Comparison of global algorithms for Minkowski tensor estimation

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Abstract

We present a comparison of two global digital algorithms for estimation of Minkowski tensors of sets $A \subset \mathbb{R}^2$ with positive reach given only a digitisation A_0 of A. We give recommendations for the choice of variables utilised by the algorithms and examine accuracy of the latter in the realistic setting of finite resolution.

Minkowski tensors and digitisations

Let $A \subset \mathbb{R}^2$ be compact, let $p_A(x)$ be the metric projection of $x \in \mathbb{R}^2$ on A (whenever well-defined), and let

$$A^R = \{x \in \mathbb{R}^2 \mid \inf_{a \in A} \|x - a\| \le R\}$$

be the R-parallel set of A for $R < 0$.

Definition 1. *If R' is the largest R (possibly ∞) such that $p_A(x)$ exists for all $x \in A^R$, A is said to have* ***reach*** *R' denoted by* $\mathrm{Reach}(A) = R'$.

For $r \ge 0$, the r**th volume tensor of** A is

$$\Phi_2^{r,0}(A) = \frac{1}{r!} \int_A x^r \, \mathrm{d}x.$$

If A has positive reach, more general tensors can be defined for $k = 0, 1, 2$ and $r, s \ge 0$ by

$$\Phi_k^{r,s}(A) = \frac{1}{r!s!} \frac{\omega_{2-k}}{\omega_{2-k+s}} \int_{\mathbb{R}^2 \times S^1} x^r u^s \Lambda_k(A; \mathrm{d}(x,u)),$$

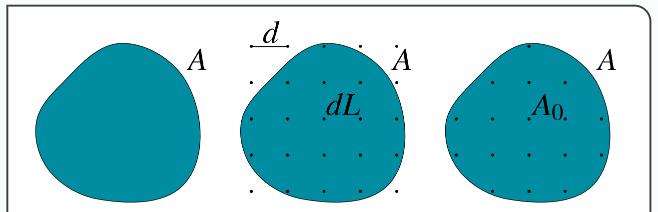
where $\Lambda_k(A; \cdot)$ is the generalised curvature measure [3]. The tensors $\Phi_k^{r,s}(A)$ are the **Minkowski tensors of** A. For $R < \mathrm{Reach}(A)$, the **Voronoi tensor measures of** A, first defined in [2], are given by

$$\mathcal{V}_R^{r,s}(A) = \int_{A^R} p_A(x)^r (x - p_A(x))^s \mathrm{d}x. \qquad (1)$$

Using a Steiner formula on (1), we obtain

$$\mathcal{V}_R^{r,s}(A) = r!s! \sum_{k=0}^{2} \kappa_{s+k} R^{s+k} \Phi_{2-k}^{r,s}(A). \qquad (2)$$

In many applications, only a *digitisation* of A is available. For $d > 0$, let $dL \subset \mathbb{R}^2$ be a square lattice. Then $A_0 = dL \cap A$ is the **digitisation** of A, and d is the **resolution**. The digitisation process is illustrated below.



Obtaining a digitisation A_0 from a set $A \subset \mathbb{R}^2$

Estimators

For A_0 a digitisation of A, we obtain estimators

$$\mathcal{V}_R^{r,s}(A_0) = \sum_{x \in A_0} x^r \int_{\mathcal{V}_{A_0}(x) \cap B_R(x)} (y - x)^s \mathrm{d}y$$

for the Voronoi tensor measures via (1). Here, $\mathcal{V}_{A_0}(x)$ is the Voronoi cell of x with respect to A_0, i.e. $\mathcal{V}_{A_0}(x) = \{y \in \mathbb{R}^2 \mid |y-x| \le |y-\chi| \text{ for all } \chi \in A_0 \setminus \{x\}\}$.

Algorithm 1. For $0 < R_0 < R_1 < R_2 < \mathrm{Reach}(A)$, we use (2) to obtain estimators

$$\begin{pmatrix} \hat{\Phi}_2^{r,s}(A_0) \\ \hat{\Phi}_1^{r,s}(A_0) \\ \hat{\Phi}_0^{r,s}(A_0) \end{pmatrix} = \left(M^{r,s}_{R_0,R_1,R_2}\right)^{-1} \begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(A_0) \\ \mathcal{V}_{R_1}^{r,s}(A_0) \\ \mathcal{V}_{R_2}^{r,s}(A_0) \end{pmatrix}$$

for the Minkowski tensors. This is our first algorithm.

Algorithm 2. Using

$$\tilde{\Phi}_2^{r,0}(A_0) = \frac{d^2}{r!} \sum_{x \in A_0} x^r$$

as estimators for the volume tensors and, for $0 < R_0 < R_1 < \mathrm{Reach}(A)$,

$$\begin{pmatrix} \tilde{\Phi}_1^{r,s}(A_0) \\ \tilde{\Phi}_0^{r,s}(A_0) \end{pmatrix} = \left(N^{r,s}_{R_0,R_1}\right)^{-1} \begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(A_0) - r!\tilde{\Phi}_2^{r,s}(A_0) \\ \mathcal{V}_{R_1}^{r,s}(A_0) - r!\tilde{\Phi}_2^{r,s}(A_0) \end{pmatrix}$$

as estimators for the remaining Minkowski tensors, we obtain our second algorithm.

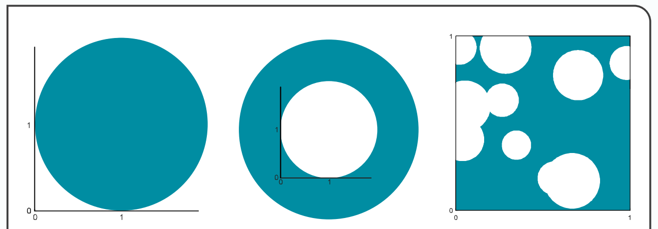
Theorem 1 ([2]). *Algorithms 1 and 2 are multigrid convergent, i.e. the estimators converge towards the true Minkowski tensors as the resolution tends to infinity.*

Aim

We have implemented algorithms 1 and 2 in MATLAB. Via simulations on certain test sets, we wish to be able to answer the following questions:

- What is the recommended choice for the radii R_i?
- Are the algorithms (equally) accurate?

The test sets which we explore are given below.



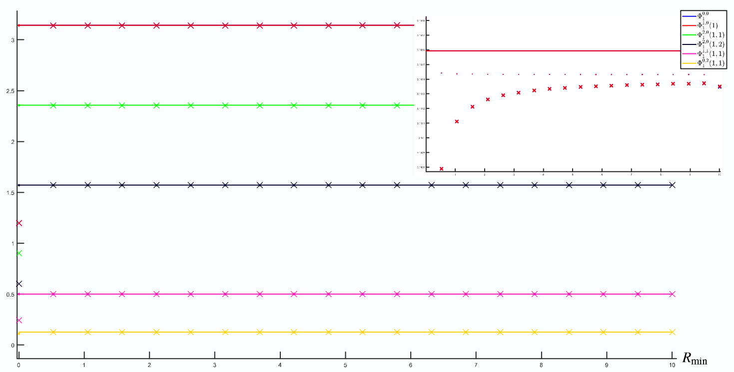
$B^2 + (1,1)^T$ $\quad$ $\overline{2B^2 \setminus B^2} + (1,1)^T$ $\quad$ $\gamma = 10$, $r_{\min} = 0.08$, $r_{\max} = 0.16$

Simulations

We run the algorithms on the suggested test sets. We first let the radii vary over the relevant interval, then fix $R_{\max}$ based on our results. Secondly, we let the smaller radii vary. Finally, best choices of radii are deduced from our plots. For the best choices of radii, we examine the algorithms for varying resolutions.

• :	Algorithm 1
× :	Algorithm 2

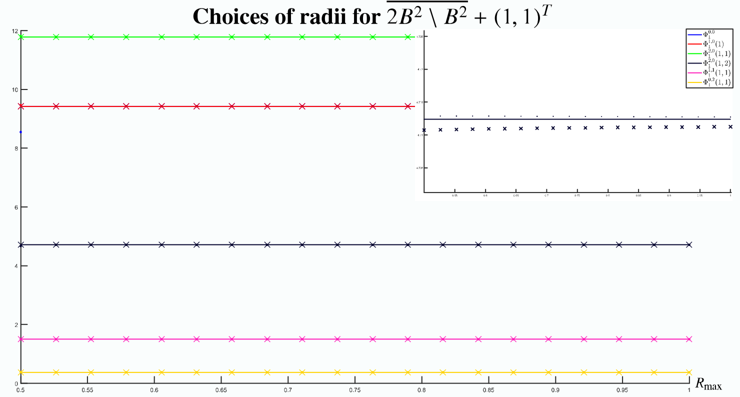
Choices of radii for $B^2 + (1,1)^T$



Resolution = 1000; $R_{\max}$ = 10; $R_{\mathrm{mid}} = (R_{\min} + R_{\max})/2$

Algorithm 1 is not affected as much by the choices of radii as Algorithm 2.

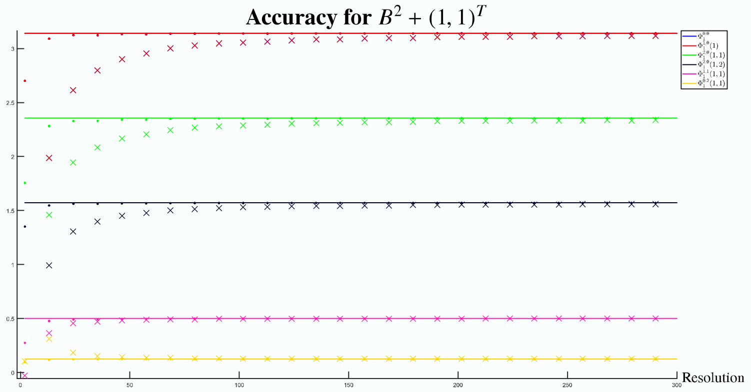
Choices of radii for $\overline{2B^2 \setminus B^2} + (1,1)^T$



Resolution = 1000; $R_{\min}$ = 0.5; $R_{\mathrm{mid}} = (R_{\min} + R_{\max})/2$

The radii should not be chosen too close to one another for accurate results.

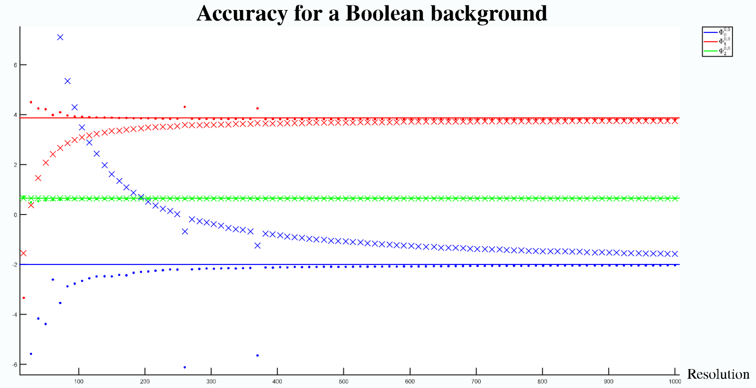
Accuracy for $B^2 + (1,1)^T$



$R_{\min} = 0.1$; $R_{\max} = 10$; $R_{\mathrm{mid}} = (R_{\min} + R_{\max})/2$

The error of estimators of algorithm 1 is below 1% at resolutions $d \approx 100$; algorithm 2 at $d \approx 200$.

Accuracy for a Boolean background



$R_{\min} = 0.035$; $R_{\max} = 0.05$; $R_{\mathrm{mid}} = (R_{\min} + R_{\max})/2$

This advanced model requires high resolutions for accurate results.

Discussion

Simulations indicate that accuracy of the algorithms depends strongly on the choice of suitable radii R_0, R_1, and R_2: R_0 should be chosen well above $d/\sqrt{2}$. Algorithm 2 is particularly inaccurate for small radii. The maximal radius $R_{\max} < \mathrm{Reach}(A)$ can be chosen quite close to the reach of A with good results. Finally, the radii should not be chosen too close to one another but rather be spread out over the possible interval.

Both algorithms yield satisfactory estimators for finite resolution as long as the resolution is chosen large enough: the more complicated the object, the higher the necessary resolution. Algorithm 1 is accurate for smaller resolutions than algorithm 2 except in the case of the volume tensors.

References

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[2] D. Hug, M. Kiderlen, and A.M. Svane, *Voronoi-based estimation of Minkowski tensors*, submitted, 2015

[3] M. Zähle, *Integral and current representation of Federer's curvature measures*, Archiv der Mathematik, 46(6):557–67, 1986

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