THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Zooming in on a Lévy process at its supremum

Jevgenijs Ivanovs

Research Report

No. 05 | November 2016

Zooming in on a Lévy process at its supremum

Jevgenijs Ivanovs

Department of Mathematics, Aarhus University, Denmark

Abstract

Let M and τ be the supremum and its time of a Lévy process X on some finite time interval. It is shown that zooming in on X at its supremum, that is, considering $(a_\eta(X_{\tau+t/\eta}-M))_{t\in\mathbb{R}}$ as $\eta, a_\eta \to \infty$, results in $(\xi_t)_{t\in\mathbb{R}}$ constructed from two independent processes corresponding to some self-similar Lévy process S conditioned to stay positive and negative. This holds when X is in the domain of attraction of S under the zooming-in procedure as opposed to the classical zooming-out of Lamperti (1962). As an application of this result we provide a limit theorem for the discretization errors in simulation of supremum and its time, which extends the result of Asmussen, Glynn, and Pitman (1995) for the Brownian motion. Moreover, a general invariance principle for Lévy processes conditioned to stay negative is given.

Keywords: Discretization error, Euler scheme, high frequency statistics, self-similarity, conditioned to stay positive, invariance principle, functional limit theorem

1 Introduction

The law of the supremum of a Lévy process X over a fixed time interval $[0, T]$ plays a key role in various areas of applied probability such as risk theory, queueing, finance and environmental since, to name a few. In particular, it is closely related to first passage (ruin) times, as well as to the distribution of the reflected (queue workload) process. Furthermore, this law is essential in pricing path-dependent options such as lookback and barrier options (Broadie et al., 1997). There are only few examples, however, where the law of the supremum is available in explicit form. More examples are known when T is an independent exponential random variable, see, e.g., (Lewis and Mordecki, 2008) and (Kuznetsov, 2010), but this essentially corresponds to taking transform over time horizon T.

An obvious way to obtain the law of the supremum is to perform Monte Carlo simulation using a random walk approximation of the Lévy process. In other words, the Lévy process is simulated on a grid with a fixed time increment $1/\eta$ for some large η which, of course, assumes that $X_{1/\eta}$ can be simulated efficiently. Even though alternative simulation methods exist (Ferreiro-Castilla et al., 2014), we focus on this obvious discretization scheme and aim at characterizing the limiting behaviour of the discretization or monitoring error. Further motivation comes from the fact that discrete-time models may be more natural in practice, whereas related continuous-time models may admit an explicit solution, see (Broadie et al., 1999) considering such approximations of discrete-time option payoffs. Finally, this setup is consistent with an influential field of high frequency statistics where it is normally assumed that an Itô semimartingale is observed at equidistant times tending to zero (Jacod and Protter, 2012).

Define the supremum and its discretized counterpart

$$M = \sup\{X_t : t \in [0,T]\}, \qquad M_\eta = \max\{X_{i/\eta} : i = 0, \dots, \lfloor \eta T \rfloor\}$$

and let $\epsilon_\eta = M - M_\eta \geq 0$ be the discretization error. The (last) times of the supremum and the maximum are denoted by τ and τ_η, respectively. In the case when X is a Brownian motion with variance σ and drift μ, Asmussen et al. (1995) showed the following weak convergence:

$$\sqrt{\eta}\epsilon_\eta \quad \Rightarrow \quad \sigma V, \tag{1.1}$$

where V is defined using two independent copies of a 3-dimensional Bessel processes and an independent uniform time shift. It is intuitive that (1.1) continues to hold if X is replaced by an independent sum of a Brownian motion and a compound Poisson process, which is indeed true as shown by Dia and Lamberton (2011). Despite numerous follow-up works and importance of (1.1) in various applications, the limiting behaviour of ϵ_η is not known for a general Lévy process X. In fact, most of the related works are concerned with asymptotic expansions of the expected error $\mathbb{E}\epsilon_\eta$, see (Janssen and Van Leeuwaarden, 2009), (Dia, 2010), (Chen, 2011) and (Dia and Lamberton, 2011).

In this paper we establish a functional limit theorem for $a_\eta(X_{\tau+t/\eta} - M), t \in \mathbb{R}$ as $\eta, a_\eta \to \infty$, which corresponds to zooming in on the Lévy process X at its supremum, see Theorem 6. The limit process ξ for positive times has the law of a certain self-similar Lévy process S conditioned to be negative, whereas for negative times it is the negative of S conditioned to be positive. It is required for this limit theorem that X is in the domain of attraction of S (with a scaling sequence a_η) under the zooming-in procedure as opposed to the classical zooming-out of Lamperti (1962). It is noted that zooming-in and zooming-out domains are very different, and the former is determined by the behaviour of X at 0. In Corollary 7 we provide a general version of (1.1) which additionally includes the scaled difference of suprema times $\eta(\tau - \tau_\eta)$. In particular, we show that (1.1) holds whenever the Brownian component is present, i.e. $\sigma > 0$ in the Lévy-Khintchine formula (2.1).

Let us briefly discuss some related literature. In the study of extremes of Gaussian processes (Piterbarg, 1996) it is standard to assume that the process of interest locally behaves as a fractional Brownian motion or, more generally, as a self-similar centered Gaussian process. In the context of Lévy processes, Barczy and Bertoin (2011) obtained a somewhat related functional limit theorem by starting the process (with a negative drift) at $x \to -\infty$, conditioning on having a positive supremum, and shifting at the instant of the supremum. Concerning the classical zooming-out procedure, Caravenna and Chaumont (2008) considered a rescaled random walk converging to a self-similar Lévy process and provided an invariance principle for conditioning to stay positive. Finally, let us also mention that our problem does not fit into the standard framework of high frequency statistics (Jacod and Protter, 2012), because the discretization error ϵ_η can not be easily retrieved from the difference of X and its discretized version.

This paper is organized as follows. Section 2 is devoted to preliminaries on Lévy processes, self-similar processes, processes conditioned to stay negative, as well as post-supremum processes. A general invariance principle for Lévy processes conditioned to stay negative is proven in Section 3. In Section 4 we present the result of Lamperti (1962) but for zooming in instead of zooming out, and then specialize to the case of Lévy processes. The main results of this paper are given in Section 5, and the domains of attraction of self-similar Lévy processes under zooming-in are studied in Section 6. We conclude by giving some remarks in Section 7.

2 Preliminaries

2.1 Canonical notation

Let Ω be a set of two-sided cádlág paths $\omega : \mathbb{R} \mapsto \mathbb{R} \cup \{\dagger\}$, where $\dagger$ is an isolated additional point added to $\mathbb{R}$. For a usual path defined on $[0, \infty)$ we put $\omega_t = 0$ for all $t < 0$ which will be convenient in the following. Additionally, we may want to terminate the path ω at some non-negative time T, and then we put $\omega_t = \dagger$ for all $t \geq T$.

We endow Ω with the extended Skorokhod J_1 topology (Whitt, 1980), so that a sequence of two-sided paths converges to some $\omega \in \Omega$ if the restrictions to $[a, b]$ converge for all $a < b$ such that a, b are the continuity points of ω. We let X be the canonical process: $X_t(\omega) = \omega_t$, and let $\mathbb{P}$ be a probability measure on Ω under which $(X_t)_{t\geq 0}$ is a Lévy process. Additionally, we write $\mathbb{P}_x$ for the law of this process issued from x.

2.2 Lévy processes

Consider a Lévy process $(X_t)_{t\geq 0}$ and let $\psi(\theta)$ be its Lévy exponent: $\mathbb{E}e^{\theta X_t} = e^{\psi(\theta)t}$, $t \geq 0$ for at least purely imaginary θ. The Lévy-Khintchine formula states that

$$\psi(\theta) = a\theta + \tfrac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{\theta x} - 1 - \theta x 1_{\{|x|<1\}}\right) \nu(dx), \tag{2.1}$$

where $a \in \mathbb{R}, \sigma \geq 0$ and $\nu(dx)$ is a measure on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(dx) < \infty$. When $\int_{-1}^{1} |x|\nu(dx) < \infty$ then this formula can be rewritten as

$$\psi(\theta) = d\theta + \tfrac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{\theta x} - 1\right) \nu(dx), \tag{2.2}$$

which corresponds to an independent sum of a drifted Brownian motion with mean d and variance σ^2, and a pure jump process of bounded variation on compacts.

In order to avoid uninteresting complications, throughout this work we exclude compound Poisson processes, i.e., the case of $d = 0, \sigma = 0, \nu(\mathbb{R}) < \infty$. Concerning the behaviour of X for large t, we recall that only the following three possibilities can occur as $t \to \infty$: (i) $X_t \to \infty$, (ii) $\liminf_t X_t = -\infty$ and $\limsup_t X_t = \infty$, (iii) $X_t \to -\infty$ a.s., where in case (ii) we say that X *oscillates*.

Often it is convenient to consider a Lévy process X *killed* (sent to $\dagger$) at an independent exponential time e_q of rate $q > 0$. This is the only way of killing which preserves stationarity and independence of increments (up to the killing time when using the convention $\dagger - x = \dagger$), and so it leads to a natural generalization of a Lévy process. We often keep $q \geq 0$ implicit, but write $\mathbb{P}^q, \psi^q$ when it is necessary to stress that the corresponding Lévy process is killed at rate q. The Lévy-Khintchine formula (2.1) is extended to killed Lévy processes by putting $\psi^q(\theta) = \psi(\theta) - q$ so that $\mathbb{E}^q(e^{\theta X_t}; X_t \neq \dagger) = e^{\psi^q(\theta)t}$.

Finally, we define the overall supremum and its time:

$$\overline{X} = \sup_{t\geq 0}\{X_t : X_t \neq \dagger\}, \qquad \overline{G} = \sup\{t \geq 0 : X_t = \overline{X} \text{ or } X_{t-} = \overline{X}\}$$

with the convention that $\overline{G} = \infty$ when $\overline{X} = \infty$. The latter occurs when X drifts to ∞ or oscillates,in which case X must be non-killed. Additionally, we let $\underline{X} = \inf_{t\geq 0}\{X_t : X_t \neq \dagger\}$ to denote the overall infimum.

2.3 Self-similar processes

A process $(X_t)_{t\geq 0}$ with $X_0 = 0$ is called *self-similar* if there exists $\alpha > 0$ such that for all $u > 0$

$$(u^{-1/\alpha} X_{ut})_{t\geq 0} \stackrel{d}{=} (X_t)_{t\geq 0}. \tag{2.3}$$

It will be always assumed that X is non-trivial in which case the index α is unique. If X is a self-similar Lévy process then necessarily $\alpha \in (0, 2]$ and $q = 0$ (no-killing). Moreover, $\alpha = 2$ corresponds to the Brownian motion with drift 0 and variance σ^2, and $\alpha \in (0, 2)$ corresponds to a *strictly α-stable* Lévy process with $\sigma = 0$ and the following Lévy measure:

$$\nu(\mathrm{d}x) = 1_{\{x>0\}} c_+ x^{-1-\alpha}\mathrm{d}x + 1_{\{x<0\}} c_- |x|^{-1-\alpha}\mathrm{d}x \tag{2.4}$$

for some $c_+, c_- \geq 0$. Moreover, the following must hold true:

$$\begin{aligned} a &= (c_+ - c_-)/(1-\alpha), \qquad && \text{if } \alpha \neq 1, \\ c_+ &= c_-, && \text{if } \alpha = 1. \end{aligned} \tag{2.5}$$

Additionally, the linear drift process $X_t = dt$ for $d \neq 0$ is self-similar with $\alpha = 1$. The above are all the possible examples of non-trivial self-similar Lévy processes.

Suppose X is a self-similar Lévy process which is not a linear drift process. Then X has paths of bounded variation on compacts if and only if $\alpha \in (0, 1)$, in which case we may use the representation (2.2) with $d = 0$ and $\sigma = 0$. In particular, if X is monotone then necessarily $\alpha < 1$, and so it is a pure jump process with all the jumps of the same sign. Finally, if X is not monotone then the point 0 is regular for $(0, \infty)$ and $(-\infty, 0)$, see (Kyprianou, 2006, Thm. 6.5). In this case, by self-similarity, the process X must be oscillating and so $\overline{X} = \infty$ and $\underline{X} = -\infty$.

2.4 Processes conditioned to stay negative

For any $x < 0$ we may define the law of a Lévy process X started in x and conditioned to stay negative:

$$\mathbb{P}^{\downarrow}_x(\,\cdot\,) = \mathbb{P}_x(\,\cdot\,|\overline{X} < 0)$$

unless $\mathbb{P}(\overline{X} = \infty) = 1$, because then we would condition on the event of zero probability. In general, we first consider a killed process and then take the limit (the corresponding events must be in $\mathcal{F}_T$ for some T):

$$\mathbb{P}^{\downarrow}_x(\,\cdot\,) = \lim_{q\downarrow 0} \mathbb{P}^q_x(\,\cdot\,|\overline{X} < 0), \tag{2.6}$$

which defines a probability law (Chaumont and Doney, 2005). It is well known that the process under $\mathbb{P}^{\downarrow}_x$ is a Markov process on $(-\infty, 0)$ with a Feller semigroup, say $p^{\downarrow}_t(x, \mathrm{d}y)$. This process has infinite life time if and only if the original Lévy process X satisfies $\underline{X} = -\infty$, i.e. X either drifts to $-\infty$ or oscillates. Finally, it is standard to express the semigroup $p^{\downarrow}_t(x, \mathrm{d}y)$ as Doob's h-transform of X killed at the entrance time into $[0, \infty)$, see (3.1) for a precise expression.

It is crucial to take the limit in (2.6) along independent exponential times, that is, the limit of conditioned killed Lévy processes, because deterministic times may result in a different limit law. In particular, when $X \to \infty$ the life time of the process under $\mathbb{P}^{\downarrow}_x$ is finite, whereas deterministic times necessarily lead to an infinite lifetime if the corresponding limit law exists, see also (Hirano, 2001). When X oscillates, we may alternatively condition on X exiting $(-y, 0)$ through $-y$ and then letting $y \to \infty$, see (Chaumont and Doney, 2005, Rem. 1). Finally, according to (Chaumont, 1996, Rem. 1), for a non-monotone self-similar Lévy process we may also take the limit along deterministic times:

$$\mathbb{P}^{\downarrow}_x(\,\cdot\,) = \lim_{t\to\infty} \mathbb{P}_x(\,\cdot\,|X_s < 0\,\forall s \leq t).$$

2.5 Post-supremum processes

Unless $\overline{X} = \infty$ we consider the post-supremum process $(X_{\overline{G}+t} - \overline{X})_{t\geq 0}$, and denote its law by $\mathbb{P}^{\downarrow}$ (there is no subscript as compared to the conditional law $\mathbb{P}^{\downarrow}_x$). In general, we consider X on a finite time interval $[0, T]$ and the corresponding post-supremum process. Then we take $T \to \infty$ to define the law $\mathbb{P}^{\downarrow}$, see (Bertoin, 1993), where it is also shown that the process under $\mathbb{P}^{\downarrow}$ is Markov with transition semigroup $p^{\downarrow}_t(x, \mathrm{d}y)$ for any $x, y < 0$ and $t \geq 0$. This explains the notation for the law of the post-supremum process. If X is such that 0 is regular for $(-\infty, 0)$ then the process under $\mathbb{P}^{\downarrow}$ starts at 0 and leaves it immediately, but otherwise it starts at a negative value having a certain distribution, see (Chaumont and Doney, 2005). In the latter case the post-supremum process may also be identically † with positive probability.

It should be noted that some of the cited results are stated for non-killed processes, but their extension to killed Lévy processes is straightforward. Furthermore, in the analogous way we define the laws $\mathbb{P}^{\uparrow}_x, x > 0$ and $\mathbb{P}^{\uparrow}$ corresponding to the Lévy process conditioned to stay positive and the post-infimum process, respectively. One may easily obtain these laws by considering $-X$ process. Here we recall that we excluded compound Poisson processes, but otherwise we would need to make a distinction between conditioning to stay non-negative and positive, as well as first and last times of the infimum.

Remark 1. *In this paper we will focus on a self-similar Lévy process S with law $\mathbb{Q}$ arising as a weak limit when zooming in on X. Recall that such S oscillates when non-monotone and hence both $\mathbb{Q}^{\uparrow}$ and $\mathbb{Q}^{\downarrow}$ are defined as the limit laws of finite time post-infimum and post-supremum processes, respectively.*

3 Invariance principle for Lévy processes conditioned to stay negative

Recall from Section 2.1 that we work with two-sided paths such that $\omega_t = 0$ for all $t < 0$. This trick allows us to provide a clean formulation of the following functional limit theorem which is central for this study.

Theorem 2. *Let $X^{(\eta)}$ be a sequence of (possibly killed) Lévy processes weakly converging to a Lévy process X (which is not a compound Poisson process). Then ${\mathbb{P}^{(\eta)}}^{\downarrow}_x \Rightarrow \mathbb{P}^{\downarrow}_x$ for all $x < 0$ and, moreover, ${\mathbb{P}^{(\eta)}}^{\downarrow} \Rightarrow \mathbb{P}^{\downarrow}$.*

If the process X has a finite supremum then the above statement follows immediately from the continuous mapping theorem and the fact that X has a unique time of the supremum. The main difficulty lies in the other case, where the law $\mathbb{P}^{\downarrow}$ is defined as a limit. A similar invariance statement was proven by Bryn-Jones and Doney (2006) and Caravenna and Chaumont (2008), but with $X^{(\eta)}$ being a rescaled random walk converging to a Brownian motion (Donsker's theorem) and to an α-stable Lévy process, respectively. Some other related works include (Bolthausen, 1976), (Iglehart, 1974), and (Doney, 1985) considering the invariance principle for the meander in the setting of Donsker's theorem and its extension.

The assumption of two-sided paths with $\omega_t = 0$ for all $t < 0$ allows us to avoid the following problem. Suppose that X is such that 0 is irregular for $(-\infty, 0)$, but $X^{(\eta)}$ are such that 0 is regular for $(-\infty, 0)$. For example, we may add to X a Brownian motion with zero drift and diminishing variance. For simplicity, assume that all the processes are killed so that the times of suprema are finite. Then X leads to the post-supremum process starting (at time 0) at a negative level, whereas for $X^{(\eta)}$ such processes start at 0 and then quickly jump to a negative level when η is large. The assumption that these processes are fixed at 0 for negative times ensures the claimed convergence in the Skorokhod's J_1 topology. A similar problem but with a different solution appears in (Chaumont and Doney, 2005, Thm. 2).

Proof of Theorem 2. The proof consists of three steps, where in steps (ii) and (iii) we use particular representations of the laws $\mathbb{P}^{\downarrow}_x$ and $\mathbb{P}^{\downarrow}$ avoiding double limits. In the following we define some quantities for the process X and assume that the analogous quantities are defined for each $X^{(\eta)}$ without explicitly writing them.

(i) Consider the ascending ladder processes (L^{-1}, H), where L^{-1} denotes the inverse local time at the supremum and $H_t = X_{L^{-1}_t}$. The corresponding Laplace exponent is denoted by $k(\alpha,\beta)$ and normalized so that $k(1,0)=1$, see (Bertoin, 1996, Ch. VI) or (Kyprianou, 2006, Ch. 6). By the continuous mapping theorem we get convergence of the Wiener-Hopf factors, which then implies convergence of the bivariate exponents and hence also weak convergence of the ladder processes:

$$k^{(\eta)}(\alpha,\beta)\to k(\alpha,\beta), \qquad ({L^{(\eta)}}^{-1},H^{(\eta)})\Rightarrow(L^{-1},H).$$

It is noted that in the above textbooks the results are formulated for non-killed Lévy processes, but they extend to killed Lévy processes in a straightforward way. In particular, such Wiener-Hopf factors concern killed Lévy processes observed up to another independent exponential time. Alternatively, we may use the obvious relation $k^q(\alpha,\beta)=k(\alpha+q,\beta)/k(1+q,0)$.

(ii) The following representation of the semigroup of the conditioned process is standard (Chaumont and Doney, 2005):

$$p_t^{\downarrow}(x,\mathrm{d}y)=\frac{h(y)}{h(x)}\mathbb{P}_x(X_t\in\mathrm{d}y,\overline{X}_t<0), \qquad x<0, \tag{3.1}$$

where $\overline{X}_t=\sup_{s\le t}X_s$ and

$$h(x)=\mathbb{E}\int_{[0,\infty)}1_{\{\overline{X}_t<-x\}}\mathrm{d}L_t=\mathbb{E}\int_0^\infty 1_{\{H_t<-x\}}\mathrm{d}t=:U(-x)$$

is a finite, continuous, increasing function. Note that $\mathbb{P}_x(\overline{X}_t=0)=0$ for $x<0$, because we assumed that X is not a compound Poisson process, and so

$$\mathbb{P}_x^{(\eta)}(X_t\in\mathrm{d}y,\overline{X}_t<0) \quad\Rightarrow\quad \mathbb{P}_x(X_t\in\mathrm{d}y,\overline{X}_t<0).$$

Furthermore, observe that $U(y)$ is the distribution function of $U(\mathrm{d}y)=\int_0^\infty\mathbb{P}(H_t\in\mathrm{d}y)\mathrm{d}t$ which is the potential measure of the ladder height process. It is well-known and easy to see that $\int_{[0,\infty)}e^{-\beta y}U(\mathrm{d}y)=1/k(0,\beta)$ for $\beta>0$. Thus according to step (i) the Laplace transform of $U^{(\eta)}(\mathrm{d}y)$ converges to that of $U(\mathrm{d}y)$ for all $\beta>0$, and so the corresponding distribution functions converge:

$$h^{(\eta)}(x)=U^{(\eta)}(-x) \quad\to\quad U(-x)=h(x),$$

because the latter is continuous. The above convergence result for infinite measures can be found in e.g. (Mimica, 2015, Thm. 2.1).

We have established convergence of the semigroup given in (3.1), and so according to (Ethier and Kurtz, 1986, Thm. 4.2.5) we obtain

$$\mathbb{P}^{(\eta)\downarrow}_x \quad\Rightarrow\quad \mathbb{P}^{\downarrow}_x, \qquad \text{for } x<0$$

because the corresponding processes are Feller and the initial distributions coincide.

(iii) Finally, we recall (Chaumont and Doney, 2005, Thm. 1) that $\mathbb{P}^{\downarrow}$ is also the law of the post-supremum process under $\mathbb{P}^{\downarrow}_x$ for any $x<0$. Under the latter law the time of the supremum is finite and unique, and so we can apply the continuous mapping theorem to establish that

$$\mathbb{P}^{(\eta)\downarrow} \quad\Rightarrow\quad \mathbb{P}^{\downarrow}.$$

This completes the proof. □

Let us remark that the assumption of Theorem 2 that X is not a compound Poisson process is essential, and a counter example can be easily provided by considering $X_t - t/\eta$ so that the limit of $\mathbb{P}^{(\eta)\downarrow}$ is the law of X conditioned to stay non-positive rather than negative. Nevertheless, it is no problem if some or all of $X^{(\eta)}$ are compound Poisson processes.

Finally, one may conjecture that $\mathbb{P}^{(\eta)\downarrow}_{x_\eta} \Rightarrow \mathbb{P}^{\downarrow}_{x}$, where $x, x_\eta \le 0$ are such that $x_\eta \to x$ and by convention $\mathbb{P}^{\downarrow}_0 = \mathbb{P}^{\downarrow}$. This result would include Theorem 2 and (Chaumont and Doney, 2005, Thm. 2) as special cases. This generalization is rather obvious when $x < 0$, but requires some effort when $x_\eta \to 0$. In the latter case we need to show that the pre-supremum process under $\mathbb{P}^{(\eta)\downarrow}_{x_\eta}$ collapses into a point in the limit, see also (Chaumont and Doney, 2005, Eq. (3.1)). This extension is not needed in this work and thus is omitted.

4 The result of Lamperti for zooming in

Consider an arbitrary stochastic process X. The famous result of Lamperti (1962) states that the class of all possible non-degenerate limits of $((X_{\eta t}+b_\eta)/a_\eta)_{t\ge 0}$, where $\eta \to \infty$ and $0 < a_\eta \to \infty$, is given by self-similar processes, see Section 2.3. The above rescaling may be seen as zooming out on the process X. In this work, however, we are interested in the opposite scaling of time and space, that is, in zooming in on the process X:

$$\lim_{\eta\to\infty} \left(a_\eta X_{t/\eta}\right)_{t\ge 0} = (S_t)_{t\ge 0}, \qquad \text{where } 0 < a_\eta \to \infty, \tag{4.1}$$

and the convergence is in the sense of finite dimensional distributions, see also Remark 4. By a slight adaptation of the arguments in (Lamperti, 1962, Thm. 2) we get the following result.

Theorem 3. *Assume that* (4.1) *holds for continuous in probability non-trivial process S. Then S is self-similar with some index $\alpha > 0$ as defined in* (2.3) *and*

$$a_\eta = \eta^{1/\alpha}\ell_\eta, \qquad X_0 = 0 \text{ a.s.},$$

where ℓ_η is a function slowly varying at ∞.

Remark 4. *Instead of* (4.1) *we may consider a seemingly more general sequence $a_\eta(X_{t/\eta} + b_\eta)$ of processes. Then it is not hard to see that there is a constant c such that $b_\eta \to c, a_\eta(b_\eta - c) \to 0$ and $\mathbb{P}(X_0 = -c) = 1$ showing that we may simply take X_t+c and apply Theorem 3. Furthermore, there are two differences from the original result of Lamperti (1962). Firstly, we necessarily have $S_0 = 0$, and, secondly, the distribution of X_0 must be concentrated at a point.*

It is worth mentioning that the sequence a_η and the limit process S are essentially unique up to a scaling factor (convergence to types result). More precisely, if the limit processes S and S' are non-trivial then S' and aS are versions of each other for some constant $a > 0$ where $a'_\eta/a_\eta \to a$.

Let us specialize (4.1) to the case when X is a Lévy process with the Lévy exponent ψ. It is clear that stationarity and independence of increments must be preserved by the limit process, and so S must be a Lévy process with Lévy exponent ψ_S, say. Now the convergence in (4.1) extends to the weak convergence on the Skorokhod space, and it is equivalent to

$$\psi^{(\eta)}(\theta) = \psi(\theta a_\eta)/\eta \to \psi_S(\theta), \qquad \eta \to \infty \tag{4.2}$$

for purely imaginary θ, where $\psi^{(\eta)}$ is the Lévy exponent of the Lévy process $X^{(\eta)}{}_t = a_\eta X_{t/\eta}$. According to Theorem 3, if S is non-trivial then it is α-self-similar Lévy process and a_η is regularly varying with index $1/\alpha$, see also Section 2.3. In Section 6 we provide a rich class of examples of Lévy processes X satisfying (4.2) and identify the associated self-similar Lévy process S, as well as the scaling sequence a_η.

We conclude this section by the following simple observation.

Lemma 5. *There is a trichotomy with respect to* (4.2)*:*

(i) X is such that 0 is regular for $(-\infty, 0)$ and for $(0, \infty)$ then S is oscillating;

(ii) X is such that 0 is irregular for $(0, \infty)$ then S is decreasing;

(iii) X is such that 0 is irregular for $(-\infty, 0)$ then S is increasing.

Proof. The given three cases are exhaustive, because we assumed that X is not a compound Poisson process. Use self-similarity of S to obtain the statements in each of the cases. □

5 Zooming in on the supremum

In the following we consider a Lévy process X satisfying (4.2) for some sequence $0 < a_\eta \to \infty$ and some non-trivial self-similar Lévy process S. In other words, X is in the domain of attraction of S when zooming in, see Section 6 providing sufficient conditions for this.

Letting $\mathbb{Q}$ be the law of S, we consider the process ξ on $\mathbb{R}$ such that

$$(\xi_t)_{t\geq 0} \text{ has the law } \mathbb{Q}^{\downarrow}, \qquad (-\xi_{(-t)-})_{t\geq 0} \text{ has the law } \mathbb{Q}^{\uparrow} \tag{5.1}$$

and both parts are independent, see Section 2.5. Note that on the right hand side we reverse both time and space. In other words, when looking at ξ from the point $(0, 0)$ backwards in time and down in space we see the law $\mathbb{Q}^{\uparrow}$. Furthermore, the laws $\mathbb{Q}^{\downarrow}$ and $\mathbb{Q}^{\uparrow}$ inherit self-similarity from $\mathbb{Q}$, and so they correspond to self-similar Markov processes, where the former is negative and the latter is positive (when started away from 0). Such processes are well-studied and, in particular, they enjoy the Lamperti representation using the associated Lévy process, see (Caballero and Chaumont, 2006, Cor. 2) specifying the latter.

Finally, according to the trichotomy of Lemma 5 we have the following cases:

(i) If S is oscillating then ξ has doubly infinite life time, i.e. $\xi_t \neq \dagger$ for all $t \in \mathbb{R}$;

(ii) If S is decreasing then

$$\xi_t = \dagger 1_{\{t<0\}} + S_t 1_{\{t\geq 0\}}, \qquad t \in \mathbb{R};$$

(iii) If S is increasing then

$$\xi_t = -S_{(-t)-} 1_{\{t<0\}} + \dagger 1_{\{t\geq 0\}}, \qquad t \in \mathbb{R}.$$

Note that in the latter two cases the non-trivial conditional law $\mathbb{Q}^\uparrow$ or $\mathbb{Q}^\downarrow$ is essentially given by the law $\mathbb{Q}$ of the self-similar Lévy process S.

Theorem 6. *Let X be a Lévy process satisfying* (4.2) *for some sequence $0 < a_\eta \to \infty$ and a non-trivial process S, which then must be a self-similar Lévy process. Consider X on $[0, T)$ for any $T > 0$, and let M and τ be the supremum and its time, respectively. Then*

$$(a_\eta(X_{\tau+t/\eta} - M))_{t\in\mathbb{R}} \quad \Rightarrow \quad (\xi_t)_{t\in\mathbb{R}} \text{ as } \eta \to \infty, \tag{5.2}$$

where ξ is defined in (5.1) *with respect to the law $\mathbb{Q}$ of S characterized by $\psi_S(\theta)$.*

Proof. Note that X can not be a compound Poisson process, because then the limit $S \equiv 0$ is trivial. Restriction of X to $[0, T)$ is achieved by putting $X_t = \dagger$ for all $t \notin [0, T)$. Instead of a deterministic T we first consider the case $T = e_q$ of an independent exponential time of rate $q > 0$. By doing so we obtain a killed Lévy process, which satisfies (4.2) with the same a_η and ψ_S, and hence the corresponding killed Lévy process $X^{(\eta)}$ converges to S. Observe that

$$a_\eta(X_{\tau+t/\eta} - M)1_{\{t\geq 0\}}$$

is the post-supremum process corresponding to $X^{(\eta)}$, and so its law converges to $\mathbb{Q}^\downarrow$ according to Theorem 2. Moreover, it is well known that the pre-supremum process

$$-a_\eta(X_{(\tau-t/\eta)-} - M)1_{\{t\geq 0\}}$$

is independent of the post-supremum process and has the law of the post-infimum process, which follows from time reversal and splitting (Greenwood and Pitman, 1980). Another application of Theorem 2, but for conditioning to stay positive, shows that the limit law is given by $\mathbb{Q}^\uparrow$, completing the proof for a random $T = e_q$.

Consider a bounded continuous functional F on the Skorokhod space of two-sided paths. Let $F_T^{(\eta)}$ and $F^{(\infty)}$ denote F applied to the left hand side of (5.2) and the right hand side, respectively. The first part of the proof shows that

$$q\int_0^\infty e^{-qT}\mathbb{E}F_T^{(\eta)}\mathrm{d}T \to \mathbb{E}F^{(\infty)} = q\int_0^\infty e^{-qT}\mathbb{E}F^{(\infty)}\mathrm{d}T,$$

that is, the transforms in T converge. Hence $\mathbb{E}F_T^{(\eta)} \to \mathbb{E}F^{(\infty)}$ for almost all $T > 0$. If X is such that 0 is regular for $(-\infty, 0)$ then $\tau \neq T$ a.s. for any $T > 0$. In this case the extension of the result to an arbitrary T is straightforward. In the other case we use time reversal to translate our supremum problem into infimum problem, and observe that the infimum can not be achieved at the end point T. Swapping the direction of time completes the proof. □

With respect to Theorem 6 and trichotomy of Lemma 5 it will be useful to observe that in case (ii) the convergence result of Theorem 6 holds on the event $\{\tau \neq 0\}$, because it trivially holds on the complementary event as well as on the whole Ω. The analogous observation can be made about the case (iii) and the event $\{\tau \neq T\}$. The following result establishes the joint convergence of $(M_\eta - M, \tau_\eta - \tau)$ according to the above trichotomy.

Corollary 7. *Let U be an independent uniform $(0,1)$ random variable. Under the conditions of Theorem 6 the sequence of random pairs*

$$(a_\eta(M_\eta - M), \eta(\tau_\eta - \tau)), \qquad \eta \to \infty$$

weakly converges to

(i) $(\max_{i\in\mathbb{Z}} \xi_{U+i}, U + \mathrm{argmax}_{i\in\mathbb{Z}}\xi_{U+i})$ *if S is oscillating,*

(ii) (S_U, U) *if S is decreasing,*

(iii) $-(S_U, U)$ *if S is increasing,*

where in (ii) and (iii) the convergence holds on the event $\{\tau \neq 0\}$ and $\{\tau \neq T\}$, respectively.

Proof. It is well known that the distribution of τ has a Lebesgue density on $(0, T)$ and an atom at 0 and at T according to (ii) and (iii) of the above trichotomy, see (Chaumont, 2013, Thm. 6). The old result by Kosulajeff (1937) states that the fractional part $\mathrm{frac}(\tau\eta)$ weakly converges to U if the distribution of τ is absolutely continuous. Moreover, it is well known, see (Jacod and Protter, 2012, Thm. 4.3.1), that we may take an independent U on the same probability space and such that $\mathrm{frac}(\tau\eta) \to U$ a.s. (this construction is related to stable convergence in law).

Note that observing X_t at the time instants $i/\eta, i \in \mathbb{Z}$ corresponds to observing $X_{\tau+t/\eta}$ at the time instants $\mathbb{Z} - \mathrm{frac}(\tau\eta)$. From Theorem 6 and the above discussion we have

$$\left(-\mathrm{frac}(\tau\eta), (a_\eta(X_{\tau+t/\eta} - M))_{t\in\mathbb{R}}\right) \quad \Rightarrow \quad (U, (\xi_t)_{t\in\mathbb{R}}),$$

where U and ξ are independent. It is easy to see that $\mathbb{P}(\xi_t = \xi_s) = 0$ for any $s \neq t$ (i $\xi_t \neq \dagger$), and $\lim_{|t|\to\infty} \xi_t \in \{-\infty, \dagger\}$. Therefore, ξ_t observed at times $i+U, i \in \mathbb{Z}$ has a unique maximum. Furthermore, ξ is continuous at each of the observation instants a.s., and so the continuous mapping theorem completes the proof. □

Let us comment on the cases (ii) of Corollary 7; the case (iii) being similar. This case corresponds to X being such that 0 is irregular for $(0, \infty)$, and so the process X achieves its maximum by a jump, unless $\tau = 0$. Hence on the event $\{\tau \neq 0\}$ the time τ_η of the discrete maximum must approach τ from the right, which is reflected by the fact that $\eta(\tau_\eta - \tau)$ converges to a positive U. Moreover, in the limit the first observation succeeding time τ yields the maximal value.

6 Domains of attraction

In this section we explore the convergence of Lévy exponents in (4.2). We assume that X has the Lévy triplet $a, \sigma, \nu(\mathrm{d}x)$ as specified in Section 2.2. Furthermore, in the bounded variation case we use the linear drift d.

Proposition 8. *The following two results hold.*

(i) If $\sigma > 0$ then (4.2) *holds with*

$$a_\eta = \sqrt{\eta}, \quad \text{and} \quad \psi_S(\theta) = \sigma^2\theta^2/2,$$

i.e. S is a Brownian motion with zero drift and variance σ^2.

(ii) If X is a bounded variation process then (4.2) *holds with*

$$a_\eta = \eta \quad \text{and} \quad \psi_S(\theta) = d\theta,$$

i.e. S is a linear drift process: $S_t = dt$, which is trivial if $d = 0$.

Proof. (i) It is well known (Bertoin, 1996, Prop. I.2) that $\psi(\theta)/\theta^2 \to \sigma^2/2$ as $|\theta| \to \infty$. Hence $\psi(\theta\sqrt{\eta})/\eta \to \sigma^2\theta^2/2$ as $\eta \to \infty$ establishing the result. The result in (ii) follows similarly. □

Proposition 9. *Assume that $\sigma = 0$ and $\nu(\mathrm{d}x)$ has a density f on some neighbourhood of 0. Suppose that*

$$\lim_{x\downarrow 0} \frac{f(-x)}{f(x)} = c \in [0,\infty), \qquad f(x) = x^{-\alpha-1}\ell(x) \quad \text{for small } x > 0,$$

where ℓ is a function slowly varying at 0, and $\alpha \in (0,1)\cup(1,2)$. Moreover, let $d = 0$ if $\alpha < 1$. Then (4.2) *holds for $a_\eta \to \infty$ satisfying*

$$\frac{a_\eta^\alpha \ell(\frac{1}{a_\eta})}{\eta} \to c_+ \in (0,\infty) \tag{6.1}$$

with S being a strictly α-stable Lévy process with $c_- = cc_+$, see (2.4).

Proof. It is assumed throughout that $\theta \in \mathrm{i}\mathbb{R}$ is fixed.

(i) Consider the case of $\alpha < 1$. Then for any fixed $\epsilon > 0$ we have

$$\psi(\theta a_\eta)/\eta = \int_{\mathbb{R}} \frac{1}{\eta}(\mathrm{e}^{\theta a_\eta x} - 1)\nu(\mathrm{d}x) = \int_{-\epsilon}^{\epsilon} \frac{1}{\eta}(\mathrm{e}^{\theta a_\eta x} - 1)\nu(\mathrm{d}x) + O(1/\eta).$$

Next observe that

$$\begin{aligned}\int_0^\epsilon (\mathrm{e}^{\theta a_\eta x} - 1)/\eta f(x)\mathrm{d}x &= \int_0^{\epsilon a_\eta} (e^{\theta x} - 1)\frac{f(x/a_\eta)}{\eta a_\eta}\mathrm{d}x \\ &= \int_0^{\epsilon a_\eta} (e^{\theta x} - 1)x^{-\alpha-1}\frac{a_\eta^\alpha \ell(1/a_\eta)}{\eta}\frac{\ell(x/a_\eta)}{\ell(1/a_\eta)}\mathrm{d}x \to c_+\int_0^\infty (e^{\theta x} - 1)x^{-1-\alpha}\mathrm{d}x\end{aligned}$$

given that the dominated convergence theorem applies which is shown in the following. Analysis of the contribution of negative jumps is now trivial. Use (2.4) to conclude.

Concerning the dominated convergence theorem, we split the second last integral in two according to $x < 1$ and $x > 1$, and use the bounds $|\mathrm{e}^{\theta x} - 1| \leq C_1 x$ and $|\mathrm{e}^{\theta x} - 1| \leq C_1$ for some $C_1 > 0$, respectively. According to Lemma 10 with $\delta = 1/a_\eta$ the first integrand is bounded by $C_2 x^{-\alpha-v}$ and the second by $C_2 x^{-\alpha-1+v}$ for large enough a_η and any $v > 0$. It is left to pick v such that $v < 1 - \alpha$ in the first case, and $v < \alpha$ in the second.

(ii) Consider the case $\alpha \in (1, 2)$. It is well known (and can be easily derived from the representation theorem) that $\ell(\delta)\delta^{-v} \to \infty$ for any $v > 0$ as $\delta \to 0$. Therefore, we must have $a_\eta/\eta \to 0$ as $\eta \to \infty$, showing that a can be arbitrary and so we fix it at 0. Similarly, to (i) for an arbitrary $\epsilon > 0$ we consider

$$\begin{aligned}&\int_0^\epsilon \frac{1}{\eta}(e^{\theta a_\eta x} - 1 - \theta a_\eta x 1_{\{x<1\}}) f(x) \mathrm{d}x \\ &\quad = \int_0^{\epsilon a_\eta} (e^{\theta x} - 1 - \theta x 1_{\{x<a_\eta\}}) x^{-\alpha-1} \frac{a_\eta^\alpha \ell(1/a_\eta)}{\eta} \frac{\ell(x/a_\eta)}{\ell(1/a_\eta)} \mathrm{d}x \\ &\quad \to c_+ \int_0^\infty (e^{\theta x} - 1 - \theta x 1_{\{x<1\}}) x^{-1-\alpha} \mathrm{d}x - c_+ \theta \int_1^\infty x^{-\alpha} \mathrm{d}x,\end{aligned}$$

where we used the dominated convergence theorem twice based on Lemma 10 with $v < 2 - \alpha$ and $v < \alpha - 1$, respectively. The latter term evaluates to $-\theta c_+/(\alpha - 1)$. Finally, analysis of the contribution of negative jumps allows us to conclude in view of (2.4) and (2.5). □

Let us note that there always exists a sequence a_η satisfying (6.1) for any c_+, and, moreover, it must be regularly varying at ∞ with index $1/\alpha$, see Theorem 3. For the boundary case $\alpha = 1$ the result is more complicated. Additionally, to the assumptions of Proposition 9 we need to ensure that

$$\frac{a_\eta}{\eta}\Big(a - \int_{1/a_\eta}^1 x\nu(\mathrm{d}x) + \int_{-1}^{-1/a_\eta} x\nu(\mathrm{d}x)\Big)$$

has a finite limit as $\eta \to \infty$ and, in particular, $c = 1$.

7 Final remarks

Let us reconsider the setting of Theorem 6. According to Proposition 8 and Proposition 9 we have three essentially different cases:

(a) X has a Brownian part: $\sigma > 0$. Then $a_\eta = \sqrt{\eta}$ and $S_t = \sigma W_t$, where W_t is a standard Brownian motion. Therefore, the limiting process ξ is the same as in the pure Brownian case analysed by Asmussen et al. (1995). It is symmetric around 0, that is, $\mathbb{Q}^\downarrow(\mathrm{d}\omega) = \mathbb{Q}^\uparrow(-\mathrm{d}\omega)$ and $\mathbb{Q}^\uparrow$ is the law of $(\sigma B_t)_{t\geq 0}$, where B_t is the three-dimensional Bessel process. In particular, (1.1) holds.

(b) X is of bounded variation with $d \neq 0$. Then $a_\eta = \eta$ and $S_t = dt$ leading to the conditioned laws $\mathbb{Q}^\downarrow$ and $\mathbb{Q}^\uparrow$ of a very simple form, and, in particular,

$$\eta\epsilon_\eta \quad \Rightarrow \quad |d|U \qquad \text{on the event } \tau \notin \{0, T\},$$

where U is a uniform $(0, 1)$ random variable.

(c) X is in the domain of attraction (under zooming in) of a strictly α-stable Lévy process (in particular, $\sigma = 0$ and $d = 0$ in bounded variation case), see Proposition 9. Then $a_\eta = \eta^{1/\alpha}\ell_\eta$, where ℓ_η is a function slowly varying at ∞. In this case S may still be monotone when $\alpha < 1$, yielding simple expressions for $\mathbb{Q}^\downarrow$ and $\mathbb{Q}^\uparrow$.

It is noted that the above cases are not exhaustive, even though they cover most of the interesting examples in practice.

As mentioned in Section 1, there is quite some interest in the literature in determining the rate of convergence of the expected error $\mathbb{E}\epsilon_\eta = \mathbb{E}(M - M_\eta)$ to 0. Our results provide a hint on this rate, but do not readily determine it. The reason is that proving uniform integrability of $a_\eta\epsilon_\eta$ seems to be a hard task in general. In some cases the representation of $\mathbb{E}\epsilon_\eta$ based on Spitzer's identity, see (Asmussen et al., 1995, Eq. (3.3)), may be useful. Furthermore, we anticipate that uniform integrability does not hold when the attractor S is a strictly α-stable Lévy process with $\alpha < 1$, which is clearly true when S is monotone and $\mathbb{E}|S_U| = \infty$.

Finally, it is possible to apply our results to study the behaviour of X around its first passage and last exit times, instead of the time of supremum. The key result here is the well known path decomposition of the Lévy process at these times (Duquesne, 2003). For example, on the event of continuous last exit from some interval $(-\infty, x)$, the post-exit process is independent from the pre-exit process and the former has the law $\mathbb{P}^\uparrow$, whereas the latter when time-reversed has the original law (up to the last exit). Hence using the tools of this paper, and in particular Theorem 2, we may provide, e.g., a limit result for zooming in on X at its last exit time.

Appendix

The following lemma is a slight extension of Potter's bounds for a slowly varying function.

Lemma 10. *Let ℓ be slowly varying at 0. Then for any $u, v > 0$ there exists $\epsilon > 0$ such that*

$$\frac{\ell(x\delta)}{\ell(\delta)} < (1+u) \times \begin{cases} x^{-v}, & x \in (0,1) \\ x^{v}, & x \in (1, \epsilon/\delta) \end{cases}$$

for all $\delta < \epsilon$.

Proof. This result is based on Karamata's representation theorem (Resnick, 2007, Cor. 2.1): there exists $h > 0$ such that for all $x \le h$

$$\ell(x) = \exp\left(u(x) + \int_x^h \frac{v(t)}{t}\mathrm{d}t\right),$$

where u, v are bounded functions, such that $\lim_{x\downarrow 0} u(x) \in (-\infty, \infty)$ and $\lim_{x\downarrow 0} v(x) = 0$.

Choose $\epsilon \in (0, h)$ so small that $\mathrm{e}^{u(x\delta)-u(\delta)} < 1 + u$ for all $x < \epsilon/\delta, \delta < \epsilon$, and $|v(t)| < v$ for all $t < \epsilon$. Thus we have

$$\frac{\ell(x\delta)}{\ell(\delta)} \le (1+u)\exp\left(v\int_{x\delta}^{\delta} \frac{\mathrm{d}t}{t}\right), \qquad x < \epsilon/\delta.$$

For $x < 1$ the integral evaluates to $\log \delta - \log(x\delta) = -\log x$, and for $x > 1$ it evaluates to $\log x$. □

Acknowledgements

I am thankful to Søren Asmussen for his suggestion to reconsider (Asmussen et al., 1995). Furthermore, I would like to express my gratitude to Mark Podolskij, Sebastian Engelke and Lanpeng Ji for pointing out various related works.

References

S. Asmussen, P. Glynn, and J. Pitman. Discretization error in simulation of one-dimensional reflecting Brownian motion. *Ann. in Appl. Probab.*, 5(4):875–896, 1995.

M. Barczy and J. Bertoin. Functional limit theorems for Lévy processes satisfying Cramér's condition. *Electron. J. Probab.*, 16:2020–2038, 2011.

J. Bertoin. Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stochastic Process. Appl.*, 47(1):17–35, 1993.

J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

E. Bolthausen. On a functional central limit theorem for random walks conditioned to stay positive. *Ann. Probab.*, 4(3):480–485, 1976.

M. Broadie, P. Glasserman, and S. Kou. A continuity correction for discrete barrier options. *Math. Finance*, 7(4):325–349, 1997.

M. Broadie, P. Glasserman, and S. G. Kou. Connecting discrete and continuous path-dependent options. *Finance Stoch.*, 3(1):55–82, 1999.

A. Bryn-Jones and R. A. Doney. A functional limit theorem for random walk conditioned to stay non-negative. *J. London Math. Soc.*, 74(2):244–258, 2006.

M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.*, 43(4):967–983, 2006.

F. Caravenna and L. Chaumont. Invariance principles for random walks conditioned to stay positive. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44(1):170–190, 2008.

L. Chaumont. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.*, 64(1):39–54, 1996.

L. Chaumont. On the law of the supremum of Lévy processes. *Ann. Probab.*, 41(3A):1191–1217, 2013.

L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.*, 10:948–961, 2005.

A. Chen. *Sampling error of the supremum of a Lévy process.* PhD thesis, University of Illinois at Urbana-Champaign, 2011.

E. H. A. Dia. *Exotic options under exponential Lévy model.* PhD thesis, Université Paris-Est, 2010.

E. H. A. Dia and D. Lamberton. Connecting discrete and continuous lookback or hindsight options in exponential Lévy models. *Adv. Appl. Probab.*, 43(4):1136–1165, 2011.

R. A. Doney. Conditional limit theorems for asymptotically stable random walks. *Z. Wahrsch. Verw. Gebiete*, 70(3):351–360, 1985.

T. Duquesne. Path decompositions for real Lévy processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(2):339–370, 2003.

S. N. Ethier and T. G. Kurtz. *Markov processes. Characterization and convergence.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986.

A. Ferreiro-Castilla, A. E. Kyprianou, R. Scheichl, and G. Suryanarayana. Multilevel Monte Carlo simulation for Lévy processes based on the Wiener-Hopf factorisation. *Stochastic Process. Appl.*, 124(2):985–1010, 2014.

P. Greenwood and J. Pitman. Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. in Appl. Probab.*, 12(4):893–902, 1980.

K. Hirano. Lévy processes with negative drift conditioned to stay positive. *Tokyo J. Math.*, 24(1):291–308, 2001.

D. L. Iglehart. Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probab.*, 2:608–619, 1974.

J. Jacod and P. Protter. *Discretization of processes*, volume 67 of *Stochastic Modelling and Applied Probability.* Springer, Heidelberg, 2012.

A. J. E. M. Janssen and J. S. H. Van Leeuwaarden. Equidistant sampling for the maximum of a Brownian motion with drift on a finite horizon. *Electron. Commun. Probab.*, 14: 143–150, 2009.

P. Kosulajeff. Sur la répartition de la partie fractionnaire d'une variable. *Math. Sbornik*, 2 (5):1017–1019, 1937.

A. Kuznetsov. Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Probab.*, 20(5):1801–1830, 2010.

A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications.* Universitext. Springer-Verlag, Berlin, 2006.

J. Lamperti. Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, 104:62–78, 1962.

A. L. Lewis and E. Mordecki. Wiener-Hopf factorization for Lévy processes having positive jumps with rational transforms. *J. Appl. Probab.*, 45(1):118–134, 2008.

A. Mimica. Laplace transform, 2015. Lecture notes, `https://web.math.pmf.unizg.hr/~amimica/lt.pdf`.

V. I. Piterbarg. *Asymptotic methods in the theory of Gaussian processes and fields*, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996.

S. I. Resnick. *Heavy-tail phenomena. Probabilistic and statistical modeling.* Springer Series in Operations Research and Financial Engineering. Springer, New York, 2007.

W. Whitt. Some useful functions for functional limit theorems. *Math. Oper. Res.*, 5(1): 67–85, 1980.