified multicut that also belongs to the root set of the inequality.

This implies that $P^i \subset V_k^1$, for $i = 1, \ldots, r'$.

Finally, we will show that $I = \{i_1, i_2\}$ belongs to node set V_k^2. Let $p_1 \in P^1$ and consider multicut B_{p_1} with shores $S_1 = \{i_2\} \cup P \cup Q^1$ and $S_2 = \{i_1\} \cup Q \setminus Q^1$ (and more shores, when $k > 2b$). Because $|S_1 \cup S_2| < 2b$ by assumption, it follows that $|S_2| \leq b - 1$. As $Q \subset V_k^1$, it then follows that $i_1 \in V_k^2$. By symmetry, it can be established in a similar manner that $i_2 \in V_k^2$. We have now shown that $I \subseteq V_k^2$ and $V \setminus I \subseteq V_k^1$ such that $V = V_k^1 \cup V_k^2$. This completes the proof. □

5. Concluding remarks

In this paper, we study the facet-defining properties of path-block-cycle inequalities for node-capacitated multicut polytopes. We first provide some necessary conditions for the facet-defining properties of PBC inequalities in the general case. Subsequently, we restrict attention to the special case of PBCs with two common nodes. The facet-defining properties of the corresponding inequalities are established for the bisection cut polytope and the node-capacitated multicut polytope, and in both cases the polytopes that are associated with the PBC graphs as well as complete graphs are considered.

We view this work as a first attempt to determine the facet-defining properties of PBC inequalities in the general case, where the cycles of the PBCs intersect in more than two nodes. We will pursue this topic in future work.

Another direction for future work lies in separation algorithms for the PBC inequalities. This is important, since, without separators, the inequalities cannot be used as cutting planes in actual computations (e.g., in a branch-and-cut algorithm). Unfortunately, the separation problem for PBC inequalities is believed to be NP-hard [8]; the separation problem for the cycle inequalities is already NP-hard. Therefore, heuristic separators must be used in practical problem solving. This topic will be treated in a forthcoming paper.

Supplement

A supplement to this article is available with the online version. It contains some parts of the proofs of Theorem 19 and Theorem 22 that we have chosen to skip here.

References

[1] K. Aardal, R. Weismantel, Polyhedral combinatorics, in: M. Dell'Amico, F. Maffioli, S. Martello (Eds.), Annotated Bibliographies in Combinatorial Optimization, Wiley, New York, 1997.

[2] M. Armbruster, C. Helmberg, M. Fügenschuh, A. Martin, On the graph bisection cut polytope, SIAM J. Discrete Math. 22 (2008) 1073–1098.

[3] M. Conforti, M.R. Rao, A. Sassano, The equipartition polytope I: Formulations, dimension and basic facets, Math. Program. 49 (1990) 49–70.

[4] M. Conforti, M.R. Rao, A. Sassano, The equipartition polytope II: Valid inequalities and facets, Math. Program. 49 (1990) 71–90.

[5] C.C. De Souza, M. Laurent, Some new classes of facets for the equicut polytope, Discrete Appl. Math. 62 (1995) 167–191.

[6] U. Faigle, R. Schrader, R. Suletzki, A cutting plane algorithm for optimal graph partitioning, Methods Oper. Res. 57 (1986) 109–116.

[7] C.E. Ferreira, A. Martin, C.C. de Souza, R. Weismantel, L.A. Wolsey, Formulations and valid inequalities for the node capacitated graph partitioning problem, Math. Program. 74 (1996) 247–266.

[8] C.E. Ferreira, A. Martin, C.C. de Souza, R. Weismantel, L.A. Wolsey, The node capacitated graph partitioning problem: A computational study, Math. Program. 81 (1998) 229–256.

[9] M.R. Garey, D.S. Johnson, Computers and Intractability, Freeman, San Francisco, 1979.

[10] W.W. Hager, D.T. Phan, H. Zhang, An exact algorithm for graph partitioning, Math. Program. 137 (2013) 531–556.

[11] M. Labbé, A. Özsoy, Size-constrained graph partitioning polytopes, Discrete Math. 310 (2010) 3473–3493.

[12] G.L. Nemhauser, L.A. Wolsey, Integer and Combinatorial Optimization, Wiley, New York, 1988.

[13] M.M. Sørensen, *b*-Tree facets for the simple graph partitioning polytope, J. Comb. Optim. 8 (2004) 151–170.

[14] M.M. Sørensen, Facet-defining inequalities for the simple graph partitioning polytope, Discrete Optim. 4 (2007) 221–231.