The Universal Moduli Space of pairs of a Riemann surface and a Holomorphic Vector Bundle and Its Hitchin Connection

Ph.D-Thesis

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Abstract

The goal of my thesis is to understand the curvature of the Hitchin connection in a vector bundle over Teichmüller space. The bundle is the holomorphic sections of a certain line bundle over the moduli space of rank n and degree k holomorphic vector bundles of fixed determinant on the Riemann surface given by the point in Teichmüller space. This is interesting as it is related to the topological quantum field theory, however, we will not explore this aspect.

The thesis consists of two parts. The first part consists of the first four chapters, where we introduce the methods and objects we will be studying in the second part. First, we introduce the moduli space of rank n degree k holomorphic vector bundles over a closed Riemann surface of genus greater than one, and we explain the connection between unitary representations and vector bundles, see [Seshadri, 1982], [Mehta and Seshadri, 1980], [Narasimhan and Seshadri, 1965] and [Narasimhan and Seshadri, 1964]. Then we briefly review [Andersen, 2012] on the general construction of a Hitchin connection. The second chapter explains the relation between the Selberg zeta function and the determinant of the Laplace operator on a holomorphic vector bundle. This relation is central to understanding the Ricci potential on the moduli space of holomorphic vector bundles. The third chapter explains, how [Zograf and Takhtadzhyan, 1989] construct coordinates and find the Ricci potential for the moduli space of holomorphic vector bundles. We only add a small observation about the Levi-Civita connection in the moduli space. To finish the first part we follow [Zograf and Takhtadzhyan, 1987] in working on the Teichmüller space. We alter our approach slightly from theirs to take into account, that we have a bundle on the Riemann surface represented by a point in Teichmüller space. These coordinates allow us to work with the Ricci potential on the moduli space of holomomorphic vector bundles and explore, how it depends on the underlying Riemann surface.

The second part contains my own work, and work with my advisor Jørgen E. Andersen. The fifth chapter is a detailed draft for a joint paper. In the paper we construct coordinates, on the moduli space of pairs of a Riemann surface and a holomorphic bundle of rank n and degree k. The construction is inspired by the previously introduced coordinates. We actually provide two different sets of coordinates and prove that they are different. Finally, Chapter 6 contains local calculations related to the Hitchin connection, and there we calculate the $(1, 1)$-part of the curvature. This allows us to modify the definition of the Hitchin connection from [Andersen, 2012] in such a way that its curvature only consists of the $(1, 1)$-part, under certain assumptions I believe are true for the genus of the surface heigh enough.

Resume

Målet med min afhandling er at forstå krumningen af Hitchin-konneksionen på et bundt over Teichmüller-rummet. Bundtet er de holomorfe snit af et bestemt linjebundt over modulirummet af rang n grad k holomorfe vektorbundter med given determinant over Riemann-fladen, som punktet i Teichmüller-rummet specificere. Dette er interessant, blandt andet fordi det er relateret til klassiske konstruktioner af topologiske kvantefeltteorier. Vi vil ikke undersøge denne forbindelse nærmere.

Afhandlingen består af to dele. Den første del består af de første fire kapitler. I disse fire kapitler introducerer vi de metoder og objekter, vi vil undersøge i den anden del. Først introducerer vi modulirummet af rang n og grad k holomorfe vektorbundter over en lukket Riemann-flade af genus større end to, see [Seshadri, 1982], [Mehta and Seshadri, 1980], [Narasimhan and Seshadri, 1965] og [Narasimhan and Seshadri, 1964]. Så giver vi en hurtig gennemgang af [Andersen, 2012]'s konstruksion af en Hitchin konneksion. Det andet kapitel forklarer, hvordan determinanten af Laplace-operatoren på et holomorft vektorbundt kan udtrykkes ved hjælp af Selbergs zeta-funktion. Denne måde at beskrive determinanten af Laplace-operatoren på er central for at forstå Ricci-potentialet på modulirummet af holomorfe vektorbundter. Det tredje kapitel forklarer, hvordan [Zograf and Takhtadzhyan, 1989] konstruerer koordinater på modulirummet af holomorfe vektorbundter, og hvordan de finder Ricci-potentialet på dette rum. Vi tilføjer en lille observation om Levi-Civita konneksionen på modulirummet. Som afslutning på den første del præsenteres [Zograf and Takhtadzhyan, 1987]'s måde at arbejde i koordinater på Teichmüller-rummet. Disse koordinater tillader os at arbejde med Ricci-potentialet på modulirummet af holomorfe vektorbundter og undersøge, hvordan det afhænger af den underliggende flade.

Den anden del indeholder hoveddelen af det nye arbejde lavet i samarbejde med min vejleder Jørgen E. Andersen. Det femte kapitel er et detalieret udkast til en fælles artikel. I denne artikel konstruerer vi koordinater på modulirummet af par af en Riemann-flade og et holomorft vectorbundt på denne af grad k og rang n. Konstruksionen er inspireret af de koordinater, der blev introduceret in den foregående del af afhandlingen. Vi konstruerer også mere direkte koordinater fra deres og viser, at de to sæt af koordinater er forskellige. Det sidst kapitel indeholder udregninger, der skal hjælpe med af forstå Hitchin konneksionen fra [Andersen, 2012]. Vi udregner $(1,1)$-delen af krumningen for denne konneksion. Dette tillader os at modificere konneksionen således, at dette er hele den krumning der er, under nogle antagelser jeg mener er rigtige, hvis genus er stor nok.

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Contents

Introduction

The work in this thesis is inspired mainly by a desire to understand a Hitchin connection, [Hitchin, 1990]. One of the interesting applications of the Hitchin connection is to construct $(2+1)$-TQFT using geometric quantization. The Hitchin connection is needed to construct the vector space associated to a surface. To use geometric quantization we need a prequantized line bundle over a Kähler manifold. The choice we are interested in occurs, when the Kähler manifold is the moduli space of holomorphic vector bundles on our surface. However, for this to make sense we need to give our surface a complex structure. If we take the family of complex structures on our surface given by the Teichmüller space, $\mathcal{T}$, this gives a family of Kähler structures on the moduli space of degree k rank n stabel vector bundles. Here the moduli space of degree k rank n stable vector bundles with fixed determinant, $\mathcal{M}_{\mathbf{SU}(n),k}$, is the symplectic manifold underlying all the Kähler structures given by identifying this moduli space with the moduli space of degree k rank n holomorphic vector bundles with fixed determinant, $\mathcal{M}_{n,k}^0$. The symplectic manifold $\mathcal{M}_{\mathbf{SU}(n),k}$ has a prequantum line bundle over it, call it $\mathcal{L}$. Now we can look at the trivial bundle $\hat{\mathcal{H}} = \mathcal{T} \times C^\infty(\mathcal{M}_{\mathbf{SU}(n),k}, \mathcal{L})$ which contains $\mathcal{T} \times H^0(\mathcal{M}_{n,k}^0, \mathcal{L})$. A Hitchin connection is a connection in $\hat{\mathcal{H}}$ of the form $\nabla^t + u$, where ∇^t is the trivial connection and u is a 1-form on $\mathcal{T}$ with values in differential operators on $C^\infty(\mathcal{M}_{\mathbf{SU}(n),k}, \mathcal{L})$. We say it is a Hitchin connection, if it preserves the subbundle $\mathcal{T} \times H^0(\mathcal{M}_{n,k}^0, \mathcal{L})$. Hitchin connnections can be shown to exist and to be projectivly flat, allowing us to define the vector space associated to a surface as the projectivly preserved section.

The aim of my thesis is to further our understanding of the Hitchin connection. We will be concerned with local calculations. To get a first grip on the Hitchin connection [Andersen, 2012] gives an explicit formula for the 1-form u in terms of the Ricci potential on $\mathcal{M}_{n,k}^0$ and the tensor valued 1-form G, which measure the variation of the complex structure by $V'[I] = G(V)\omega$. Additional refinement can be found in [Andersen and Gammelgaard, 2011], where they give an expression for the curvature of the Hitchin connection, however, these expressions are not as specific as we would like. So in my thesis I will work in local coordinates, inspired by [Zograf and Takhtadzhyan, 1989] and [Zograf and Takhtadzhyan, 1987].

Now that we have the setting of my thesis, we will go through what the thesis broadly speaking contains. The thesis consists of six chapters of which the first four mostly gather background material and the last two primarily contain my results and computations. In Chapter 1 I introduce the moduli space and its complex structure from fairly basic notions. The second half of Chapter 1 briefly describes the results of [Andersen, 2012] and [Hitchin, 1990, Lemma 2.15], they will help us understand $G(V)$. Chapter 2 is mainly a brief review of the Selberg zeta function. Following [Sarnak, 1987] the chapter ends in proving that the determinant of the Laplace operator is closely related to the derivative of the Selberg zeta function. After these introductory chapters follow Chapter 3 and Chapter 4 in which the material of [Zograf and Takhtadzhyan, 1989] and [Zograf and Takhtadzhyan, 1987] are presented. In Chapter 3, in addition to the results of [Zograf and Takhtadzhyan, 1989], I also prove that a slight modification (Proposition 3.2.3) of their Lie derivative gives me a way of calculating the Levi-Civita connection on $\mathcal{M}_{n,k}$. In Chapter 4 I also deviate slightly from [Zograf and Takhtadzhyan, 1987, Theorem 2], they are working with a Laplace operator on tensors, while my Theorem 4.3.3 is concerned with the Laplace operator in a certain family of vector bundles on the Riemann surface, this family is also what in Proposition 5.2.3 is used (and justified) to compare two complex structures. The Theorem 4.3.3 follows the same logic as [Zograf and Takhtadzhyan, 1987, Theorem 2].

The last two chapters present my main results. Chapter 5 is a detailed draft for a paper with my advisor Jørgen Ellegaard Andersen, we also thank Peter Zograf for discussing it with us. In this paper we construct two sets of coordinates on the moduli space of pairs of a Riemann surface and a holomorphic bundle of rank n and degree k over the surface. The highlights are the following five results:

Theorem 0.0.1 (Theorem 5.1.1)
For all sufficiently small $\mu \oplus \nu \in H^{0,1}(X_{\rho_0}, TX_{\rho_0}) \oplus H^{0,1}(X_{\rho_0}, EndE)$ there exist a unique bundle map $\Phi^{\mu\oplus\nu}$ such that

1. *$\Phi^{\mu\oplus\nu}$ solves*
$$\bar{\partial}_{\mathbb{H}}\Phi^{\mu\oplus\nu} = \partial\Phi^{\mu\oplus\nu}(\mu \oplus \nu),$$
where ν is considered a left-invariant vector field on $\mathbf{GL}(n, \mathbb{C})$ at each point in $\mathbb{H}$.

2. *The base map extends to the boundary of $\mathbb{H}$ and fixes $0, 1$ and ∞.*

3. *The pair of representations $(\rho^{\mu}, \rho_E^{\mu\oplus\nu})$ defined by equation (5.2) represents a point in $\mathcal{T} \times M'$.*

4. *$p_{\mathbf{GL}(n,\mathbb{C})}(\Phi^{\mu\oplus\nu}(z_0, e))$ has determinant 1 and is positive definite.*

Theorem 0.0.2 (Theorem 5.1.2)
Mapping all sufficiently small pairs

$$\mu \oplus \nu \in H^{0,1}(X_\rho, TX_\rho) \oplus H^{0,1}(X_\rho, EndE)$$

to

$$(\rho^\mu, \rho_E^{\mu\oplus\nu}) \in \mathcal{T} \times M'$$

provides local analytic coordinates centered at $(\rho_0, \rho_E) \in \mathcal{T} \times M'$.

We identify the algebraic complex structure on the moduli space with the complex structure given to the moduli space in [Hitchin, 1990].

Proposition 0.0.3 (Proposition 5.2.3)
We have that the map

$$\Psi : (\mathcal{T} \times M', J) \to \mathcal{M}^s$$

is complex analytic, e.g. J *is in fact the complex analytic structure this space gets from the Narasimhan-Seshadri diffeomorphism* Ψ.

We also construct another set of coordinates by first using the Bers's coordinates and then the coordinates on the moduli space [Zograf and Takhtadzhyan, 1989] to flow along the fiber. We prove that these fibred coordinates resemble the previous constructed coordinates.

Theorem 0.0.4 (Theorem 5.1.3)
The fibered coordinates and the universal coordinates agree to second-order, but not the third-order at the center of the coordinates.

And we end the Chapter with:

Theorem 0.0.5 (Theorem 5.5.5)
The coordinates of 5.1.1 and the fibered coordinates agree to second-order, but differ at third-order in the derivatives at the center point.

In the final Chapter 6 I have collected various results obtained by local calculations, the main highlight being the following four which give a good understanding of the curvature:

Lemma 0.0.6 (Lemma 6.1.1)
In coordinates on $\mathcal{T} \times \mathcal{M}^0_{n,k}$*, centered at* $(\sigma, E) \in \mathcal{T} \times \mathcal{M}^0_{n,k}$*, we have for a vector field* V *on* $\mathcal{T}$*, identified by the Kodaira-Spencer map at* $\sigma \in \mathcal{T}$ *with the Beltrami differential* $\mu \in \mathcal{H}^{0,1}(X_\sigma, TX_\sigma)$*, and two cotangent vectors on* $(\mathcal{M}^0_{n,k}, (-\star)_\sigma)$*, represented by harmonic forms* $\bar{\nu}_i^T, \bar{\nu}_j^T \in \mathcal{H}^{1,0}(X_\sigma, EndE)$*, that at* (σ, E)*:*

$$G(V)(\bar{\nu}_i^T, \bar{\nu}_j^T) = -2i \int_X \mu \mathrm{tr} \bar{\nu}_i^T \bar{\nu}_j^T.$$

Lemma 0.0.7 (Lemma 6.2.1)
On the complex space $\mathcal{T} \times \mathcal{M}^0_{n,k}$ *we have:*

$$2i\bar{\partial}_\sigma V'[F_\sigma] = trG(V)\partial_\sigma F_\sigma \omega - \frac{1}{2} tr\nabla^{(1,0)}G(V)\omega.$$

Corollary 0.0.8 (Corollary 6.3.2)
The following identity holds

$$\theta(V,W) - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F(V,W) = \frac{\mathrm{Rank}E}{6\pi}\omega_{WP}(V,W) = \frac{n}{6\pi}\omega_{WP}(V,W)$$

and

$$F^{(1,1)}(V,W) = \frac{ikn}{12\pi(k+n)}\omega_{WP}(V,W).$$

And finally we modify the Hitchin connection in [Andersen, 2012] to have no curvature of type $(2,0)$, under certain assumptions I believe to be true for g big enough.

Theorem 0.0.9 (Theorem 6.3.3)
The connection $\hat{\nabla} = \nabla^t + u + \frac{k}{(2n+k)^2}\tilde{c}$ *on* $H^k \to \mathcal{T}$ *is a Hitchin connection with curvature* $\frac{-ikn}{12\pi(k+n)}\omega_{WP}$.

Chapter 1

Moduli Space

In this first chapter of the thesis we will introduce some of the central concepts, which we will investigate in greater detail later in the thesis. We start with a review of the construction of the moduli space of stable vector bundles in [Narasimhan and Seshadri, 1964], however, we twist it a little to suit our purpose better. As part of the construction we will see, that bundles also can be described by a conjugacy class of irreducible representations, both ways of seeing the space will be important. In the second section we follow [Andersen, 2012] in describing the Hitchin connection on a family of Kähler structures of a symplectic manifold. While much can be said in this general setting, not all we would like has been done yet. In this thesis we will work with refining the general formulas of [Andersen and Gammelgaard, 2011] to the case, where the underlying symplectic manifold is the moduli space of connections on a fixed bundle E over Σ of rank n, which is flat on $\Sigma - p$ and has the holonomy $e^{\frac{-2\pi ik}{n}}I$ around p. This is symplectic with the Goldman symplectic form. The moduli space of connections is diffeomorphic to the space of stable holomorphic vector bundles of rank n and degree k, however, only the latter carries a canonical complex structure that is in fact also Kähler with respect to the Goldman symplectic form. While we will only study a special case, it is an important case, since the moduli space of connections is used to construct a bundle over Teichmüller space, with a Hitchin connection that can be used to construct the Witten-Reshetikhin-Tureav topological quantum field theory of the underlying surface. It does so through geometric quantization in the spirit of [Witten, 1989]. We will end the chapter with a lemma from [Hitchin, 1990] about the variation of the complex structure on the moduli space of connections, which we will later need to refine in order to calculate the curvature of the Hitchin connection.

1.1 Construction of the Moduli Space of Stable Vector Bundles of Rank n and Degree k

The moduli space of stable holomorphic vector bundles of rank n and degree k over a Riemann surface X will be our main object of study. From Narasimhan's and Seshadri's theorem we know that this is closely related to a subset of unitary representations of a central extension of the fundamental group, the space of representations will be denoted $\mathcal{R}$. Here the central extension of $\pi_1(X)$ will be given by puncturing the surface at p. The representations of $\pi_1(X-p)$ can be interpreted as a subset of $\mathbf{U}(n)^{|\pi_1(X-p)|}$, this gives the representations a topology as a subset. The topology can also be described differently using that the group $\pi_1(X-p)$ is generated by $a_1, b_1, \dots a_g, b_g, \gamma$ where g is the genus of X. The the only relation in this representation of $\pi_1(X-p)$ is $\prod_{i=1}^{g}[a_i, b_i] = \gamma$. Now the topology of the representations we just introduced, can also be described by looking only at what happens to the generators a_i and b_i, this gives a map $\mathcal{R} \to \mathbf{U}(n)^{2g} := \Omega$ which is injective. The topology of the image of $\mathcal{R}$ in $\mathbf{U}(n)^{2g}$ is homeomorphic to $\mathcal{R}$, and so the subset topology from the image is the same topology as the first topology we gave $\mathcal{R}$.

1.1.1 The Tangent Space of the Representation Variety

The map introduced above, $\mathcal{R} \to \Omega$, we will use indirectly. For a representation $\rho \in \mathcal{R}$ we will denote the image of the generators as follows $A_i = \rho(a_i)$ and $B_i = \rho(b_i)$. We are interested in the subset of representations such that $\prod_{i=1}^{g}[A_i, B_i] = e^{\frac{-2\pi ik}{n}} I$. This equation cuts out a real analytic variety of Ω. We will only be interested in the irreducible representations. The irreducible points are simple and the equation gives the structure of a real analytic manifold on the subset of simple points. Before moving on to study the space and showing that the irreducible representations form a manifold, we note the following:

Proposition 1.1.1 ([Narasimhan and Seshadri, 1964, Prop. 2.1]) *For a genus $g \geq 2$ surface, Σ, there exists irreducible representations of $\pi_1(\Sigma - p)$ into $\mathbf{U}(n)$, with the loop around p being sent to $e^{\frac{-2\pi ik}{n}} I$ for every $n \geq 1$.*

While this proposition is slightly different from their version, the proof works unchanged.

Our space of representations is cut from Ω by the equation

$$\prod_{i=1}^{g}[A_i, B_i] = e^{\frac{-2\pi ik}{n}} I,$$

so we need to understand the differential of the map $f(\rho) = \prod_{i=1}^{g}[A_i, B_i]$. Let $\rho \in f^{-1}(e^{\frac{-2\pi ik}{n}} I)$ and denote the differential of f at ρ by $D_1 : \Omega_\rho \to$

$\mathbf{su}(n)_{e^{\frac{-2\pi i k}{n}}}$. We want to find the kernel of D_1, it can be described using the right-invariant Maurer-Cartan form $\omega = dA \cdot A^{-1}$. The Maurer-Cartan form is zero only on the zero vector, hence the kernel of the pullback of ω with f is the same as the kernel of D_1:

$$\ker D_1 = \{v \in \Omega_\rho | df \cdot f^{-1}(v) = 0\}.$$

We can rewrite the requirement $df \cdot f^{-1}(v) = 0$ in terms of a 1-cocycle on F, the free group on $2g$ generators (a_i's and b_i's) with values in Adρ. For this purpose write a tangent vector v in Ω_ρ by right translation as $(\alpha_i, \beta_i)_{i=1}^g$ with $\{\alpha_i, \beta_i\} \in T_{\{e\}}\Omega$. In fact we will show that the unique 1-cocycle given by $\delta_v(a_i) = \alpha_i$ and $\delta_v(b_i) = \beta_i$ fulfills $\delta_v(\prod_{i=1}^g [a_i, b_i]) = df \cdot f^{-1}(v)$. To see this consider $\Omega \times F$ as an analytic manifold, where we have given F the discrete topology. Every point in Ω defines a representation of F, and so we have the map $\Phi : \Omega \times F \to U(n)$ given by $\Phi(\tilde{\rho}, \gamma) = \tilde{\rho}(\gamma)$. We can define a function on F for $v \in \Omega_\rho$ given by $\delta_v = d\Phi \cdot \Phi^{-1}(v, \cdot) : F \to \mathbf{u}(n)$. Finally, we need to show that this construction of δ_v satisfies the three things claimed below.

The first claim states that δ_v is a 1-cocycle: Let $\rho \in \Omega$ and $\gamma_1, \gamma_2 \in F$, then we have that $\Phi(\rho, \gamma_1\gamma_2) = \Phi(\rho, \gamma_1)\Phi(\rho, \gamma_2)$, since these are representations of F. This implies that:

$$\begin{aligned} d\Phi(\rho, \gamma_1\gamma_2)\Phi(\rho, \gamma_1\gamma_2)^{-1} &= (d\Phi(\rho, \gamma_1)\Phi(\rho, \gamma_2) + \Phi(\rho, \gamma_1)d\Phi(\rho, \gamma_2))\Phi(\rho, \gamma_1\gamma_2)^{-1} \\ &= (d\Phi(\rho, \gamma_1)\Phi(\rho, \gamma_2) + \Phi(\rho, \gamma_1)d\Phi(\rho, \gamma_2))\Phi(\rho, \gamma_2)^{-1}\Phi(\rho, \gamma_1)^{-1} \\ &= d\Phi(\rho, \gamma_1)\Phi(\rho, \gamma_1)^{-1} + \Phi(\rho, \gamma_1)d\Phi(\rho, \gamma_2)\Phi(\rho, \gamma_2)^{-1}\Phi(\rho, \gamma_1)^{-1}. \end{aligned}$$

We have $\delta_v(\gamma_1\gamma_2) = \delta_v(\gamma_1) + \rho(\gamma_1)\delta_v(\gamma_2)\rho(\gamma_1)^{-1}$ which implies δ_v is a 1-cocycle for F with respect to Adρ.

The second claim states that $\delta_v(a_i) = \alpha_i$ and $\delta_v(b_i) = \beta_i$: this follows directly from the definition as $\delta_v(a_i) = dp_i \cdot p_i^{-1}(v)$, where p_i is the projection on Ω corresponding to a_i. But $dp_i \cdot p_i^{-1}(v)$ is the right translation of the component corresponding to a_i of v, and so we get α_i. The case with the b_i's is similar.

The third claim states that $\delta_v(\prod_{i=1}^g [a_i, b_i]) = df \cdot f^{-1}(v)$: by definition we have

$$\delta_v(\prod_{i=1}^g [a_i, b_i]) = d\Phi(\rho, \prod_{i=1}^g [a_i, b_i]) \cdot \Phi(\rho, \prod_{i=1}^g [a_i, b_i])^{-1}(v),$$

but $f(\rho) = \Phi(\rho, \prod_{i=1}^g [a_i, b_i])$, and so this proves the third claim.

We have a map from $F \to \pi_1(X - p)$ given by sending what we called a_i, b_i in F to the corresponding element in $\pi_1(X - p)$ which we called a_i, b_i at the start of the chapter. This map identifies cocycles for F with respect to Adρ, which vanish on $\prod_{i=1}^g [a_i, b_i]$, with cocycles of $\pi_1(X - p)$ with respect to Adρ.

The Dimension of the Tangent Space and Regular Points

We have now reduced the problem of understanding the tangent space to a question of understanding cohomology.

$$\dim\ker D_1 = \dim H^1(\pi_1(X-p), \mathrm{Ad}\rho) + \dim \mathbf{u}(n) - \dim H^0(\pi_1(X-p), \mathrm{Ad}\rho),$$

since we have that the kernel is isomorphic to the 1-cocycles for the group $\pi_1(\Sigma - p)$ with respect to the representation $\mathrm{Ad}\rho$. The first term above gives the dimension of the quotient by exact 1-cocycles, and the two last terms give the dimension of the exact 1-cocycles.

Next, we need to understand the dimension of the cohomology groups. The following two propositions will give us the relevant information:

Proposition 1.1.2 ([Narasimhan and Seshadri, 1964, Prop. 2.2]) *Let ρ be a $\mathbf{U}(n)$ representation of $\pi_1(X-p)$ and let $\mathrm{Ad}\rho$ be the representation obtained by composing with the adjoint representation of $\mathbf{U}(n)$ on $\mathbf{u}(n)$ then:*

$$\dim H^1(\pi_1(X-p), Ad\rho) = 2n^2(g-1) + 2\dim H^0(\pi_1(X-p), Ad\rho).$$

Proposition 1.1.3 ([Narasimhan and Seshadri, 1964, Prop. 2.3]) *An unitary representation, ρ, of $\pi_1(X-p)$ into $\mathbf{U}(n)$ is irreducible if and only if*

$$\dim H^0(\pi_1(X-p), Ad\rho) = 1.$$

Both proofs can be shown just as in [Narasimhan and Seshadri, 1964] since the loop around p acts trivially, and so $\mathrm{Ad}\rho$ is in fact a representation of $\pi_1(X)$.

Returning to $\ker D_1$ we have that for f to be of maximal rank $\dim\ker D_1 + \dim \mathbf{SU}(n) = \dim\Omega$. By the above propositions we have:

$$\begin{aligned}\dim\ker D_1 &= \dim H^1(\pi_1(X-p), \mathrm{Ad}\rho) + \dim \mathbf{u}(n) - \dim H^0(\pi_1(X-p), \mathrm{Ad}\rho)\\ &= 2n^2(g-1) + 2\dim H^0(\pi_1(X-p), \mathrm{Ad}\rho) + n^2\\ &\qquad - \dim H^0(\pi_1(X-p), \mathrm{Ad}\rho)\\ &= 2n^2 g - n^2 + \dim H^0(\pi_1(X-p), \mathrm{Ad}\rho).\end{aligned}$$

Since $\dim \mathbf{SU}(\mathbf{n}) = n^2 - 1$ and $\dim\Omega = 2gn^2$ this implies that

$$\dim H^0(\pi_1(X-p), \mathrm{Ad}\rho) = 1,$$

and so ρ must be irreducible. From the implicit function theorem it follows that the irreducible representations form a manifold.

1.1.2 The Manifold Structure on Equivalence Classes of Irreducible Representations

In order to get the moduli space of stable holomorphic vector bundles of rank n and degree k, $\mathcal{M}_{n,k}$, we need to take the quotient by the conjugation right action of $\mathbf{PU}(n)$ on representations. In the next section we will show that it is in fact the moduli space, but for the rest of this section we will just let $\mathcal{M}_{n,k} = \mathcal{R}/\mathbf{PU}(n)$. Since the group $\mathbf{PU}(n)$ is compact we have that the orbit space is Hausdorff, further the action is free and analytic, and so we have that the orbit space inherits an analytic structure from the space of irreducible representations, $\mathcal{R}^{irr}$. From the construction we have a map $p : \mathcal{R}^{irr} \to \mathcal{M}_{n,k}$ given by sending the representation to its conjugacy class. This gives $\mathcal{R}$ the structure of a $\mathbf{PU}(n)$-principal bundle over $\mathcal{M}_{n,k}$. To study $\mathcal{M}_{n,k}$ we need a description of its tangent space, we will argue it can be identified with $H^1(\pi_1(X-p), \mathrm{Ad}\rho)$. We have identified $\mathcal{R}^{irr}$'s tangent space at ρ with the 1-cocycles, and since the tangent space of $\mathcal{M}_{n,k}$ at $p(\rho)$ can be described as the quotient of $\mathcal{R}^{irr}$'s tangent space by the kernel of p, we will be done when we have shown these are the boundaries. The kernel of p is exactly the image of the differential of the conjugation action of $\mathbf{U}(n)$ at I. The conjugation map in Ω is given by $(T^{-1}A_1T, T^{-1}B_1T \dots T^{-1}A_gT, T^{-1}B_gT)$ and the differential is $Y \to (A_1Y - YA_1, B_1Y - YB_1, \dots A_gY - YA_g, B_gY - YB_g), y \in \mathbf{u}(n)$. However, we used the identification where we right translated back to I, and so the tangent map is $Y \to (A_1YA_1^{-1} - Y, B_1YB_1^{-1} - Y, \dots A_gYA_g^{-1} - Y, B_gYB_g^{-1} - Y)$, which is a 1-coboundary, and this also gives all the 1-coboundaries. Hence we conclude that $T_{[\rho]}\mathcal{M}_{n,k} \cong H^1(\pi_1(X-p), \mathrm{Ad}\rho)$.

1.1.3 Construction of Bundles from Representations

Given a representation, ρ, of $\pi_1(X-p)$ on a finite dimensional vector space, V, we have a local coefficient system, $L(\rho)$, of vector spaces on X, and we will denote the i'th cohomology group with coefficients in $L(\rho)$ by $H^i(X-p, L(\rho))$.

Now given a unitary representation, ρ, of $\pi_1(X-p)$ with the loop around p being sent to $e^{\frac{-2\pi ik}{n}}I$, we can construct a sheaf of holomorphic sections on X that gives rise to a holomorphic vector bundle, $W(\rho)$. We follow [Mehta and Seshadri, 1980] and [Seshadri, 1982] in the construction of parabolic degree zero bundles for the special case, where all the weights are the same, $e^{\frac{-2\pi ik}{n}}$, and there is only one parabolic point. In this case the bundle is defined by specifying a sheaf by pullback from the universal cover of $X-p$, we note that we only consider genus greater than two. In order to construct the vector bundle we first consider the universal cover of $X-p$, which is $\mathbb{H}$ with an action of $\pi_1(X-p)$, we can choose the puncture to correspond to ∞. We then have that $\mathbb{H} \times \mathbb{C}^n$ has an action of $\pi_1(X-p)$. And we can define the a sheaf on $X-p$ by associating to an open set $U \subset X-p$ the invariant sections of $\mathbb{C}^n$ on $p^{-1}(U) \subset \mathbb{H}$, all these are locally free of rank n. To finish the construction of the sheaf on X we need to specify the value on a small neighborhood, V, of p. We look at $p^{-1}(V)$ and take the component near ∞. We can assume that V is such that this component is given by $\{z \in \mathbb{H} | \text{Im} z > c\}$. Then we define the sheaf over V to be the bounded invariant sections over $\{z \in \mathbb{H} | \text{Im} z > c\}$. We can choose a small enough V, so that for invariant sections only the loop around p needs to be considered. This loop lifts to the transformation $z \to z+1$ on $\mathbb{H}$ and acts on $\mathbb{C}^n$ by $e^{\frac{-2\pi ik}{n}} I$. We can see that for $\{e_j\}_{j=1}^n$, a basis of $\mathbb{C}^n$, the invariant sections are given and spanned by $e^{\frac{-2\pi ik}{n} z} f(p_X(z))e_j$, where $f(z) \in (O(V))$. This implies that the sheaf is a free sheaf of rank n even at p. We have now defined the vector bundle $W(\rho)$, by specifying a sheaf. Direct calculations, or comments in [Seshadri, 1982], will show that it has degree k. We also observe that the transition functions for the neighborhood of p will depend on the representation in the usual way, only we also need to modify them by multiplying with a function which is independent of the chosen representation.

Because this is a special case where the order of $e^{\frac{-2\pi ik}{n}} I$ is of order n, we can find a central extension

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{\pi} \to \pi_1(X) \to 0$$

and a representation of $\tilde{\pi}$ making it a ramified cover of X, with ramification point p. We can define the sheaf directly by the invariant section on $\mathbb{H} \times \mathbb{C}^n$ under $\tilde{\pi}$. This is the approach which will be used later. This will be in the spirit of [Narasimhan and Seshadri, 1965, Proposition 6.2], we note they argue the transition functions near p depend on the representation with a modifications which is the same for all representations. The modifiaction does not disturb constructing universal bundles.

Finally, we note that the prinipal $\mathbf{GL}(n, \mathbb{C})$-bundle $P(\rho)$ associated with $W(\rho)$ can be constructed using the same transition functions, and so it also depends holomorphically on the representation.

1.1.4 The Moduli Space of Stable Bundles

First, we need to relate the different cohomology groups, as this will be the basis for showing that conjugate representations give equivalent bundles.

Proposition 1.1.4 ([Narasimhan and Seshadri, 1964, Prop. 4.1]) *Let X be a compact connected complex manifold and $\rho : \pi_1(X) \to AutV$ be a representation in a complex vector space V. Assume further ρ leaves a Hermitian form, H on X, invariant. Then the natural map*

$$H^0(\pi_1(X), \rho) \to H^0(X, W(\rho))$$

is an isomorphism.

PROOF: Any element in $H^0(\pi_1(X), \rho)$ will give an element of $H^0(X, W(\rho))$, which is seen clearly from the description of the sheaf.

Let $f \in H^0(X, W(\rho))$, then this is by construction identified with a function $\tilde{f} : \tilde{X} \to V$ satisfying $\rho(\gamma)^{-1} f(z) = f(z\gamma)$. Since $H(f(z), f(z))$ is left-invariant by ρ, the action of $\pi_1(X)$ leaves $H(f(z), f(z))$ invariant, which implies it is a function on X. Since this is a plurisubharmonic function and X is compact, $H(f(z), f(z))$ is constant. But this means f has a constant norm and is a holomorphic function, this is only possible if f is constant. Hence we have $f(z) = C \in V$ and $\rho(\gamma)^{-1} C = C, \forall \gamma \in \pi_1(X)$. But this says $C \in H^0(\pi_1(X), \rho)$. ■

Proposition 1.1.5 ([Narasimhan and Seshadri, 1964, Prop. 4.2]) *Let ρ_1 and ρ_2 be two $\mathbf{U}(n)$ representations of $\pi_1(X - p)$ with the loop around p being represented by $e^{\frac{-2\pi i k}{n}} I$. Then the two holomorphic $\mathbf{GL}(n, \mathbb{C})$-principal bundles $P(\rho_1)$ and $P(\rho_2)$ are isomorphic if and only if the representations ρ_1 and ρ_2 are conjugate.*

PROOF: If the representations are conjugate we can use this conjugation as an isomorphism between the two bundles. Conversely assume the bundles are isomorphic. Then the isomorphism is given by a function $f : \tilde{X} \to \mathbf{GL}(n, \mathbb{C})$ which transforms as $\rho_1(\gamma)^{-1} f(z) \rho_2(\gamma) = f(z\gamma), \forall \gamma \in \pi_1(X - p)$. Since $\mathbf{GL}(n, \mathbb{C}) \subset \mathbf{gl}(n, \mathbb{C})$ we can define a representation on the vector space $\mathbf{gl}(n, \mathbb{C})$ by $R(\gamma)A = \rho_1(\gamma) A \rho_2(\gamma)^{-1}$, this leaves the Hermitian form $\mathrm{tr}(A^* A)$ invariant. We can now apply the previous proposition, since the loop around p is in $\ker R$, to conclude that $f(z) = C \in \mathbf{GL}(n, \mathbb{C})$. But then

$$\rho_1(\gamma)^{-1} f(z) \rho_2(\gamma) = f(z\gamma) \Rightarrow \rho_2(\gamma) = C^{-1} \rho_1(\gamma) C.$$

Finally, we observe that since the ρ's are unitary, we can find a unitary C_1 which will also conjugate the two representations. ■

If our representation, ρ, is irreducible then by Proposition 1.1.4

$$\dim H^0(X, \mathrm{Ad}P(\rho)) = 1,$$

and so $W(\rho)$ must be indecomposable by Schur's lemma, since $\mathrm{Ad}P(\rho)$ is the endomorphism bundle.

The rest of this section will be used to reformulate, once again, what the tangent space to the moduli space is by relating group cohomology groups to cohomology groups with values in a complex vector bundle. The cohomology groups with values in a complex vector bundle have a complex structure, and it will turn out to give the moduli space the structure of a complex manifold.

Proposition 1.1.6 ([Narasimhan and Seshadri, 1964, Prop. 4.3])
Let X be a compact, connected, Kähler manifold. Let ρ be a representation of $\pi_1(X-p)$ with the loop around p represented by $e^{\frac{-2\pi ik}{n}}I$. Then the natural map

$$H^1(X, L(Ad\rho)) \to H^1(X, AdP(\rho))$$

is an isomorphism.

Although the proposition is stated a little differently here since the loop around p is in the kernel of $\mathrm{Ad}\rho$, the proof from [Narasimhan and Seshadri, 1964] can be used without changes.

1.1.5 The Complex Structure on the Moduli Space

To construct the complex structure we will use a proposition of S. Nakano [Nakano, 1961, Proposition 1]. The setup is as follows. Let $\mathcal{V} \to M$ be a family of complex compact manifolds and let $\mathcal{P} \to \mathcal{V}$ be a family of principal G-bundles for some Lie group G. Then there is a map $\eta_t : T_tM \to H^1(X, \Sigma_t)$, where Σ_t is the sheaf of germs of holomorphic vector fields with values in $\mathrm{Ad}\mathcal{P}_t$.

Proposition 1.1.7 ([Nakano, 1961, Prop. 1])
If $\dim H^1(X, \Sigma_t)$ is independent of t and η_t is an injection with the image of η_t a complex subspace of $\dim H^1(X, \Sigma_t)$ for all t, then M is a complex manifold with the almost complex structure we gave T_tM.

We can construct a family of principal $\mathbf{GL}(n, \mathbb{C})$-bundles over the family of compact complex manifolds $\mathcal{R}^{irr} \times X$, which we denote $\mathcal{P}$. This is done by letting $\mathcal{P}$ be the automorphism bundle of a vector bundle given by the representation in $\mathcal{R}^{irr}$, as of Section 1.1.3. This is a complex manifold since the transition functions are nicely behaved.

We have seen that $T_m\mathcal{M}_{n,k} \overset{I(\rho)}{\cong} H^1(\pi, \mathrm{Ad}(\rho)) \overset{J(\rho)}{\cong} H^1(X, \mathrm{Ad}P(\rho))$. Since $\mathrm{Ad}P(\rho)$ is a complex vector space this last cohomology group is a complex vector space, and so this gives pointwise a complex structure. However, here we used a representative for the conjugacy class, so we need to show that the complex structure is independent of this choice. Further, we will have to show it is smooth and gives rise to the structure of a complex manifold $\mathcal{M}_{n,k}$.

Let ρ_1 and ρ_2 be two equivalent representations and let T be the element conjugating ρ_1 to ρ_2. Then this means that $\mathrm{Ad}\rho_2 = \mathrm{Ad}(T)\mathrm{Ad}\rho_1\mathrm{Ad}(T)^{-1}$, and so T induces isomorphisms

$$H^1(\pi, \mathrm{Ad}(\rho_1)) \cong H^1(X, \mathrm{Ad}(\rho_2))$$

and

$$H^1(\pi, \mathrm{Ad}P(\rho_1)) \cong H^1(X, \mathrm{Ad}P(\rho_2)).$$

The latter of the two comes from the holomorphic map induced by T between the two principal bundles $P(\rho_1)$ and $P(\rho_2)$, and so it is a complex isomorphism of vector spaces, and so the complex structure does not depend on the representative we chose.

For the last part we will need to calculate the deformation map of our family of principal $\mathbf{GL}(n, \mathbb{C})$ bundles over $\mathcal{R}^{irr} \times X$.

Proposition 1.1.8 ([Narasimhan and Seshadri, 1964, Prop. 5.1]) *The infinitesimal deformation map (Kodaira map) $\tau_\rho : \mathcal{R}^{irr} \to H^1(X, AdP(\rho))$ is the composition of the following natural maps:*

$$T_\rho \mathcal{R}^{irr} \xrightarrow{\delta} Z^1(\pi_1(X-p), Ad\rho) \xrightarrow{q} H^1(\pi_1(X-p), Ad\rho) \\ \to H^1(X, L(Ad(\rho))) \to H^1(X, AdP(\rho)).$$

We will skip the proof, from the sheaf description we get transition functions for $\mathcal{P}$, we can calculate just as in [Narasimhan and Seshadri, 1964] to verify the result.

Since the $U(n)$ action on the bundle $\mathcal{P}$ is effective we can reduce this to a bundle over $\mathcal{M}_{n,k} \times X$. The deformation map will be the same, and so this is a smooth map. We will now show, that this is the same map as the isomorphism we used to define the complex structure on $\mathcal{M}_{n,k}$ at every point, and so our pointwise complex structures are chosen smoothly and give an almost complex structure. We work locally on $U \subset \mathcal{M}_{n,k}$ and choose a section on the bundle $\mathcal{R}^{irr} \to \mathcal{M}_{n,k}$ over U, call it σ. This gives us the map $\tau_{\sigma(\rho)} \circ d\sigma : T_{[\rho]}\mathcal{M}_{n,k} \to H^1(X, \mathrm{Ad}P(\sigma(\rho)))$, which is smooth. We look at the definition of $\tau_\rho = J(\rho) \circ q \circ \delta$, since the map $I(\rho) = q \circ \delta \circ d\sigma$, and so we see that this is exactly the map we used to define the complex structure and that it is smooth.

In order to show the complex structure is integrable, we verify the conditions of Proposition 1.1.7 in the following lemma.

Lemma 1.1.9 ([Narasimhan and Seshadri, 1964, Lemma 6.1]) *Let ρ be an irreducible $\mathbf{U}(n)$ representation of $\pi_1(X-p)$ with the loop around p represented by $e^{\frac{-2\pi i k}{n}} I$. Let $\Sigma(\rho)$ be the holomorphic vector bundle on X of invariant tangent vector fields on $P(\rho)$, then:*

1. $\dim_{\mathbb{C}} H^1(X, \Sigma(\rho))$ *is independent of* ρ
2. *the natural map $H^1(X, AdP(\rho)) \to H^1(X, \Sigma(\rho))$ is injective and a complex linear map.*

Proof: We have the exact sequence of vector bundles

$$0 \to \mathrm{Ad}P(\rho) \to \Sigma(\rho) \to \Theta \to 0,$$

where Θ is the holomorphic tangent bundle. Now $H^0(X, \Theta) = 0$, and since X only has one complex dimension $H^2(X, \mathrm{Ad}P(\rho)) = 0$, and so from the long exact sequence of cohomology groups, we get:

$$0 \to H^1(X, \mathrm{Ad}P(\rho)) \to H^1(X, \Sigma(\rho)) \to H^1(X, \Theta) \to 0. \tag{1.1}$$

Equation (1.1) shows the first claim, since $H^1(X, \Theta)$ is independent of ρ and from Proposition 1.1.2 we know $\dim H^1(X, \mathrm{Ad}P(\rho))$ is independent of ρ. The second claim is just what the first half of (1.1) means. ■

The complex structure we just gave $\mathcal{M}_{n,k}$ can also be described from the complex structure on the manifold X as an almost complex structure on the underlying differential manifold, the moduli space of flat vector bundles. This is what [Hitchin, 1990] does, and the two complex structures are the same.

1.2 The Hitchin Connection

In this chapter we describe the construction of the Hitchin connection which can be used for the moduli space of connections, as described in [Andersen, 2012].

We will let (M, ω) be a compact symplectic manifold, and $(\mathcal{L}, (\cdot, \cdot), \nabla)$ be a prequantum line bundle over M. Here $(\cdot, \cdot)$ is a Hermitian inner product, and ∇ a compatible connection with the curvature given by $F_\nabla = \frac{-i}{2\pi}\omega$. Both (M, ω) and $\mathcal{L}$ will be fixed, we will consider holomorphic families of Kähler structures on M.

Definition 1.2.1 ([Andersen and Gammelgaard, 2011, Defn. 2.2]) *Let $\mathcal{T}$ be a complex manifold with a map, I, to the space of complex structures on M such that (M, ω, I_σ) is a Kähler manifold for all $\sigma \in \mathcal{T}$. Then we call I holomorphic if:*

$$V'[I] = (V[I])' \quad V''[I] = (V[I])'' \quad \forall V \in \mathcal{X}(\mathcal{T})$$

Where V' denote the $(1,0)$-part of the vector field V and V'' the $(0,1)$-part. Also we have that $V[I] \in C^\infty(M, T_\sigma \otimes (\bar{T}_\sigma)^* \oplus \bar{T}_\sigma \otimes (T_\sigma)^*)$ and so $(V[I])'$ is a projection to the first factor and $(V[I]'')$ is a projection to the second factor.

We define the 1-form $\tilde{G}$ on $\mathcal{T}$ with values in two tensors on M by:

$$V[I] = \tilde{G}(V)\omega \quad V \in \Gamma(T\mathcal{T}). \tag{1.2}$$

Here V differentiates I as a section of the trivial bundle $\mathcal{T} \times C^\infty(M, \mathrm{End}(TM_\mathbb{C}))$, and the tensors on the right-hand side are contracted. We will later see that in fact $\tilde{G}(V) \in C^\infty(M, S^2(TM_\mathbb{C}))$. Since each M has a complex structure we can consider the complexified tangent space $TM_\mathbb{C} \cong T_\sigma \oplus \bar{T}_\sigma$. And so the map I being holomorphic means we can split $\tilde{G}$ as

$$\tilde{G}(V) = G(V') + \bar{G}(V'').$$

Here $G(V') \in C^\infty(M, S^2(T_\sigma))$.

Definition 1.2.2 ([Andersen, 2012, Defn. 1.1])
A complex family of Kähler structures, I, on (M,ω) is called rigid if

$$\nabla^{0,1}_\sigma(G_\sigma(V)) = 0, \quad \forall V \in \Gamma(T\mathcal{T}), \forall \sigma \in \mathcal{T}$$

Consider the trivial bundle $\mathcal{H}^{(k)} := \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$, we will assume that the the subspaces of holomorphic sections $H^{(k)} := \{s \in C^\infty(M, \mathcal{L}^k) | \nabla^{0,1} s = 0\}$ for a smooth finite dimensional subbundle.

Definition 1.2.3 ([Andersen, 2012, defn. 1.2])
A connection in $\mathcal{H}^{(k)}$ over $\mathcal{T}$ of the form

$$\hat{\nabla} = \hat{\nabla}^t - u,$$

where $\hat{\nabla}^t$ is the trivial connection in $\mathcal{H}^{(k)}$ and u is a 1-form with values in differential operators on $C^\infty(M, \mathcal{L}^k)$, is called a Hitchin connection if it preserves $H^{(k)}$.

Now we can state the theorem.

Theorem 1.2.4 ([Andersen, 2012, Theorem 1.3])
If $(\mathcal{T}, I)$ is a rigid family of Kähler structures on the symplectic prequantizable compact manifold (M,ω), which satisfies that there exists $n \in \mathbb{Z}$ such that the first Chern class of (M,ω) in $n[\omega] \in H^2(M,\mathbb{Z})$ and $H^1(M,\mathbb{R}) = 0$. Then there exists a Hitchin connection, $\hat{\nabla}$ in $\mathcal{H}^{(k)}$. The connection is given as

$$\hat{\nabla}_V = \hat{\nabla}^t_V - u(V)$$

With $\hat{\nabla}^t$ the trivial connection in $\mathcal{H}^{(k)}$, V a smooth vector field on $\mathcal{T}$ and $u(V)$ the second order differential operator given by:

$$u(V)(s) = \frac{1}{2k+n}\left(\frac{1}{2}\Delta_{G(V)}(s) - \nabla_{G(V)dF_\sigma}(s) + 2kV'[F_\sigma]s\right).$$

Where F_σ is a smooth family of Ricci potentials (Ric$=$ $n\omega + 2i\partial\bar{\partial}F_\sigma$) for M_σ and V' is the $(1,0)$-part of the vector field V on $\mathcal{T}$. And $\Delta_{G(V)}$ is the composition:

$$\Delta_{G(V)} : C^\infty(M,\mathcal{L}^k) \xrightarrow{\nabla^{1,0}_\sigma} C^\infty(M, T^*_\sigma \otimes \mathcal{L}^k) \xrightarrow{G\otimes Id} C^\infty(M, T_\sigma \otimes \mathcal{L}^k)$$
$$\xrightarrow{\nabla^{1,0}_\sigma \otimes Id + Id \otimes \nabla^{1,0}_\sigma} C^\infty(M, T^*_\sigma \otimes T_\sigma \otimes \mathcal{L}^k) \xrightarrow{tr} C^\infty(M,\mathcal{L}^k).$$

Theorem 1.2.4 can be used for the moduli space of flat $\mathbf{SU}(n)$-connections. We will study a smooth family Ricci potentials in the coming chapters in order to understand the Hitchin connection.

The proof of Theorem 1.2.4 splits into three lemmas. The first lemma gives an equation for $u(V)$. The second lemma calculates what happens to the equation, if we assume $u(V)$ is a second order operator. Finally, the third lemma shows that the second order operator with the symbol $G(V)$ and the Ricci potential can be combined into a solution of the equation for $u(V)$.

Lemma 1.2.5 ([Andersen, 2012, Lemma 2.2])
The connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$ from Theorem 1.2.4 induces a connection in $H^{(k)}$ if and only if

$$\frac{i}{2}V[I]\nabla^{1,0}s + \nabla^{0,1}u(V)s = 0 \tag{1.3}$$

for all smooth sections of $H^{(k)}$ and all vector fields V on $\mathcal{T}$.

So to show $\hat{\nabla}$ is a Hitchin connection we only need to show the equation (1.3).

For the statement of the following lemma let $G \in H^0(M_\sigma, S^2(T_\sigma))$, then by the procedure given at the end of Theorem 1.2.4 we construct a second order differential operator we call Δ_G.

Lemma 1.2.6 ([Andersen, 2012, Lemma 2.6])
Assume the first Chern class of (M,ω) is $n[\omega] \in H^2(M,\mathbb{Z})$, for $\sigma \in \mathcal{T}$, $s \in H^{(k)}$ and for any $G \in H^0(M_\sigma, S^2(T_\sigma))$ we have:

$$\nabla^{0,1}_\sigma(\Delta_G(s) - 2\nabla_{GdF_\sigma}(s)) = -i(2k+n)G\omega\nabla^{1,0}_\sigma(s) \\ - iktr(\nabla^{1,0}_\sigma(G)\omega - 2G\partial_\sigma F\omega)s.$$

And finally the third lemma is:

Lemma 1.2.7 ([Andersen, 2012, Lemma 2.8])
The following identity holds:

$$4i\bar{\partial}_\sigma(V'[F_\sigma]) = 2tr(G(V)\partial_\sigma(F_\sigma)\omega - \nabla^{1.0}(G(V)\omega))$$

if $H^1(M,\mathbb{R}) = 0$.

We will show this for the $\mathbf{SU}(n)-$moduli space later Lemma 6.2.1 without assuming $H^1(M,\mathbb{R}) = 0$.

If we combine Lemma 1.2.6 and Lemma 1.2.7 it turns out, that $u(V)$, as defined in Theorem 1.2.4, solves the equation in Lemma 1.2.5 and so $u(V)$ defines a Hitchin connection.

1.2.1 $G(V)$ for the Moduli Space of Vector Bundles

We follow Hitchin [Hitchin, 1990] in deriving properties for $G(V)$ in the general case and formulas for the moduli space of flat vector bundles. A complex structure, $I \in C^\infty(M, \mathrm{End}TM)$, gives a Kähler stucture on a symplectic manifold (M,ω) if:

$$I^2 = -Id, \tag{1.4}$$

$$[IX, IY] = [X, Y] + I[X, IY] + I[IX, Y], \tag{1.5}$$
$$\omega(X, IY) = -\omega(IX, Y), \tag{1.6}$$
$$\omega(X, IX) > 0 \quad \forall X \neq 0. \tag{1.7}$$

Here X and Y are tangent vectors on M.

We want to study a family of Kähler structures. Let us first consider a path of Kähler structures I_t. Differentiating in (1.4) with respect to t we have

$$\dot{I}I + I\dot{I} = 0,$$

where $\dot{I}$ is the derivative of I_t with respect to t. This means that $\dot{I}$ takes the $-i$ eigenspace to the i eigenspace of I, and this map contains the information to reconstruct $\dot{I}$, therefore we can think of $\dot{I} \in \Omega^{0,1}(M, T_I M)$ as a $(0,1)$-form with values in $(1,0)$-tangent vectors on (M, I). Expressed in local coordinates we get:

$$\dot{I} = \sum_{i,j} a_{ij} \frac{\partial}{\partial z_i} \otimes d\bar{z}_j. \tag{1.8}$$

Next we turn our attention to the moduli space of vector bundles. The tangent space can be identified with the harmonic 1-forms with values in $\mathrm{End}_0 E$. For a 1-form, α, to be harmonic we have:

$$d_A \alpha = 0$$
$$d_A \star \alpha = 0,$$

where $\star$ is the Hodge star and d_A the flat connection in the bundle $\mathrm{End}_0 E$. This implies that the Hodge star preserves the harmonic 1-form, and we can define a complex structure as:

$$I\alpha = - \star \alpha.$$

For a surface the complex stucture is described by the Hodge star $\star$, and from the considerations (1.8) we have that

$$\dot{\star} = a \frac{\partial}{\partial z} \otimes d\bar{z} \in \Omega^{0,1}(\Sigma, K^{-1})$$

is a Beltrami differential. For the moduli space of vector bundles the $(1,0)$-tangent space can be identified with $H^1(\Sigma, \mathrm{End}_0 E)$ at the bundle E and the $(0,1)$-tangent space is identified with $H^0(\Sigma, \mathrm{End}_0 E \otimes K)$. Using these identifications we have the following description of $\dot{I}$:

Lemma 1.2.8 ([Hitchin, 1990, Lemma 2.13])
If $X \in T^{0,1}$ is represented by a holomorphic section α of $\mathrm{End}_0 E \otimes K$, then $\dot{I}X \in T^{1,0}$ is represented in $H^1(\Sigma, \mathrm{End}_0 E)$ by $-\dot{\star}\alpha \in \Omega^{0,1}(\Sigma, \mathrm{End}_0 E)$.

PROOF: We let $\star(t)$ be a 1-parameter family of complex structures on Σ and then we have a corresponding family $I(t)$ on the moduli space of vector bundles. In order to calculate $I\alpha$ we choose a harmonic representative h with respect to the conformal (complex) structure $\star$:

$$\alpha = h + d_A\psi.$$

Differentiating this equation at $t = 0$ gives $0 = \dot{h} + d_A\dot{\psi}$. Further we have, that $\alpha = h$ for $t = 0$. By the definition of I we have

$$I[\alpha] = [- \star h] \in H^1_A(\Sigma, \mathrm{End}_0 E).$$

When we differentiate at $t = 0$, we get:

$$\dot{I}[\alpha] = [-\dot{\star}h - \star\dot{h}] = [-\dot{\star}\alpha + \star d_A\dot{\psi}].$$

Since $\dot{I}[\alpha]$ represents an element of $T^{1,0}$ there exists a $(0,1)$-form β and a $\varphi \in \Omega^0(\Sigma, \mathrm{End}_0 E)$ such that:

$$-\dot{\star}\alpha + \star d_A\dot{\psi} + d_A\varphi = \beta \in \Omega^{0,1}(\Sigma, \mathrm{End}_0 E).$$

Now since we have $\dot{\star} \in \Omega^{0,1}(\Sigma, k^{-1})$ and $\alpha \in \Omega^0(\Sigma, \mathrm{End}_0 E \otimes K)$ both β and $\dot{\star}\alpha$ are of type $(0,1)$ and we must have:

$$\beta = -\dot{\star}\alpha + \bar{\partial}(\varphi - i\dot{\psi}),$$

which means $[\beta] = [-\dot{\star}\alpha] \in H^1(\Sigma, \mathrm{End}_0 E)$. ■

Using Lemma 1.2.8 and ω we can describe $G(V)$ explicitly, since $G(V) = \dot{I}\omega^{-1}$ when we view ω as a map from $(T^{1,0})^* \to T^{0,1}$. We can also view both ω^{-1} and $G(V)$ as quadratic functions on the cotangent spaces $(T^{1,0})^* \otimes (T^{0,1})^*$ and $(T^{1,0})^* \otimes (T^{1,0})^*$ respectively. In this interpretation $\omega(\alpha, \beta) = \int_\sigma \mathrm{tr}\alpha \wedge \beta$, and so we have:

$$G(V)(\alpha, \alpha) = \int_X V[-\star]\mathrm{tr}\alpha^2. \tag{1.9}$$

It is clear from (1.9) that $G(V)$ is holomorphic on the moduli space of vector bundles. We will calculate $V[-\star]$ in Chapter 6.

Chapter 2

The Laplace Operator on Quotients of the Upper Half-plane

We will be interested in studying the Ricci potential on the moduli space of stable holomorphic vector bundles with fixed determinant over a Riemann surface, $X = (\Sigma, \sigma)$ and degree k. This function is closely related to the Laplace operator on the bundle over X. We will study the case where X is compact and of genus greater than 1. In this case X can be identified with a quotient of the upper half-plane $\mathbb{H}$ by a discreet subgroup of $\mathbf{PSL}(2, \mathbb{R})$, a Fuchsian group of the first kind. The bundles can be constructed using unitary representations of $\pi_1(\Sigma)$ or $\pi_1(\Sigma - p)$ with certain restrictions. In this section we will review results about automorphic forms and relate them to sections of the vector bundle over X constructed from the unitary representation.

After these preliminary considerations we will review elements of the theory of automorphic forms, in particular the Selberg trace formula and the Selberg zeta function. We shall present the results without proofs. We will end the section following [Sarnak, 1987] by proving the close relation between ζ-regularized determinants of the Laplace operator and values of the Selberg zeta function. This result allows us to interpret the determinant as the Selberg zeta function in further studies.

2.1 Structural Results

For our convenience we will begin by mentioning some usefull structural results about the subgroups of $\mathbf{PSL}(2, \mathbb{R})$, we will encounter, and their fundamental domains. The discrete subgroups can be described by the following theorem.

Theorem 2.1.1 ([Venkov, 1982, Theorem 1.2.1])
A Fuchsian group of the first kind is determined by a finite set of generators

$A_1, B_1, \ldots, A_g, B_g, S_1, \ldots, S_r, R_1, \ldots, R_l$ *with the defining relations:*

$$[A_1, B_1] \cdot \ldots \cdot [A_g B_g] \cdot S_1 \cdot \ldots \cdot S_r \cdot R_1 \cdot \ldots \cdot R_l = I$$
$$R_1^{m_1} = \ldots = R_l^{m_l} = I.$$

Here the elements R are elliptic ($|\mathrm{tr}R| < 2$), S parabolic ($|\mathrm{tr}S| = 2$) and A and B hyperbolic ($|\mathrm{tr}A| > 2$).

We will mainly consider the case where there are no parabolic elements. Groups with no parabolic elements give a compact fundamental domain, whereas groups with parabolic elements give a non-compact finite measure fundamental domain. Also, most of the time we are interested in groups without elliptic elements, as these give genus g closed surfaces. However, we may want to consider an orbifold with a single n-fold branching point, this corresponds to having an elliptic element of order n.

Associated to the group Γ we can find a fundamental domain that is connected, and whose interior only contains Γ inequivalent points, while the closure contains representatives of all Γ equivalence classes. Further, we can find the fundamental domain as a hyperbolic polygon with sides pairwise identified. If the group Γ contains no parabolic elements we can choose the fundamental domain to be compact. We denote such a choice $\mathcal{F}$ in the following.

These results let us work with objects on the manifold X by considering functions on $\mathbb{H}$ which are equivariant with respect to an action of the related group Γ. We will introduce one such object in the next section, simply writing about it from the perspective of X being a quotient of $\mathbb{H}$ by a discrete group Γ.

2.2 Automorphic Forms

Given a discrete subgroup of $\Gamma \subset \mathbf{PSL}(2, \mathbb{R})$ and a $\mathbf{U}(n)$-representation, ρ, of Γ, a vector valued automorphic form of weight (n, m) is a smooth function on $f : \mathbb{H} \to \mathbb{C}^n$ such that

$$f(\gamma z) = \rho(\gamma) f(z) (\gamma' z)^n (\overline{\gamma' z})^m.$$

The group Γ acts on $\mathbb{H}$ as isometries of the hyperbolic plane and on $\mathbb{C}^n$ with the unitary representation ρ. This means that Γ acts on $\mathbb{H} \times \mathbb{C}^n$ and this action is discrete and faithful, and so the quotient space gives a vector bundle over the surface $\mathbb{H}/\Gamma$. The automorphic forms will correspond to sections of the bundle associated with ρ by the above construction tensored with $(T_\sigma X)^{-\otimes n} \otimes (\bar{T} X)^{-\otimes m}$, where T_σ denotes the subbundle of $T_{\mathbb{C}} X$ on which the complex structure acts as i. Conversely, given a section of this bundle over X we can pull it back to $\mathbb{H}$, and it will be a function with the required transformation properties.

Since we have a unitary representation, there is a naturally associated inner product on these sections given by:

$$\langle f, g\rangle = \int_{\mathcal{F}} (f(z), g(z))_{\mathbb{C}^n} \frac{dzd\bar{z}}{2iy^{2-n-m}}.$$

This inner product is well-defined on functions whose restriction to the closure of $\mathcal{F}$ is compactly supported, and the inner product is independent of the fundamental domain chosen. We will work with the completion of the smooth automorphic forms of weight (n, m) with finite norm (the L^2 space). We will mostly be interested in the Laplace operator on functions in $L^2(\Gamma, \rho)$ that is given by $-y^2 \frac{d^2}{dzd\bar{z}}$. The Laplace operator is a symmetric operator and extends to a positive self-adjoint operator.

2.2.1 Kernel Operator Associated with the Laplace Operator

We want to develop tools to help us understand the spectrum of the Laplace operator, specifically we want to be able to manipulate the ζ-regularized determinant. In this section, we will take some function, h, defined in a neighborhood of the real axis and construct an operator with eigenvalues $h(\lambda)$ for $\lambda \in \sigma(\Delta)$. First, on the upper half-plane model of hyperbolic space $\mathbb{H}$ we can consider the following equation:

$$-y^2 \frac{d^2}{dzd\bar{z}} f_1(z) - s(s-1) f_1(z) = f_2(z). \tag{2.1}$$

The following theorem then holds:

Theorem 2.2.1 ([Venkov, 1982, Lemma 1.1.1])
For $f_2 : \mathbb{H} \to \mathbb{C}$ a smooth and bounded function, there exists a kernel $k(z, z', s), z, z' \in \mathbb{H}, s \in \mathbb{C}$, such that for $\mathrm{Re}(s) > 1$ *we have:*

$$f_1(z) = \int_{\mathbb{H}} k(z, z', s) f_2(z')\rho$$

is a bounded smooth function solving (2.1).

Further, $k(z, z', s)$ is given by:

$$k(z, z', s) = k(u(z, z'), s), \quad u(z, z') = \frac{|z - z'|^2}{yy'} = \cosh(\mathrm{dist}_{\mathrm{hyp}}(z, z')) - 1 \tag{2.2}$$

$$k(u, s) = \frac{1}{4\pi} \int_0^1 [t(1-t)]^{s-1} \left(t + \frac{u}{4}\right)^{-s} dt \tag{2.3}$$

for $\mathrm{Re}(s) > 0$.

Finally, $k(u, s)$ is analytic in s and smooth in u for $\mathrm{Re}(s) > 0$ *and $u > 0$. It has the following assymptotics:*

$$k(u, s) = -\frac{1}{4\pi} \log u + O(1), \quad u \to 0$$

and the bound:

$$k(u,s) = O(u^{-\mathrm{Re} s}), \quad u \to \infty.$$

This Theorem gives us formulas for the resolvent kernel of $(\Delta - s(s-1))$. In the case where $s = 1$, $k(u,1) = \frac{-1}{4\pi} \ln u$. This is the kernel associated to Δ we will encounter most often. If we have a Riemann surface, X, the kernel of the Laplace operator on X is:

$$G(z,z') = \sum_{\gamma \in \rho} k(u(z, \rho_{\mathbb{H}}(\gamma)z'), 1), \quad z \notin \rho_{\mathbb{H}}(\pi_1(X))z'. \tag{2.4}$$

2.3 The Selberg Trace Formula

In this section we describe the Selberg trace formula. The trace formula reformulates the trace of nicely behaved functions h used on the Laplace operator in geometric terms. As we have seen earlier, it is convenient to know the integral kernel of an operator for these purposes. It is obtained using the Selberg transform:

Theorem 2.3.1 ([Venkov, 1982, Theorem 3.3.3])
If $h : [0; \infty) \to \mathbb{C}$ is measurable and bounded then the operator

$$h(\Delta) : L^2(\Gamma, \chi) \to L^2(\Gamma, \chi)$$

is defined, and the kernel as integral operator has the spectral decomposition:

$$h(z, z', \Delta) = \sum_j h(\lambda_j) w(z, \lambda_j) \otimes w(z' \lambda_j).$$

If we further assume that there is a function $\tilde{h} : \mathbb{C} \to \mathbb{C}$ which fulfills:

1. *$\tilde{h}(s) = h(s(s-1))$ is analytic in the strip $-\varepsilon < \mathrm{Re}(s) < 1 + \varepsilon$ for some $\varepsilon > 0$.*

2. *That there exists a $\delta > 0$ such that $\tilde{h}$ has the following growth estimate*

$$\tilde{h}(s) = O((|\mathrm{Im}(s)| + 1)^{-2-\delta}), \quad -\varepsilon < \mathrm{Re}(s) < 1 + \varepsilon.$$

Then the kernel can be written $h(z, z', \Delta) = \sum_{\gamma \in \Gamma} \chi(\gamma) k(u(z, \gamma z'))$ with u as in (2.2), and $k : [0; \infty) \to \mathbb{C}$ and h related by:

$$\int_\omega^\infty \frac{k(t)}{\sqrt{t - \omega}} dt = Q(\omega), \quad k(t) = \frac{1}{\pi} \int_1^\infty \frac{dQ(\omega)}{\sqrt{\omega - t}}, \tag{2.5}$$

$$Q(e^u + e^{-u} - 2) = g(u), \tag{2.6}$$

$$h\left(\frac{1}{4} + r^2\right) = \int_{-\infty}^{\infty} e^{-iru} g(u) du, \quad g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} h\left(\frac{1}{4} + r^2\right) dr, \tag{2.7}$$

where the series $\sum_{\gamma \in \Gamma} \chi(\gamma) k(u(z, \gamma z'))$ converge absolutely in the norm on V and uniformly on compact subsets for $\mathbb{H} \times \mathbb{H}$ giving a continuous map to V.

Now we are ready to state the Selberg trace formula. Let χ be a unitary representation of Γ. Then from the identity $h(z,z',\Delta) = \sum_{\gamma\in\Gamma}\chi(\gamma)k(u(z,\gamma z'))$ and the spectral decomposition of h, we have that:

$$\sum_j h(\lambda_j) = \int_{\mathcal{F}} \sum_{\gamma\in\Gamma} \mathrm{tr}\chi(\gamma)k(u(z,\gamma z))d\mu(z). \tag{2.8}$$

Since $\mathcal{F}$ is compact we can interchange summation and integration. We get:

$$\begin{aligned}\sum_j h(\lambda_j) &= \sum_{\gamma\in\Gamma} \mathrm{tr}\chi(\gamma)\int_{\mathcal{F}} k(u(z,\gamma z))d\mu(z)\\ &= \sum_{\{\gamma\}_\Gamma}\sum_{\gamma'\in\Gamma_\gamma\backslash\Gamma} \mathrm{tr}\chi(\gamma)\int_{\mathcal{F}} k(u(z,\gamma'^{-1}\gamma\gamma' z))d\mu(z)\\ &= \sum_{\{\gamma\}_\Gamma} \mathrm{tr}\chi(\gamma)\int_{\mathcal{F}_\gamma} k(u(z,\gamma z))d\mu(z),\end{aligned}$$

where $\mathcal{F}_\gamma$ is the fundamental domain of Γ_γ, the centralizer of γ. Now each term in the sum reduces to calculating the integral over certain nicely described regions, and we have the following lemma.

Lemma 2.3.2 ([Venkov, 1982, Lemma 4.3.4])
The following equalities hold with h, g and k related as in (2.5):

$$\int_{\mathcal{F}_E} k(u(z,z))d\mu = \frac{\mathrm{Vol}(\mathcal{F})}{4\pi}\int_{-\infty}^{\infty} r(\tanh \pi r)h\left(\frac{1}{4}+r^2\right)dr,$$

where E is the identity element.

$$\int_{\mathcal{F}_\gamma} k(u(z,\gamma z))d\mu(z) = \frac{\ln(N(P))}{N(P)^{\frac{k}{2}} - N(P)^{-\frac{k}{2}}}g(k\ln N(P)),$$

where $\gamma = P^k$ and P is a primitive hyperbolic element with norm $N(P)$(the square of the biggest eigenvalue).

$$\int_{\mathcal{F}_\gamma} k(u(z,\gamma z))d\mu(z) = \frac{1}{2m\sin\left(\frac{k\pi}{m}\right)}\int_{-\infty}^{\infty}\frac{e^{\frac{-2\pi rk}{m}}}{1-e^{-2\pi r}}h\left(\frac{1}{4}+r^2\right)dr,$$

where $\gamma = R^k$ and R is an elliptic element of order $m \geq 2$.

And this lets us conclude, that for a discrete group with no parabolic elements we have:

Theorem 2.3.3 ([Venkov, 1982, Theorem 4.3.6])
Let $h(s(1-s))$ satisfy:

1. $h(s(1-s)) = h_1(s(1-s))^2$.

2. *$\tilde{h}_1(s) = h_1(s(1-s))$ is analytic for $-\varepsilon < \mathrm{Re}(s) < 1+\varepsilon$ for some $\varepsilon > 0$.*

3. *In the strip $-\varepsilon < \mathrm{Re}(s) < 1+\varepsilon$ we have $\tilde{h}_1 = O((1+|\mathrm{Im}(s)|)^{-4-\delta})$ for some $\delta > 0$.*

4. *$h_1(\mathbb{R}_+) \subset \mathbb{R}$.*

Then the following identity holds:

$$\begin{aligned}\sum_l h(\frac{1}{4}+r_j^2) =& \frac{\mathrm{Vol}(\mathcal{F})}{2\pi}\int_{-\infty}^{\infty} r(\tanh \pi r) h\left(\frac{1}{4}+r^2\right) dr \\ &+2 \sum_{\gamma_\Gamma \text{ primimitive hyperbolic}} \sum_{k=1}^{\infty} \frac{\ln(N(P))}{N(P)^{\frac{k}{2}} - N(P)^{-\frac{k}{2}}} g(k \ln N(P)) \\ &+ \sum_{\gamma_\Gamma \text{ primitive eliptic}} \sum_{k=1}^{m-1} \frac{1}{m \sin\left(\frac{k\pi}{m}\right)} \int_{-\infty}^{\infty} \frac{e^{\frac{-2\pi r k}{m}}}{1-e^{-2\pi r}} h\left(\frac{1}{4}+r^2\right) dr.\end{aligned}$$

2.4 The Selberg Zeta Function

Definition 2.4.1 (Selberg's Zeta Function)
For $\mathrm{Re}(s) > 1$ we have the convergent product:

$$Z(s,\rho_{\mathbb{H}},\rho_E) = \prod_{\gamma \text{ simple closed geodesic}} \prod_{k=0}^{\infty} \det(I - \rho_E(\gamma) e^{-(k+s)|\gamma|_{\rho_{\mathbb{H}}}}) \tag{2.9}$$

$$= \prod_{\gamma \in \rho_{\mathbb{H}}(\pi_1(\Sigma)) \text{primitive hyperbolic}} \prod_{k=0}^{\infty} \det(I - \rho_E(\gamma) N(\gamma)^{-s-k}). \tag{2.10}$$

*Here $N(\gamma)$ is the norm of the biggest eigenvalue of $\gamma \in$ **PSL**$(2,\mathbb{R})$ squared. The Selberg zeta function* (2.9) *extends to a meromorphic function for $s \in \mathbb{C}$.*

In order to prove the claim in the definition that the Selberg zeta function is meromorphic, we would need to connect the derivative to the Selberg trace formula. We will not discuss how this is done, but will still rewrite the derivative of the Selberg zeta function to an expression convenient later:

$$\begin{aligned}\frac{d}{ds}\log Z(s,\rho_{\mathbb{H}},\rho_E) =& \frac{d}{ds} \sum_{\gamma\in\pi_1(\Sigma)}' \sum_{k=0}^{\infty} \ln\det(I - \rho_E(\gamma) N(\rho_{\mathbb{H}}(\gamma))^{-s-k}) \\ =& \frac{d}{ds} \sum_{\gamma\in\pi_1(\Sigma)}' \sum_{k=0}^{\infty} \mathrm{tr}\ln(I - \rho_E(\gamma) N(\rho_{\mathbb{H}}(\gamma))^{-s-k}) \\ =& \frac{d}{ds} \sum_{\gamma\in\pi_1(\Sigma)}' \sum_{k=0}^{\infty} \mathrm{tr}\left(- \sum_{l=1}^{\infty} \frac{\rho_E(\gamma^l) N(\rho_{\mathbb{H}}(\gamma))^{-(s+k)l}}{l} \right.\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\gamma\in\pi_1(\Sigma)}' \sum_{k=0}^{\infty}\sum_{l=1}^{\infty} \mathrm{tr}(\rho_E(\gamma^l))N(\rho_{\mathbb{H}}(\gamma))^{-(s+k)l}\ln N(\rho_{\mathbb{H}}(\gamma)) \\
&= \sum_{\gamma\in\pi_1(\Sigma)}' \sum_{l=1}^{\infty} \frac{\mathrm{tr}(\rho_E(\gamma^l))N(\rho_{\mathbb{H}}(\gamma))^{-sl}\ln N(\rho_{\mathbb{H}}(\gamma))}{1-N(\rho_{\mathbb{H}}(\gamma))^{-l}} \\
&=(2s-1) \sum_{\gamma\in\pi_1(\Sigma)}' \sum_{l=1}^{\infty} \frac{\mathrm{tr}(\rho_E(\gamma^l))\ln N(\rho_{\mathbb{H}}(\gamma))}{N(\rho_{\mathbb{H}}(\gamma))^{-l/2}-N(\rho_{\mathbb{H}}(\gamma))^{-l/2}} \\
&\qquad \cdot g(k\ln N(\rho_{\mathbb{H}}(\gamma)),s) \\
&=(2s-1)\int_{\mathbb{H}\backslash\Gamma} \sum_{\gamma \text{ hyperbolic}} \mathrm{tr}\rho_E(\gamma)Q_s^{\varepsilon\mu}(z,\rho_{\mathbb{H}}(\gamma)z)\frac{dxdy}{y^2},
\end{aligned}
\tag{2.11}
$$

where the prime over the sums indicates that the sum is only over primitive elements, and $g(u,s)=\frac{e^{-(s-\frac{1}{2})u}}{2s-1}$ is the Fourier transform of $((s-\frac{1}{2})^2+r^2)^{-1}$ in r. The last equality is a consequence of 2.3.2, which was an intermediate part of the derivation of the Selberg trace formula.

2.4.1 The Relation to Determinants

The Selberg zeta function is closely related to determinants of the Laplace operators on automorphic forms, see [D'Hoker and Phong, 1986]. We will only need the result for the Laplace operator on functions and present the argument of [Sarnak, 1987]. For us the important result is, that for compact manifolds $\det\Delta$ is a constant multiple of $Z'(1)$. We will differentiate the logarithm of $\det\Delta$, and so the exact constant factor will be unimportant for us.

Theorem 2.4.2 ([Sarnak, 1987, Theorem 1])
We have that

$$
\det(\Delta+s(s-1)) = Z(s,\Gamma,\chi)(e^{E-s(s-1)\frac{\Gamma_2(s)^2}{\Gamma(s)}(2\pi)^s})^{2g-2}. \tag{2.12}
$$

The constant E is given by $E=-\frac{1}{4}-\frac{1}{2}\log 2\pi+2\zeta'(-1)$, g is the genus of the compact surface given by $\mathbb{H}/\Gamma$. Finally, Γ_2 is the Barnes double gamma function.

PROOF: We know that the Laplace operator has a discrete spectrum with only an accumulation point at infinity. We will denote the eigenvalues $\lambda_0\le\lambda_1\le\ldots\le\lambda_n\le\ldots$. It is well known that

$$
\theta(t)=\sum_{n=0}^{\infty} e^{-(\lambda_n-\lambda_0)t} = \mathrm{tr}(e^{-(\Delta-\lambda_0 I)t}), \quad t>0
$$

fulfills the following asymptotic as t goes to 0:

$$\theta(t) \sim \frac{\alpha}{t} + \beta + \gamma t + \dots .$$

This is used to prove Weyl's law. Now, for $s(s-1) \geq \lambda_0$ we define $\det(\Delta + s(s-1))$ as follows:

$$\begin{aligned} H(w,s) &= \sum_{k=0}^{\infty} (\lambda_k + s(s-1))^{-w} \qquad \mathrm{Re}(w) > 0 \\ &= \frac{1}{\Gamma(w)} \int_0^\infty \theta(t) e^{-t(s(s-1)+\lambda_0)} t^w \frac{dt}{t}. \\ \det(\Delta + s(s-1)) &:= e^{-\frac{\partial H}{\partial w(0,s)}} \end{aligned}$$

That the latter is well-defined will follow once we show, that H is meromorphic and regular at $w = 0$. This is seen from splitting H up as:

$$\begin{aligned} H(w,s) =& \frac{1}{\Gamma(w)} \int_0^1 (\theta(t) - \frac{\alpha}{t} - \beta) e^{-t(s(s-1)+\lambda_0)} t^w \frac{dt}{t} \\ &+ \frac{1}{\Gamma(w)} \int_1^\infty \theta(t) e^{-t(s(s-1)+\lambda_0)} t^w \frac{dt}{t} \\ &+ \frac{\alpha}{\Gamma(w)(s(s-1)+\lambda_0)^{w-1}} \left(\Gamma(w-1) - \int_{(s(s-1)+\lambda_0}^\infty e^{-y} y^{w-1} \frac{dy}{y} \right) \\ &+ \frac{\beta}{\Gamma(w)(s(s-1)+\lambda_0)^{w}} \left(\Gamma(w) - \int_{(s(s-1)+\lambda_0}^\infty e^{-y} y^{w} \frac{dy}{y} \right). \end{aligned}$$

Here we used, that the integral representation of $\Gamma(w) = \int_0^\infty t^{w-1} e^{-t} dt$ and the change of variable $y = t(s(s-1)+\lambda_0)$. From this expression of H we can see, that it is regular at $w = 0$ and smooth for $s(s-1) > -\lambda_0$. Let $\prod_{n=1}^{\infty}{}' \mu_n$ denote the zeta regularized product, then we have the following is true:

$$\det(\Delta+s(s-1)) = \prod_{k=0}^{\infty}{}' (\lambda_k+s(s-1)) = \prod_{k=0}^{R} (\lambda_k+s(s-1)) \prod_{k=R+1}^{\infty}{}' (\lambda_k+s(s-1)).$$

And the last zeta regularized product is well-defined and smooth for $s(s-1) > -\lambda_{R+1}$, this will allow us to define the determinant for all real s. Now we differentiate with respect to s:

$$\begin{aligned} \frac{1}{2s-1} \frac{\partial H}{\partial s} &= \sum_{k=0}^{\infty} (-w)(\lambda_k + s(s-1))^{-w-1} \\ \frac{1}{2s-1} \frac{\partial}{\partial s} \left(\frac{1}{2s-1} \frac{\partial H}{\partial s} \right) &= w(w+1) \sum_{k=0}^{\infty} (\lambda_k + s(s-1))^{-w-2}. \end{aligned}$$

Differentiating with respect to w and evaluating at $w = 0$, where the expression above converges, we get:

$$\frac{\partial}{\partial s}\left(\frac{1}{2s-1}\frac{\partial}{\partial s}(-\log\det(\Delta + s(s-1)))\right) = (2s-1)\sum_{k=0}^{\infty}(\lambda_k + s(s-1))^{-2}.$$

We will also want to know the asymptotics of $\det(\Delta + s(s-1))$ as $s \to \infty$. We have that there are constants $\tilde{\alpha}$ and $\tilde{\beta}$ such that:

$$\tilde{\theta}(t) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \sim \frac{\tilde{\alpha}}{t} + \tilde{\beta} + O(t), \quad t \to 0.$$

This impliës:

$$\begin{aligned}\frac{\partial H}{\partial w}(0,s) &= \frac{\partial}{\partial w}|_{w=0}\frac{1}{\Gamma(w)}\int_0^{\infty}\theta(t)e^{-s(s-1)t}t^w\frac{dt}{t} \\ &= \int_0^1 f(t)e^{-ts(s-1)}dt - \tilde{\alpha}s(s-1) + \tilde{\alpha}s(s-1)\log((s(s-1))) \\ &\quad - \tilde{\beta}\log(s(s-1)) - \tilde{\alpha}s(s-1)\int_{s(s-1)}^{\infty} e^{-y}\frac{dy}{y^2} - \tilde{\beta}\int_{s(s-1)}^{\infty} e^{-y}\frac{dy}{y^2}\end{aligned}$$

with $f(t) = \frac{1}{t}(\tilde{\theta}(t) - \frac{\tilde{\alpha}}{t}\beta)$. So we have that all the integrals goes to 0 as $s \to \infty$. Hence

$$\begin{aligned}-\log\det(\Delta + s(s-1)) = -\tilde{\alpha}s(s-1) + \tilde{\alpha}s(s-1)\log((s(s-1))) \\ - \tilde{\beta}\log(s(s-1)) + o(1) \text{ as } s \to \infty,\end{aligned}$$

where the constants $\tilde{\alpha} = g - 1$ and $\tilde{\beta} = -\frac{g-1}{12}$.

Next, we will use the trace formula and Barnes's digamma function to construct another solution to the differential equation with the same asymptotics.

The trace formula for $h = \frac{1}{r^2+(s-\frac{1}{2})^2} - \frac{1}{r^2+\beta^2}$ gives:

$$\begin{aligned}\frac{1}{2s-1}\frac{Z'}{Z}(s) = \frac{1}{2\beta}\frac{Z'}{Z}(\frac{1}{2}+\beta) + \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n + s(s-1)} - \frac{1}{\lambda_n - \frac{1}{4} + \beta^2}\right) \\ + (2g-2)\sum_{k=0}^{\infty}\left(\frac{1}{\beta + \frac{1}{2} + k} - \frac{1}{s+k}\right).\end{aligned}$$

We have also that the Barnes digamma function, $\Gamma_2(s)$, fulfills:

$$\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\left(-\log\frac{\Gamma_2(s)^2(2\pi)^s}{\Gamma(s)}\right) = \frac{1}{2(s-\frac{1}{2})^2} + \sum_{k=1}^{\infty}\frac{1}{(k+s-\frac{1}{2})^2}.$$

Therefore we have that:

$$\frac{\partial}{\partial s}\left(\frac{1}{2s-1}\frac{\partial}{\partial s}\frac{Z'}{Z}(s)+\frac{d}{ds}\log\left(\frac{\Gamma_2(s)^2(2\pi)^s}{\Gamma(s)}\right)^{2g-2}\right)=\sum_{n=0}^{\infty}\frac{2s-1}{(\lambda_n+s(s-1))^2}$$
$$=\frac{\partial}{\partial s}(\frac{1}{2s-1}\frac{\partial}{\partial s}\log\det(\Delta+s(s-1)).$$

And we conclude

$$\det(\Delta+s(s-1))=e^{E+Fs(s-1)}Z(s)\frac{\Gamma_2(s)^2(2\pi)^s}{\Gamma(s)},$$

with $F=-(2g-2)$ and $E=(2g-2)(2\zeta'(-1)'\frac{1}{4}-\frac{1}{2}\log 2\pi)$. ■

Chapter 3

The Moduli Space of Stable Vector Bundles of Rank n

In this chapter, we follow the article [Zograf and Takhtadzhyan, 1989] fairly closely as their results are central to our further thinking and calculations. First, we introduce coordinates for the moduli space of stable holomorphic bundles of rank n and degree k, $\mathcal{M}_{n,k}$, over the Riemann surface, X. The coordinates, we explain, are complex analytic with respect to the complex structure induced on $\mathcal{M}_{n,k}$ from X. After introducing the coordinates we introduce a connection, which is referred to as a the Lie derivative in [Zograf and Takhtadzhyan, 1989]. This allows us to make calculations, and we present the first and second variation of the metric following [Zograf and Takhtadzhyan, 1989], observing however, that the connection, if composed with an appropiate projection, becomes the Levi-Civita connection on $\mathcal{M}_{n,k}$. We end the chapter with presenting the calculation of the Ricci potential in [Zograf and Takhtadzhyan, 1989]. The Ricci potential is a central object in the rest of the thesis.

3.1 Coordinates on $\mathcal{M}_{n,k}$

As we have seen earlier, $T_E\mathcal{M}_{n,k}$ is naturally identified with

$$H^1(X, \mathrm{Ad}P(\rho)) \cong \mathcal{H}^{0,1}(X, \mathrm{End}E),$$

the space of harmonic $(0,1)$-forms, through Kodaira-Spencer theory (see Chapter 1 Proposition 1.1.8). In this section we explain the construction of coordinates in [Zograf and Takhtadzhyan, 1989]. Fix some vectorbundle E over a compact Riemann surface X, we call the differential manifold underlying X for Σ. In a neighborhood of E we now construct coordinates for $\mathcal{M}_{n,k}$.

We choose to work with an orbitfold n-fold covering of Σ, where the only singularity is over a fixed point p. This can be represented as a quotient of $\mathbb{H}$. This quotient is best described by a representation of the central extension of

$\pi_1(\Sigma)$ by an element, $\tilde{\gamma}$, of order n, we denote the extention $\tilde{\pi}_1(\Sigma)$. We then have a representation $\rho_{\mathbb{H}} : \tilde{\pi}_1(\Sigma) \to \mathbf{PSL}(2, \mathbb{R})$. The image under $\rho_{\mathbb{H}}$ of $\tilde{\gamma}$ is an elliptic element which has a unique fixpoint we denote $\tilde{z}$. Now our bundle E can be described by Narasimhan and Seshadri's theorem as an irreducible $U(n)$ representation

$$\rho : \tilde{\pi}_1(\Sigma) \to \mathbf{U}(n),$$

with $\rho(\tilde{\gamma}) = e^{\frac{-2\pi i k}{n}} I$, we call these representations admissible.

To construct the coordinates we find maps $f^\nu : \mathbb{H} \to \mathbf{GL}(n, \mathbb{C})$ for each $\nu \in \mathcal{H}^{0,1}(X, \mathrm{End}E)$ such that:

1. $\bar{\partial} f^\nu = f^\nu \nu$.
2. $\rho_{E^\nu}(\gamma) = f^\nu(\gamma z)\rho_E(\gamma)(f^\nu(z))^{-1}$ where both ρ's are admissible representations corresponding to vector bundles, and the result is independent of z.
3. $f^\nu(\tilde{z})$ is a positive definite matrix with determinant 1.

These f^ν can be constructed in two steps. First, find the antiholomorphic solution to the differential equation $\bar{\partial} f^\nu = f^\nu \nu$ that is I at $\tilde{z}$. It is possible to do this since ν is closed. We call this function f_-^ν. For small ν we have that the bundle given by the representation $\chi(\gamma) = f_-^\nu(\gamma z)\rho_E(\gamma)(f_-^\nu(z))^{-1}$ corresponds to a stable bundle and therefore to an admissible representation. This is so, since stability is an open condition and we have a differentiable family of holomorphic vector bundles. For ν close to zero the representation corresponds to an admissible representation, and so we can find a bundle isomorphism between the two bundles. This isomorphism is given by a holomorphic function $f_+ : \mathbb{H} \to \mathbf{GL}(n, \mathbb{C})$ conjugating χ_ν to ρ_{E^ν}, the corresponding admissible representation. The function $f_+^\nu \cdot f_-^\nu$ conjugates ρ to an admissible representation for E_{ρ^ν}. However this is not unique, since so does $U f_+^\nu \cdot f_-^\nu g$ for any unitary matrix U and function $g \in C^\infty(\mathbb{H})$. Now the Cartan decomposition decomposes a matrix as a unique unitary matrix times a positive definite matrix, hence the requirement that f^ν is a positive definite matrix at $\tilde{z}$ fixes U's value. And when we further require the determinant to be one at $\tilde{z}$, there is only one choice of g such that $f^{\varepsilon\nu} \to I$ for $\varepsilon \to 0$.

Finally, we note that had we chosen ρ as an $\mathbf{SU}(n)$-representation and our $\nu \in \mathcal{H}^{0,1}(X, \mathrm{End}_0 E)$, where $\mathrm{End}_0 E$ are the traceless endomorphisms, then this construction provides coordinates on the moduli space of holomorphic vector bundles of rank n and degree k with fixed determinant.

3.1.1 Tangent Vectors

Having introduced coordinates we get a local identification around $E \in U$ of $T_{[\rho^\varepsilon]}\mathcal{M}_{n,k}$ with $\mathcal{H}^{0,1}(X, \mathrm{End}E)$. From Kodaira-Spencer theory we also have that $T_{[\rho^\varepsilon]}\mathcal{M}_{n,k}$ is identified with $\mathcal{H}^{0,1}(X, \mathrm{End}E_{\rho^\varepsilon})$. These two identifications agree at $\varepsilon = 0$, and so we can use the Kodaira-Spencer way of identifying the tangent vector to verify that the complex structure the coordinates have from $\mathcal{H}^{0,1}(X, \mathrm{End}E)$ agree with the complex structure from Section 1.1.5. We will now find the identification our coordinates give with the Kodaira-Spencer identification of tangent vectors at a point ρ^{ν_2}. The tangent vector ν_1 at ρ^{ν_2} is the tangent vector of the curve $\rho^{\nu_2+\varepsilon\nu_1}$ at $\varepsilon = 0$. We calculate the Kodaira-Spencer class of the curve $\rho^{\nu_2+\varepsilon\nu_1}$ at $\varepsilon = 0$:

We choose a finite covering of a neighborhood of a fundamental domain for X in $\mathbb{H}$, such that no open set in the cover contains $\pi_1(X)$ equivalent points. Let (U_i, φ_i) be the chart corresponding to the i'th open set in the covering. Then if $U_{ij} = U_i \cap U_j \neq \emptyset$ there is an element of Γ such that $\rho_{\mathbb{H}}(\gamma_{ij})\varphi_i U_{ij} = \varphi_j U_{ij}$. The transition functions for the bundle $E_{\rho^{t\nu_1+\nu_2}}$ from $U_{ij} \times \mathbb{C}^n$ in the chart U_i to U_j is $\rho_{\mathbb{H}}(\gamma_{ij}) \times \rho^{\varepsilon\nu_1+\nu_2}(\gamma_{ij})$. And so the Kodaira-Spencer class is given by the 1-cocycle $\theta_{ij}(t) = \frac{d}{d\varepsilon}\big|_{\varepsilon=t} \rho^{\varepsilon\nu_1+\nu_2}(\gamma_{ij})$ in the sheaf of holomorphic sections of $\mathrm{End}E_{\rho^{t\nu_1}}$. Using the short exact sequence of sheafs:

$$\mathcal{O}(\mathrm{End}E_{\rho^{t\nu_1+\nu_2}}) \to \Omega^0(\mathrm{End}E_{\rho^{t\nu_1+\nu_2}}) \xrightarrow{\bar{\partial}} \Omega^{(0,1)}(\mathrm{End}E_{\rho^{t\nu_1+\nu_2}}), \tag{3.1}$$

where $\Omega^{(p,q)}(F)$ is the germ of smooth (p,q)-forms with values in the bundle F, and $\mathcal{O}(F)$ is the germ of holomorphic functions with values in F, we get a harmonic $(0,1)$-form representing the Kodaira-Spencer class. The connecting homomorphism of the cohomology groups

$$H^1(X, \mathcal{O}(\mathrm{End}E_{\rho^{t\nu_1+\nu_2}})) \xrightarrow{\delta^*} H^{0,1}(X, \mathrm{End}E_{\rho^{t\nu_1+\nu_2}}) \overset{P^{t\nu_1+\nu_2}}{\cong} \mathcal{H}^{0,1}(X, \mathrm{End}E_{\rho^{t\nu_1+\nu_2}})$$

is an isomorphism. Let $g_i = \{\frac{d}{d\varepsilon}\big|_{\varepsilon=t} f^{\varepsilon\nu_1+\nu_2}\}(f^{t\nu_1+\nu_2})^{-1}\varphi_i$ be a section over $U_i \times \mathbb{C}^n$. Then $\rho^{t\nu_1+\nu_2}(\gamma_{ij})(g_i \circ \rho_{\mathbb{H}}(\gamma_{ij}))\rho^{t\nu_1+\nu_2}(\gamma_{ij})^{-1} - g_j = \theta_{ij}$. This implies that

$$\begin{aligned}
\delta^*(\theta_{ij}) &= \bar{\partial}\left(\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=t} f^{\varepsilon\nu_1+\nu_2}\right)\left(f^{t\nu_1+\nu_2}\right)^{-1}\right) \circ \varphi_i \\
&= \mathrm{Ad}(f^{\nu_2})\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=t} ((f^{\varepsilon\nu_1+\nu_2})^{-1}(\bar{\partial} f^{\varepsilon\nu_1+\nu_2})) \circ \varphi_i = \mathrm{Ad}(f^{\nu_2})\nu_1 \circ \varphi_i.
\end{aligned}$$

Now we have that the tangent vector ν_1 corresponds to $P^{\nu_2}(\mathrm{Ad}(f^{\nu_2})\nu_1)$ at ν_2. Since the map $\nu_1 \to P^{\nu_2}(\mathrm{Ad}(f^{\nu_2})\nu_1)$ is complex linear we see that the two complex structures agree, and we have constructed complex analytic coordinates.

3.2 Derivatives

We want to calculate the curvature of the canonical metric on $\mathcal{M}_{n,k}$. In order to do this we need to calculate the derivatives locally. It is convenient to introduce the connection L on sections of $\Omega^{p,q}(X, \mathrm{End}E_{\rho^\varepsilon})$ over $\mathcal{M}_{n,k}$ given by

$$L_\nu g = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} (f^{\varepsilon\nu(0)})^{-1} g(\varepsilon) f^{\varepsilon\nu(0)}, \tag{3.2}$$

$$L_{\bar\nu} g = \frac{\partial}{\partial \bar\varepsilon}\bigg|_{\varepsilon=0} (f^{\varepsilon\nu(0)})^{-1} g(\varepsilon) f^{\varepsilon\nu(0)}, \tag{3.3}$$

at the point E_{ρ^0}. This gives a connection on operators on these bundles as well. Let

$$F^\varepsilon : \Omega^{p,q}(X, \mathrm{End}E_{\rho^\varepsilon}) \to \Omega^{p',q'}(X, \mathrm{End}E_{\rho^\varepsilon})$$

then we have:

$$L_\nu F^\varepsilon|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1} F^\varepsilon \mathrm{Ad}(f^{\varepsilon\nu}). \tag{3.4}$$

Lemma 3.2.1
The operator $L : \Omega^{p,q}(X, \mathrm{End}E_{\rho^\varepsilon}) \otimes \mathcal{T}(\mathcal{M}_{n,k}) \to \Omega^{p,q}(X, \mathrm{End}E_{\rho^\varepsilon})$ *is a connection.*

PROOF: We need to verify three equations

$$\begin{aligned} &L_\nu a g = a L_\nu g \quad a \in \mathbb{R} \\ &L_\nu h g = h L_\nu g + \nu(h) \quad h \in C^\infty(\mathcal{M}_{n,k}) \\ &L_{h\nu} g = h L_\nu g \quad h \in C^\infty(\mathcal{M}_{n,k}). \end{aligned}$$

Choose a basis $\{\nu_i\}$ for $\mathcal{H}^{0,1}(X, \mathrm{End}E)$, then we denote a point in $\mathcal{M}_{n,k}$ in these coordinates by the vector ε. Also let ν be a vector valued function corresponding to the vector field $\hat\nu = \sum_i \nu_i(\varepsilon)\nu_i$. In the following $\varepsilon\hat\nu(0) = \sum_i \nu_i(0)\nu_i\varepsilon_i$.

First

$$L_\nu a g = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} (f^{\varepsilon\hat\nu(0)})^{-1} a g(\varepsilon\nu(0)) f^{\varepsilon\hat\nu(0)} = a L_\nu g,$$

and secondly,

$$\begin{aligned} L_\nu h g = &\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} (f^{\varepsilon\hat\nu(0)})^{-1} h(\varepsilon\nu(0)) g(\varepsilon\nu(0)) f^{\varepsilon\hat\nu(0)} \\ =& h(0) \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} (f^{\varepsilon\hat\nu(0)})^{-1} g(\varepsilon) f^{\varepsilon\hat\nu(0)} + \left(\frac{\partial}{\partial \varepsilon\nu(0)}\bigg|_{\varepsilon=0} h(\varepsilon\nu(0)) \right) g \\ =& h(0) L_\nu g + \nu(h)(0). \end{aligned}$$

The third equation follows directly from the definition

$$L_{h\nu} g = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} (f^{\varepsilon h(0)\hat\nu(0)})^{-1} g(\varepsilon h(0)\nu(0)) f^{\varepsilon h(0)\hat\nu(0)} = h(0) L_\nu g.$$

This shows L is a connection. ■

Now consider the universal bundle of endomorphisms $\mathcal{E} \to X \times \mathcal{M}_{n,k}$. This bundle has a canonical metric given by:

$$h_{\mathcal{E}}(\xi, \eta) = \mathrm{tr}(\xi \overline{\eta}^T) \text{ for } \xi, \eta \in \Omega^0(X, \mathrm{End} E_\rho).$$

The following lemma follows is a reformulation of the first half of [Zograf and Takhtadzhyan, 1989, Lemma 1], in our language:

Lemma 3.2.2
For $h_{\mathcal{E}}$, $\nu \in \mathcal{H}^{1,0}(X, \mathrm{End} E)$ and $\xi, \eta \in \Omega^0(X, \mathrm{End} E)$we have:

$$(L_\nu h_{\mathcal{E}})(\xi, \eta) = L_{\bar{\nu}} h_{\mathcal{E}}(\xi, \eta) = 0.$$

PROOF: By definition

$$\begin{aligned} L_\nu h_{\mathcal{E}}(\xi, \eta) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathcal{E}}(\mathrm{Ad}(f^{\varepsilon\nu})\xi, \mathrm{Ad}(f^{\varepsilon\nu})\eta) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{tr}(f^{\varepsilon\nu}\xi(f^{\varepsilon\nu})^{-1}(\overline{(f^{\varepsilon\nu})^{-1}})^T \overline{\eta^T (f^{\varepsilon\nu})^T}) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{tr}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu} \xi (\overline{(f^{\varepsilon\nu})}^T f^{\varepsilon\nu})^{-1} \overline{\eta^T}) = tr(\mathrm{ad}(\Phi_\nu)(\xi)\bar{\eta}^T), \end{aligned}$$

where $\Phi_\nu = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \overline{(f^{\varepsilon\nu})}^T f^{\varepsilon\nu}$. We see that:

$$\begin{aligned} \Phi_\nu(\gamma z) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{(f^{\varepsilon\nu})}^T (\gamma z) f^{\varepsilon\nu}(\gamma z) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{(\rho_{\varepsilon\nu}(\gamma)(f^{\varepsilon\nu})\rho(\gamma)^{-1})}^T (z) \rho_{\varepsilon\nu}(\gamma) f^{\varepsilon\nu}(z) \rho(\gamma)^{-1} \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \rho(\gamma) \overline{(f^{\varepsilon\nu})}^T (z) f^{\varepsilon\nu}(z) \rho(\gamma)^{-1} = \rho(\gamma)\Phi_\nu \rho(\gamma)^{-1}, \end{aligned}$$

since the representation is unitary. This transformation relationship means that Φ_ν descends to an element of $\Omega^0(X, \mathrm{End} E)$. Furthermore, after carrying out the differentiation we see, that Φ_ν is a sum of holomorphic and anti-holomorphic functions and therefore is a harmonic function. But $\mathrm{End} E = \mathrm{ad} E \oplus \mathbb{C} I$, and $\mathrm{ad} E$ is stable and therefore has no holomorphic sections nor any antiholomorphic sections. The surface X is compact, this implies that Φ_ν is a constant multiple of I, but condition 2 on the $f^{\varepsilon\nu}$ requires determinant 1 and hence $\mathrm{tr}\Phi^\nu = \frac{d}{d\varepsilon}_{\varepsilon=0} \det(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu}(\tilde{z})) = \frac{d}{d\varepsilon} 1 = 0$. This proves that the first variation of $L_\nu h_{\mathcal{E}}$ is zero. The antiholomorphic variation is calculated similarly. ■

The fact that $\Phi_\nu = 0$ gives us useful information about $f_+^{\varepsilon\nu}$.

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} ((\overline{f^{\varepsilon\nu}})^T f^{\varepsilon\nu}) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\overline{f_-^{\varepsilon\nu}})^T + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\overline{f_+^{\varepsilon\nu}})^T$$

$$+ \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f_+^{\varepsilon\nu} + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f_-^{\varepsilon\nu},$$

and

$$\bar{\partial}\Phi_\nu = \overline{(\left.\frac{d}{d\bar{\varepsilon}}\right|_{\varepsilon=0} \partial f_+^{\varepsilon\nu})^T} + \nu,$$

hence $\left.\frac{d}{d\bar{\varepsilon}}\right|_{\varepsilon=0} \partial f_+^{\varepsilon\nu} = -\bar{\nu}^T = -i * \nu$. Similarly we have $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \partial f_+^{\varepsilon\nu} = 0$. From this it follows that:

$$\begin{aligned}
&\left(\frac{\partial}{\partial z}\left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} \mathrm{Ad} f^{\varepsilon\nu}\right)\eta = \frac{\partial}{\partial z}\left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} f^{\varepsilon\nu}\eta(f^{\varepsilon\nu})^{-1} \\
&= \left(\frac{\partial}{\partial z}\left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} f^{\varepsilon\nu}\right)\eta(f^{0\nu})^{-1} - f^{0\nu}\eta(f^{0\nu})^{-1}\left(\frac{\partial}{\partial z}\left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} f^{\varepsilon\nu}\right)(f^{0\nu})^{-1} \\
&= -i\mathrm{ad} * \nu = -\mathrm{ad}\bar{\nu}^T, \quad (3.5)
\end{aligned}$$

where the second equality follows from $f^0(z) = I$ and $\partial f^0(z) = 0$. This implies that the term differentiated with repect to z also must be differentiated with respect to $\bar{\varepsilon}$, otherwise the contribution is zero.

With this information we can calculate the variation of $\bar{\partial}$ and its adjoint $- * \bar{\partial} * = \bar{\partial}^*$.

$$\begin{aligned}
L_\nu \bar{\partial} &= \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}\bar{\partial}\mathrm{Ad} f^{\varepsilon\nu} \\
&= \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}\mathrm{Ad} f^{\varepsilon\nu}\bar{\partial} \\
&+ \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}\left(\bar{\partial}\mathrm{Ad} f^{\varepsilon\nu}\right) \\
&= \left(\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}\right)\left(\bar{\partial}\mathrm{Ad} f^{0\nu}\right) \\
&+ (\mathrm{Ad} f^{0\nu})^{-1}\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \left(\bar{\partial}\mathrm{Ad} f^{\varepsilon\nu}\right) \\
&= 0 + \mathrm{ad}(\nu),
\end{aligned}$$

where the first term is zero, since $\bar{\partial}\mathrm{Ad} f^{0\nu} = \bar{\partial}\mathrm{Ad} I = 0$. We find

$$\begin{aligned}
L_{\bar{\nu}} \bar{\partial}^* &= \left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}\bar{\partial}^*\mathrm{Ad} f^{\varepsilon\nu} \\
&= \left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}\mathrm{Ad}(f^{\varepsilon\nu})\bar{\partial}^* \\
&+ \left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}(-*)\left(\bar{\partial}\mathrm{Ad}\overline{(f^{\varepsilon\nu})^{-1}}^T\right) *
\end{aligned}$$

$$
\begin{aligned}
&= (\mathrm{Ad}f^{0\nu})^{-1}(-*) \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \left(\bar\partial \mathrm{Ad}\overline{(f^{\varepsilon\nu})^{-1}}^T\right) * \\
&= - * \overline{\left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} (\partial \mathrm{Ad}((f^{\varepsilon\nu})^{-1})^T)}* \\
&= - * \,\mathrm{ad}\nu * .
\end{aligned}
$$

The last equality follows from (3.5), $\partial(\mathrm{Ad}(f^{0\nu})^{\pm 1}) = 0$ and the calculation:

$$
\begin{aligned}
0 =& \frac{\partial}{\partial z} \left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} (\mathrm{Ad}(f^{\varepsilon\nu})\mathrm{Ad}(f^{\varepsilon\nu})^{-1}) \\
=& \left(\frac{\partial}{\partial z} \left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})\right) \mathrm{Ad}(f^{0\nu})^{-1} + \left(\frac{\partial}{\partial z}\mathrm{Ad}(f^{0\nu})\right) \left(\left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}\right) \\
+& \left(\left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})\right) \left(\frac{\partial}{\partial z}\mathrm{Ad}(f^{0\nu})^{-1}\right) + \mathrm{Ad}(f^{0\nu}) \left(\frac{\partial}{\partial z} \left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}\right) \\
=& - \mathrm{ad}\bar\nu^T + \left(\frac{\partial}{\partial z} \left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}\right).
\end{aligned}
$$

Similarly we find

$$L_{\bar\nu}\bar\partial = L_\nu \bar\partial^* = 0.$$

Now we have all the information we need to calculate the variation of the Laplace operator and the harmonic projection.

$$
\begin{aligned}
L_\nu \Delta &= L_\nu \bar\partial^* \bar\partial = \bar\partial^* \mathrm{ad}\nu, \\
L_{\bar\nu} \Delta &= L_{\bar\nu} \bar\partial^* \bar\partial = - * \mathrm{ad}\nu * \bar\partial, \\
L_\nu P &= L_\nu (I - \bar\partial \Delta_0^{-1} \bar\partial^*) = -\mathrm{ad}\nu \Delta_0^{-1}\bar\partial^* + \bar\partial\Delta_0^{-1}\bar\partial^* \mathrm{ad}\nu \Delta_0^{-1}\bar\partial^* \\
&= -P \mathrm{ad}\nu \Delta_0^{-1} \bar\partial^*
\end{aligned}
$$

and

$$L_{\bar\nu} P = \bar\partial \Delta_0^{-1} * \mathrm{ad}\nu * P.$$

Finally, we can differentiate $\tilde\nu_1(\varepsilon) = P^{\varepsilon\nu_2}(\mathrm{Ad}f^{\varepsilon\nu_2}\nu_1)$ as a family of elements of $\Omega^{(0,1)}(X, \mathrm{End}E_{\rho^{\varepsilon\nu_2}})$.

$$
\begin{aligned}
L_{\bar{\nu_2}} \tilde{\nu_1} &= \left.\frac{\partial}{\partial \bar\varepsilon}\right|_{\varepsilon=0} (f^{\varepsilon\nu_2})^{-1} P^{\varepsilon\nu_2}(f^{\varepsilon\nu_2}\nu_1(f^{\varepsilon\nu_2})^{-1}) f^{\varepsilon\nu_2} && (3.6) \\
&= (L_{\bar{\nu_2}}P)(\nu_1) = \bar\partial \Delta_0^{-1}(*[*\nu_1, \nu_2]), && (3.7)
\end{aligned}
$$

and likewise we find

$$L_{\nu_2}\tilde{\nu_1} = (L_{\nu_2}P)(\nu_1) = -P\mathrm{ad}\nu_2\Delta_0^{-1}\partial^*\nu_1 = 0. \qquad (3.8)$$

Now, if we consider the connection $\nabla = P^{0,1}L_v$ on $\mathcal{H}^{0,1}(X, \mathrm{End}E) \cong T_E\mathcal{M}_{n,k}$, then from Lemma 3.2.2 we see that ∇ is compatible with the metric. From (3.6) and (3.8) it follows that ∇ is torsion free, since for coordinate vector fields $\nabla\tilde\nu = 0$ for $\nu \in \mathcal{H}^{0,1}(X, \mathrm{End}E)$, and so $\nabla_{\nu_1}\tilde\nu_2 - \nabla_{\nu_2}\tilde\nu_1 - [\nu_1, \nu_2] = 0$ at E. This proves:

Proposition 3.2.3
The connection $\nabla = P^{0,1}L_v : T_E\mathcal{M}_{n,k} \times \Gamma(T\mathcal{M}_{n,k}) \to \Gamma(T\mathcal{M}_{n,k})$ *is well defined on* $\mathcal{M}_{n,k}$ *and is the Levi-Civita connection for the Kähler metric.*

3.3 The Kähler Metric on the Moduli Space

We have the metric, $h_{\mathcal{E}}$, on fibers of End$E$. This fiber metric induces a metric on the tangent bundle of $\mathcal{M}_{n,k}$ as follows. Let $\nu_1(z)d\bar{z}, \nu_2(z)d\bar{z} \in \mathcal{H}^{0,1}(X, \mathrm{End}E)$ then

$$g(\nu_1(z)d\bar{z}, \nu_2(z)d\bar{z}) = \int_X \mathrm{tr}(\nu_1(z)d\bar{z} \wedge *(\nu_2(z)d\bar{z})) = \int_X h_{\mathcal{E}}(\nu_1(z), \nu_2(z))\frac{dzd\bar{z}}{i}. \tag{3.9}$$

We now look at a coordinate neighborhood of E and choose a basis $\{\nu_i\}$ of $\mathcal{H}^{0,1}(X, \mathrm{End}E)$. We will write a point in coordinates as $\varepsilon\nu = \sum_i \varepsilon_i\nu_i$. The tangent vector in coordinates corresponding to ν_i will be written ∂_i for the holomorphic and $\partial_{\bar{i}}$ for the antiholomorphic derivative. In these coordinates the metric is:

$$\begin{aligned} g_{i\bar{j}}(\varepsilon\nu) &= g(\partial_i, \partial_{\bar{j}})(\varepsilon\nu) \\ &= \int_X h_{\mathcal{E}}^{\varepsilon\nu}(\tilde{\nu}_i^{\varepsilon\nu}, \tilde{\nu}_j^{\varepsilon\nu}) = \int_X h_{\mathcal{E}}^{\varepsilon\nu}(P^{\varepsilon\nu}\mathrm{Ad}(f^{\varepsilon\nu})\nu_i, P^{\varepsilon\nu}\mathrm{Ad}(f^{\varepsilon\nu})\nu_j) \\ &= \int_X h_{\mathcal{E}}^{\varepsilon\nu}(P^{\varepsilon\nu}\mathrm{Ad}(f^{\varepsilon\nu})\nu_i, \mathrm{Ad}(f^{\varepsilon\nu})\nu_j) \\ &= \int_X h_{\mathcal{E}}^{0}(\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T})P^{\varepsilon\nu}\mathrm{Ad}(f^{\varepsilon\nu})\nu_i, \nu_j), \end{aligned}$$

since P is an orthogonal projection.

Inspired by [Zograf and Takhtadzhyan, 1989, Lemma 1] we prove a slightly reformulated version:

Lemma 3.3.1
The metric g *on* $\mathcal{M}_{n,k}$ *is a Kähler metric and in coordinates around* E *we have*

$$\partial_k g_{i\bar{j}}(0) = \partial_{\bar{k}} g_{i\bar{j}}(0) = 0.$$

And the second-order derivatives are:

$$\begin{aligned} \partial_k\partial_{\bar{l}} g_{i\bar{j}}(0) = &- \int_X h_{\mathcal{E}}([\nu_k, \Delta_0^{-1}(*[*\nu_i, \nu_l])], \nu_j) \\ &+ \int_X h_{\mathcal{E}}(\mathrm{ad}(\Delta_0^{-1}(*[*\nu_k, \nu_l]))(\nu_i), \nu_j) \end{aligned}$$

Proof: We know $\frac{d}{d\varepsilon}|_{\varepsilon=0}\overline{(f^{\varepsilon\nu})^T}f^{\varepsilon\nu} = 0$, and so we find:

$$\partial_k g_{i\bar{j}}(0) = \frac{\partial}{\partial\varepsilon_k}\bigg|_{\varepsilon=0} \int_X h_{\mathcal{E}}^{0}(\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T})P^{\varepsilon\nu}\mathrm{Ad}(f^{\varepsilon\nu})\nu_i, \nu_j)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \varepsilon_k}\Big|_{\varepsilon=0} \int_X h^0_{\mathcal{E}}(\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})\mathrm{Ad}(f^{\varepsilon\nu})^{-1}\tilde{\nu}_i^{\varepsilon\nu}, \nu_j) \\
&= \int_X h^0_{\mathcal{E}}(\frac{\partial}{\partial \varepsilon_k}\Big|_{\varepsilon=0} \mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})\tilde{\nu}_i^0, \nu_j) \\
&\quad + \int_X h^0_{\mathcal{E}}(\frac{\partial}{\partial \varepsilon_k}\Big|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}\tilde{\nu}_i^{\varepsilon\nu}, \nu_j) \\
&= \int_X h^0_{\mathcal{E}}(\bar{\partial}\Delta_0^{-1} * [*\nu_i, \nu_k], \nu_j) = 0.
\end{aligned}
$$

Since ν_j is in the orthogonal complement of the image of $\bar{\partial}$.

It only remains to calculate:

$$
\begin{aligned}
\partial_k \partial_{\bar{l}} g_{i\bar{j}}(0) &= \frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} \int_X h^0_{\mathcal{E}}(\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})\mathrm{Ad}(f^{\varepsilon\nu})^{-1}\tilde{\nu}_i^{\varepsilon\nu}, \nu_j) \\
&= \int_X h^0_{\mathcal{E}}(\frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} (\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu}))\nu_i, \nu_j) \\
&\quad + \int_X h^0_{\mathcal{E}}(\frac{\partial}{\partial \varepsilon_k}\Big|_{\varepsilon=0} (\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})) \frac{\partial}{\partial \bar{\varepsilon}_l}\Big|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}\tilde{\nu}_i^{\varepsilon\nu}, \nu_j) \\
&\quad + \int_X h^0_{\mathcal{E}}(\frac{\partial}{\partial \bar{\varepsilon}_l}\Big|_{\varepsilon=0} (\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})) \frac{\partial}{\partial \varepsilon_k}\Big|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}\tilde{\nu}_i^{\varepsilon\nu}, \nu_j) \\
&\quad + \int_X h^0_{\mathcal{E}}(\frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}\tilde{\nu}_i^{\varepsilon\nu}, \nu_j) \\
&= \int_X h^0_{\mathcal{E}}(\frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} (\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu}))\nu_i, \nu_j) \\
&\quad + \int_X h^0_{\mathcal{E}}(\frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} \tilde{\nu}_i^{\varepsilon\nu}, \nu_j).
\end{aligned}
$$

We calculate the first term. Since the first derivatives at $\varepsilon\nu = 0$ of $\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})$ are zero and $\overline{(f^0)^T} f^0) = I$, we have

$$
\frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} (\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})) = \mathrm{ad}\ \frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} (\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu}).
$$

Define $\Phi_{\nu_k, \bar{\nu}_l} = \frac{d^2}{d\bar{\varepsilon}_l d\varepsilon_k}\Big|_{\varepsilon=0} \overline{(f^{\varepsilon_l\nu_l + \varepsilon_k\nu_k})}^T f^{\varepsilon_l\nu_l + \varepsilon_k\nu_k}$. Then we find that

$$
\begin{aligned}
\Delta\Phi_{\nu_k, \bar{\nu}_l} &= -2y^2 \frac{d^2}{dz d\bar{z}} \frac{d^2}{d\bar{\varepsilon}_l d\varepsilon_k}\Big|_{\varepsilon=0} \overline{(f^{\varepsilon_l\nu_l + \varepsilon_k\nu_k})}^T f^{\varepsilon_l\nu_l + \varepsilon_k\nu_k} \\
&= -2y^2(-\overline{\nu_l^T}\nu_k - \overline{\nu_l^T}\nu_k + \overline{\nu_l^T}\nu_k + \nu_k\overline{\nu_l^T}) = -2y^2[\nu_k, \bar{\nu}_l^T].
\end{aligned}
$$

So $\Phi_{\nu_k, \bar{\nu}_l} = -\Delta_0^{-1}(*[*\nu_k, \nu_l]) + \varphi_{\nu_k, \bar{\nu}_l}$, with $\varphi_{\nu_k, \bar{\nu}_l}$ a harmonic function. Since $\varphi_{\nu_k, \bar{\nu}_l}$ is harmonic it must be a constant multiple of I. And since $\mathrm{ad}(I) = 0$ we have

$$
\frac{\partial^2}{\partial \bar{\varepsilon}_l \partial \varepsilon_k}\Big|_{\varepsilon=0} (\mathrm{Ad}(\overline{(f^{\varepsilon\nu})^T} f^{\varepsilon\nu})) = -\mathrm{ad}\Delta_0^{-1} * [*\nu_k, \nu_l].
$$

Having calculated the first term, let us continue with the second term. We get

$$\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon_k\partial\bar{\varepsilon}_l}|_{\varepsilon=0}(\mathrm{Ad}f^{\sum_i\varepsilon_i\nu_i})^{-1}P^{\sum_i\varepsilon_i\nu_i}\mathrm{Ad}f^{\sum_i\varepsilon_i\nu_i}\\
&=\frac{\partial^2}{\partial\varepsilon_k\partial\bar{\varepsilon}_l}|_{\varepsilon=0}(I-(\mathrm{Ad}f^{\sum_i\varepsilon_i\nu_i})^{-1}\bar{\partial}^{\sum_i\varepsilon_i\nu_i}(\Delta_0^{-1})^{\sum_i\varepsilon_i\nu_i}(\bar{\partial}^*)^{\sum_i\varepsilon_i\nu_i}\mathrm{Ad}f^{\sum_i\varepsilon_i\nu_i})\\
&=\mathrm{ad}(\nu_k)\Delta_0^{-1}*\mathrm{ad}(\nu_l)*-\mathrm{ad}(\nu_k)\Delta_0^{-1}*\mathrm{ad}(\nu_l)*\bar{\partial}\Delta_0^{-1}\bar{\partial}^*+\text{terms from }\mathrm{Im}(\bar{\partial})\\
&=\mathrm{ad}(\nu_k)\Delta_0^{-1}*\mathrm{ad}(\nu_l)*P+\text{terms from }\mathrm{Im}(\bar{\partial})\\
&=-(L_{\nu_k}\bar{\partial})\Delta_0^{-1}(L_{\nu_l}\bar{\partial}^*)P+\text{terms from }\mathrm{Im}(\bar{\partial}),
\end{aligned}$$

since $\frac{\partial^2}{\partial\varepsilon_k\partial\bar{\varepsilon}_l}|_{\varepsilon=0}(\mathrm{Ad}f^{\sum_i\varepsilon_i\nu_i})^{-1}\bar{\partial}^{\sum_i\varepsilon_i\nu_i}\mathrm{Ad}f^{\sum_i\varepsilon_i\nu_i}=\frac{\partial^2}{\partial\varepsilon_k\partial\bar{\varepsilon}_l}|_{\varepsilon=0}\sum_i\varepsilon_i\nu_i=0$ and $\frac{\partial}{\partial\bar{\varepsilon}_l}|_{\varepsilon=0}\sum_i\varepsilon_i\nu_i=0$. ■

3.3.1 Geodesic Coordinates and the Curvature Tensor

As we have seen in the proof of Lemma 3.3.1, the first derivatives of the metric are zero at E in the coordinates around E. And so our coordinates are geodesic coordinates for the Kähler metric. This means, that we can calculate the curvature of the metric as

$$\begin{aligned}
R_{i\bar{j}k\bar{l}}(0)&=(\partial_{\nu_k}\bar{\partial}_{\nu_l}g_{i\bar{j}})(0) &(3.10)\\
&=-h((\mathrm{ad}\Delta_0^{-1}(*[*\nu_k,\nu_l])+\mathrm{ad}(\nu_k)\Delta_0^{-1}*\mathrm{ad}(\nu_l)*)\nu_i,\nu_j). &(3.11)
\end{aligned}$$

This can be rewritten as

$$R_{i\bar{j}k\bar{l}}(0)=-\langle\Delta_0^{-1}(*[*\nu_i,\nu_l]),*[*\nu_j,\nu_k]\rangle-\langle\Delta_0^{-1}(*[*\nu_i,\nu_j]),*[*\nu_l,\nu_k]\rangle, \quad (3.12)$$

where $\langle a,b\rangle=\int_X h_{\mathcal{E}}(a,b)\rho, a,b\in\Omega^0(X,\mathrm{End}E)$. Where ρ denote the density of the hyperbolic volume on X.

3.3.2 Variation of the Metric in the Determinant Line Bundle

Choose a basis $\nu_i\in\mathcal{H}^{0,1}(X,\mathrm{End}E_\rho)$. The metric in the determinant line bundle, $\det\mathrm{ind}\bar{\partial}\cong\det T^*N$ on $\nu_1\wedge\nu_2\ldots\wedge\nu_d$, is then given by $\det G$ for $G(E_{\rho^{\varepsilon\nu}})=\{g_{ij}\}_{i,j}$ for g_{ij}, the metric pairing of ν_i and ν_j.

Lemma 3.3.2 ([Zograf and Takhtadzhyan, 1989, Lemma 2]) *For $E\in N$ we have the following formula for the curvature with respect to the basis $\{\nu_i\}$:*

$$\begin{aligned}
\Theta_{\mathrm{End}}(\partial_{\nu_n},\overline{\partial_{(\nu_m)}})&=\Theta_{\mathrm{ad}}(\partial_{\nu_n},\overline{\partial_{(\nu_m)}})\\
&=-tr((\mathrm{ad}\Delta_0^{-1}(*[*\nu_n,\nu_m])+(L_{\nu_n^T}\bar{\partial})\Delta_0^{-1}(L_{\bar{\nu}_m}\bar{\partial}^*))P^{0,1}).
\end{aligned}$$

PROOF: In our coordinates around E the norm of the holomorphic section of the cotangent bundle, $\bigwedge_i *\tilde{\nu}_i$, is given by:

$$|\bigwedge_i *\tilde{\nu}_i|^2 = \det\{G\}.$$

Hence, the curvature form of the metric connection is given by:

$$\begin{aligned}
\Theta_{\mathrm{End}}(\partial_{\nu_n}, \overline{\partial_{(\nu_m)}}) &= \frac{\partial^2}{\partial\varepsilon_n \partial\bar{\varepsilon}_m}|_{\varepsilon=0} \log |\bigwedge_i *\tilde{\nu}_i|^2 = \frac{\partial^2}{\partial\varepsilon_n \partial\bar{\varepsilon}_m}|_{\varepsilon=0} \log\det G \\
&= \frac{\partial}{\partial\varepsilon_n}|_{\varepsilon=0} \mathrm{tr} G^{-1} \frac{\partial}{\bar{\partial}\varepsilon_m}|_{\varepsilon_m=0} G \\
&= \mathrm{tr} G^{-1} \frac{\partial^2}{\partial\varepsilon_n \partial\bar{\varepsilon}_m}|_{\varepsilon=0} G \\
&\quad - G^{-1}\left(\frac{\partial}{\bar{\partial}\varepsilon_m}|_{\varepsilon_m=0} G\right) G^{-1}\left(\frac{\partial}{\partial\varepsilon_n}|_{\varepsilon_m=0} G\right) \\
&= \mathrm{tr} G^{-1} \frac{\partial^2}{\partial\varepsilon_n \partial\bar{\varepsilon}_m}|_{\varepsilon=0} G.
\end{aligned}$$

Where the last equality is a consequence of the first derivatives of g_{ij} being zero. Now we use that $\mathrm{tr}AP = \sum_{i,j} h_E(A\nu_i, \nu_j)(G^{-1})_{j,i}$ and (3.11) which shows:

$$\Theta_{\mathrm{End}}(\partial_{\nu_n}, \overline{\partial_{(\nu_m)}}) = -\mathrm{tr}(\mathrm{ad}(\Delta_o^{-1}(*[*\nu_n, \nu_m]))P) + \mathrm{tr}(\mathrm{ad}(\nu_n)\Delta_0^{-1} * \mathrm{ad}(\nu_m) * P),$$

which concludes the proof. ∎

3.3.3 Green's Function and ζ-regularized Determinants

In this section we recall two results from Chapter 2 that are relevant for our further considerations. First, recall we worked with the Laplace operator on automorphic forms of weight (n, m) with respect to a group Γ and a $\mathbf{U}(n)$-representation χ. If we let Γ be the group $\pi_1(\Sigma)$ represented by deck transformations on the universal cover of Σ, $\mathbb{H}$ and we let χ be a $\mathbf{U}(n)$-representation which corresponds to the vector bundle E, then an automorphic form is equivalent to a section of

$$E \otimes (T_\sigma\Sigma)^{-n} \otimes (\bar{T}_\sigma\Sigma)^{-m}.$$

This means that the Laplace operator on automorphic forms corresponds to the Laplace operator on Σ with values in E. We can pull back the objects on Σ to $\mathbb{H}$, and they will have the following properties as functions on $\mathbb{H}$.

The Green's function of the Laplace operator on Σ is pulled back to a $\mathrm{EndEnd}E$ valued function on $\mathbb{H} \times \mathbb{H}$, with a logarithmic singularity on the diagonal, which transforms as:

$$G(\gamma_1 z, \gamma_2 z') = \mathrm{Ad}(\rho(\gamma_1))G(z, z')\mathrm{Ad}(\rho(\gamma_2))^{-1} \quad \forall z, z' \in \mathbb{H} \quad \forall \gamma_1\gamma_2 \in \tilde{\Gamma}.$$

The function $G(z, z')$ can be expressed as a sum over Γ of the form:

$$G(z, z') = \sum_{\gamma \in \Gamma} \mathrm{Ad}\rho(\gamma) Q(z, \gamma z'), \quad z \neq z'.$$

Here Q is the resolvent of the Laplace operator on $\mathbb{H}$, with values in $\mathrm{End}\mathbb{C}^n$, which is well know to be:

$$Q(z, z') = -\frac{1}{\pi} \log \left| \frac{z - z'}{\bar{z} - z'} \right| I_{\mathrm{End}\mathbb{C}^n}.$$

Secondly, in Chapter 2 we also introduced the Selberg zeta function, which fulfills:

$$\begin{aligned} \frac{1}{1-2s} \frac{d}{ds} \log Z(s) &= \int_\Sigma \sum_{\gamma \text{ hyperbolic}} \mathrm{Ad}\rho(\gamma) Q_s(z, \gamma z) \frac{dxdy}{y^2} \\ &= \int_\Sigma (G_s(z, z') - Q_s(z, z'))|_{z=z'} \frac{dxdy}{y^2} \\ \left.\frac{d}{ds}\right|_{s=1} \log Z(s) &= \log(k \det \Delta_0). \end{aligned}$$

Where Q_s is the integral kernel for $(\Delta + s(s-1))^{-1}$, se (2.11), and k is a constant.

3.4 Variations of the Determinant of the Laplace Operator

We will continue to use coordinates around E with a fixed basis $\{\nu_i\} \subset \mathcal{H}^{0,1}(X, \mathrm{End}E)$. For a $\nu \in \mathcal{H}^{0,1}(X, \mathrm{End}E)$ the derivative with respect to ν is denoted, ∂_ν, and for the corresponding coordinate vector field in a neighborhood, we write $\partial_{\tilde{\nu}}$, then the first variation is:

Lemma 3.4.1 ([Zograf and Takhtadzhyan, 1989, Lemma 3])
For $\nu \in \mathcal{H}^{0,1}(X, \mathrm{End}E)$, the formula:

$$\partial_\nu \log \det \Delta = -i \int_X tr(\mathrm{ad}\nu \wedge \psi)$$

holds at the point E in N, where

$$\psi(z) = \frac{\partial}{\partial z'} (G(z, z') - Q(z, z'))|_{z=z'} \in \Omega^{0,1}(X, \mathrm{End}\mathrm{End}E).$$

Or stated in terms of forms

$$\partial \log \det \Delta = -\sum_j i \int_X tr(\mathrm{ad}\nu_j \wedge \psi) d\nu_j \in T^*_{[\sigma,E]} \mathcal{M}_{n,k}.$$

Proof: We have that

$$\partial_\nu \log\det\Delta = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \log\det\Delta^{\varepsilon\nu} = -\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \frac{d}{ds}|_{s=0}\zeta(\Delta^{\varepsilon\nu}, s).$$

For $\mathrm{Re}(s) > 1$ and $\lambda > 1$ we have that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \zeta_\lambda(\Delta^{\varepsilon\nu}, s) &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{tr}((\Delta+\lambda)^{-s}) \\ &= \mathrm{tr}\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\mathrm{Ad} f^{\varepsilon\nu})^{-1}(\Delta+\lambda)^{-s}\mathrm{Ad} f^{\varepsilon\nu}\right) \\ &= -s\,\mathrm{tr}((\Delta+\lambda)^{-s-1}L_\nu\Delta),\end{aligned}$$

where the last equality follows from the Taylor expansion and the cyclic property of the trace. Again using the cyclic property of the trace we have

$$-s\,\mathrm{tr}((\Delta+\lambda)^{-s-1}L_\nu\Delta) = -s\,\mathrm{tr}(\mathrm{ad}\nu(\Delta+\lambda)^{-s-1}\bar\partial^*).$$

Now the kernel of $(\Delta+\lambda)^{-s-1}\bar\partial^*$ is found using Stokes's theorem and

$$*\bar\partial * f(z)d\bar z = *\bar\partial\overline{f(z)^T}(-i)dz = i * \overline{\partial f(z)^T} dz\wedge d\bar z = 2y^2\partial f(z).$$

We find

$$\begin{aligned}(\Delta+\lambda)^{-s-1}\bar\partial^*(f(z')d\bar z') &= \int_X K_{s+1,\lambda}(z,z')(-*\bar\partial' * f(z')d\bar z')\frac{dz'\wedge d\bar z'}{2iy^2} \\ &= -\int_X 2K_{s+1,\lambda}(z,z')\partial' f(z')\frac{dz'\wedge d\bar z'}{2i} \\ &= \int_X 2\partial' K_{s+1,\lambda}(z,z')f(z')\frac{dz'\wedge d\bar z'}{2i}.\end{aligned}$$

With the kernel we can calculate the trace to be

$$\partial_\nu\zeta_\lambda(s) = -2s\int_X \mathrm{tr}(\mathrm{ad}\nu(z)\partial' K_{s+1,\lambda}(z,z'))|_{z=z'}\frac{dz\wedge d\bar z}{2i}.$$

From the explicit description of $K_{s,\lambda}$ as a sum over the elements of Γ we see, that the ∂ of the identity term vanishes at the diagonal, and therefore we have:

$$\begin{aligned}\partial_\nu\zeta_\lambda(s) &= si\int_X \mathrm{tr}(\mathrm{ad}\nu \wedge \psi_{s+1,\lambda}(z)) \\ \psi_{s+1,\lambda}(z) &= \partial'(K_{s+1,\lambda}(z,z') - k_{s+1,\lambda}(z,z'))|_{z=z'}.\end{aligned}$$

This last kernel expression makes sense for $\mathrm{Re}(s) > -\frac{1}{2}$ and by analytic continuation holds for these s.

$$\partial_\nu \log\det(\Delta+\lambda) = -\frac{d}{ds}_{s=0}\partial_\nu\zeta_\lambda(\Delta, s) = -i\int_X \mathrm{tr}(\mathrm{ad}\nu\wedge\psi_{1,\lambda}(z)),$$

Now since $\log\det(\Delta) = \lim_{\lambda\to 0^+}\log\det(\Delta+\lambda) - \log\lambda$ and $\partial_\nu\log\lambda = 0$. We conclude the proof by observing $k_{1,0} = Q$ and $\partial' K_{1,0} = \partial' G$. ■

Using the coordinate transformations to move this result in coordinates centered at $E^{\varepsilon\nu}$ to our coordinates around E we find, that:

$$\partial \log \det \Delta^{\varepsilon\nu} = -\sum_j i \int_X \mathrm{tr}(\mathrm{ad}\nu_j^{\varepsilon\nu} \wedge (\partial'^{\varepsilon\nu}(G^{\varepsilon\nu}(z,z') - Q(z,z')))|_{z=z'})d\nu_j.$$

The second variation can now be calculated:

Theorem 3.4.2 ([Zograf and Takhtadzhyan, 1989, Theorem 2])
For $\nu_j, \nu_i \in \mathcal{H}^{0,1}(X, EndE)$ we have:

$$\bar\partial\partial \log \det \Delta(\partial_{\bar i}, \partial_j) = \Theta_{\mathrm{End}}(\partial_{\bar i}, \partial_j) + \frac{1}{2\pi}\int_X tr(\mathrm{ad}\nu_j \wedge \mathrm{ad} * \nu_i).$$

PROOF: From Lemma 3.4.1 we have:

$$\begin{aligned}
\bar\partial\partial \log \det \Delta(\partial_{\bar i}, \partial_j) &= -i\frac{d}{d\bar\varepsilon}|_{\varepsilon=0}\int_X \mathrm{tr}((\mathrm{Ad}f^{\varepsilon\nu_i})^{-1}\mathrm{ad}(\tilde{\nu_j} \wedge \psi)\mathrm{Ad}f^{\varepsilon\nu_i})\\
&= -i\frac{d}{d\bar\varepsilon}|_{\varepsilon=0}\int_X \mathrm{tr}((\mathrm{Ad}f^{\varepsilon\nu_i})^{-1}\mathrm{ad}\tilde{\nu_j}\mathrm{Ad}f^{\varepsilon\nu_i}\\
&\qquad \wedge (\mathrm{Ad}f^{\varepsilon\nu_i})^{-1}\psi\mathrm{Ad}f^{\varepsilon\nu_i})\\
&= -i\int_X \mathrm{tr}((L_{\bar{\nu_i}}\mathrm{ad}\tilde{\nu_j}) \wedge \psi)\\
&\quad - i\int_X \mathrm{tr}(\mathrm{ad}\nu_j \wedge L_{\bar{\nu_i}}(\partial'(G(z,z') - Q(z,z'))|_{z=z'})).
\end{aligned}$$

We take each term by itself. First the Q term:

$$\begin{aligned}
L_{\bar{\nu_i}}\partial' Q(z,z') &= L_{\bar{\nu_i}}\partial'\frac{-1}{2\pi}\log\frac{|z-z'|^2}{|\bar z - z'|^2}I\\
&= -\frac{\partial}{\partial\bar\varepsilon}\Big|_{\varepsilon=0}\frac{\mathrm{Ad}(f^{\varepsilon\nu_i}(z))^{-1}\mathrm{Ad}(f^{\varepsilon\nu_i}(z'))}{2\pi}\left(\frac{1}{z-z'} - \frac{1}{\bar z - z'}\right)\\
&= \frac{1}{2\pi}\left(\frac{1}{z-z'} - \frac{1}{\bar z - z'}\right)\left(\frac{\partial}{\partial\bar\varepsilon}\Big|_{\varepsilon=0}\mathrm{Ad}(f^{\varepsilon\nu_i}(z)) - \frac{\partial}{\partial\bar\varepsilon}\Big|_{\varepsilon=0}\mathrm{Ad}(f^{\varepsilon\nu_i}(z'))\right)\\
&= \frac{1}{2\pi}\left(\frac{1}{z-z'} - \frac{1}{\bar z - z'}\right)\left(\frac{\partial}{\partial\bar\varepsilon}\Big|_{\varepsilon=0}\mathrm{Ad}(f_+^{\varepsilon\nu_i}(z)) - \frac{\partial}{\partial\bar\varepsilon}\Big|_{\varepsilon=0}\mathrm{Ad}(f_+^{\varepsilon\nu_i}(z'))\right),
\end{aligned}$$

since $\frac{d}{d\bar\varepsilon}\big|_{\varepsilon=0}\bar\partial f_-^{\varepsilon\nu_i} = 0$ from the definition of $f_-^{\varepsilon\nu_i}$. This means $\frac{d}{d\bar\varepsilon}\big|_{\varepsilon=0} f_-^{\varepsilon\nu_i}$ is a constant and so does not contribute. We are interested in what happens on the diagonal, and since we know that $\partial\,\frac{\partial}{\partial\bar\varepsilon}\big|_{\varepsilon=0}\mathrm{Ad}(f_+^{\varepsilon\nu_i}(z)) = -i * \nu_i$, we can calculate the limit and find:

$$i\int_X \mathrm{tr}(\mathrm{ad}\nu_j \wedge L_{\bar{\nu_i}}\partial' Q(z,z')|_{z=z'}) = \frac{1}{2\pi}\int_X \mathrm{tr}(\mathrm{ad}\nu_j \wedge \mathrm{ad} * \nu_i).$$

Since the contribution from the Q term is finite, so must the contribution from $\partial' G$ be.

The next term is Green's function. Since $-i\partial' G$ is the kernel of $\Delta_0^{-1}\bar\partial^*$, we can vary this operator instead:

$$L_{\bar\nu_i}\Delta_0^{-1}\bar\partial^* = \Delta_0^{-1} * \mathrm{ad}\nu_i * - \Delta_0^{-1} * \mathrm{ad}\nu_i * \bar\partial\Delta_0^{-1}\bar\partial^* = \Delta_0^{-1} * \mathrm{ad}\nu_i * P.$$

This implies:

$$-i\int_X \mathrm{tr}(\mathrm{ad}\nu_j \wedge L_{\bar\nu_i}\partial' G(z,z')|_{z=z'}) = \mathrm{tr}((L_{\nu_j}\bar\partial)\Delta_0^{-1}L_{\bar\nu_i}\bar\partial^* P).$$

The final term we need to calculate is

$$\begin{aligned}-i\int_X \mathrm{tr}(L_{\bar\nu_i}\mathrm{ad}\nu_j \wedge \psi) &= -i\int_X \mathrm{tr}(\mathrm{ad}\bar\partial\Delta_0^{-1}(*[*\nu_j,\nu_i]) \wedge \psi)\\ &= i\int_X \mathrm{tr}(\mathrm{ad}\Delta_0^{-1}(*[*\nu_j,\nu_i]) \wedge \bar\partial\psi),\end{aligned}$$

where the last equality follows from Stokes's theorem. Now we have

$$\bar\partial\psi(z) = ((\bar\partial + \bar\partial')\partial'(G(z,z') - Q(z,z')))|_{z=z'}.$$

The kernel $G(z,z') - Q(z,z')$ is a regular function, and so we can calculate

$$\begin{aligned}((\bar\partial + \bar\partial')\partial'(G(z,z') - Q(z,z')))|_{z=z'} &= \lim_{z\to z'}(\bar\partial + \bar\partial')\partial'(G(z,z') - Q(z,z'))\\ &= \lim_{z\to z'}(\bar\partial + \bar\partial')\partial'(G(z,z'))\\ &\quad - (\bar\partial + \bar\partial')\partial'(Q(z,z')).\end{aligned}$$

By direct calculation using that

$$Q(z,z') = \frac{1}{\pi}\log\frac{|z-z'|}{|\bar z - z'|} I_{\mathrm{End}\mathbb{C}^{n^2}},$$

we find

$$\bar\partial'\partial' Q(z,z') = \frac{1}{y^2}\Delta' Q(z,z') = 0 \quad \bar\partial\partial' Q(z,z') = \frac{1}{8\pi y^2} I,$$

when not on the diagonal. The term $-i\bar\partial'\partial' G(z,z')$ is the kernel of $\Delta_0^{-1}\Delta = I - P_0$, where P_0 is the projection on the kernel of Δ. The kernel of Δ is spanned by the section I. This means

$$\begin{aligned}G(z,z') &= \delta(z,z') - P_0(z,z') \overset{z\neq z'}{=} -P_0(z,z')\\ &= \frac{1}{h_{\mathcal{E}}(I,I)\mathrm{Area}(X)2yy'} I = \frac{1}{n^2 4\pi(g-1)2yy'} I.\end{aligned}$$

The term $-i\bar\partial\partial' G(z,z')$ is the kernel of $\bar\partial\Delta_0^{-1}\bar\partial^* = I - P$. This implies that

$$\bar\partial\partial' G(z,z') = \delta(z,z') - P(z,z') \overset{z\neq z'}{=} -P(z,z').$$

Hence:

$$\bar{\partial}\psi(z) = -P(z,z') - \frac{1}{8\pi y^2}\left(1 + \frac{1}{n^2(g-1)}\right) I.$$

Therefore, since $\mathrm{tr}(\mathrm{ad}(A)I) = 0$:

$$-i\int_X \mathrm{tr}(L_{\bar{\nu}_i}\mathrm{ad}\nu_j \wedge \psi) = \mathrm{tr}(\mathrm{ad}\Delta_0^{-1}(*[*\nu_j,\nu_i])P),$$

and by Lemma 3.3.2 the conclusion follows. ■

In the next section, we will see that $\frac{-1}{2}\log\det\Delta$ is the Ricci potential when we restrict to $\mathcal{M}^{n,k}_{VB,0}$.

3.4.1 The Symplectic Form

On the moduli space the symplectic form is given by

$$\Omega\left(\partial_{\nu_j}, \bar{\partial}_{\nu_j}\right) = \frac{i}{2}\langle \nu_i, \nu_j\rangle = \frac{1}{2}\int_X \mathrm{tr}(\nu_i \wedge \bar{\nu}_j^T).$$

Now we have that

$$\mathrm{tr}(\mathrm{ad}a\mathrm{ad}b) = 2n\mathrm{tr}ab - 2\mathrm{tr}a\mathrm{tr}b.$$

And so from Theorem 3.4.2 and since $\det\Delta = \det\Delta_{\mathrm{ad}}\det\Delta_X$:

$$\Theta_{\mathrm{Ad}E}(\partial_{\nu_i}, \bar{\partial}_{\nu_j}) = \partial\bar{\partial}\log\det\Delta|_{\mathrm{Ad}E}\left(\partial_{\nu_i}, \bar{\partial}_{\nu_j}\right) + \frac{2ni}{\pi} g\left(\partial_{\nu_i}, \bar{\partial}_{\nu_j}\right),$$

because $\partial_{\mathcal{M}^0_{n,k}}\Delta_X = 0$. Now we want to understand the Ricci form so we compose with the complex structure and get:

$$\mathrm{Ric}_{\mathrm{Ad}E}(\partial_{\nu_i}, \bar{\partial}_{\nu_j}) = i\partial\bar{\partial}\log\det\Delta|_{\mathrm{ad}}\left(\partial_{\nu_i}, \bar{\partial}_{\nu_j}\right) + \frac{2n}{\pi}\omega\left(\partial_{\nu_i}, \bar{\partial}_{\nu_j}\right).$$

In other words, the Ricci potential is $\frac{-1}{2}\log\det\Delta$.

Chapter 4

Teichmüller Variations

In the previous chapters we have mainly talked about the moduli space of rank n degree k stable vector bundles. Our interest in this space was to study the Hitchin connection and related structures on the bundle of holomorphic sections over a family of Kähler structures. We have a naturally induced complex structure on the moduli space from the underlying Riemann surface. Hence, one natural candidate for complex structures is the space of complex structures on the surface. In fact each complex structure on the surface gives a Kähler structure on the moduli space. In this chapter we will introduce coordinates for the cover of the Riemann moduli space known as the Teichmüller space.

The Teichmüller space can be defined as:

$$\mathcal{T}(\Sigma) := \{ (X_f, f : \Sigma \to X_f) | \, f \in \mathrm{Diff}(\Sigma, X_f), X_f \text{ a Riemann surface} \} /(R), \tag{4.1}$$

where R is the relation given by fRg if there is a biholomophism $\Phi : X_f \to X_g$ such that $g^{-1} \circ \Phi \circ f$ is isotopic to the identity.

We know from Kodaira-Spencer theory that the maximal possible tangent space at (X_f, f) is $H^1(X, \Theta)$, the first Čech cohomology group with values in the germ of holomorphic vector fields. This can be identified with harmonic sections of

$$T_\sigma X \otimes \overline{T_\sigma^* X},$$

here $T_\sigma X$ is the holomorphic tangent space inside $T\Sigma_{\mathbb{C}}$. We will denote the set of smooth sections of

$$(T_\sigma X)^n \otimes (\overline{T_\sigma X})^{-m}$$

by $\Omega^{0,m}(X, (T_\sigma X)^n)$ for $n, m \geq 0$. We will now introduce local coordinates constructed from the Beltrami differentials.

4.1 Bers's Coordinates

In this section we introduce Bers's coordinates [Ahlfors and Bers, 1960]. These are constructed from $\Omega^{0,1}(X, T_\sigma X)$. First, we use uniformization to represent

X as a quotient of $\mathbb{H}$ by $\rho_{\mathbb{H}} : \pi_1(\Sigma) \to \Gamma \subset \mathbf{PSL}(2,\mathbb{R})$. Then there is a correspondence between tensors on X and a subset of functions on $\mathbb{H}$ given by

$$\Omega^{0,m}(X,(T_\sigma X)^n) \cong \{f : \mathbb{H} \to \mathbb{C} | f(\gamma z)(\gamma' z)^{-n}\overline{(\gamma' z)}^m = f(z) \forall \gamma \in \Gamma\}. \quad (4.2)$$

Let $\mu \in \Omega^{0,1}(X, T_\sigma X)$ then, from the transformation properties, we see that $\|\mu\|_\infty$ is well-defined. Now use the correspondence (4.2) and consider μ as a function on $\mathbb{H}$. If $\|\mu\|_\infty < 1$ then we can solve the equation

$$\bar\partial g^\mu(z) = \mu(z)\partial g^\mu(z)$$

on $\mathbb{H}$. This solution is unique if we require that it fixes $0, 1$ and ∞. We define the new complex structure on Σ by changing the representation of π_1 into $\mathbf{PSL}(2,\mathbb{R})$ to $\rho^\mu_{\mathbb{H}}(\gamma) = (g^\mu(\rho_{\mathbb{H}}(\gamma)((g^\mu)^{-1})))$. This gives us a map from a neighborhood of zero in $\Omega^{0,1}(X, T_\sigma X)$ into Teichmüller space. We will now calculate the differential and see that the restriction to the harmonic Beltrami differentials, $\mathcal{H}^{0,1}(X, T_\sigma X)$, has maximal rank at zero and is injective, and hence define coordinates on a neighborhood of X in Teichmüller space. These coordinates will be holomorphic coordinates for Teichmüller space.

4.1.1 The Differential

We calculate the Kodaira-Spencer map corresponding to $\nu \in \mathcal{H}^{0,1}(X, T_\sigma X)$ at X^μ for $\mu \in \mathcal{H}^{0,1}(X, T_\sigma X)$. That is the Kodaira-Spencer map of the curve $\rho^{\mu+\varepsilon\nu}_{\mathbb{H}}$. This will allow us to identify our coordinate vector fields as Kodaira-Spencer classes.

We consider an open covering, $\mathcal{U}$, of a fundamental domain of X^μ in it's universal cover $\mathbb{H}$. The Kodaira 1-cocycle is given by $\theta_{ij} = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^{\mu+\nu\varepsilon})^{-1} \frac{\partial}{\partial z}$ for $U_i, U_j \in \mathcal{U}$ and $\gamma_{ij} U_j \cap U_i \neq \emptyset$. We have that:

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^{\mu+\nu\varepsilon})^{-1} \frac{\partial}{\partial z} \\
&\quad = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^{\mu})^{-1} \frac{\partial}{\partial z} + \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu} \circ \gamma_{ij} \circ (g^{\mu+\nu\varepsilon})^{-1} \frac{\partial}{\partial z} \\
&\quad = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^{\mu})^{-1} \frac{\partial}{\partial z} \\
&\qquad + \left(\partial g^\mu \circ \gamma_{ij} \circ (g^\mu)^{-1}\right) \cdot \gamma'_{ij} \circ (g^\mu)^{-1} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (g^{\mu+\nu\varepsilon})^{-1} \frac{\partial}{\partial z} \\
&\qquad + \left(\bar\partial g^\mu \circ \gamma_{ij} \circ (g^\mu)^{-1}\right) \cdot \overline{\gamma'_{ij}} \circ (g^\mu)^{-1} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\overline{g^{\mu+\nu\varepsilon}})^{-1} \frac{\partial}{\partial z}.
\end{aligned}$$

Before continuing this calculation we consider:

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (g^{\mu+\nu\varepsilon})^{-1} \circ g^{\mu+\nu\varepsilon} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (g^{\mu+\nu\varepsilon})^{-1} \circ g^{\mu}$$

$$+ (\partial(g^\mu)^{-1}) \circ g^\mu \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon}$$
$$+ (\bar{\partial}(g^\mu)^{-1}) \circ g^\mu \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{g^{\mu+\nu\varepsilon}}.$$

In what follows we will need the following two relations:

$$\begin{aligned}
0 &= \bar{\partial}(g^\mu \circ (g^\mu)^{-1}) = (\partial g^\mu) \circ (g^\mu)^{-1} \bar{\partial}(g^\mu)^{-1} + (\bar{\partial} g^\mu) \circ (g^\mu)^{-1} \overline{\partial(g^\mu)^{-1}} \\
&= (\partial g^\mu) \circ (g^\mu)^{-1} \bar{\partial}(g^\mu)^{-1} + (\mu \partial g^\mu) \circ (g^\mu)^{-1} \overline{\partial(g^\mu)^{-1}} \\
&\Rightarrow \bar{\partial}(g^\mu)^{-1} = -\mu \circ (g^\mu)^{-1} \overline{\partial(g^\mu)^{-1}}, \\
1 &= \partial(g^\mu \circ (g^\mu)^{-1}) = (\partial g^\mu) \circ (g^\mu)^{-1} \partial(g^\mu)^{-1} + (\bar{\partial} g^\mu) \circ (g^\mu)^{-1} \partial\overline{(g^\mu)^{-1}} \\
&= (\partial g^\mu) \circ (g^\mu)^{-1} \partial(g^\mu)^{-1} - (\mu \partial g^\mu) \circ (g^\mu)^{-1} \bar{\mu} \circ (g^\mu)^{-1} \partial(g^\mu)^{-1} \\
&\Rightarrow \partial(g^\mu)^{-1} = \frac{1}{1-|\mu|^2} \frac{1}{\partial g^\mu} \circ (g^\mu)^{-1}.
\end{aligned}$$

Using this we have:

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^{\mu+\nu\varepsilon})^{-1} \frac{\partial}{\partial z} \\
&\quad = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^\mu)^{-1} \frac{\partial}{\partial z} - (\partial g^\mu) \circ \gamma_{ij} \circ (g^\mu)^{-1} \gamma'_{ij} \circ (g^\mu)^{-1} \\
&\quad \Bigg(\partial(g^\mu)^{-1} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \\
&\qquad\qquad + (\bar{\partial}(g^\mu)^{-1}) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{g^{\mu+\nu\varepsilon}} \circ (g^\mu)^{-1} \Bigg) \frac{\partial}{\partial z} \\
&\quad - (\bar{\partial} g^\mu) \circ \gamma_{ij} \circ (g^\mu)^{-1} \overline{\gamma'_{ij}} \circ (g^\mu)^{-1} \\
&\quad \Bigg((\overline{\bar{\partial}(g^\mu)^{-1}}) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{g^{\mu+\nu\varepsilon}} \circ (g^\mu)^{-1} \\
&\qquad\qquad + (\overline{\partial(g^\mu)^{-1}}) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \Bigg) \frac{\partial}{\partial z} \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^\mu)^{-1} \frac{\partial}{\partial z} - (\partial g^\mu) \circ \gamma_{ij} \circ (g^\mu)^{-1} \gamma'_{ij} \circ (g^\mu)^{-1} \\
&\quad \Bigg((\partial(g^\mu)^{-1}) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \\
&\qquad\qquad - (\mu \overline{\partial(g^\mu)^{-1}}) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{g^{\mu+\nu\varepsilon}} \circ (g^\mu)^{-1} \Bigg) \frac{\partial}{\partial z} \\
&\quad - (\mu \partial g^\mu) \circ \gamma_{ij} \circ (g^\mu)^{-1} \overline{\gamma'_{ij}} \circ (g^\mu)^{-1} \Bigg((\overline{\partial(g^\mu)^{-1}}) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{g^{\mu+\nu\varepsilon}} \circ (g^\mu)^{-1} \\
&\qquad\qquad - (\bar{\mu} \partial(g^\mu)^{-1}) \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \Bigg) \frac{\partial}{\partial z}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^\mu)^{-1} \frac{\partial}{\partial z} - (\partial g^\mu) \circ \gamma_{ij} \circ (g^\mu)^{-1} \\
&\quad \left(\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \right) \partial (g^\mu)^{-1} \gamma'_{ij} \circ (g^\mu)^{-1} \right. \\
&\quad \left(\gamma'_{ij} \circ (g^\mu)^{-1} - \bar{\gamma}'_{ij} \circ (g^\mu)^{-1} \left(\mu \circ \gamma_{ij} \circ (g^\mu)^{-1} \right) \bar{\mu} \circ (g^\mu)^{-1} \right) \\
&\quad + \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \overline{g^{\mu+\nu\varepsilon}} \circ (g^\mu)^{-1} \\
&\quad \left. \left(\mu \circ \gamma_{ij} \circ (g^\mu)^{-1} \bar{\gamma}'_{ij} \circ (g^\mu)^{-1} - \mu \circ (g^\mu)^{-1} \gamma'_{ij} \circ (g^\mu)^{-1} \right) \overline{\partial (g^\mu)^{-1}} \right) \frac{\partial}{\partial z} \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ \gamma_{ij} \circ (g^\mu)^{-1} \frac{\partial}{\partial z} - \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \\
&\quad ((\partial g^\mu) \circ \gamma_{ij} \circ (g^\mu)^{-1})(1 - |\mu|^2) \partial (g^\mu)^{-1} (\gamma'_{ij} \circ (g^\mu)^{-1}) \frac{\partial}{\partial z} \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \circ g^\mu \circ \gamma_{ij} \circ (g^\mu)^{-1} \frac{\partial}{\partial z} \\
&\quad - \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^{\mu+\nu\varepsilon} \circ (g^\mu)^{-1} \left(\frac{\partial}{\partial z} \circ (g^\mu \circ \gamma_{ij} \circ (g^\mu)^{-1}) \right).
\end{aligned}
$$

And so we have that $\theta_{ij} = \delta^*(g_i)$ for $g_i = \frac{dg^{\mu+\varepsilon\nu}}{d\varepsilon}\Big|_{\varepsilon=0} \circ (g^\mu)^{-1} \frac{\partial}{\partial z}$. To finish the calculation we have to calculate:

$$
\begin{aligned}
\bar{\partial} g_i &= \frac{d\partial g^{\mu+\varepsilon\nu}}{d\varepsilon}\bigg|_{\varepsilon=0} \circ (g^\mu)^{-1} (\bar{\partial} (g^\mu)^{-1}) \frac{\partial}{\partial z} \\
&\quad + \frac{d\bar{\partial} g^{\mu+\varepsilon\nu}}{d\varepsilon}\bigg|_{\varepsilon=0} \circ (g^\mu)^{-1} (\overline{\partial (g^\mu)^{-1}}) \frac{\partial}{\partial z} \\
&= \frac{d\partial g^{\mu+\varepsilon\nu}}{d\varepsilon}\bigg|_{\varepsilon=0} \circ (g^\mu)^{-1} (-\mu \circ (g^\mu)^{-1} \overline{\partial (g^\mu)^{-1}}) \frac{\partial}{\partial z} \\
&\quad + \frac{d(\mu + \varepsilon\nu) \partial g^{\mu+\varepsilon\nu}}{d\varepsilon}\bigg|_{\varepsilon=0} \circ (g^\mu)^{-1} (\overline{\partial (g^\mu)^{-1}}) \frac{\partial}{\partial z} \\
&= \nu \circ (g^\mu)^{-1} \partial g^\mu \circ (g^\mu)^{-1} (\overline{\partial (g^\mu)^{-1}}) \frac{\partial}{\partial z} \\
&= \frac{\nu}{1 - |\mu|^2} \frac{\partial g^\mu}{\overline{\partial g^\mu}} \circ (g^\mu)^{-1} \frac{\partial}{\partial z}.
\end{aligned}
$$

So the tangent vector corresponding to ν at μ is $P_{0,1}(\frac{\nu}{1-|\mu|^2} \frac{\partial g^\mu}{\overline{\partial g^\mu}} \circ (g^\mu)^{-1})$. This map is holomorphic in ν, and so we have holomorphic coordinates. Further, we see that only the different harmonic forms give rise to different complex structures.

4.2 Variations of the $\bar{\partial}$-Operator on X

We will now introduce a connection like we did for the $\mathbf{SU}(n)$ moduli space. Let δ_μ be defined as

$$\delta_\mu(s) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (g^{\varepsilon\mu})^* s^{\varepsilon\mu} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (s^{\varepsilon\mu} \circ g^{\varepsilon\mu})(\partial g^{\varepsilon\mu})^n (\overline{\partial g^{\varepsilon\mu}})^m$$

for a section $s^{\varepsilon\mu} \in \Omega^{n,m}(X^{\varepsilon\mu})$ and $\mu \in \mathcal{H}^{0,1}(X, T_\sigma X)$. And similarly we have for the operators:

$$\delta_v F^\varepsilon = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (g^{\varepsilon\mu})^* \circ F^\varepsilon \circ ((g^{\varepsilon\mu})^*)^{-1}.$$

This definition also works for E-valued sections for some holomorphic bundle E on X for a fixed $\mathbf{U}(n)$-representation ρ_E of $\pi_1(X)$. In order to work with the Ricci potential, we need to understand the derivatives along the Teichmüller space of $\bar{\partial}$ on $\Omega^{n,0}$ and $\bar{\partial}^*$ on $\Omega^{n,1}$, or if we want to work with the Ricci potential on the moduli space of bundles, we need to understand their endomorphism valued versions. However, as long as the bundle comes from a fixed representation both alternatives are calculated by:

$$\begin{aligned}
\delta_\mu(\bar{\partial}_n h) &= (\delta_\mu \bar{\partial}_n) h + \bar{\partial}_n \delta_\mu h \\
\delta_\mu(\bar{\partial}_n h) - \bar{\partial}_n \delta_\mu h &= \frac{d}{dt}\bigg|_{t=0} (h_{\bar{z}}(g^{t\mu})(\partial g^{t\mu})^n (\overline{\partial g^{t\mu}}) - \bar{\partial}_n h(g^{t\mu})(\partial g^{t\mu})^n) \\
&= -\frac{d}{dt}\bigg|_{t=0} (h_z(g^{t\mu})(\partial g^{t\mu})^n (\bar{\partial} g^{t\mu}) \\
&\qquad + n h(g^{t\mu})(\partial g^{t\mu})^{n-1} (\bar{\partial}\partial g^{t\mu})) \\
&= -h_z(z)\mu(z) - n h(z) \partial\mu(z) \\
&= -\left(h_z(z)\mu(z) - n\mu(z) h(z) \frac{\partial \rho}{\rho} \right) \\
&= -(\mu(z)\rho^n \partial_n \rho^{-n}) h(z) = -(\mu(z)\bar{\partial}^*_{n+1}\rho) h(z).
\end{aligned}$$

And so $\delta_\mu \bar{\partial}_n = -(\mu(z)\bar{\partial}^*_{n+1}\rho)$, where ρ is the volume form. Similarly, we find $\bar{\delta}_\mu \bar{\partial}_n = \delta_\mu \bar{\partial}_n{}^* = 0$ and $\bar{\delta}_\mu \bar{\partial}_n{}^* = \bar{\mu}\bar{\partial}_{n-1}\rho^{-1}$. We see that the differential operators are on new spaces now, but the volume form ρ also changes tensor type.

4.3 Variation of the Selberg Zeta Function

As a short reminder, we have that the Selberg Zeta function is given by

$$Z(s, \rho_{\mathbb{H}}, \rho_E) = \prod_{\gamma \text{ primitive hyperbolic}} \prod_{k=0}^{\infty} \det(I - \rho_E(\gamma) e^{-(k+s)|\gamma|_{\rho_{\mathbb{H}}}}).$$

And from this follows:

$$\frac{1}{2s-1}\frac{d}{ds}\log Z(s,\rho_{\mathbb{H}},\rho_E) = \int_{\mathbb{H}\backslash\Gamma}\sum_{\gamma\text{ hyperbolic}} \mathrm{tr}\rho_E(\gamma)Q_s^{\varepsilon\mu}(z,\gamma z)\frac{dxdy}{y^2} \tag{4.3}$$

when $\mathrm{Re}(s)>1$.

Lemma 4.3.1 ([Takhtajan and Zograf, 1991, Lemma 3])
For $\mu\in\mathcal{H}^{0,1}(X_\sigma,T_\sigma X)$ and $Re(s)>1$ we have:

$$\begin{aligned}\partial_\mu \log Z(s,\rho_{\mathbb{H}},\rho_E) &= \frac{\partial}{\partial\varepsilon}\log Z(s,\rho_{\mathbb{H}}^{\varepsilon\mu},\rho_E)\\ &= -i\int_X \mu(\partial\partial' tr(G_s(z,z')-Q_s(z,z'))|_{z=z'},\end{aligned}$$

at the point $\sigma\in\mathcal{T}$. Where $\partial\partial'(G_s(z,z')-Q_s(z,z')))|_{z=z'}$ is a $(2,0)$-tensor with values in EndE.

Note that i have an i infront of my expression, I believe this is a mistake in the original formulation.

Proof: Now differentiating in (4.3) we find:

$$\begin{aligned}&\frac{1}{2s-1}\frac{d}{ds}\frac{\partial}{\partial\varepsilon_\mu}\log Z(s,\rho_{\mathbb{H}}^{\varepsilon\mu},\rho_E)\\ &= \frac{\partial}{\partial\varepsilon_\mu}\int_{\mathbb{H}/\Gamma}\sum_{\gamma\text{ hyperbolic}}\mathrm{tr}\rho_E(\gamma)Q_s(z,\gamma z)\frac{dxdy}{y^2}\\ &= \frac{\partial}{\partial\varepsilon_\mu}\int_{\mathbb{H}/\Gamma}\mathrm{tr}(G_s(z,z')-Q_s(z,z'))|_{z=z'}\frac{dxdy}{y^2}\\ &= \int_{\mathbb{H}/\Gamma}\mathrm{tr}(\delta_\mu G_s(z,z')-\delta_\mu Q_s(z,z'))|_{z=z'}\frac{dxdy}{y^2}\\ &= \int_{\mathbb{H}/\Gamma}\mathrm{tr}\left(\int_{\mathbb{H}/\Gamma}G_s(z,z'')(\delta_\mu\Delta)G_s(z'',z')\frac{dx''dy''}{y''^2}\right. \qquad (4.4)\\ &\left.-\int_{\mathbb{H}}Q_s(z,z'')(\delta_\mu\Delta)Q_s(z'',z')\frac{dx''dy''}{y''^2}\right)\Bigg|_{z=z'}\frac{dxdy}{y^2},\end{aligned}$$

since the derivative of the area vanishes. Now we have that

$$\begin{aligned}&\int_{\mathbb{H}/\Gamma}G_s(z,z'')(\delta_\mu\Delta)G_s(z'',z')\frac{dx''dy''}{y''^2}\\ &= \int_{\mathbb{H}/\Gamma}\mu(z'')\frac{d}{dz''}G_s(z,z'')\frac{d}{dz''}G_s(z'',z')dx''dy''\\ &= -i\int_{\mathbb{H}/\Gamma}\mu(z'')\frac{d}{dz}G_s(z,z'')\frac{d}{dz'}G_s(z'',z')dx''dy'',\end{aligned}$$

and similarly for the Q's remembering that here we have to integrate over all of $\mathbb{H}$. Going back to (4.4) and changing the order of integration we have:

$$\int_{\mathbb{H}/\Gamma} \mathrm{tr}\left(\int_{\mathbb{H}/\Gamma} G_s(z,z'')(\delta_\mu\Delta)G_s(z'',z') - Q_s(z,z'')(\delta_\mu\Delta)Q_s(z'',z')\right)\Big|_{z=z'}$$
$$= -i\int_{\mathbb{H}/\Gamma}\mu(z)\mathrm{tr}\left(\frac{d^2}{dzdz'}\int_{\mathbb{H}/\Gamma} G_s(z,z'')G_s(z'',z')\frac{dx''dy''}{y''^2}\right.$$
$$\left. + i\int_{\mathbb{H}} Q_s(z,z'')Q_s(z'',z')\ \frac{dx''dy''}{y''^2}\right)\Big|_{z=z'}$$
$$= -i\int_{\mathbb{H}/\Gamma}\mu(z)\left(\frac{d^2}{dzdz'}\frac{1}{2s-1}\frac{d}{ds}\mathrm{tr}(G_s(z,z') - Q_s(z,z'))\right)\Big|_{z=z'} dxdy.$$

Now multiplying by $(2s-1)$ and then integrating this from $1 < t < b$ with respect to s we take the limit $b \to \infty$. Since $\sum_{\gamma \text{ hyperbolic}} \mathrm{tr}\rho(\gamma)Q_s(z,\gamma z) \to 0$ and $\log Z(s) \to 0$ as $s \to \infty$, we have the desired conclusion:

$$\frac{\partial}{\partial \varepsilon_\mu}\log Z(s,\rho_{\mathbb{H}}^{\varepsilon\mu},\rho_E). \qquad \blacksquare$$

Corollary 4.3.2
Let $\mu \in \mathcal{H}^{0,1}(X,T_\sigma X)$ then

$$\partial_\mu \log\det\Delta_{E,\sigma} = -i\int_X \mu tr(\partial'\partial\psi(z,z'))|_{z=z'},$$

with $\psi(z,z') = G_1(z,z') - Q_1(z,z')$. Here $\psi(z,z')|_{z=z'} \in \Omega^0(X, EndE)$.

This is the specialization to $s=1$ of Lemma 4.3.1 using that

$$\log\det\Delta = \log Z'(1,\rho_{\mathbb{H}}^{\varepsilon\mu},\rho_E) = \lim_{s\to 1}\delta_\mu \log Z(s,\rho_{\mathbb{H}}^{\varepsilon\mu},\rho_E),$$

where the last equality holds because Z has a simple zero at $s=1$. If we want to compare to the Laplace operator of the previous chapter, then we should choose E to be the endomorphism bundle of the point in the moduli space of bundles, which corresponds to the representation $\mathrm{Ad}\rho$.

The lemma 4.3.1 and corollary 4.3.2 only give formulas for the holomorphic derivatives, but since we have that $Z(s,\rho_{\mathbb{H}}^{\varepsilon\mu},\rho_E)$ is a real valued function for real $s \geq 1$, we can conjugate the formulas in order to get the antiholomorphic 1-forms. To say more than this we need a better understanding of the quasi-conformal maps g^μ. Define the following functions:

$$G_\mu = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g^{\varepsilon\mu}, \quad \Phi_\mu = \frac{d}{d\bar\varepsilon}\Big|_{\varepsilon=0} g^{\varepsilon\mu}, \quad \mu\in\Omega^{0,1}(X,T_\sigma X), \quad \varepsilon\in\mathbb{C} \qquad (4.5)$$

Following the formulation in [Zograf and Takhtadzhyan, 1987] of the results of [Ahlfors, 1961] we then have, that Φ_μ is a holomorphic Eichler integral of weight 2 for the Riemann surface given by $\rho_{\mathbb{H}}$ and:

$$\Phi_\mu'''(z) = -\frac{1}{2}y^{-2}\bar{\mu}(z) \in \Omega^{2,0}(X), \tag{4.6}$$

and further:

$$\begin{aligned}\bar{\partial}G_\mu &= \mu,\\ G_\mu &= \frac{(z-\bar{z})^2}{2}\overline{\Phi_\mu''(z)} + (z-\bar{z})\overline{\Phi'(z)} + \overline{\Phi(z)}.\end{aligned}$$

Now, for the mixed second derivatives we have the following result inspired by [Zograf and Takhtadzhyan, 1987, Theorem 2]:

Theorem 4.3.3
We have the following identity:

$$\begin{aligned}\bar{\partial}\partial \log\det \Delta_{EndE}(\mu_1, \bar{\mu_2}) = \frac{i\text{RankE}}{12\pi}\int_X \mu_1\bar{\mu_2}\rho + tr((\mu_1\bar{\mu_2}\\ + \mu_1\partial\Delta_0^{-1}\bar{\mu_2}\bar{\partial}\rho^{-1})P^{0,1}).\end{aligned}$$

Proof: We have:

$$\begin{aligned}\bar{\partial}\partial\log\det\Delta_{\text{End}E}(\mu_1,\bar{\mu_2}) =& -\delta_{\bar{\mu_2}} i\int_X \mu_1(z)\text{tr}(\partial\partial'(G-Q)(z,z'))|_{z=z'}\\ =& -i\int_X(\delta_{\bar{\mu_2}}\mu_1(z))\text{tr}(\partial\partial'(G-Q)(z,z'))|_{z=z'}\\ & -i\int_X \mu_1(z)\text{tr}(\delta_{\bar{\mu_2}}(\partial\partial'(G-Q)(z,z')))|_{z=z'}.\end{aligned}$$

The last term will actually split into two since both the G and the Q term will be finite, as is seen from calculating:

$$\begin{aligned}\delta_{\bar{\mu_2}}\partial\partial' Q =& \frac{d}{d\bar{\varepsilon}}\bigg|_{\varepsilon=0}\frac{(\partial g^{\varepsilon\mu_2}(z))(\partial' g^{\varepsilon\mu_2}(z'))}{2\pi(g^{\varepsilon\mu_2}(z)-g^{\varepsilon\mu_2}(z'))^2}\\ =&\frac{1}{(z-z')^2}\left(\partial\Phi_{\mu_2}(z)+\partial'\Phi_{\mu_2}(z') - 2\frac{\Phi_{\mu_2}(z)-\Phi_{\mu_2}(z')}{z-z'}\right)\\ =&\frac{-1}{6\pi}\Phi_{\mu_2}'''(z) = \frac{\bar{\mu_2}(z)}{12\pi y^2}I_E,\end{aligned}$$

where we expand Φ_{μ_2} in a power series using its complex analytic properties. Now we can calculate the G contribution since $-i\partial\partial' G$ is the kernel of $\bar{\partial}_1^*\rho\Delta_0^{-1}\bar{\partial}_0^*$, and we have:

$$\delta_{\bar{\mu_2}}\bar{\partial}_1^*\rho\Delta_0^{-1}\bar{\partial}_0^* = \bar{\mu_2}\bar{\partial}\rho^{-1}\rho\Delta_0^{-1}\bar{\partial}_0^* + \bar{\partial}_1^*\rho(-\Delta_0^{-1}\bar{\mu_2}\bar{\partial}\rho^{-1}\bar{\partial}\Delta_0^{-1}\bar{\partial}_0^* + \Delta_0^{-1}\bar{\mu_2}\bar{\partial}\rho^{-1})$$

$$= \bar{\mu_2}(I - P^{0,1}) + \bar{\partial}_1^* \rho \Delta_0^{-1} \bar{\mu_2} \bar{\partial} \rho^{-1} P^{0,1}.$$

Since we know the integral is finite, we have:

$$-i \int_X \mu_1(z) \mathrm{tr}(\delta_{\bar{\mu_2}}(\partial \partial' G))|_{z=z'} = \mathrm{tr}((-\mu_1 \bar{\mu_2} - \mu_1 \partial \Delta_0^{-1} \bar{\mu_2} \bar{\partial} \rho^{-1}) P^{0,1}).$$

These two terms give the theorem once we have checked that

$$\int_X \delta_{\bar{\mu_2}}(\mu_1(z)) \mathrm{tr}(\partial \partial' (G - Q)(z, z'))|_{z=z'} = 0$$

does not contribute:

$$\begin{aligned} &\int_X \delta_{\bar{\mu_2}}(\mu_1(z)) \mathrm{tr}(\partial \partial' (G - Q)(z, z'))|_{z=z'} \\ &\qquad = \int_X \bar{\partial} \rho^{-1} \bar{\partial} \left(\Delta_0 + \frac{1}{2} \right)^{-1} (\mu_1(z) \bar{\mu_2}) \mathrm{tr}(\partial \partial' (G - Q)(z, z'))|_{z=z'} \\ &\qquad = -\int_X \bar{\partial} \left(\Delta_0 + \frac{1}{2} \right)^{-1} (\mu_1(z) \bar{\mu_2}) \rho^{-1} \bar{\partial} (\mathrm{tr}(\partial \partial' (G - Q)(z, z'))|_{z=z'}). \end{aligned}$$

And we have that

$$\begin{aligned} \bar{\partial}(\mathrm{tr}(\partial \partial' (G - Q)(z, z'))|_{z=z'} &= \mathrm{tr}((\bar{\partial} + \bar{\partial}') \partial \partial' (G - Q)(z, z'))|_{z=z'}) \\ &= \mathrm{tr}(\partial \Delta' (G - Q)(z, z') + \partial' \Delta (G - Q)(z, z'))|_{z=z'} = 0, \end{aligned}$$

as it is the kernel of $\bar{\partial}^* \Delta_0^{-1} \Delta_0 - \Delta_0 \Delta_0^{-1} \bar{\partial}^* = 0$. ■

And so we have the following identity of forms:

$$\begin{aligned} -i \mathrm{tr}((\mu_1 \bar{\mu_2} + \mu_1 \partial \Delta_0^{-1} \bar{\mu_2} \bar{\partial} \rho^{-1}) P^{0,1}) + i \bar{\partial} \partial \log \det \Delta_{\mathrm{End} E}(\mu_1, \bar{\mu_2}) \\ = \frac{\mathrm{Rank} E}{6\pi} \omega_{WP}(\mu_1, \bar{\mu_2}). \end{aligned}$$

Chapter 5

Coordinates on the Moduli Space of Pairs of a Riemann Surface and a Holomorphic Vector Bundle

Abstract

In this paper we provide two ways of constructing complex coordinates on the moduli space of pairs of a Riemann surface and a stable holomorphic vector bundle centered around any such pair. We compute the transformation between the coordinates to second-order at the center of the coordinates. We conclude that they agree to second-order, but not the third-order at the center.

5.1 Introduction

Fix $g, n > 1$ to be integers and let $d \in \{0, \dots n-1\}$. Let Σ be a closed oriented surface of genus g. Consider the universal moduli space, $\mathcal{M}$, consisting of equivalence classes of pairs $(\varphi : \Sigma \to X, E)$, where X is a Riemann surface of genus g, $\varphi : \Sigma \to X$ is a diffeomorphism and E is a semi-stable bundle over X of rank n and degree d. Let $\mathcal{M}^s$ be the open dense subset of $\mathcal{M}$ consisting of equivalence classes of such pairs $(\varphi : \Sigma \to X, E)$ with E stable. The main objective of this paper is to provide coordinates in a neighborhood of the equivalence class of any pair $(\varphi : \Sigma \to X, E)$ in $\mathcal{M}^s$. There is an obvious forgetful map

$$\pi_{\mathcal{T}} : \mathcal{M} \to \mathcal{T}$$

where $\mathcal{T}$ is the Teichmüller space of Σ, whose fiber over $[\varphi : \Sigma \to X] \in \mathcal{T}$ is the moduli space of semi-stable bundles for that Riemann surface structure on

Σ. Let $\pi^s_{\mathcal{T}} : \mathcal{M}^s \to \mathcal{T}$ denote the restriction of $\pi_{\mathcal{T}}$ to $\mathcal{M}^s$, and we denote a point $[\varphi : \Sigma \to X]$ in $\mathcal{T}$ by σ.

We recall that locally around any $\sigma \in \mathcal{T}$ there are the Bers coordinates [Ahlfors and Bers, 1960]. Further, for any point $[E]$ in some fiber $(\pi^s_{\mathcal{T}})^{-1}(\sigma)$ we have the Zograf and Takhtadzhyan coordinates near $[E]$ along that fiber of $\pi_{\mathcal{T}}$ [Zograf and Takhtadzhyan, 1989].

In order to describe our coordinates on $\mathcal{M}^s$ we recall the Narasimhan-Seshadri theorem. Let $\tilde{\pi}_1(\Sigma)$ be the universal central $\mathbb{Z}/n\mathbb{Z}$ extension of $\pi_1(\Sigma)$, and let M be the moduli space of representations of $\tilde{\pi}_1(\Sigma)$ to $\mathbf{U}(n)$ such that the central generator goes to $e^{2\pi i d/n}\,\mathrm{Id}$. Let M' be the subset of M consisting of equivalence classes of irreducible representations. The Narasimhan-Seshadri theorem gives us a diffeomorphism

$$\Psi : \mathcal{T} \times M' \to \mathcal{M}^s,$$

which we use to induce a complex structure on $\mathcal{T} \times M'$ such that Ψ is complex analytic. We will now represent a point in $\mathcal{T}$ by a representation

$$\rho_0 : \tilde{\pi}_1(\Sigma) \to \mathbf{PSL}(2)$$

and denote the corresponding point in Teichmüller space by X_{ρ_0}. Here, ρ_0 is really a representation of $\pi_1(\Sigma)$ pulled back to $\tilde{\pi}_1(\Sigma)$. A point in M' will be represented by a representation

$$\rho_E : \tilde{\pi}_1(\Sigma) \to \mathbf{U}(n),$$

which corresponds to the stable holomorphic bundle E on X_{ρ_0}.

We build complex analytic coordinates around any such $(\rho_0, \rho_E) \in \mathcal{T} \times M'$ by providing a complex analytic isomorphism from a small neighborhood around zero in the vector space $H^{0,1}(X_{\rho_0}, TX_{\rho_0}) \oplus H^{0,1}(X_{\rho_0}, \mathrm{End}E)$ to a small open subset containing (ρ_0, ρ_E) in $\mathcal{T} \times M'$.

The coordinates are given by constructing a certain family

$$\Phi^{\mu\oplus\nu} : \mathbb{H} \times \mathbf{GL}(n, \mathbb{C}) \to \mathbb{H} \times \mathbf{GL}(n, \mathbb{C}) \tag{5.1}$$

of bundle maps of the trivial $\mathbf{GL}(n, \mathbb{C})$-principal bundles over $\mathbb{H}$ indexed by pairs of sufficiently small elements

$$\mu \oplus \nu \in H^{0,1}(X_{\rho_0}, TX_{\rho_0}) \oplus H^{0,1}(X_{\rho_0}, \mathrm{End}E).$$

These bundle maps will uniquely determine representations $(\rho^\mu, \rho_E^{\mu\oplus\nu}) \in \mathcal{T} \times M'$ such that

$$\rho^\mu(\gamma) \times \rho_E^{\mu\oplus\nu}(\gamma) = \Phi^{\mu\oplus\nu} \circ (\rho_0(\gamma) \times \rho_E(\gamma)) \circ (\Phi^{\mu\oplus\nu})^{-1} \tag{5.2}$$

for all $\gamma \in \tilde{\pi}_1(X)$ by the following theorem. Pick a base point $z_0 \in \mathbb{H}$, and let $p_{\mathbf{GL}(n,\mathbb{C})}$ be the projection onto $\mathbf{GL}(n, \mathbb{C})$ of the trivial bundle $\mathbb{H} \times \mathbf{GL}(n, \mathbb{C})$.

Theorem 5.1.1
For all sufficiently small $\mu \oplus \nu \in H^{0,1}(X_{\rho_0}, TX_{\rho_0}) \oplus H^{0,1}(X_{\rho_0}, EndE)$ *there exist a unique bundle map* $\Phi^{\mu\oplus\nu}$ *such that*

1. $\Phi^{\mu\oplus\nu}$ *solves*
$$\bar{\partial}_{\mathbb{H}}\Phi^{\mu\oplus\nu} = \partial\Phi^{\mu\oplus\nu}(\mu \oplus \nu), \tag{5.3}$$
where ν *is considered a left-invariant vector field on* $\mathbf{GL}(n, \mathbb{C})$ *at each point in* $\mathbb{H}$.

2. *The base map extends to the boundary of* $\mathbb{H}$ *and fixes* $0, 1$ *and* ∞.

3. *The pair of representations* $(\rho^\mu, \rho_E^{\mu\oplus\nu})$ *defined by equation (5.2) represents a point in* $\mathcal{T} \times M'$.

4. $p_{\mathbf{GL}(n,\mathbb{C})}(\Phi^{\mu\oplus\nu}(z_0, e))$ *has determinant* 1 *and is positive definite.*

From this theorem we easily derive our main theorem of this paper.

Theorem 5.1.2
Mapping all sufficiently small pairs
$$\mu \oplus \nu \in H^{0,1}(X_\rho, TX_\rho) \oplus H^{0,1}(X_\rho, EndE)$$
to
$$(\rho^\mu, \rho_E^{\mu\oplus\nu}) \in \mathcal{T} \times M'$$
provides local analytic coordinates centered at $(\rho_0, \rho_E) \in \mathcal{T} \times M'$.

Our second coordinate construction provides fibered coordinates, which along $\mathcal{T}$ uses Bers's coordinates, [Ahlfors and Bers, 1960], and which uses Zograf's and Takhtadzhyan's coordinates [Zograf and Takhtadzhyan, 1989] along the fibers. We refer to section 5.4 for the precise description of these fibered coordinates.

Finally, we compare the two sets of coordinates by computing the infinitesimal transformation of the coordinates up to second order at the center of both coordinates.

Theorem 5.1.3
The fibered coordinates and the universal coordinates agree to second-order, but not the third-order at the center of the coordinates.

We refer to Theorem 5.5.5, for the details of how the two set of coordinates differ at third-order.

Remark 5.1.4
If we perform our construction using elements of $\mathcal{H}^{0,1}(X, (End_0E))$ *where* (End_0E) *is the subspace of traceless endomorphisms, we get coordinates on the universal* $\mathbf{SU}(n)$ *moduli space in a completely similar way.*

5.2 The Complex Structure on $\mathcal{M}^s$ from a Differential Geometric Perspective

Recall, that we endow the space $\mathcal{T} \times M$ with the structure of a complex manifold by using the Narasimhan-Seshadri theorem to provide us with the diffeomorphism

$$\Psi : \mathcal{T} \times M' \to \mathcal{M}^s,$$

and then declaring it to be complex analytic. There is the following alternative construction of this complex manifold structure.

Recall the general setting of [Andersen et al., 2012] in the context of geometric quantization and the Hitchin connection, namely $\tilde{\mathcal{T}}$ is a general complex manifold and $(\tilde{M}, \omega)$ is a general symplectic manifold. In that paper a construction of a complex structure on $\tilde{\mathcal{T}} \times \tilde{M}$ is provided via the following proposition. But first we need the following definition.

Definition 5.2.1 ([Andersen and Gammelgaard, 2011, Defn. 2.2])
A family of Kähler structures on $(\tilde{M}, \omega)$ parametrized by $\tilde{\mathcal{T}}$ is called holomorphic if it satisfies:

$$V'[J] = V[J]' \quad \textit{and} \quad V''[J] = V[J]''$$

for all vector fields V on $\tilde{\mathcal{T}}$. Here, the single prime on V denotes projection on the $(1,0)$-part, and the double prime on V denotes projection on the $(0,1)$-part of the vector field V. Further $V[J] \in T_\sigma \otimes (\bar{T}_\sigma)^* \oplus \bar{T}_\sigma \otimes T_\sigma^*$, and we let $V[J]'$ denote the projection on the first and $V[J]''$ the projection on the second factor.

Proposition 5.2.2 ([Andersen et al., 2012, Proposition 6.2])
The family J_σ of Kähler structures on $\tilde{M}$ is holomorphic, if and only if the almost complex structure J, given by

$$J(V \oplus X) = IV \oplus J_\sigma X, \quad \forall(\sigma, [\rho_E]) \in \tilde{\mathcal{T}} \times \tilde{M}, \forall (V, X) \in T_{\sigma,[\rho_E]}(\tilde{\mathcal{T}} \times \tilde{M}),$$

is integrable.

The family of complex structures on M' considered in [Hitchin, 1990], see also [Andersen et al., 2012], [Andersen, 2012] and [Andersen and Gammelgaard, 2011], and given by the hodge star, $-\star_\sigma$, $\sigma \in \mathcal{T}$, fulfills the requirements of the proposition with respect to the Atiyah-Bott symplectic form ω on M'. We will denote the complex structure which $\mathcal{T} \times M'$ has as J.

Proposition 5.2.3
We have that the map

$$\Psi : (\mathcal{T} \times M', J) \to \mathcal{M}^s$$

is complex analytic, e.g. J is in fact the complex analytic structure this space gets from the Narasimhan-Seshadri diffeomorphism Ψ.

PROOF: In order to understand the complex structure of $\mathcal{T} \times M'$ from the algebraic geometric perspective we want to construct holomorphic horizontal sections of $\mathcal{T} \times M' \to \mathcal{T}$. We will use the universal property of the space of holomorphic bundles to show that the sections $\mathcal{T} \to \mathcal{T} \times \{\rho_E\} \subset \mathcal{T} \times M'$ are holomorphic for all $[\rho_E]$ in M'.

Our first objective is to construct a holomorphic family of vector bundles over Teichmüller space, where each bundle corresponds to the same unitary representation of $\tilde{\pi}_1(\Sigma)$. We start from the universal curve, $\mathcal{T} \times \Sigma$, and its universal cover, $\mathcal{T} \times \tilde{\Sigma}$. Both of these spaces are complex analytic, and we get the universal curve $\mathcal{T} \times \Sigma$ as the quotient of $\mathcal{T} \times \tilde{\Sigma}$ by the holomorphic $\pi_1(\Sigma)$-action.

This allows us to construct the vector bundles over $\mathcal{T}$ as the sheaf theoretic quotient of

$$\mathcal{T} \times \tilde{\Sigma} \times \mathbb{C}^n$$

by the $\tilde{\pi}_1(\Sigma)$-action, given by the $\pi_1(\Sigma)$-action on $\mathcal{T} \times \tilde{\Sigma}$, and the unitary action on $\mathbb{C}^n$ given by our fixed representation $\rho_E : \tilde{\pi}_1(\Sigma) \to \mathbf{U}(n)$ (See [Mehta and Seshadri, 1980] for details on this construction, here we are simply composing representations with the natural quotient map from $\pi_1(\Sigma - \{p\})$ to $\tilde{\pi}_1(\Sigma)$ to match the setting of our paper to a special case of the setting in [Mehta and Seshadri, 1980]). The action is of course holomorphic, and so the quotient (fiberwise invariant sections over $\mathcal{T}$) is a family of Riemann surfaces with a holomorphic vector bundle over it of rank n and degree d. The universal property of $\mathcal{M}^s$ implies, that this family therefore induced a holomorphic section

$$\iota_{\rho_E} : \mathcal{T} \to \mathcal{M}^s.$$

This shows that the horizontal sections are holomorphic submanifolds, and so the tangent space must split at every point as $I \oplus J_\sigma$. Here J_σ must be $-\star_\sigma$, since it comes from the structure of the fibers. ■

The conclusion is, that the algebraic complex structure on the moduli space of pairs of a Riemann surface and a holomorphic vector bundle over it and the complex structure from [Andersen et al., 2012] on $\mathcal{T} \times M'$ are the same.

5.3 Coordinates for the Universal Moduli Space of Holomorphic Vector Bundles

In this section we prove Theorem 5.1.1.

We will need the composition of the map $\Phi^{\mu\oplus\nu}$ with the projection on each of the two factors, which we denote as follows:

$$\Phi_1^{\mu\oplus\nu} : \mathbb{H} \times \mathbf{GL}(n, \mathbb{C}) \to \mathbb{H},$$
$$\Phi_2^{\mu\oplus\nu} : \mathbb{H} \times \mathbf{GL}(n, \mathbb{C}) \to \mathbf{GL}(n, \mathbb{C}).$$

In fact $\Phi_1^{\mu\oplus\nu}$ is the projection onto $\mathbb{H}$ followed by the induced map on the base by (5.6) below.

The equation (5.3) is equivalent to the following two equations on $\Phi_i^{\mu\oplus\nu}$:

$$\bar{\partial}_{\mathbb{H}}\Phi_1^{\mu\oplus\nu}(z,g) = \mu\partial_{\mathbb{H}}\Phi_1^{\mu\oplus\nu}(z,g), \tag{5.4}$$

$$\bar{\partial}_{\mathbb{H}}\Phi_2^{\mu\oplus\nu}(z,g) = \mu\partial_{\mathbb{H}}\Phi_2^{\mu\oplus\nu}(z,g) + \partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_2^{\mu\oplus\nu}(z,g)\nu, \tag{5.5}$$

since $\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_1^{\mu\oplus\nu}(z,g) = 0$. With this simplification the first equation is exactly Bers's equation for

$$\Phi_1^{\mu}(z) = \Phi_1^{\mu\oplus\nu}(z,g), \tag{5.6}$$

and so we can solve it using the techniques in [Ahlfors and Bers, 1960], and we obtain a Riemann surface, X_{ρ_μ}, corresponding to a representation, ρ_μ.

The second equation (5.5) we solve in two steps. First, we identify ν with an endomorphism valued 1-form using the standard identification of left-invariant vector fields and elements of the Lie algebra. To solve the equation we consider the antiholomorphic solution of the equation

$$\bar{\partial}_{\mathbb{H}}\Phi_-^{\nu}(z,e) = \partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^{\nu}(z,g)(\nu)|_{g=e} = \Phi_-^{\nu}(z,e)\cdot\nu$$

and extend it equivariantly to the rest of $\mathbb{H}\times\mathbf{GL}(n,\mathbb{C})$. We observe that $\partial_{\mathbb{H}}\Phi_-^{\nu} = 0$, since it is antiholomorphic. And so it follows, by adding zero to the defining equation of Φ_-^{ν}, that:

$$\bar{\partial}_{\mathbb{H}}\Phi_-^{\nu}(z,g) = \partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^{\nu}(z,g)(\nu) + \mu\partial_{\mathbb{H}}\Phi_-^{\nu}(z,g).$$

The vector bundle on X_{ρ_μ} corresponding to the representation

$$\chi^{\nu}(\gamma) = \Phi_-^{\nu}(\rho_0(\gamma)z,e)\rho_E^{0\oplus 0}(\gamma)(\Phi_-^{\nu}(z,e))^{-1}$$

is stable, if $\mu\oplus\nu$ is small enough. This means, we can find a holomorphic gauge transformation on the universal cover of X_{ρ_μ}, $\Phi_+^{\mu\oplus\nu}:\mathbb{H}\to\mathbf{GL}(n,\mathbb{C})$, such that

$$\rho_E^{\mu\oplus\nu}(\gamma) = \Phi_+^{\mu\oplus\nu}(\rho_\mu(\gamma)z)\chi^{\mu\oplus\nu}(\gamma)(\Phi_+^{\mu\oplus\nu}(z))^{-1} \tag{5.7}$$

is an admissible $\mathbf{U}(n)$-representation and independent of z by the Narasimhan-Seshadri theorem [Narasimhan and Seshadri, 1964]. Now we use the basemap to define $\tilde{\Phi}_+^{\mu\oplus\nu} = \Phi_+^{\mu\oplus\nu}\circ\Phi_1^{\mu}$.

The following computation shows that the map $\tilde{\Phi}_+^{\mu\oplus\nu}$ is in the kernel of $\bar{\partial}_{\mathbb{H}} - \mu\partial_{\mathbb{H}}$:

$$\begin{aligned}(\bar{\partial}_{\mathbb{H}} - \mu\partial_{\mathbb{H}})\tilde{\Phi}_+^{\mu\oplus\nu} &= (\bar{\partial}_{\mathbb{H}}\Phi_+^{\mu\oplus\nu})\circ\Phi_1^{\mu}\bar{\partial}_{\mathbb{H}}\bar{\Phi}_1^{\mu} + (\partial_{\mathbb{H}}\Phi_+^{\mu\oplus\nu})\circ\Phi_1^{\mu}\bar{\partial}_{\mathbb{H}}\Phi_1^{\mu}\\ &\quad - \mu(\bar{\partial}_{\mathbb{H}}\Phi_+^{\mu\oplus\nu})\circ\Phi_1^{\mu}\partial_{\mathbb{H}}\bar{\Phi}_1^{\mu} - \mu(\partial_{\mathbb{H}}\Phi_+^{\mu\oplus\nu})\circ\Phi_1^{\mu}\partial_{\mathbb{H}}\Phi_1^{\mu}.\end{aligned}$$

We then use the differential equation $\bar{\partial}\Phi_1^\mu = \mu\partial\Phi_1^\mu$ and that $\bar{\partial}_{\mathbb{H}}\Phi_+^{\mu\oplus\nu} = 0$ to get that

$$(\bar{\partial}_{\mathbb{H}} - \mu\partial_{\mathbb{H}})\tilde{\Phi}_+^{\mu\oplus\nu} = \partial_{\mathbb{H}}\Phi_+^{\mu\oplus\nu} \circ \Phi_1^\mu \mu\partial_{\mathbb{H}}\Phi_1^\mu - \mu\partial_{\mathbb{H}}\Phi_+^{\mu\oplus\nu} \circ \Phi_1^\mu\partial_{\mathbb{H}}\Phi_1^\mu = 0.$$

Define $\Phi_2^{\mu\oplus\nu}(z,g) = \tilde{\Phi}_+^{\mu\oplus\nu}(z,g)\Phi_-^\nu(z,g)$. We see that $\Phi_2^{\mu\oplus\nu}$ fulfills equation (5.5) by the following calculation

$$\begin{aligned}\bar{\partial}_{\mathbb{H}}\Phi_2^{\mu\oplus\nu} &= (\bar{\partial}_{\mathbb{H}}\tilde{\Phi}_+^{\mu\oplus\nu})(\Phi_-^\nu) + (\tilde{\Phi}_+^{\mu\oplus\nu})(\bar{\partial}_{\mathbb{H}}\Phi_-^\nu) \\ &= (\bar{\partial}_{\mathbb{H}}\tilde{\Phi}_+^{\mu\oplus\nu})(\Phi_-^\nu) + (\tilde{\Phi}_+^{\mu\oplus\nu})(\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^\nu\nu),\end{aligned}$$

since $\tilde{\Phi}_+^{\mu\oplus\nu} \in \ker(\bar{\partial}_{\mathbb{H}} - \mu\partial_{\mathbb{H}})$ we get that

$$\bar{\partial}_{\mathbb{H}}\Phi_2^{\mu\oplus\nu} = (\mu\partial_{\mathbb{H}}\tilde{\Phi}_+^{\mu\oplus\nu})(\Phi_-^\nu) + (\tilde{\Phi}_+^{\mu\oplus\nu})(\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^\nu\nu).$$

To finish the calculation we use that Φ_+ and Φ_1 are independent of the $\mathbf{GL}(n,\mathbb{C})$ factor, and therefore so is $\tilde{\Phi}_+^{\mu\oplus\nu}$. Also $\Phi_-^{\mu\oplus\nu}$ is antiholomorphic, so we have that

$$\begin{aligned}\bar{\partial}_{\mathbb{H}}\Phi_2^{\mu\oplus\nu} &= \mu\partial_{\mathbb{H}}(\tilde{\Phi}_+^{\mu\oplus\nu}\Phi_-^\nu) + \partial_{\mathbf{GL}(n,\mathbb{C})}(\tilde{\Phi}_+^{\mu\oplus\nu}\Phi_-^\nu)\nu \\ &= \mu\partial_{\mathbb{H}}\Phi_2^{\mu\oplus\nu} + (\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_2^{\mu\oplus\nu})\nu.\end{aligned}$$

To show that we still get an admissible representation, we use that (5.7) is independent of which z we choose. This lets us conclude that

$$\begin{aligned}\rho_E^{\mu\oplus\nu}(\gamma) =& \Phi_+^{\mu\oplus\nu}(\rho_\mu(\gamma)\Phi_1^\mu(z))\chi^{\mu\oplus\nu}(\gamma)(\Phi_+^{\mu\oplus\nu}(\Phi_1^\mu(z)))^{-1} \\ =& \Phi_+^{\mu\oplus\nu}(\Phi_1^\mu(\rho_0(\gamma)(\Phi_1^\mu)^{-1}(\Phi_1^\mu(z))))\chi^{\mu\oplus\nu}(\gamma)(\Phi_+^{\mu\oplus\nu}(\Phi_1^\mu(z)))^{-1} \\ =& \tilde{\Phi}_+^{\mu\oplus\nu}(\rho_0(\gamma)z)\chi^{\mu\oplus\nu}(\gamma)(\tilde{\Phi}_+^{\mu\oplus\nu}(z))^{-1},\end{aligned}$$

and so

$$\rho_E^{\mu\oplus\nu}(\gamma) = \Phi_2^{\mu\oplus\nu}(\rho_0(\gamma)z,g)\rho_E^{0\oplus 0}(\gamma)(\Phi_2^{\mu\oplus\nu}(z,g))^{-1}$$

is an admissible $\mathbf{U}(n)$-representation. Finally, the requirement that $\Phi_2^{\mu\oplus\nu}(z_0,e)$ is a positive definite matrix of determinant 1 fixes all remaining indeterminacy as in [Zograf and Takhtadzhyan, 1989].

5.3.1 The Tangent Map from Kodaira-Spencer Theory

We will now analyse the tangential map of our coordinates. The only problematic part is what happens in the tangent directions parallel to the fibers. We can calculate the Kodaira-Spencer map of the family of representations $\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}, t \in \mathbb{C}$. However, to ease the computation we first prove the following lemma.

Lemma 5.3.1
We let X_{ρ_0} be a Riemann surface and ρ_0 the corresponding representation of $\pi_1(X_{\rho_0})$. For a family of representations of $R_t : \tilde{\pi}_1(X_{\rho_0}) \to \mathbf{U}(n)$, where

$$R_t(\gamma) = \Upsilon(t, \rho_0(\gamma)z)\rho_E(\gamma)\Upsilon(t, z)^{-1}$$

with both ρ_0 and ρ_E independent of t and Υ any smooth map

$$\Upsilon : \mathbb{C} \times \mathbb{H} \to \mathbf{GL}(n, \mathbb{C}),$$

we have that the Kodaira-Spencer class's harmonic representative of the family R_t at $t = 0$ is:

$$P^{0,1}_{\rho_0,E}\left(Ad\Upsilon(0,z)\left(\left.\frac{d}{dt}\right|_{t=0}\Upsilon(t,z)^{-1}\bar{\partial}_{\mathbb{H}}\Upsilon(t,z)\right)\right) \in H^{0,1}(X_0, EndE_{R_0}).$$

Here $P^{0,1}_{\rho_0,E}$ denotes the projection on the harmonic forms on X_{ρ_0} with values in $\mathrm{End}E_{R_0}$.

Proof: To compute the Kodaira-Spencer map we first consider $\left.\frac{d}{dt}\right|_{t=0} R_t$ and note, this is an element of $H^1(X, \mathrm{End}(E))$. However, this cohomology group is isomorphic to $H^{0,1}(X, \mathrm{End}E)$. The isomorphism is constructed by finding a Čech chain with values in the sheaf $\Omega^1(\mathrm{End}E)$, say φ_i, such that

$$\delta^*(\varphi)_{ij} = \varphi_i - \varphi_j = \left.\frac{d}{dt}\right|_{t=0} R_t(\gamma_{ij}),$$

for open sets $U_i \cap U_j \neq \emptyset$ which are related by the transformation $\gamma_{ij} \in \tilde{\pi}_1(\Sigma)$ on the universal cover. Once φ_i has been found, $P^{0,1}_{\rho_0,E}(\bar{\partial}_{\mathbb{H}}\varphi_i)$ will give a harmonic representative of the Kodaira-Spencer class.

We can now calculate that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} R_t(\gamma_{ij}) &= \left.\frac{d}{dt}\right|_{t=0} \Upsilon(t, \rho_0(\gamma)z)\rho_E(\gamma)\Upsilon(t, z)^{-1} \\
&= \left.\frac{d}{dt}\right|_{t=0} \Upsilon(t, \rho_0(\gamma)z)\rho_E(\gamma)\Upsilon(0, z)^{-1} \\
&\quad + \Upsilon(0, \rho_0(\gamma)z)\rho_E(\gamma) \left.\frac{d}{dt}\right|_{t=0} \Upsilon(t, z)^{-1} \\
&= \left.\frac{d}{dt}\right|_{t=0} (\Upsilon(t, \rho_{\mathbb{H}}(\gamma_{ij})z)\Upsilon(0, \rho_0(\gamma_{ij})z)^{-1})R_0(\gamma_{ij}) \\
&\quad - R_0(\gamma_{ij}) \left.\frac{d}{dt}\right|_{t=0} (\Upsilon(t, z)\Upsilon(0, z)^{-1}) \\
&= R_0(\gamma_{ij})\delta(\left.\frac{d}{dt}\right|_{t=0} (\Upsilon(t, z)\Upsilon(0, z)^{-1}))_{ij}.
\end{aligned}$$

The Kodaira-Spencer class is then:

$$\bar{\partial}_{\mathbb{H}} \frac{d}{dt}\Big|_{t=0} (\Upsilon(t,z)\Upsilon(0,z)^{-1}) = \mathrm{Ad}\Upsilon(0,z)\left(\frac{d}{dt}\Big|_{t=0} \Upsilon(t,z)^{-1}\bar{\partial}_{\mathbb{H}}\Upsilon(t,z)\right).$$

We compose with the harmonic projection to get the harmonic representative. ∎

We have the following proposition.

Proposition 5.3.2
The Kodaira-Spencer map of $\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}, t \in \mathbb{C}$, at $\mu\oplus\nu \in H^{0,1}(X,TX)\oplus H^{0,1}(X,\mathrm{End}E)$,

$$KS_{\mu\oplus\nu} : H^{0,1}(X,TX)\oplus H^{0,1}(X,\mathrm{End}E) \to H^{0,1}(X_{\rho_\mu},TX_{\rho_\mu})\oplus H^{0,1}(X_{\rho_\mu},\mathrm{End}E_{\rho_E^{\mu\oplus\nu}})$$

is given by

$$KS_{\mu\oplus\nu}(\tilde{\mu}\oplus\tilde{\nu}) = P_\mu\tilde{\mu}^\mu \oplus P^{0,1}_{\mu\oplus\nu}\left((\Phi_1^\mu)_*^{-1}\left(Ad\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}+\tilde{\nu}\right)\right)\right).$$

Here $\tilde{\mu}^\mu = (\frac{\tilde{\mu}}{1-|\mu|^2}\frac{\partial\Phi_1^\mu}{\overline{\partial\Phi_1^\mu}})\circ(\Phi_1^\mu)^{-1}$, and P_μ and $P^{0,1}_{\mu\oplus\nu}$ are the L^2-projections on the harmonic forms $H^{0,1}(X_{\rho_\mu},TX_{\rho_\mu})$ respectively $H^{0,1}(X_{\rho_\mu},\mathrm{End}E_{\rho_E^{\mu\oplus\nu}})$.

Proof: By using that the defining equation (5.2) for $\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}$ is independent of z, we get that

$$\begin{aligned}\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}} &= \Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}(\rho_0(\gamma)z,e)\rho_E^{0\oplus 0}(\gamma)\Phi_2^{\mu\oplus\nu}(z,e)^{-1}\\ &= \Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}((\Phi_1^\mu)^{-1}(\rho_\mu(\gamma)z),e)\rho_E^{0\oplus 0}(\gamma)\\ &\quad\cdot\Phi_2^{\mu\oplus\nu}((\Phi_1^\mu)^{-1}(z),e)^{-1}.\end{aligned}$$

And so to find the Kodaira-Spencer class, by Lemma 5.3.1 we only need to calculate:

$$\begin{aligned}&\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\circ(\Phi_1^\mu)^{-1}\frac{d}{dt}|_{t=0}(\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}\circ(\Phi_1^\mu)^{-1})^{-1}\bar{\partial}(\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}\circ(\Phi_1^\mu)^{-1})\\ &=\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\circ(\Phi_1^\mu)^{-1}\\ &\quad\cdot\frac{d}{dt}|_{t=0}((t\tilde{\mu}(\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}})^{-1}\partial\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}})\circ\Phi_1^\mu)^{-1}\bar{\partial}(\bar{\Phi}_1^\mu)^{-1}\\ &\quad+\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\circ(\Phi_1^\mu)^{-1}\frac{d}{dt}|_{t=0}(\nu+t\tilde{\nu})\circ(\Phi_1^\mu)^{-1}\bar{\partial}(\bar{\Phi}_1^\mu)^{-1}\\ &=(\Phi_1^\mu)_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}+\tilde{\nu}\right)\right).\end{aligned}\tag{5.8}$$

Now, to get the Kodaira-Spencer map we project on the harmonic (0,1)-forms and remark, that in the Teichmüller directions we can apply the usual arguments from the classical case of Bers's coordinates. ∎

We see the map is injective and complex linear in both $\tilde{\mu}$ and $\tilde{\nu}$. Since we know $\mathcal{T} \times M'$ is a manifold the Implicit Function Theorem now implies that the coordinates we constructed are in fact holomorphic coordinates in a small neighborhood. This completes the proof of Theorem 5.1.2.

5.4 The Fibered Coordinates

In this section we will fuse Zograf's and Takhtadzhyan's coordinates with Bers's coordinates in a kind of fibered manner in order also to produce coordinates on $\mathcal{T} \times M'$, which are complex analytic with respect to J.

Since through any stable bundle we have a copy of $\mathcal{T}$ embedded as a complex submanifold, we can construct fibered coordinates once we identify the tangent spaces in the fiber direction locally along these copies of $\mathcal{T}$. We identify them by the maps

$$H^{0,1}(X_{\rho_0}, \mathrm{End}E_{\rho_E^0}) \ni \nu \to \nu^\mu = P^{0,1}_{\mu,E}((\Phi_1^\mu)^{-1}_*(\nu)) \in H^{0,1}(X_{\rho_\mu}, \mathrm{End}E_{\rho_E^{0\mu}}).$$

This identification gives us coordinates taking (μ, ν) to

$$(\rho_\mu, \rho_E^{\nu^\mu}) = \left(\Phi_1^\mu \circ \rho_0(\gamma) \circ (\Phi_1^\mu)^{-1}, f^{\nu^\mu}(\rho_\mu(\gamma)z)\rho_E^{0^0}(\gamma)(f^{\nu^\mu}(z))^{-1}\right).$$

These are complex coordinates, since ν^μ are local holomorphic sections of the tangent bundle.

Before we calculate the Kodaira-Spencer maps for these coordinate curves, we will need to understand the derivatives of $(\Phi_1^\mu)^{-1}$.

Lemma 5.4.1
We have the following two identities for $(\Phi_1^\mu)^{-1} : \mathbb{H} \to \mathbb{H}$

$$\bar{\partial}(\Phi_1^\mu)^{-1} = -\mu \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}}, \tag{5.9}$$

$$\bar{\partial}\overline{(\Phi_1^\mu)^{-1}} = \left(\frac{1}{1-|\mu|^2} \frac{1}{\overline{\partial \Phi_1^\mu}}\right) \circ (\Phi_1^\mu)^{-1}. \tag{5.10}$$

Proof: We consider the identity $\Phi_1^\mu \circ (\Phi_1^\mu)^{-1}(z) = z$. And we use the differential equation for Φ_1^μ, which is

$$\bar{\partial}\Phi_1^\mu = \mu \partial \Phi_1^\mu,$$

to calculate:

$$\begin{aligned} 0 = \bar{\partial}(\Phi_1^\mu \circ (\Phi_1^\mu)^{-1}) &= (\bar{\partial}\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial \Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}(\Phi_1^\mu)^{-1} \\ &= (\mu \partial \Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial \Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}(\Phi_1^\mu)^{-1}. \end{aligned}$$

Now $\partial \Phi_1^\mu \neq 0$ for μ small, since we perturb the map $z : \mathbb{H} \to \mathbb{H}$, we have that

$$-\mu \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} = \bar{\partial}(\Phi_1^\mu)^{-1},$$

which is (5.9). We can use (5.9) to describe $\bar{\partial}\overline{(\Phi_1^\mu)^{-1}}$ by differentiating $\Phi_1^\mu \circ (\Phi_1^\mu)^{-1} = z$:

$$\begin{aligned} 1 = \partial(\Phi_1^\mu \circ (\Phi_1^\mu)^{-1}) &= (\bar{\partial}\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1}\partial\overline{(\Phi_1^\mu)^{-1}} + (\partial\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1}\partial(\Phi_1^\mu)^{-1} \\ &= -\mu(\partial\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1}\bar{\mu}\bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1}\partial(\Phi_1^\mu)^{-1} \\ &= ((1 - |\mu|^2)(\partial\Phi_1^\mu)) \circ (\Phi_1^\mu)^{-1}\partial(\Phi_1^\mu)^{-1}, \end{aligned}$$

and so conjugating and isolating $\bar{\partial}\overline{(\Phi_1^\mu)^{-1}}$ we find:

$$\bar{\partial}(\overline{\Phi_1^\mu})^{-1} = \left(\frac{1}{1 - |\mu|^2}\frac{1}{\overline{\partial\Phi_1^\mu}}\right) \circ (\Phi_1^\mu)^{-1},$$

which proves (5.10). ■

Let κ_μ be an (n, m)-tensor with values in a holomorphic bundle, $E_{\rho_E^{0\mu}}$, on the Riemann surface, X_{ρ_μ}, i.e.

$$\kappa_\mu \in C^\infty(X_{\rho_\mu}, T^{-n}X_{\rho_\mu} \otimes \overline{T}^{-m}X_{\rho_\mu} \otimes \mathrm{End}E_{\rho_E^{0\mu}}).$$

Then we define

$$(\Phi_1^\mu)_*(\kappa_\mu) = (\kappa_\mu \circ \Phi_1^\mu)(\partial\Phi_1^\mu)^n(\overline{\partial\Phi_1^\mu})^m,$$

and so

$$(\Phi_1^\mu)_*^{-1}(\kappa_0) = (\kappa_0 \circ (\Phi_1^\mu)^{-1})(\partial\Phi_1^\mu)^{-n}(\overline{\partial\Phi_1^\mu})^{-m}.$$

We have the families of unbounded operators

$$\begin{aligned} \bar{\partial}_{\mu,E_{\rho_E^{0\mu}}} &: L^2(X_{\rho_\mu}, \mathrm{End}E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, \Omega^{0,1} \otimes \mathrm{End}E_{\rho_E^{0\mu}}), \\ \bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}} &: L^2(X_{\rho_\mu}, \Omega^{0,1} \otimes \mathrm{End}E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, \mathrm{End}E_{\rho_E^{0\mu}}), \\ \Delta_{\mu,E_{\rho_E^{0\mu}}} &= \bar{\partial}^*\bar{\partial} : L^2(X_{\rho_\mu}, \mathrm{End}E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, \mathrm{End}E_{\rho_E^{0\mu}}), \end{aligned}$$

and the finite range operator

$$\begin{aligned} P^{0,1}_{\mu,E_{\rho_E^{0\mu}}} &: L^2(X_{\rho_\mu}, \Omega^{0,1} \otimes \mathrm{End}E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, \Omega^{0,1} \otimes \mathrm{End}E_{\rho_E^{0\mu}}) \\ P^{0,1}_{\mu,E_{\rho_E^{0\mu}}} &= I - \bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}, \end{aligned}$$

where $\Delta_{0,\mu,E_{\rho_E^{0\mu}}}$ is the restriction of $\Delta_{\mu,E_{\rho_E^{0\mu}}}$ to the complement of the constant function, and $P^{0,1}$ is the projection on the harmonic $(0, 1)$-forms. We will also need the following results:

Lemma 5.4.2 ([Takhtajan and Zograf, 1991])
We have the following variational formulae for the derivative at X_{ρ_0}:

$$\frac{d}{dt}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1} = -\tilde{\mu}\partial_{0,E}, \qquad \frac{d}{d\bar{t}}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1} = 0,$$

$$\frac{d}{dt}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}^*_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1}=0,\qquad \frac{d}{dt}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}^*_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1}=-\partial^*_{0,E}\overline{\tilde{\mu}}.$$

And at $(X_{\rho_\mu},E_{\rho_E^{0\mu}})$ *we also have that:*

$$\begin{aligned}
&\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad=(\Phi_1^{\mu})_*P^{0,1}_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad\quad(\Phi_1^{\mu})_*\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\\
&\qquad+(\Phi_1^{\mu})_*\bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}|_{t=0}\\
&\qquad(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}^*_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\frac{d}{dt}(\Phi_1^{\mu})_*P^{0,1}_{\mu,E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu})_*^{-1}.
\end{aligned}$$

Proof: The first identities are proven in [Takhtajan and Zograf, 1991, Equation (2.6)] (without the End$E$, which makes no difference), the last statement is seen straightforwardly:

$$\begin{aligned}
&\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad=\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta^{-1}_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}^*_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}.
\end{aligned}$$

We can then use the following identities

$$\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta^{-1}_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}=-(\Phi_1^{\mu})_*\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}$$

and

$$\begin{aligned}
&\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\Phi_1^{\mu})_*\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\\
&\qquad\qquad\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&=\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\Phi_1^{\mu})_*\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\\
&\quad+\frac{d}{dt}|_{t=0}(\Phi_1^{\mu})_*\bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}^*_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}.
\end{aligned}$$

Now, putting this together and using that

$$P^{0,1}_{\mu,E_{\rho_E^{0\mu}}}=I-\bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}},$$

we have the last identity. ■

Proposition 5.4.3
The Kodaira-Spencer map of the curve $\rho_E^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}$ *at* $t=0$ *is:*

$$\begin{aligned} KS_{\nu^\mu}(\tilde{\mu}\oplus\tilde{\nu}) =& P_\mu\tilde{\mu}^\mu \oplus P^{0,1}_{\nu^\mu}\left(Ad(f^{\nu^\mu})((f^{\nu^\mu})^{-1}\cdot(\partial f^{\nu^\mu})\tilde{\mu}^\mu+\tilde{\nu}^\mu)\right.\\ &+Adf^{\nu^\mu}(\Phi_1^\mu)^{-1}_*\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu}}(\Phi_1^{\mu+t\tilde{\mu}})^{-1}_*(\nu)(1-|\mu\circ(\Phi_1^\mu)^{-1}|^2)), \end{aligned}$$

with $\tilde{\mu}^\mu = (\frac{\tilde{\mu}}{1-|\mu|^2}\frac{\partial\Phi_1^\mu}{\overline{\partial\Phi_1^\mu}})\circ(\Phi_1^\mu)^{-1}$ *and* P_μ *and* $P^{0,1}_{\nu^\mu}$ *the* L^2*-projections on the harmonic forms* $\mathcal{H}^{0,1}(X_{\rho_\mu},TX_{\rho_\mu})$ *respectively* $\mathcal{H}^{0,1}(X_{\rho_\mu},\mathrm{End}E_{\rho_E^{\nu^\mu}})$.

PROOF: First, we observe that the Teichmüller direction is unchanged from the classical case. Then we want to use Lemma 5.3.1, and so using that $\rho_E^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}$ is independent of z we find:

$$\begin{aligned} \rho_E^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(\gamma) &= f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(\rho_{\mu+t\tilde{\mu}}(\gamma)z)\rho_E(\gamma)(f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(z))^{-1}\\ &= f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(\Phi_1^{\mu+t\tilde{\mu}}((\Phi_1^\mu)^{-1}(\rho_\mu(\gamma)z)))\rho_E(\gamma)\\ &\quad (f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(\Phi_1^{\mu+t\tilde{\mu}}((\Phi_1^\mu)^{-1}(z))))^{-1}. \end{aligned}$$

Next, we have to calculate:

$$\begin{aligned} &\mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\left(\left((f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}\circ\Phi_1^{\mu+t\tilde{\mu}})\circ(\Phi_1^\mu)^{-1}\right)\right.\\ &\qquad\left.\bar{\partial}\left((f^{\nu^{\mu+t\tilde{\mu}}})^{-1}\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\right)\right)\\ &=\mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\left(\left((\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\right)\right.\\ &\qquad\left.\overline{(\partial(\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}))}\right)\\ &\quad+\mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\Big(\left(f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\right)^{-1}\\ &\qquad\cdot(\partial f^{\nu^\mu})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}(\bar{\partial}(\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}))\Big). \end{aligned}$$

For the first term we find:

$$\begin{aligned} &\mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\left(\left(((\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\right)\overline{(\partial(\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}))}\right)\\ &=\frac{d}{dt}|_{t=0}\mathrm{Ad}(f^{\nu^\mu})((\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\\ &\quad\cdot\overline{\left((\partial\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\Phi_1^\mu)^{-1}+(\bar{\partial}\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\bar{\Phi}_1^\mu)^{-1}\right)}. \end{aligned}$$

We can now rewrite the last factor using (5.9) and (5.10) and their conjugates:

$$\begin{aligned} &(\partial\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\Phi_1^\mu)^{-1}+(\bar{\partial}\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\bar{\Phi}_1^\mu)^{-1}\\ &\quad=(\partial\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\Phi_1^\mu)^{-1} \end{aligned}$$

$$+ ((((\mu + t\tilde{\mu})\partial\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^{\mu})^{-1})(-\bar{\mu} \circ (\Phi_1^{\mu})^{-1}\partial(\Phi_1^{\mu})^{-1})$$
$$= (\partial\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1})\partial(\Phi_1^{\mu})^{-1}(1 - (((\mu + t\tilde{\mu})\bar{\mu}) \circ (\Phi_1^{\mu})^{-1})).$$

We have that the pushforward of a $(0,1)$-form $(\Phi_1^{\mu})_*(\nu) = \nu \circ \Phi_1^{\mu}\overline{\partial\Phi_1^{\mu}}$, and so using this for the first term we have:

$$\begin{aligned}
&\mathrm{Ad}(f^{\nu^{\mu}})\frac{d}{dt}|_{t=0}\left(\left(((\nu + t\tilde{\nu})^{\mu+t\tilde{\mu}}) \circ \Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1}\right)\left(\overline{\partial(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1})}\right)\right)\\
&= \mathrm{Ad}(f^{\nu^{\mu}})\frac{d}{dt}|_{t=0}\left((\Phi_1^{\mu})_*^{-1}\left((\Phi_1^{\mu+t\tilde{\mu}})_*\left((\nu + t\tilde{\nu})^{\mu+t\tilde{\mu}}\right)(1 - |\mu|^2 - \bar{t}\mu\overline{\tilde{\mu}})\right)\right)\\
&= \mathrm{Ad}(f^{\nu^{\mu}})(\Phi_1^{\mu})_*^{-1}\left((1 - |\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\right.\\
&\qquad\qquad\left.\left(P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu + t\tilde{\nu})\right)\right)\\
&= \mathrm{Ad}(f^{\nu^{\mu}})P^{0,1}_{\mu,E_{\rho_E^{0\mu}}}\left((\Phi_1^{\mu})_*^{-1}(\tilde{\nu}) + \tilde{\mu}^{\mu}\partial_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}(\nu)\right)\\
&\qquad\qquad\left((1 - |\mu|^2)\right) \circ (\Phi_1^{\mu})^{-1}.
\end{aligned}$$

Here we used the result from Lemma 5.4.2 to calculate the derivative of the projection.

For the second term we rewrite

$$\begin{aligned}
&\overline{\partial}(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1})\\
&= (\bar{\partial}\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^{\mu})^{-1}\bar{\partial}\overline{(\Phi_1^{\mu})^{-1}} + (\partial\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^{\mu})^{-1}\bar{\partial}(\Phi_1^{\mu})^{-1}\\
&= ((\mu + t\tilde{\mu})\partial\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^{\mu})^{-1}\bar{\partial}\overline{(\Phi_1^{\mu})^{-1}} + (\partial\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^{\mu})^{-1}\bar{\partial}(\Phi_1^{\mu})^{-1} \quad (5.11)
\end{aligned}$$

using (5.10) and (5.9) in (5.11) and find:

$$\overline{\partial}(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1}) = \left(\frac{t\tilde{\mu}}{1 - |\mu|^2}\frac{\partial\Phi_1^{\mu+t\tilde{\mu}}}{\overline{\partial\Phi_1^{\mu}}}\right) \circ (\Phi_1^{\mu})^{-1},$$

which implies that:

$$\begin{aligned}
&\mathrm{Ad}(f^{\nu^{\mu}})\frac{d}{dt}|_{t=0}\Big(\left(f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}} \circ \Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1}\right)^{-1}\\
&\qquad\qquad \cdot (\partial f^{\nu^{\mu}}) \circ \Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1}(\overline{\partial}(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^{\mu})^{-1}))\Big)\\
&\qquad = \mathrm{Ad}(f^{\nu^{\mu}})((f^{\nu^{\mu}})^{-1} \cdot (\partial f^{\nu^{\mu}})(\tilde{\mu}^{\mu})),
\end{aligned}$$

where $\tilde{\mu}^{\mu} = \left(\frac{\tilde{\mu}}{1-|\mu|^2}\frac{\partial\Phi_1^{\mu}}{\overline{\partial\Phi_1^{\mu}}}\right) \circ (\Phi_1^{\mu})^{-1}$. And so we have

$$\mathrm{Ad}(f^{\nu^{\mu}})\frac{d}{dt}|_{t=0}\left(\left((f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}} \circ \Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^{\mu})^{-1}\right)\right.$$

$$\begin{aligned}
&\bar{\partial}\left((f^{\nu^{\mu+t\tilde{\mu}}})^{-1}\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^{\mu})^{-1}\right)\\
=&\mathrm{Ad}(f^{\nu^{\mu}})\Big((\Phi_1^{\mu})_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}\\
&(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu)(1-|\mu|^2)\circ(\Phi_1^{\mu})^{-1})\Big)\\
&+\mathrm{Ad}(f^{\nu^{\mu}})\tilde{\nu}^{\mu}+\mathrm{Ad}(f^{\nu^{\mu}})((f^{\nu^{\mu}})^{-1}\cdot(\partial f^{\nu^{\mu}})(\tilde{\mu}^{\mu})). \qquad (5.12)
\end{aligned}$$

We have shown that composing with the projection gives us the harmonic representative. ■

5.4.1 Comparison of the Two Tangent Maps and a Proof of the First Part of Theorem 5.1.3

We compare

$$P^{0,1}_{\mu\oplus\nu,}\left((\Phi_1^{\mu})_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}+\tilde{\nu}\right)\right)\right)$$

and

$$\begin{aligned}
&P^{0,1}_{\nu^{\mu}}\left(\mathrm{Ad}(f^{\nu^{\mu}})((f^{\nu^{\mu}})^{-1}\cdot(\partial f^{\nu^{\mu}})\tilde{\mu}^{\mu}+\tilde{\nu}^{\mu}\right)\\
+&\mathrm{Ad}f^{\nu^{\mu}}(\Phi_1^{\mu})_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu)(1-|\mu\circ(\Phi_1^{\mu})^{-1}|^2)).
\end{aligned}$$

First, we observe that

$$\mathrm{Ad}f^{\nu^{\mu}}(\Phi_1^{\mu})_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu)(1-|\mu\circ(\Phi_1^{\mu})^{-1}|^2)$$

is a term that vanishes in ν and μ to first-order at the center, since either we differentiate with respect to μ and set $\nu=0$ or we differentiate with respect to ν and then we find, when we evaluate at $\mu=0$, that $\bar{\partial}^*_{0,E}\nu=0$, from the expression in Lemma 5.4.2.

Next, we compare

$$(\Phi_1^{\mu})_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}\right)\right)$$

with

$$\mathrm{Ad}(f^{\nu^{\mu}})((f^{\nu^{\mu}})^{-1}\cdot(\partial f^{\nu^{\mu}})\tilde{\mu}^{\mu}).$$

We observe, that since $\partial I=0$ both $(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}$ and $(f^{\nu^{\mu}})^{-1}\cdot(\partial f^{\nu^{\mu}})$ vanish unless we differentiate them with respect to the moduli space direction or the Teichmüller direction. If we differentiate with respect to μ we get $\frac{\partial}{\partial\varepsilon}\nu^{\varepsilon\mu}$, but at $\nu=0$ this is zero. This means we can compare the two after evaluating $\mu=0$, and then we have $f^{\nu^0}=\Phi^{0\oplus\nu}$, and so they agree to first-order.

The last terms to consider are $(\Phi_1^\mu)_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}(\tilde{\nu})\right)$ and $\mathrm{Ad}(f^{\nu^\mu})\tilde{\nu}^\mu$. Now, if we put $\mu=0$ the terms agree. If we differentiate with respect to μ, we can put $\nu=0$ first. We are differentiating a term of the form

$$\bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^\mu)_*^{-1}\tilde{\nu}$$

with respect to μ. The result is an exact term, which is killed by the harmonic projection $P^{0,1}$ out in front of our expressions, plus a term containing $\bar{\partial}^*_{0,E}\nu=0$. This proves the first part of Theorem 5.1.3. The second part will be proved in the following section.

5.5 Variation of the Metric

In order to settle the question whether our new coordinates are the same as the fibered coordinates discussed above, we shall consider the variation of the metric in both sets of coordinates and use the resulting formulae to demonstrate that they are not identical to third-order.

5.5.1 Variation in the Universal Coordinates

With the newly introduced coordinates we get a new tool to analyse the metric and the curvature of the moduli space. As a first step in understanding the curvature, we will calculate the second variation of the metric in local coordinates, at the center point. This section will use the coordinates from Theorem 5.1.1. In the next section we will do the same calculation for the fibered coordinates, and use this to show that the two sets of coordinates differ at third-order.

So first we consider the function $(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}$. This transforms as a function on X with values in $\mathrm{End}E$, our reference point. Now, to further understand this function, we look at $\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}$. Then we find:

$$\Delta\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}=\Delta(\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_+^{\varepsilon(\mu\oplus\nu)})^T+\frac{d}{d\varepsilon}|_{\varepsilon=0}\Phi_-^{\varepsilon(\mu\oplus\nu)}$$
$$+\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_-^{\varepsilon(\mu\oplus\nu)})^T+\frac{d}{d\varepsilon}|_{\varepsilon=0}\Phi_+^{\varepsilon(\mu\oplus\nu)}),$$

and since $\Phi_-^{\varepsilon(\mu\oplus\nu)}$ is antiholomorphic we get:

$$\Delta\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}=\frac{d}{d\varepsilon}|_{\varepsilon=0}\Delta(\bar{\Phi}_+^{\varepsilon(\mu\oplus\nu)})^T+\frac{d}{d\varepsilon}|_{\varepsilon=0}\Delta\Phi_+^{\varepsilon(\mu\oplus\nu)}.$$

We now use that $(\bar{\partial}-\varepsilon\mu\partial)\Phi_+^{\varepsilon(\mu\oplus\nu)}=0$ and $\Delta=y^{-2}\partial\bar{\partial}$ to see:

$$\Delta\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}=y^{-2}\bar{\mu}\bar{\partial}\bar{\partial}(\bar{\Phi}_+^{0(\mu\oplus\nu)})^T+y^{-2}\mu\partial\partial\Phi_+^{0(\mu\oplus\nu)}=0,$$

since $\Phi^0_+ = I$, and so the derivative is zero. This allows us to conclude, that $\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}$ is a constant multiple of the identity element in End$E$, and because of the determinant criteria in Theorem 5.1.1 we have

$$\begin{aligned}0 &= \frac{d}{d\varepsilon}|_{\varepsilon=0}(\det(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}) \\ &= \operatorname{tr}|_{\varepsilon=0}((\bar{\Phi}_2^0)^T\Phi_2^0)^{-1}\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)} = \operatorname{tr}\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)},\end{aligned}$$

and so $\frac{d}{d\varepsilon}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)} = 0$. We see that this immediately implies that $\frac{d}{d\varepsilon}|_{\varepsilon=0}\partial\Phi_+^{\varepsilon(\mu\oplus\nu)} = -\overline{\frac{d}{d\bar{\varepsilon}}|_{\varepsilon=0}\bar{\partial}\Phi_-^{\varepsilon(\mu\oplus\nu)}}^T = 0$.

We can study $\frac{d}{d\bar{\varepsilon}}|_{\varepsilon=0}(\bar{\Phi}_2^{\varepsilon(\mu\oplus\nu)})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}$ similarly and conclude that

$$\frac{d}{d\bar{\varepsilon}}|_{\varepsilon=0}\partial\Phi_+^{\varepsilon(\mu\oplus\nu)} = -\overline{\frac{d}{d\varepsilon}|_{\varepsilon=0}\bar{\partial}\Phi_-^{\varepsilon(\mu\oplus\nu)}}^T = -\bar{\nu}^T.$$

Next, we want to understand the variation of the $\bar{\partial}_{\mu,E_{\rho(\mu\oplus\nu)}}$-operator on functions and $\bar{\partial}^*_{\mu,E_{\rho(\mu\oplus\nu)}}$-operator on $(0,1)$-forms, since they play a central role in understanding the tangent spaces over the universal moduli space. We work on the universal cover and pull back our family of differential operators from the universal cover of $(X_{\rho_\mu}, E_{\rho(\mu\oplus\nu)})$ to (X_{ρ_0}, E), in terms of representations. Then $\bar{\partial}_{\mu,E_{\rho(\mu\oplus\nu)}}$ is just represented by $\bar{\partial}$ on $\mathbb{H}$:

$$\begin{aligned}&\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu}(\Phi_1^{\varepsilon\mu})_*\bar{\partial}(\Phi_1^{\varepsilon\mu})_*^{-1}(\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu})^{-1} \\ &\qquad= \frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu}\frac{1}{1-|\varepsilon\mu|^2}(\bar{\partial}-\mu\partial)(\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu})^{-1} \\ &\qquad= \frac{d}{d\varepsilon}|_{\varepsilon=0}\frac{1}{1-|\varepsilon\mu|^2}\big(\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu}(\mathrm{ad}(\bar{\partial}-\varepsilon\mu\partial)\Phi_2^{\varepsilon\mu\oplus\nu})(\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu})^{-1} \\ &\qquad\qquad+(\bar{\partial}-\varepsilon\mu\partial)\big) \\ &\qquad= \frac{d}{d\varepsilon}|_{\varepsilon=0}\frac{1}{1-|\varepsilon\mu|^2}(\varepsilon\mathrm{ad}\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu}\nu+(\bar{\partial}-\varepsilon\mu\partial)) \\ &\qquad= \mathrm{ad}\nu-\mu\partial. \end{aligned} \tag{5.13}$$

Likewise we find that the variation of $\bar{\partial}^* = -\rho^{-1}\partial$. We begin by observing that on $(0,1)$-forms we have:

$$\begin{aligned}(\Phi_1^{\varepsilon\mu})_*\partial(\Phi_1^{\varepsilon\mu})_*^{-1}\alpha &= (\Phi_1^{\varepsilon\mu})_*\partial(\alpha\circ(\Phi_1^{\varepsilon\mu})^{-1}\frac{1}{\overline{\bar{\partial}\Phi_1^{\varepsilon\mu}\circ(\Phi_1^{\varepsilon\mu})^{-1}}}) \\ &= (\Phi_1^{\varepsilon\mu})_*\Bigg((\partial\alpha)\circ(\Phi_1^{\varepsilon\mu})^{-1}\frac{\partial(\Phi_1^{\varepsilon\mu})^{-1}}{\overline{\bar{\partial}\Phi_1^{\varepsilon\mu}\circ(\Phi_1^{\varepsilon\mu})^{-1}}} \\ &\qquad+(\bar{\partial}\alpha)\circ(\Phi_1^{\varepsilon\mu})^{-1}\frac{\partial\overline{(\Phi_1^{\varepsilon\mu})^{-1}}}{\overline{\bar{\partial}\Phi_1^{\varepsilon\mu}\circ(\Phi_1^{\varepsilon\mu})^{-1}}}\end{aligned}$$

$$
\begin{aligned}
&- \alpha \circ (\Phi_1^{\varepsilon\mu})^{-1} \frac{\overline{(\partial(\bar\partial\Phi_1^{\varepsilon\mu})) \circ (\Phi_1^{\varepsilon\mu})^{-1} (\partial\bar\Phi_1^{\varepsilon\mu})^{-1}}}{(\bar\partial\overline{\Phi_1^{\varepsilon\mu} \circ (\Phi_1^{\varepsilon\mu})^{-1}})^2} \\
&- \alpha \circ (\Phi_1^{\varepsilon\mu})^{-1} \frac{\overline{(\partial\partial\Phi_1^{\varepsilon\mu}) \circ (\Phi_1^{\varepsilon\mu})^{-1} (\bar\partial\Phi_1^{\varepsilon\mu})^{-1}}}{(\bar\partial\overline{\Phi_1^{\varepsilon\mu} \circ (\Phi_1^{\varepsilon\mu})^{-1}})^2} \Bigg) \\
&= \frac{1}{1-|\varepsilon\mu|^2}(\partial - \bar\varepsilon\bar\mu\bar\partial - \bar\varepsilon(\bar\partial\bar\mu)) = \frac{1}{1-|\varepsilon\mu|^2}(\partial - \overline{\varepsilon\partial\mu}).
\end{aligned}
$$

And so we find:

$$
\begin{aligned}
&\frac{d}{d\bar\varepsilon}|_{\varepsilon=0} \mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu}(\Phi_1^{\varepsilon\mu})_* \bar\partial^* (\Phi_1^{\varepsilon\mu})_*^{-1} (\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu})^{-1} \\
&\quad = \frac{d}{d\bar\varepsilon}|_{\varepsilon=0} \mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu} \frac{-\rho^{-1}}{1-|\varepsilon\mu|^2}(\partial - \bar\varepsilon\bar\partial\bar\mu)(\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu})^{-1} \\
&\quad = \frac{d}{d\bar\varepsilon}|_{\varepsilon=0} \frac{-\rho^{-1}}{1-|\varepsilon\mu|^2} \big(\mathrm{Ad}\Phi_2^{\varepsilon\mu\oplus\nu}(\mathrm{Ad}(\partial - \bar\varepsilon\bar\partial\bar\mu)\Phi_2^{\varepsilon\mu\oplus\nu})^{-1} \\
&\qquad\qquad + (\bar\partial - \bar\varepsilon\bar\partial\bar\mu)\big) \\
&\quad = \frac{d}{d\bar\varepsilon}|_{\varepsilon=0} \frac{-\rho^{-1}}{1-|\varepsilon\mu|^2} (\mathrm{ad}(\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}) - \bar\varepsilon\bar\mu\rho^{-1}\mathrm{ad}\bar\partial\Phi_2^{\varepsilon(\mu\oplus\nu)} \\
&\qquad\qquad + (\partial - \bar\varepsilon\bar\partial\bar\mu)) \\
&\quad = - * \mathrm{ad}\nu * + \bar\mu\bar\partial\rho^{-1}, \qquad (5.14)
\end{aligned}
$$

where the equality follows from the equation $\partial\mu = 2y^{-1}\mu$, and $\rho^{-1} = y^2$. Alternativly, we could have used that the first derivative of density ρ is zero at the center point of our coordinates [Wolpert, 1986] to find this.

This is the first step in understanding the metric on the universal moduli space of pairs of a Riemann surface and a holomorphic bundle over it, given at a point (X, E) by identifying the tangent space with $\mathcal{H}^{0,1}(X, TX) \oplus \mathcal{H}^{0,1}(X, \mathrm{End}E)$. Two elements $\mu_1 \oplus \nu_1$ and $\mu_2 \oplus \nu_2$ can be paired as:

$$
g(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = \int_\Sigma (\rho_X \mu_1 \bar\mu_2 + i\mathrm{tr}\nu_1 \wedge \star\bar\nu_2^T),
$$

where ρ_X is the density of the hyperbolic metric corresponding to the complex structure on X. Since the term $\int_\Sigma \rho_X \mu_1 \bar\mu_2$, is independent of the bundle, nothing has changed compared to the situation on Teichmüller space. Let us examine the term $\int_\Sigma \mathrm{tr}\nu_1 \wedge \overline{(-\star)\nu_2^T}$, since we are evaluating the metric on tangent vectors, $-\star$ will act by $-i$, and so we replace it in the following to avoid confusion.

In coordinates around (X, E) we have, using Proposition 5.3.2, that the metric is given by:

$$
\begin{aligned}
&g^{VB}_{\varepsilon(\mu\oplus\nu)}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) \\
&\quad = -i \int_\Sigma \mathrm{tr} P^{0,1}_{\varepsilon(\mu\oplus\nu)} (\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1} \mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_1
\end{aligned}
$$

$$
\begin{aligned}
&\wedge \overline{P^{0,1}_{\varepsilon(\mu\oplus\nu)}((\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_2}^T \\
&- i\int_\Sigma \mathrm{tr}P^{0,1}_{\varepsilon(\mu\oplus\nu)}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_1(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)} \\
&\wedge \overline{P^{0,1}_{\varepsilon(\mu\oplus\nu)}((\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_2}^T \\
&- i\int_\Sigma \mathrm{tr}P^{0,1}_{\varepsilon(\mu\oplus\nu)}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_1 \\
&\wedge \overline{P^{0,1}_{\varepsilon(\mu\oplus\nu)}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}(\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_2(\Phi_2^{\varepsilon(\mu\oplus\nu)})\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T \\
&- i\int_\Sigma \mathrm{tr}P^{0,1}_{\varepsilon(\mu\oplus\nu)}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_1(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)} \\
&\wedge \overline{P^{0,1}_{\varepsilon(\mu\oplus\nu)}((\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_2(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T.
\end{aligned}
\tag{5.15}
$$

Now we can use that $P^{0,1}_{\varepsilon(\mu\oplus\nu)}$ is self-adjoint with respect to the metric to rewrite the terms as:

$$
\begin{aligned}
&\int_\Sigma \mathrm{tr}P^{0,1}_{\varepsilon(\mu\oplus\nu)}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_1 \wedge \overline{P^{0,1}_{\varepsilon(\mu\oplus\nu)}((\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_2}^T \\
&\quad = \int_\Sigma \mathrm{tr}\mathrm{Ad}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T\Phi_2^{\varepsilon(\mu\oplus\nu)})(1-|\varepsilon\mu|^2)\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1} \\
&\qquad (\Phi_1^{\varepsilon(\mu\oplus\nu)})_*P^{0,1}_{\varepsilon(\mu\oplus\nu)}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_1\wedge\overline{\nu_2}^T.
\end{aligned}
\tag{5.16}
$$

Since $(\Phi_1^{\varepsilon\mu})_*(d\bar z\wedge dz) = (|\partial\Phi_1^{\varepsilon\mu}|^2 - |\bar\partial\Phi_1^{\varepsilon\mu}|^2)d\bar z\wedge dz$. Also recall that

$$(\Phi_1^{\varepsilon\mu})_*^{-1}\nu = \left(\frac{\nu}{\bar\partial\bar\Phi_1^{\varepsilon\mu}}\right)\circ(\Phi_1^{\varepsilon\mu})^{-1}$$

and $(\Phi_1^{\varepsilon\mu})_*P^{0,1}_{\varepsilon(\mu\oplus\nu)}h = (\bar\partial\bar\Phi_1^{\varepsilon\mu})(P^{0,1}_{\varepsilon(\mu\oplus\nu)}h)\circ\Phi_1^{\varepsilon\mu}$.

From this it follows:

Lemma 5.5.1
In the coordinates around (X,E) given by Theorem 5.1.1 we have that:

$$
\begin{aligned}
\frac{d}{d\varepsilon}|_{\varepsilon=0}g^{VB}_{\varepsilon(\mu\oplus\nu)}(\mu_1\oplus\nu_1,\mu_2\oplus\nu_2) &= i\int_X tr((\bar\mu_2\nu_1)\wedge\nu) \\
\frac{d}{d\bar\varepsilon}|_{\varepsilon=0}g^{VB}_{\varepsilon(\mu\oplus\nu)}(\mu_1\oplus\nu_1,\mu_2\oplus\nu_2) &= i\int_X tr((\mu_1\bar\nu^T)\wedge\bar\nu_2^T).
\end{aligned}
$$

Proof: We calculate each term gathering the terms like (5.16). We have already seen that $\frac{d}{d\varepsilon}|_{\varepsilon=0}\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T\Phi_2^{\varepsilon(\mu\oplus\nu)} = 0$, and so these terms do not contribute. Now we consider the operators, we have left out subscripts from the calculation as their domains should be clear. The derivative of the projection is a sum of an operator ending with $\bar\partial$ and one which starts with $\bar\partial^*$, as is seen from:

$$\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_* P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\\
&\quad=\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*(-\bar\partial\Delta_0^{-1}\bar\partial^*)(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\\
&\quad=\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*(-\bar\partial)(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\\
&\qquad\quad\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*(\Delta_0^{-1})(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\\
&\qquad\quad\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*(\bar\partial^*)(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\\
&\quad=\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*(-\bar\partial)(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\Delta_0^{-1}\bar\partial^*\\
&\quad+\bar\partial\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*(\Delta_0^{-1})(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\bar\partial^*\\
&\quad+\bar\partial\Delta_0^{-1}\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*(\bar\partial^*)(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)}).
\end{aligned}$$

The first two terms are orthogonal to $\nu\in\mathcal{H}^{0,1}(X,\mathrm{End}E)$, and the second one applied to a harmonic from is zero. This means the contribution from the first term in (5.15) is zero.

Now all the remaining terms contain a $\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}$ which is zero at $\varepsilon=0$. Hence, the only contributions to the derivative arise when we derive these, and then we have that $\frac{d}{d\bar\varepsilon}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}=-\bar\nu^T$ and $\frac{d}{d\varepsilon}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}=0$. Inserting this and setting $\varepsilon=0$ we find the formulae in the lemma. ■

We proceed to calculate the second order derivatives of the metric. To do so we need to calculate $\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}$, $\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}$ and the contribution from $\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}P^{0,1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)}$ which is non-zero when applied to a harmonic form and also is not orthogonal to a harmonic form.

We now calculate the three terms. For the first term we begin by applying the Laplace operator on $\mathbb{H}$ to the expression:

$$\begin{aligned}
&\Delta\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}\\
&\quad=y^2\bar\partial\partial\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}\\
&\quad=\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}y^2\bar\partial\partial((\overline{\Phi_+^{\varepsilon(\mu\oplus\nu)}\Phi_-^{\varepsilon\mu}})^T\Phi_+^{\varepsilon(\mu\oplus\nu)}\Phi_-^{\varepsilon\mu})\\
&\quad=\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}y^2\big(\overline{\bar\partial\Phi_-^{\varepsilon\mu}}^T\overline{\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\Phi_+^{\varepsilon(\mu\oplus\nu)}\Phi_-^{\varepsilon\mu}+\overline{\bar\partial\Phi_-^{\varepsilon\mu}}^T\overline{\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\bar\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}\Phi_-^{\varepsilon\mu}\\
&\qquad+\overline{\bar\partial\Phi_-^{\varepsilon\mu}}^T\overline{\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\Phi_+^{\varepsilon(\mu\oplus\nu)}\bar\partial\Phi_-^{\varepsilon\mu}+\overline{\Phi_-^{\varepsilon\mu}}^T\overline{\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}\Phi_-^{\varepsilon\mu}
\end{aligned}$$

$$+\overline{\Phi_-^{\varepsilon\mu}}^T\overline{\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}\bar\partial\Phi_-^{\varepsilon\mu}+\overline{\Phi_-^{\varepsilon\mu}}^T\overline{\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\bar\partial\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}\Phi_-^{\varepsilon\mu}$$
$$+\overline{\Phi_-^{\varepsilon\mu}}^T\overline{\bar\partial\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\Phi_+^{\varepsilon(\mu\oplus\nu)}\Phi_-^{\varepsilon\mu}+\overline{\Phi_-^{\varepsilon\mu}}^T\overline{\bar\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}}^T\Phi_+^{\varepsilon(\mu\oplus\nu)}\bar\partial\Phi_-^{\varepsilon\mu}).$$

For all the terms where two different factors are differentiated we are only able to match the ε-derivatives in one way that is nonzero. We also have that $\bar\partial\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}=\partial\varepsilon\mu\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}$, and so we need to derive it with respect to ε and $\bar\varepsilon$ to get a nonzero contribution. For the same reason $\bar\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}$ needs to be differentiated twice to be nonzero. Since $\bar\partial\Phi_+^{\varepsilon(\mu\oplus\nu)}$ is always paired with another term we need to differentiate, these terms will not contribute:

$$\begin{aligned}\Delta_0\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)} &= y^2(\overline{(\nu_2)}^T\overline{(-\bar\nu_1^T)}^T+0+\overline{(\nu_2)}^T\nu_1+\overline{(-\bar\nu_1)^T}^T\\ &\qquad\cdot(-\bar\nu_2^T)+(-\bar\nu_2)^T\nu_1-\mu_1\bar\nu_2^T-\bar\mu_2\nu_1+0)\\ &= y^2([\nu_1,\bar\nu_2^T]-\partial\mu_1\bar\nu_2^T-\partial\bar\mu_2\nu_1).\end{aligned}$$

We conclude that

$$\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}=\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1-\star\partial\mu_1\bar\nu_2^T-\star\bar\partial\bar\mu_2\nu_1)+cI$$

for some constant c, since the kernel of Δ_0 is the constant multiple of I. In what remains this term will not contribute, as we will be looking at

$$\mathrm{ad}(\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}).$$

Next we calculate the second term $\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}$. The calculation follows directly from the previous computation:

$$\begin{aligned}\bar\partial\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1-\star\mu_1\bar\nu_2^T-\star\bar\mu_2\nu_1) &= \bar\partial\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}\\ &= \frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\bar\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}+\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}\\ &= \frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T(\Phi_2^{\varepsilon(\mu\oplus\nu)}\varepsilon\nu+\varepsilon\mu\partial\Phi_2^{\varepsilon(\mu\oplus\nu)})\\ &\quad+\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T\Phi_2^{\varepsilon(\mu\oplus\nu)}\\ &= -\mu_1\bar\nu_2^T+\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}(\overline{(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T,\end{aligned}$$

since $\frac{d}{d\varepsilon}|_{\varepsilon=0}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}})^T\Phi_2^{\varepsilon(\mu\oplus\nu)}=0$.

Finally, we need to calculate the third term

$$\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon\mu})_*P^{0,1}(\Phi_1^{\varepsilon\mu})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)},$$

but only where both $\bar{\partial}$ and $\bar{\partial}^*$ has been differentiated in $\bar{\partial}\Delta_0^{-1}\bar{\partial}^*$. This is simplified by $\bar{\partial}$ (see (5.13)) only depending on ε and not $\bar{\varepsilon}$. Using this and (5.14) we have:

$$\frac{d}{d\varepsilon_1}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon\mu})_*\bar{\partial}(\Phi_1^{\varepsilon\mu})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)}\Delta_0^{-1}$$
$$\frac{d}{d\bar{\varepsilon}_2}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon\mu})_*\bar{\partial}^*(\Phi_1^{\varepsilon\mu})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)}$$
$$= (-\mu_1\partial + \mathrm{ad}\nu_1)\Delta_0^{-1}(\partial^*\bar{\mu}_2 - \star\mathrm{ad}\nu_2\star).$$

Now we are ready to prove:

Theorem 5.5.2
Consider the second variation of the metric in the coordinates on the universal moduli space of pairs of a Riemann surface and a holomorphic bundle on it. Then we have this second variation at the center is:

$$\begin{aligned}
&\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}\bigg|_{\varepsilon=0} g_{\varepsilon(\mu\oplus\nu)}(\mu_3\oplus\nu_3,\mu_4\oplus\nu_4)\\
&\quad= -i\int_\Sigma tr(-\mu_1\partial + ad\nu_1)\Delta_0^{-1}(\partial^*\bar{\mu}_2 - \star ad\nu_2\star)\nu_3\wedge\bar{\nu}_4^T\\
&\quad - i\int_\Sigma tr(ad\Delta_0^{-1}((-\star)ad\nu_2\star\nu_1 - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1))\nu_3\wedge\bar{\nu}_4^T)\\
&\quad + i\int_\Sigma \mu_1\bar{\mu}_2 tr\nu_3\wedge\bar{\nu}_4^T\\
&\quad - i\int_\Sigma tr(ad\nu_1 + \mu_1\partial)\Delta_o^{-1}\bar{\partial}^*\mu_3\bar{\nu}_2^T\wedge\bar{\nu}_4^T\\
&\quad - i\int_\Sigma tr\mu_3\partial\Delta_0^{-1}(\star[\star v_1\nu_2] - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1))\wedge\bar{\nu}_4^T\\
&\quad - i\int_\Sigma tr\bar{\mu}_2\mu_3\nu_1\wedge\overline{\nu_4}^T\\
&\quad - i\int_\Sigma tr\bar{\partial}\Delta_o^{-1}(-\star ad\nu_2\star + \partial^*\mu_2)\nu_3\wedge\bar{\mu}_4\nu_1\\
&\quad - i\int_\Sigma tr\nu_3\wedge\overline{\mu_4\partial\Delta_0^{-1}(\star[\star v_2\nu_1] - \star(\partial\mu_2\bar{\nu}_1^T) - \star(\bar{\partial}\bar{\mu}_1\nu_2))}^T\\
&\quad - i\int_\Sigma tr\nu_3\wedge\overline{\bar{\mu}_1\mu_4\nu_2}^T - i\int_\Sigma tr\mu_3\nu_1\wedge\overline{\mu_4\nu_2}^T.
\end{aligned}$$

PROOF: Since we already have computed all the ingredients, we gather the results here:

$$-i\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}\int_\Sigma \mathrm{tr}P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_3$$
$$\wedge\overline{P^{0,1}((\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_4}^T$$

$$
\begin{aligned}
&= -i\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}\int_\Sigma \mathrm{tr}\mathrm{Ad}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T\Phi_2^{\varepsilon(\mu\oplus\nu)})(1-|\varepsilon\mu|^2)\\
&\quad \mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_* P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_3\wedge\overline{\nu_4}^T\\
&= -i\int_\Sigma \mathrm{tr}\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}\mathrm{Ad}(\overline{\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_3\wedge\overline{\nu_4}^T\\
&- i\int_\Sigma \mathrm{tr}\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*\\
&\qquad P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_3\wedge\overline{\nu_4}^T\\
&- i\int_\Sigma \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(1-|\varepsilon\mu|^2)\mathrm{tr}\nu_3\wedge\bar{\nu}_4^T\\
&= -i\int_\Sigma \mathrm{tr}(\mathrm{ad}(\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1-\star(\partial\mu_1\bar{\nu}_2^T)\\
&\qquad\qquad -\star(\bar\partial\bar{\mu}_2\nu_1)))\nu_3\wedge\bar{\nu}_4^T)\\
&- i\int_\Sigma \mathrm{tr}(-\mu_1\partial+\mathrm{ad}\nu_1)\Delta_0^{-1}(\partial^*\bar{\mu}_2-\star\mathrm{ad}\nu_2\star)\nu_3\wedge\bar{\nu}_4^T\\
&+ i\int_\Sigma \mu_1\bar{\mu}_2\mathrm{tr}\nu_3\wedge\bar{\nu}_4^T.
\end{aligned}
$$

Now, for the second term we have:

$$
\begin{aligned}
\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0} - i\int_\Sigma &\mathrm{tr}P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_3(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}\\
&\wedge\overline{P^{0,1}((\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_4}^T\\
&= -i\int_\Sigma \mathrm{tr}P^{0,1}\mu_3\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}\wedge\overline{\nu_4}^T\\
&- i\int_\Sigma \mathrm{tr}\frac{d}{d\varepsilon_1}|_{\varepsilon=0}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_* P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\\
&\qquad \mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_3\frac{d}{d\bar{\varepsilon}_2}|_{\varepsilon=0}(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}\wedge\overline{\nu_4}^T\\
&= -i\int_\Sigma \mathrm{tr}P^{0,1}\mu_3(\partial\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1-\star(\bar\partial\bar{\mu}_2\nu_1)\\
&\qquad\qquad -\star(\partial\mu_1\bar{\nu}_2^T))+\bar{\mu}_2\nu_1)\wedge\overline{\nu_4}^T\\
&- i\int_\Sigma \mathrm{tr}(-\mu_1\partial+\mathrm{ad}\nu_1)\Delta_0^{-1}\bar\partial^*\mu_3\bar{\nu}_2^T\wedge\overline{\nu_4}^T.
\end{aligned}
$$

And similarly

$$
\begin{aligned}
-i\int_\Sigma &\mathrm{tr}P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\nu_3\\
&\wedge\overline{P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}(\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_4(\Phi_2^{\varepsilon(\mu\oplus\nu)})\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T
\end{aligned}
$$

$$= -i\int_\Sigma \mathrm{tr}\nu_3 \wedge \overline{P^{0,1}\mu_4(\partial\Delta_0^{-1}((-\star)\mathrm{ad}\nu_1 \star \nu_2 - \star(\bar\partial\bar\mu_1\nu_2)}$$
$$\overline{-\star(\partial\mu_2\bar\nu_1^T)) + \bar\mu_1\nu_2)}^T$$
$$-i\int_\Sigma \mathrm{tr}\bar\partial\Delta_0^{-1}(-\partial^*\mu_2 - \star\mathrm{ad}\nu_2\star)\nu_3 \wedge \overline{\mu_4}\nu_1.$$

Finally, there is not much choice in how to differentiate:

$$-i\frac{d^2}{d\varepsilon_1 d\bar\varepsilon_2}|_{\varepsilon=0}\int_\Sigma \mathrm{tr}P^{0,1}(\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_3(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}$$
$$\wedge\overline{P^{0,1}((\Phi_1^{\varepsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\varepsilon(\mu\oplus\nu)})\mu_4(\Phi_2^{\varepsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T$$
$$= -i\int_\Sigma \mathrm{tr}P^{0,1}\mu_3\frac{d}{d\bar\varepsilon_2}|_{\varepsilon=0}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)} \wedge \overline{P^{0,1}\mu_4\frac{d}{d\bar\varepsilon_1}|_{\varepsilon=0}\partial\Phi_2^{\varepsilon(\mu\oplus\nu)}}^T$$
$$= -i\int_\Sigma \mathrm{tr}\mu_3\bar\nu_2^T \wedge \bar\mu_4\nu_1.$$

Collect all these results and we have the conclusion. ■

5.5.2 The Variation of the Metric in Fibered Coordinates

In this section we look at the fibered coordinates and do the same computations. We end the section comparing the two sets of coordinates, and we show they differ at third-order.

From the calculation of the Kodaira-Spencer map (Proposition 5.4.3) we know, that the metric restricted to the vertical tangent bundle (the moduli space of bundles direction) is given by:

$$g_{VB}^{\varepsilon\nu^{\varepsilon\mu}}(\mu_1\oplus\nu_1,\mu_2\oplus\nu_2) = -i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_1^{\varepsilon\mu}\wedge\overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_2^{\varepsilon\mu}}^T$$
$$-i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_1^{\varepsilon\mu}\wedge\overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_2^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})}^T$$
$$-i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_1^{\varepsilon\mu}$$
$$\wedge\overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\varepsilon\mu+t\mu_2})^{-1}\nu)}^T$$
$$-i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_1^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})\wedge\overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_2^{\varepsilon\mu}}^T$$
$$-i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_1^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})$$
$$\wedge\overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_2^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})}^T$$
$$-i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_1^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})$$

$$\wedge \overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\varepsilon\mu+t\mu_2})^{-1}\nu)}^T$$

$$- i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\mu_1})P^{0,1}(\Phi_1^{\varepsilon\mu+t\mu_1})^{-1}\nu)$$

$$\wedge \overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_2^{\varepsilon\mu}}^T$$

$$- i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\mu_1})P^{0,1}(\Phi_1^{\varepsilon\mu+t\mu_1})^{-1}\nu)$$

$$\wedge \overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_2^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})}^T$$

$$- i\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\mu_1})P^{0,1}(\Phi_1^{\varepsilon\mu+t\mu_1})^{-1}\nu)$$

$$\wedge \overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\varepsilon\mu+t\mu_2})^{-1}\nu)}^T .$$

While these nine terms look intimidating we can discard three of the terms, because $P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\varepsilon\mu+t\mu_2})^{-1}\nu)$ vanishes to second-order and $P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_2^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})$ vanishes to first-order, so terms containing both kind of factors or only the first kind of factors will vanish to higher order than we are interested in. Now the first variation will be the same as in Section 5.5, but to calculate it we will have to work with slightly different expressions.

First we consider

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu^\varepsilon\mu}})^T f^{\varepsilon\nu^\varepsilon\mu}) \circ \Phi_1^{\varepsilon\mu} = 0$$

and

$$\frac{d}{d\bar\varepsilon}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu^\varepsilon\mu}})^T f^{\varepsilon\nu^\varepsilon\mu}) \circ \Phi_1^{\varepsilon\mu} = 0.$$

Both of these follow from the computations in Chapter 3, where it was shown that $\frac{d}{d\bar\varepsilon}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu}})^T f^{\varepsilon\nu}) = 0$. Composing with $\nu \to \nu^{\varepsilon\mu}$ will not change it, and if we differentiate $\Phi_1^{\varepsilon\mu}$, then we can set $\varepsilon = 0$ in the rest of the terms and calculate $\frac{d}{d\varepsilon}I \circ \Phi_1^{\varepsilon\mu} = 0$. Now, for a projection the first derivative will either have harmonic forms in it's kernel or the image will be in the orthogonal complement, hence the only contributions are from the terms:

$$\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_1^{\varepsilon\mu}\overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_2^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})}^T$$

and

$$\int_\Sigma P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})(\mu_1^{\varepsilon\mu}(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}})\overline{P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}}\mathrm{Ad}f^{\varepsilon\nu^{\varepsilon\mu}}\nu_2^{\varepsilon\mu}}^T.$$

And so we have, completely analogous to the previous section:

Lemma 5.5.3

In the fibered coordinates around (X, E) we have that:

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} g^{VB}_{\varepsilon\nu^{\varepsilon\mu}}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = i \int_X \bar{\mu}_2 tr(\nu_1 \nu),$$
$$\frac{d}{d\bar{\varepsilon}}|_{\varepsilon=0} g^{VB}_{\varepsilon\nu^{\varepsilon\mu}}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = i \int_X \mu_1 tr(\bar{\nu}^T \bar{\nu}_2^T).$$

In order to get the second variation of the metric we need to calculate the two terms

$$\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu}$$

and

$$\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}((f^{\varepsilon\nu^{\varepsilon\mu}})^{-1} \partial f^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu}.$$

We calculate these as the previous set of coordinates, applying the same strategy:

$$\begin{aligned}\Delta_0 \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu} &= y^2 \partial\bar{\partial} \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu} \\ &= \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0} y^2 \partial\bar{\partial}((\overline{f_-^{\varepsilon\nu^{\varepsilon\mu}}})^T (\overline{f_+^{\varepsilon\nu^{\varepsilon\mu}}})^T f_+^{\varepsilon\nu^{\varepsilon\mu}} f_-^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu}.\end{aligned}$$

Now we use:

$$\begin{aligned}\bar{\partial}\partial(h \circ \Phi_1^{\varepsilon\mu}) &= \partial\Phi_1^{\varepsilon\mu} \partial\bar{\Phi}_1^{\varepsilon\mu} (\partial\partial h) \circ \Phi_1^{\varepsilon\mu} + \bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \partial\Phi_1^{\varepsilon\mu} (\partial\bar{\partial} h) \circ \Phi_1^{\varepsilon\mu} \\ &+ \bar{\partial}\Phi_1^{\varepsilon\mu} \partial\bar{\Phi}_1^{\varepsilon\mu} (\bar{\partial}\partial h) \circ \Phi_1^{\varepsilon\mu} + \partial\bar{\Phi}_1^{\varepsilon\mu} \bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} (\bar{\partial}\bar{\partial} h) \circ \Phi_1^{\varepsilon\mu} \\ &= \partial\Phi_1^{\varepsilon\mu} \varepsilon\mu \bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} (\partial\partial h) \circ \Phi_1^{\varepsilon\mu} + \bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \partial\Phi_1^{\varepsilon\mu} (\partial\bar{\partial} h) \circ \Phi_1^{\varepsilon\mu} \\ &+ |\varepsilon\mu|^2 \partial\Phi_1^{\varepsilon\mu} \partial\bar{\Phi}_1^{\varepsilon\mu} (\bar{\partial}\partial h) \circ \Phi_1^{\varepsilon\mu} + \overline{\varepsilon\mu} \bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} (\bar{\partial}\bar{\partial} h) \circ \Phi_1^{\varepsilon\mu},\end{aligned}$$

for $h = (\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}}$, and since we know that $\partial f^{\varepsilon\nu^{\varepsilon\mu}}$ and $\bar{\partial} f^{\varepsilon\nu^{\varepsilon\mu}}$ vanish to first-order in ε, we only have the surviving terms:

$$\Delta_0 \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu} = y^2(-\partial\mu_1 \bar{\nu}_2^T - \bar{\partial}\bar{\mu}_2 \nu_1 + [\nu_1, \bar{\nu}_2^T]),$$

this expression is similar to the previous section. We proceed to calculate

$$\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}((f^{\varepsilon\nu^{\varepsilon\mu}})^{-1} \partial f^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu},$$

and so we study

$$\begin{aligned}&\bar{\partial}\Delta_0^{-1} y^2(-\mu_1 \partial\bar{\nu}_2^T - \bar{\mu}_2 \bar{\partial}\nu_1 + [\nu_1, \bar{\nu}_2^T]) \\ &\qquad = \bar{\partial} \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}}) \circ \Phi_1^{\varepsilon\mu}\end{aligned}$$

$$\begin{aligned}
=&\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(\bar{\partial}((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\Phi_1^{\varepsilon\mu} \\
&+ \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(\partial((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\Phi_1^{\varepsilon\mu} \\
=&\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{\partial f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\Phi_1^{\varepsilon\mu} \\
&+ \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{\bar{\partial} f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\Phi_1^{\varepsilon\mu} \\
&+ \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T \bar{\partial} f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \\
&+ \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T \partial f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\Phi_1^{\varepsilon\mu}.
\end{aligned}$$

Here the second and fourth term cancel, as is seen by using that two of the factors vanish to first-order in ε, which then give $\mu_1\bar{\nu}_2^T$ and $-\mu_1\bar{\nu}_2^T$ respectively.

$$\begin{aligned}
&\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{\partial f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \\
&\qquad + \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T \bar{\partial} f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \\
&\quad = \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}}}^T \overline{f^{\varepsilon\nu^{\varepsilon\mu}}}^T f^{\varepsilon\nu^{\varepsilon\mu}})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \\
&\qquad + \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{f^{\varepsilon\nu^{\varepsilon\mu}}})^T f^{\varepsilon\nu^{\varepsilon\mu}} \varepsilon\nu^{\varepsilon\mu})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \\
&\quad = \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}}}^T) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \\
&\qquad + \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(\varepsilon\nu^{\varepsilon\mu})) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} \\
&\quad = \frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}(((\overline{(f^{\varepsilon\nu^{\varepsilon\mu}})^{-1}\partial f^{\varepsilon\nu^{\varepsilon\mu}}}^T) \circ \Phi_1^{\varepsilon\mu}\bar{\partial}\bar{\Phi}_1^{\varepsilon\mu} + \bar{\partial}\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1.
\end{aligned}$$

Note that this varies from the previous section.

We need to consider two more things. The first is the second variation of the harmonic projection. However, since the only relevant part is the contribution where both $\bar{\partial}$ and $\bar{\partial}^*$ have been differentiated in $\bar{\partial}\Delta_0^{-1}\bar{\partial}^*$ and the coordinates agree to second-order nothing will have changed, and we have it results in:

$$(-\mu_1\partial + \mathrm{ad}\nu_1)\Delta_0^{-1}(\partial^*\bar{\mu}_2 - \star\mathrm{ad}\nu_2\star).$$

The second and final term to consider is the new term in the formula for the metric, which is:

$$\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0} P^{0,1}_{\varepsilon\nu^{\varepsilon\mu}} \mathrm{Ad}(f^{\varepsilon\nu^{\varepsilon\mu}})((\Phi_1^{\varepsilon\mu})_*^{-1}(1-|\varepsilon\mu|^2)_*$$

$$\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})P^{0,1}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*^{-1}\varepsilon\nu),$$

as $\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*P^{0,1}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*^{-1}(\varepsilon\nu)$ vanishes to second-order this has to be differentiated twice:

$$\begin{aligned}
&\frac{d}{d\bar{\varepsilon}_2}|_{\varepsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*P^{0,1}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*^{-1}\nu_1\\
&\quad= -(\frac{d}{d\bar{\varepsilon}_2}|_{\varepsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*\bar{\partial}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad\qquad \Delta_0^{-1}\frac{d}{d\bar{\varepsilon}_2}|_{\varepsilon=0}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*\bar{\partial}^*(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*^{-1}\nu_1)\\
&\quad - \bar{\partial}\frac{d}{d\bar{\varepsilon}_2}|_{\varepsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*\Delta_0^{-1}\bar{\partial}^*(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*^{-1}\nu_1\\
&\quad= -\tilde{\mu}\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1 - \bar{\partial}\frac{d}{d\bar{\varepsilon}_2}|_{\varepsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*\Delta_0^{-1}\bar{\partial}^*(\Phi_1^{\varepsilon\mu+t\tilde{\mu}})_*^{-1}\nu_1.
\end{aligned}$$

We are now ready to gather all the contributions in the following:

Theorem 5.5.4
We have the following for the second variation of the metric at (X, E) in the fibered coordinates:

$$\begin{aligned}
&\frac{d^2}{d\varepsilon_1 d\bar{\varepsilon}_2}|_{\varepsilon=0}g^{VB}_{\varepsilon\nu^{\varepsilon\mu}}(\mu_3\oplus\nu_3,\mu_4\oplus\nu_4)\\
&\quad= \int_\Sigma tr((-\mu_1\partial + \mathrm{ad}\nu_1)\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star + \partial^*\bar{\mu}_2)\nu_3)\wedge\bar{\nu}_4^T\\
&\quad - i\int_\Sigma tr(\mathrm{ad}\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1 - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1))\nu_3\wedge\bar{\nu}_4^T)\\
&\quad - i\int_\Sigma tr(\mathrm{ad}\nu_1 - \mu_1\partial)\Delta_o^{-1}\bar{\partial}^*\mu_3\bar{\nu}_2^T\wedge\bar{\nu}_4^T\\
&\quad + i\int_\Sigma \mu_1\bar{\mu}_2 tr\nu_3\wedge\bar{\nu}_4^T\\
&\quad - i\int_\Sigma tr\mu_3\partial\Delta_0^{-1}(\star[\star\nu_1\nu_2] - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1))\wedge\bar{\nu}_4^T\\
&\quad + i\int_\Sigma tr\mu_3\partial\Delta_0^{-1}\bar{\partial}^*\mu_1\bar{\nu}_2^T\wedge\overline{\nu_4}^T\\
&\quad - i\int_\Sigma tr\bar{\partial}\Delta_o^{-1}(-\star\,\mathrm{ad}\nu_2\star + \partial^*\bar{\mu}_2)\nu_3\wedge(-\bar{\mu}_4\nu_1)\\
&\quad - i\int_\Sigma tr\nu_3\wedge\overline{\mu_4\partial\Delta_0^{-1}(\star[\star\nu_2\nu_1] - \star(\partial\mu_2\bar{\nu}_1^T) - \star(\bar{\partial}\bar{\mu}_1\nu_2))}^T\\
&\quad + i\int_\Sigma tr\nu_3\wedge\overline{\mu_4\partial\Delta_0^{-1}\bar{\partial}^*\mu_2\bar{\nu}_1^T}^T - i\int_\Sigma tr\mu_3\nu_1\wedge\overline{\mu_4\nu_2}^T\\
&\quad + i\int_\Sigma \mu_3\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1\wedge\bar{\nu}_4^T + i\int_\Sigma \nu_3\wedge\overline{\mu_4\partial\Delta_0^{-1}\partial^*\bar{\mu}_1\nu_2}^T.
\end{aligned}$$

Comparing this to the previous coordinates (Theorem 5.5.2) we see that there are four terms here which we did not have before and two we no longer have. The new terms are:

$$i\int_{\Sigma}\mu_3\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1\wedge\bar{\nu}_4^T,$$

$$i\int_{\Sigma}\nu_3\wedge\overline{\mu_4\partial\Delta_0^{-1}\partial^*\bar{\mu}_1\nu_2}^T,$$

$$i\int_{\Sigma}\mathrm{tr}\mu_3\partial\Delta_0^{-1}\bar{\partial}^*\mu_1\bar{\nu}_2^T\wedge\overline{\nu_4}^T$$

and

$$i\int_{\Sigma}\mathrm{tr}\nu_3\wedge\overline{\mu_4\partial\Delta_0^{-1}\bar{\partial}^*\mu_2\bar{\nu}_1^T}^T.$$

While the ones we no longer have are:

$$-i\int_{\Sigma}\mathrm{tr}\nu_3\wedge\overline{\mu_1\bar{\mu}_4\nu_2}^T$$

and

$$-i\int_{\Sigma}\mathrm{tr}\bar{\mu}_2\mu_3\nu_1\wedge\overline{\nu_4}^T.$$

Now we have that for $\nu_1=\nu_4$ and $\mu_2=\mu_3$ and the rest zero the difference between the two expressions are:

$$\begin{aligned}-i\int_{\Sigma}\mu_3\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1\wedge\bar{\nu}_4^T-i\int_{\Sigma}\mathrm{tr}\bar{\mu}_2\mu_3\nu_1\wedge\overline{\nu_4}^T\\=-i\int_{\Sigma}\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1\wedge\overline{\partial^*\bar{\mu}_2\nu_1}^T\rho-i\int_{\Sigma}\mathrm{tr}\bar{\mu}_2\mu_2\nu_1\wedge\overline{\nu_1}^T.\end{aligned}$$

And since Δ is a positive operator we have that $-i\int_{\Sigma}\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1\wedge\overline{\partial^*\bar{\mu}_2\nu_1}^T\rho\geq 0$ and, for obvious reasons, $-i\int_{\Sigma}\mathrm{tr}\bar{\mu}_2\mu_2\nu_1\wedge\overline{\nu_1}^T>0$.

Theorem 5.5.5
The coordinates of 5.1.1 and the fibered coordinates agree to second-order, but differ at third-order in the derivatives at the center point.

Chapter 6

Local Calculations for the Hitchin Connection

In this Chapter I have gathered all the remaining material. The first two sections use local coordinates, in the first we derive an explicit expression for $G(V)$ and in the second we prove [Andersen, 2012, Lemma 2.8], a lemma for the construction of the Hitchin connection.

After these preliminary excercises, the next three sections are concerned with the curvature of the Hitchin connection. We first recall the result on the curvature of the Hitchin connection from [Andersen and Gammelgaard, 2011]. The next section finishes the calculation of the $(1,1)$-part of the curvature. The final section contains an argument based on the computation of the $(1,1)$-part of the curvature which shows we can modify the Hitchin connection so that the $(2,0)$-part of the curvature is zero, under some assumptions I believe hold for $g > 2$.

The final section is a calculation of the requirements that an inner product on the bundle of holomorphic sections is projectivly compatible with the Hitchin connection.

6.1 A Refined Expression for $G(V)$

In [Hitchin, 1990] it was shown that:

$$G(V)(\alpha, \alpha) = \int_\Sigma (V[-\star])\mathrm{tr}\alpha^2,$$

where $V[-\star]$ is a Beltrami differential given by differentiating the complex structure. We will now calculate in coordinates which Beltrami differential it is. So far, when we have calculated variations of $(1,0)$-forms, we have calculated them under the condition that each is a $(1,0)$-form, however, since the complex structure does not know about these types we will have to vary them as 1-forms. Another reason we have been content to vary our objects preserving type is, that for most expressions we have been looking at integrals of the wedge of a $(1,0)$-form and a $(0,1)$-form. In this case the contribution from varying the forms as 1-forms is $|\varepsilon|^2$, and so it does not actually contribute to the previous computations. However when calculating $G(V)$, we pair two $(1,0)$-forms. Now the contribution to the $(1,1)$-part is actually order ε, which means it will be important. In short, what we do now is the correct way of varying our objects, the only reason we have not discussed it previously is because it would not have contributed.

So, looking at a fixed 1-form $\bar{\nu}^T dz \in \mathcal{H}^{1,0}(X_0, \mathrm{End}E)$ and varying it as a 1-form, rather than as a $(1,0)$-form, we then find:

$$\begin{aligned}(\Phi_1^{\varepsilon\mu})_*^{-1}(\bar{\nu}^T dz) &= \bar{\nu}^T \circ (\Phi_1^{\varepsilon\mu})^{-1} \cdot \partial(\Phi_1^{\varepsilon\mu})^{-1} dz + \bar{\nu}^T \circ (\Phi_1^{\varepsilon\mu})^{-1} \cdot \bar{\partial}(\Phi_1^{\varepsilon\mu})^{-1} d\bar{z} \\ &= \bar{\nu}^T \circ (\Phi_1^{\varepsilon\mu})^{-1} \cdot \partial(\Phi_1^{\varepsilon\mu})^{-1} dz \\ &\quad - (\varepsilon\mu\bar{\nu}^T \frac{\partial\Phi_1^{\varepsilon\mu}}{\overline{\partial\Phi_1^{\varepsilon\mu}}}) \circ (\Phi_1^{\varepsilon\mu})^{-1} \partial(\Phi_1^{\varepsilon\mu})^{-1} d\bar{z}.\end{aligned}$$

We need a harmonic representative, and so we project the dz term to $\mathcal{H}^{1,0}(X_{\varepsilon\mu}, \mathrm{End}E)$ and the $d\bar{z}$ term to $\mathcal{H}^{0,1}(X_{\varepsilon\mu}, \mathrm{End}E)$:

$$\begin{aligned}P_{\mathcal{H}}\left((\Phi_1^{\varepsilon\mu})_*^{-1}(\bar{\nu}^T dz)\right) &= P^{1,0}\left(\bar{\nu}^T \circ (\Phi_1^{\varepsilon\mu})^{-1} \cdot \partial(\Phi_1^{\varepsilon\mu})^{-1} dz\right) \\ &\quad - P^{0,1}\left((\varepsilon\mu\bar{\nu}^T \frac{\partial\Phi_1^{\varepsilon\mu}}{\overline{\partial\Phi_1^{\varepsilon\mu}}}) \circ (\Phi_1^{\varepsilon\mu})^{-1} \partial(\Phi_1^{\varepsilon\mu})^{-1} d\bar{z}\right).\end{aligned}$$

We know, how the complex structure acts on the spaces $\mathcal{H}^{1,0}(X_{\varepsilon\mu}, \mathrm{End}E)$ and $\mathcal{H}^{0,1}(X_{\varepsilon\mu}, \mathrm{End}E)$ by mutiplying the $(1,0)$-part by $-i$ and the $(0,1)$-part by i. Now we can pull these 1-forms back to X_0. Finally, we pull the computation back to $\mathcal{H}^{0,1}(X_0, \mathrm{End}E) \oplus \mathcal{H}^{1,0}(X_0, \mathrm{End}E)$:

$$\begin{aligned}&(\Phi_1^{\varepsilon\mu})_* I P_{\mathcal{H}} (\Phi_1^{\varepsilon\mu})_*^{-1} \bar{\nu}^T \\ &\quad = -iP^{1,0}\left(\bar{\nu}^T \circ (\Phi_1^{\varepsilon\mu})^{-1} \cdot \partial(\Phi_1^{\varepsilon\mu})^{-1} dz\right) \circ \Phi_1^{\varepsilon\mu}(\partial\Phi_1^{\varepsilon\mu} + \bar{\partial}\Phi_1^{\varepsilon\mu}\frac{d\bar{z}}{dz}) \\ &\quad - iP^{0,1}\left((\varepsilon\mu\bar{\nu}^T \frac{\partial\Phi_1^{\varepsilon\mu}}{\overline{\partial\Phi_1^{\varepsilon\mu}}}) \circ (\Phi_1^{\varepsilon\mu})^{-1} \partial(\Phi_1^{\varepsilon\mu})^{-1} d\bar{z}\right) \circ \Phi_1^{\varepsilon\mu}(\partial\bar{\Phi}_1^{\varepsilon\mu}\frac{dz}{d\bar{z}} + \bar{\partial}\bar{\Phi}_1^{\varepsilon\mu}).\end{aligned}$$

Next, we want to differentiate with respect to ε at $\varepsilon = 0$. The first dz term gives something in $\mathrm{Im}\bar{\partial}$ when differentiated, and it can be calculated by the method used so far. For our later application to the metric this term will not be relevant, as it is in the orthogonal complement of the harmonic forms. The second dz term is of order $|\varepsilon|^2$ and so it gives zero as well. Both $d\bar{z}$ terms are of order ε, and so we get:

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}(\Phi_1^{\varepsilon\mu})_* I P_{\mathcal{H}} (\Phi_1^{\varepsilon\mu})_*^{-1} \bar{\nu}^T = -iP^{0,1}\mu_V\bar{\nu}^T - i\mu_V P^{1,0}\bar{\nu}^T. \tag{6.1}$$

Also note, that from this calculation it is easily seen that

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}(\Phi_1^{\varepsilon\mu})_* P_{\mathcal{H}}(\Phi_1^{\varepsilon\mu})_*^{-1}\bar{\nu}^T = -P^{0,1}\mu_V\bar{\nu}^T + \mu_V P^{1,0}\bar{\nu}^T,$$

and so we conclude:

$$V_\mu[I]\bar{\nu}^T = \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}(\Phi_1^{\varepsilon\mu})_* I(\Phi_1^{\varepsilon\mu})_*^{-1}\right)\bar{\nu}^T = -2iP^{0,1}\mu_V\bar{\nu}^T.$$

We have proved the following:

Lemma 6.1.1
In coordinates on $\mathcal{T}\times\mathcal{M}^0_{n,k}$, centered at $(\sigma,E)\in\mathcal{T}\times\mathcal{M}^0_{n,k}$, we have for a vector field V on $\mathcal{T}$, identified by the Kodaira-Spencer map at $\sigma\in\mathcal{T}$ with the Beltrami differential $\mu\in\mathcal{H}^{0,1}(X_\sigma,TX_\sigma)$, and two cotangent vectors on $(\mathcal{M}^0_{n,k},(-\star)_\sigma)$, represented by harmonic forms $\bar{\nu}_i^T,\bar{\nu}_j^T\in\mathcal{H}^{1,0}(X_\sigma,EndE)$, that at (σ,E):

$$G(V)(\bar{\nu}_i^T,\bar{\nu}_j^T) = -2i\int_X \mu\mathrm{tr}\bar{\nu}_i^T\bar{\nu}_j^T. \tag{6.2}$$

The function $G(V)(\bar{\nu}_i^T,\bar{\nu}_j^T)$ in our local neighborhood is clearly holomorphic on the moduli space of holomorphic bundles, since:

$$\begin{aligned}G(V)(\bar{\nu}_i^T,\bar{\nu}_j^T) &= -2i\int_X \mu\mathrm{tr}P^{1,0}_{\varepsilon\nu}(\mathrm{Ad}(f^{\varepsilon\nu}(\bar{\nu}_i^T))\wedge P^{1,0}_{\varepsilon\nu}(\mathrm{Ad}(f^{\varepsilon\nu}(\bar{\nu}_i^T))\\ &= -2i\int_X \mathrm{tr}\mathrm{Ad}(f^{\varepsilon\nu})^{-1}(P^{0,1}_{\varepsilon\nu}(\mu P^{1,0}_{\varepsilon\nu}(\mathrm{Ad}(f^{\varepsilon\nu}(\bar{\nu}_i^T)))))\wedge\bar{\nu}_i^T.\end{aligned}$$

For $\nu\in\mathcal{H}^{0,1}(X_\sigma,\mathrm{End}E)$ we find:

$$\begin{aligned}&\frac{d}{d\bar{\varepsilon}}|_{\varepsilon=0}\mathrm{Ad}(f^{\varepsilon\nu})^{-1}P^{0,1}_{\varepsilon\nu}\mu P^{1,0}_{\varepsilon\nu}\mathrm{Ad}(f^{\varepsilon\nu})\\ &\qquad = -\bar{\partial}\Delta_0^{-1}*(\mathrm{ad}\nu)*P^{0,1}\mu P^{1,0} - P^{0,1}\mu P^{1,0}\mathrm{ad}\bar{\nu}^T\Delta_0^{-1}\partial^*.\end{aligned}$$

Now the first term is orthogonal to $\bar{\nu}_j^T$ and the second term is zero, since $\bar{\nu}_i^T\in\ker\partial^*$. Hence we see that $\frac{d}{d\bar{\varepsilon}}|_{\varepsilon=0}G(V)(\bar{\nu}_i^T,\bar{\nu}_j^T)=0$, and since $\nabla^{0,1}\nu=0$ at the center point this implies that $\nabla^{0,1}G(V)=0$ at the center point, and so since it is a tensor we have this for all points. Alternatively, we can calculate $\nabla^{0,1}G(V)$ in a small neighborhood. To do so we study:

$$\begin{aligned}\nabla^{0,1}_{\bar{\nu}^T}\bar{\nu}_i^T &= P^{1,0}\frac{d}{d\bar{\varepsilon}}\mathrm{Ad}\Phi_2^{\varepsilon 0\oplus\tilde{\nu}_\nu}P^{1,0}(\mathrm{Ad}\Phi_2^{\varepsilon 0\oplus\tilde{\nu}+0\oplus\nu})^{-1}\bar{\nu}_i^T\\ &= P^{1,0}\mathrm{ad}\overline{\tilde{\nu}_\nu^T}\Delta_0^{-1}\partial^*\bar{\nu}_i^T = \mathrm{ad}\overline{\tilde{\nu}_\nu^T}\Delta_0^{-1}\partial^*\bar{\nu}_i^T.\end{aligned}$$

The last equality follows from considering $P^{1,0} = P^{1,0}P^{1,0}$. When we differentiate we only get something nonzero from the last projection. Now we can calculate:

$$\begin{aligned}\frac{d}{d\bar{\varepsilon}}G(V)(\bar{\nu}_i^T,\bar{\nu}_j^T) &= -2i\frac{d}{d\bar{\varepsilon}}\int_X \mu \mathrm{tr} P^{1,0}_{\varepsilon\nu}(\mathrm{Ad}(\Phi_2^{\varepsilon 0\oplus\tilde{\nu}+0\oplus\nu}(\bar{\nu}_i^T)) \\ &\qquad\wedge P^{1,0}_{\varepsilon\nu}(\mathrm{Ad}(\Phi_2^{\varepsilon 0\oplus\tilde{\nu}+0\oplus\nu}(\bar{\nu}_i^T)) \\ &= -2i\frac{d}{d\bar{\varepsilon}}\int_X \mu \mathrm{tr}\mathrm{Ad}(\Phi_2^{\varepsilon 0\oplus\tilde{\nu}_\nu})P^{1,0}_{\varepsilon\nu}(\mathrm{Ad}(\Phi_2^{\varepsilon 0\oplus\tilde{\nu}+0\oplus\nu})^{-1}(\bar{\nu}_i^T)) \\ &\qquad\wedge \mathrm{Ad}(\Phi_2^{\varepsilon 0\oplus\tilde{\nu}_\nu})P^{1,0}_{\varepsilon\nu}(\mathrm{Ad}(\Phi_2^{\varepsilon 0\oplus\tilde{\nu}+0\oplus\nu})^{-1}(\bar{\nu}_i^T)) \\ &= G(V)(\nabla^{0,1}_{\bar{\tilde{\nu}}^T}\bar{\nu}_i^T,\bar{\nu}_j^T) + G(V)(\bar{\nu}_i^T,\nabla^{0,1}_{\bar{\tilde{\nu}}^T}\bar{\nu}_j^T),\end{aligned}$$

and therefore

$$\begin{aligned}(\nabla^{0,1}_{\bar{\tilde{\nu}}^T}G(V))(\bar{\nu}_i^T,\bar{\nu}_j^T) &= \frac{d}{d\bar{\varepsilon}}G(V)(\bar{\nu}_i^T,\bar{\nu}_j^T) \\ &\quad - (G(V)(\nabla^{0,1}_{\bar{\tilde{\nu}}^T}\bar{\nu}_i^T,\bar{\nu}_j^T) + G(V)(\bar{\nu}_i^T,\nabla^{0,1}_{\bar{\tilde{\nu}}^T}\bar{\nu}_j^T)) = 0.\end{aligned}$$

6.2 Calculations Concerning the Hitchin Connection on Teichmüller Space

Before we turn our attention the curvature of the Hitchin connection, we show the following lemma. It is proved in [Andersen, 2012], but here we will make no homological assumptions and we will prove it by local calculations.

Lemma 6.2.1
On the complex space $\mathcal{T}\times\mathcal{M}^0_{n,k}$ we have:

$$2i\bar{\partial}_\sigma V'[F_\sigma] = trG(V)\partial_\sigma F_\sigma\omega - \frac{1}{2}tr\nabla^{(1,0)}G(V)\omega. \tag{6.3}$$

PROOF: To prove the result we need only do so pointwise, and so we will look at $(\sigma,E)\in\mathcal{T}\times\mathcal{M}^0_{n,k}$, and we will choose a basis ν_i of $\mathcal{H}^{0,1}(X_\sigma,\mathrm{End}E)$ such that $\omega(\partial_i,\bar{\partial}_j)=\delta_i^j$ at (σ,E). In order to calculate the form at (σ,E) we work with our coordinates around this point.

We have calculated $V'[2iF]$ in Lemma 4.3.2 to be

$$V'[2iF] = V'[-i\log\det\Delta_0] = -\int_X \mu\mathrm{tr}(\partial\partial'(G(z,z')-Q(z,z'))|_{z=z'}).$$

And so we find:

$$\begin{aligned}2i\bar{\partial}_\sigma V'[F_\sigma](\bar{\partial}_k) &= -\left.\frac{d}{d\bar{\varepsilon}}\right|_{\varepsilon=0}\int_X \mu\mathrm{tr}(\partial\partial'(G^{\varepsilon\nu_k}(z,z')-Q^{\varepsilon\nu_k}(z,z'))_{z=z'}) \\ &= -\left.\frac{d}{d\bar{\varepsilon}}\right|_{\varepsilon=0}\int_X \mu\mathrm{tr}(\mathrm{Ad}f^{\varepsilon\nu_k}(z)^{-1}(\partial\partial'(G^{\varepsilon\nu_k}(z,z')\end{aligned}$$

$$- Q^{\varepsilon\nu_k}(z,z'))|_{z=z'})\mathrm{Ad}f^{\varepsilon\nu_k}(z')).$$

Recall that $G(V)\omega(\bar\partial_k) = -2iP^{0,1}(\mu_k\bar\nu_k^T)$. Using Lemma 3.4.1, which states

$$2\partial_\sigma F_\sigma(\nu) = i\int_X \mathrm{ad}\nu \wedge (\partial'(G(z,z') - Q(z,z'))|_{z=z'}),$$

we find that:

$$\begin{aligned}\mathrm{tr}G(V)\partial_\sigma F_\sigma\omega(\bar\partial_k) &= \int_X \mathrm{ad}(P^{0,1}(\mu_k\bar\nu_k^T)) \wedge (\partial'(G(z,z') - Q(z,z'))|_{z=z'})\\ &= \int_X \mathrm{ad}((\mu_k\bar\nu_k^T) - \bar\partial\Delta_0^{-1}\bar\partial^*(\mu_k\bar\nu_k^T))\\ &\qquad\qquad \wedge (\partial'(G(z,z') - Q(z,z'))|_{z=z'}).\end{aligned}$$

We can evaluate the difference of all but one of the terms with

$$Q(z,z') = \frac{-1}{2\pi}\log\frac{|z-z'|^2}{|z-\bar z'|^2}$$

on the diagonal:

$$\begin{aligned}&\mathrm{tr}(\mathrm{ad}(\mu\bar\nu_k^T)\partial' Q(z,z') + \frac{d}{d\bar\varepsilon}\Big|_{\varepsilon=0}\mathrm{Ad}f_+^{\varepsilon\nu_k}(z)^{-1}(\mathrm{Ad}f_+^{\varepsilon\nu_k}(z'))\mu\partial'\partial Q(z,z'))\\ &= \frac{-1}{2\pi}\Bigg(\frac{\mathrm{ad}(\mu\bar\nu_k^T)}{z-z'} - \frac{\mathrm{ad}(\mu\bar\nu_k^T)}{\bar z - z'}\\ &+ \left(\frac{d}{d\bar\varepsilon}\Big|_{\varepsilon=0}\mathrm{Ad}f_+^{\varepsilon\nu_k}(z) - \frac{d}{d\bar\varepsilon}\Big|_{\varepsilon=0}\mathrm{Ad}f_+^{\varepsilon\nu_k}(z')\right)(\frac{\mu}{(z-z')^2})\Bigg)\\ &\overset{z'\to z}{=} \frac{-1}{2\pi}\left(-\frac{\mathrm{ad}(\mu\bar\nu_k^T)}{2iy} + \mu\partial\frac{d}{d\bar\varepsilon}\Big|_{\varepsilon=0}\partial\mathrm{Ad}(f_+^{\varepsilon\nu_k})\right)\\ &= \frac{-1}{2\pi}\left(-\frac{\mathrm{ad}(\mu\bar\nu_k^T)}{2iy} + \mu\partial\mathrm{ad}\bar\nu_k^T\right).\end{aligned}$$

Since $\mathrm{ad}A$, $A \in \mathbf{GL}(\mathbb{C}^n)$ is traceless and the expression is non-singular, we see that the contribution from the Q terms above is zero. Now we know that the Q contribution is finite. The two contributions from the kernel G must be finite as well, and we can directly calculate the variation of these by calculating the variation of the operators.

$$\begin{aligned}&(2i\bar\partial_\sigma V'[F_\sigma] - \mathrm{tr}G(V)\partial_\sigma F_\sigma\omega)(\nu_k)\\ &= -\int_X \mathrm{tr}(\mu(\nabla_{\bar\nu}\partial\partial' G(z,z')) - \mathrm{ad}((\mu\bar\nu_k^T) \wedge \partial' G(z,z'))|_{z=z'})\\ &\quad + \int_x \mathrm{tr}(\mathrm{ad}(\bar\partial\Delta_0^{-1}\bar\partial^*(\mu\bar\nu_k^T)) \wedge (\partial'(G(z,z') - Q(z,z')))|_{z=z'})\\ &= -\int_X\int_X \mu tr\partial G(z,z') \wedge (-*\mathrm{ad}(\nu_k(z')*)P(z',w))|_{z=w}dz'dw\end{aligned}$$

$$+\int_x \operatorname{tr}(\operatorname{ad}(\bar\partial\Delta_0^{-1}\bar\partial^*(\mu\bar\nu_k^T))\wedge(\partial'(G(z,z')-Q(z,z')))|_{z=z'}).$$

Here we used that $-i\partial\partial' G(z,z')$ is the kernel of $\partial\Delta_0^{-1}\bar\partial^*$. Then we see that

$$\begin{aligned}\nabla_{\bar\nu}\partial\Delta_0^{-1}\bar\partial^* &= -\operatorname{ad}\bar\nu^T\Delta_0^{-1}\bar\partial^* - \partial\Delta_0^{-1} * \operatorname{ad}\nu * + \partial\Delta_0^{-1} * \operatorname{ad}\nu * \bar\partial\Delta_0^{-1}\bar\partial^* \\ &= -\operatorname{ad}\bar\nu^T\Delta_0^{-1}\bar\partial^* - \partial\Delta_0^{-1} * \operatorname{ad}\nu * P.\end{aligned}$$

Since our basis is orthogonal at E, we have that

$$\begin{aligned}&\int_X \partial G(z,z')\wedge(- * \operatorname{ad}(\nu_k(z')*)P(z',w)) \\ &\qquad = -\partial\Delta_0^{-1}\sum_j(*\operatorname{ad}\nu_k(z')(*\nu_j(z')))\otimes\bar\nu_j^T(z),\end{aligned}$$

and so we have:

$$\begin{aligned}&(2i\bar\partial_\sigma V'[F_\sigma] - \operatorname{tr}G(V)\partial_\sigma F_\sigma\omega)(\nu_k) \\
&= + i\sum_j\int_X \mu\operatorname{tr}\partial\Delta_0^{-1}(*[*\nu_j,\nu_k])\bar\nu_j^T \\
&\qquad + \int_X \operatorname{tr}\bar\partial\Delta_0^{-1}\bar\partial^*(\operatorname{ad}((\mu\bar\nu_k^T))(\partial'(G(z,z')-Q(z,z'))))|_{z=z'} \\
&\overset{(*)}{=} i\sum_j\int_X \mu\operatorname{tr}\partial\Delta_0^{-1}(\overline{*[*\nu_k,\nu_j]}^T)\bar\nu_j^T \\
&\qquad - \int_X \operatorname{tr}\Delta_0^{-1}\bar\partial^*(\operatorname{ad}((\mu\bar\nu_k^T)))\wedge\bar\partial(\partial'(G(z,z')-Q(z,z')))|_{z=z'} \\
&= i\sum_j\int_X \mu\operatorname{tr}\partial\Delta_0^{-1}(\overline{*[*\nu_k,\nu_j]}^T)\bar\nu_j^T - \int_X \operatorname{tr}(\Delta_0^{-1}\bar\partial^*\operatorname{ad}((\mu\bar\nu_k^T)))\wedge(-P(z,z')) \\
&= i\sum_j\int_X \mu\operatorname{tr}\partial\Delta_0^{-1}(\overline{*[*\nu_k,\nu_j]}^T)\bar\nu_j^T + \sum_j\int_X \operatorname{tr}[\Delta_0^{-1}\bar\partial^*(\mu\bar\nu_k^T),\nu_j],\bar\nu_j^T \\
&= i\sum_j\int_X \mu\operatorname{tr}\partial\Delta_0^{-1}(\overline{*[*\nu_k,\nu_j]}^T)\bar\nu_j^T + i\sum_j\int_X \operatorname{tr}\Delta_0^{-1}\bar\partial^*(\mu\bar\nu_k)(\overline{*[*\nu_j,\nu_j]}^T)\rho \\
&= i\sum_j\int_X \mu\operatorname{tr}\partial\Delta_0^{-1}(\overline{*[*\nu_k,\nu_j]}^T)\bar\nu_j^T \\
&\qquad\qquad + i\sum_j\int_X \operatorname{tr}(-\rho^{-1}\partial\mu\bar\nu_k)\Delta_0^{-1}(\overline{*[*\nu_j,\nu_j]}^T)\rho \\
&= i\sum_j\int_X \mu\operatorname{tr}\partial\Delta_0^{-1}(\overline{*[*\nu_k,\nu_j]}^T)\bar\nu_j^T + i\sum_j\int_X \mu\operatorname{tr}(\partial\Delta_0^{-1}(\overline{*[*\nu_j,\nu_j]}^T))\bar\nu_k^T.\end{aligned}$$

The equality in $(*)$ follows from the fact that $\bar\partial(\operatorname{ad}(\Delta_0^{-1}\bar\partial^*\mu\bar\nu_k^T)\wedge\partial G)$ is exact, and so the integral is zero. Further, as part of the proof of Theorem 3.4.2 we calculated that when the limit exists in the integral it is:

$$-P(z,z') + \frac{1}{8\pi y^2}\left(\frac{1}{n^2(g-1)}+1\right) I = \bar{\partial}(\partial'(G(z,z') - Q(z,z'))|_{z=z'}).$$

The final term in the lemma is

$$\begin{aligned}
-\frac{1}{2}&\mathrm{tr}\nabla^{(1,0)}G(V)\omega(\bar{\partial}_k)\\
&= +i\sum_{l,j} d\nu_l((\nabla^{(1,0)}_{\partial_l}\int_x \mu\mathrm{tr}(\bar{\nu}_l^T\bar{\nu}_j^T)\partial_l\otimes\partial_j)\delta_j^k)\\
&= +i\sum_l \frac{d}{d\varepsilon}|_{\varepsilon=0}(\int_x \mu\mathrm{tr}(\overline{\nu_l^{\varepsilon\nu_k}}^T\overline{\nu_k^{\varepsilon\nu_k}}^T))\\
&= +i\sum_l \int_X \mu\mathrm{tr}(\overline{\bar{\partial}\Delta_0^{-1} * [*\nu_l,\nu_l]}^T\bar{\nu}_k^T + \bar{\nu}_l^T\overline{\bar{\partial}\Delta_0^{-1} * [*\nu_k,\nu_l]}^T)\\
&= +i\sum_l \int_X \mu\mathrm{tr}(\partial\Delta_0^{-1}\overline{*[*\nu_l,\nu_l]}^T\bar{\nu}_k^T + \bar{\nu}_l^T\partial\Delta_0^{-1}\overline{*[*\nu_k,\nu_l]}^T),
\end{aligned}$$

since $\nabla\partial_l = 0$ at E. This concludes the argument. ■

6.3 Consideration of the Curvature of the Hitchin Connection

We will now refine the expression found in [Andersen and Gammelgaard, 2011] for the curvature of the Hitchin connection. In [Andersen and Gammelgaard, 2011, Theorem 4.8] the following theorem is proved:

Theorem 6.3.1
The curvature of the Hitchin connection acts by

$$F_\nabla^{2,0} = \frac{k}{(2k+2n)^2}P_k(\partial_\mathcal{T} c) \quad F_\nabla^{1,1} = \frac{ik}{2k+2n}(\theta - 2i\partial_\mathcal{T}\bar{\partial}_\mathcal{T} F) \quad F_\nabla^{0,2} = 0,$$

on sections of the bundle $H^{(k)}$.

Here θ is as defined below in (6.4). The form c is given by:

$$c(V) = -\Delta_{G(V)} - dFG(V)dF - 2nV'[F].$$

Finally, $P_k(\partial_\mathcal{T} c(V,W))$ is the prequantum operator associated with the function $\partial_\mathcal{T} c(V,W)$:

$$P_k(\partial_\mathcal{T} c(V,W)) = \frac{i}{k}\nabla_{X_{\partial_\mathcal{T} c(V,W)}} + \partial_\mathcal{T} c(V,W)$$

for $X_{\partial_\mathcal{T} c(V,W)}$, the Hamiltonian vector field.

6.3.1 The $(1,1)$-part of the Curvature

We know that all the dependency on $\mathcal{M}_{VB}^{n,k}$ of the form $\bar{\partial}_{\mathcal{T}}\partial_{\mathcal{T}}\log\det\Delta_0(\mu_1,\bar{\mu_2})$ comes from

$$\theta(\mu_1,\bar{\mu_2}) = \frac{1}{4}g_{\mathcal{M}_{VB}^{n,k}}(\mathcal{S}_{\mathcal{M}_{VB}^{n,k}}(G(\mu_1)\omega_{\mathcal{M}_{VB}^{n,k}}\bar{G}(\bar{\mu_2}))), \tag{6.4}$$

which we can calculate using Hitchin's formulas presented in section 1.2.1 and the refinement calculated later in (6.2). Choose an orthonormal basis at (σ,ρ), then we have that:

$$\begin{aligned} &G(\mu_1)\omega_{\mathcal{M}_{VB}^{n,k}}\bar{G}(\bar{\mu_2})_{i\bar{j}} \\ &= \left(\sum_{j,l} -2i\int_X \mu_1 \mathrm{tr}\bar{\nu}_i^T\bar{\nu}_j^T(-\int \mathrm{tr}\nu_j\wedge\bar{\nu}_l^T)2i\int_X\bar{\mu_2}\mathrm{tr}\nu_l\nu_k\right)(X_{\nu_i}\otimes X_{\bar{\nu}_j}) \\ &= -4\left(\int_X \mu_1\mathrm{tr}\bar{\nu}_i^T P^{0,1}(\bar{\mu}_2\nu_k)\right)(X_{\nu_i}\otimes X_{\bar{\nu}_j}). \end{aligned}$$

If we symmetrize and then contract with the metric the result is the same as just contracting with the metric, since the symmetrization contains an averaging, and so we have:

$$\theta(\mu_1,\bar{\mu_2}) = -i\mathrm{tr}(\mu_1 P^{1,0}\bar{\mu_2}P^{1,0}), \tag{6.5}$$

the i comes since to rewrite as a trace we need the metric rather than the symplectic form.

Since a Beltrami differential solves the equation $\partial\mu(z) = \frac{2}{z-\bar{z}}\mu(z)$, we find that $\bar{\mu}\bar{\partial}\rho^{-1} = \rho^{-1}\bar{\partial}\bar{\mu} = -\partial^*\bar{\mu}$. Inserting this in the formula from Theorem 4.3.3 we have:

Corollary 6.3.2
The following identity holds

$$\theta(V,W) - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F(V,W) = \frac{\mathrm{Rank}E}{6\pi}\omega_{WP}(V,W) = \frac{n}{6\pi}\omega_{WP}(V,W) \tag{6.6}$$

and

$$F^{(1,1)}(V,W) = \frac{ikn}{12\pi(k+n)}\omega_{WP}(V,W).$$

6.3.2 The $(2,0)$-part of the Curvature

In order to calculate the $(2,0)$-part of the curvature we need the following from [Andersen and Gammelgaard, 2011]. They argue, because the curvature has to preserve the holomorphic sections, the operators $P_k(\partial_{\mathcal{T}}c(V,W))$ must preserve the holomorphic sections. As a consequence the Hamiltonian vector field of the function $\partial_{\mathcal{T}}c(V,W)$ on $\mathcal{M}_{n,k}^0$ must be holomorphic, and as there are none of those, $\partial_{\mathcal{T}} c(V,W)$ is in fact constant and $d_{\mathcal{M}_{n,k}}\partial_{\mathcal{T}} c(V,W) = 0$. We now use the Bianchi identity and Corollary 6.3.2 to see that:

$$\begin{aligned} U''[\frac{k}{(2k+2n)^2}\partial_{\mathcal{T}} c(V',W')] + V'[\frac{ikn}{12\pi(k+n)}\omega_{WP}(W',U'')] \\ - W'[\frac{-ikn}{12\pi(k+n)}\omega_{WP}(V',U'')] \\ = (\bar{\partial}_{\mathcal{T}}\partial_{\mathcal{T}} c)(U'',V',W') + \partial(\frac{ikn}{12\pi(k+n)}\omega_{WP})(W',V',U'') = 0. \end{aligned}$$

From this we conclude $\bar{\partial}_{\mathcal{T}}\partial_{\mathcal{T}} c = 0$, since ω_{WP} is closed. We therefore have $d_{\mathcal{T}}\partial_{\mathcal{T}} c = 0$.

Before we continue the argument, we will make two assumptions, which I believe are true for genus greater than two. The assumptions are:

1. That $H^2_{dR}(\mathcal{M}_g, \mathbb{C}) \cong \mathbb{C}$ and is generated by the symplectic form.
2. There is an embedded Riemann surface, X_g, in $\mathcal{M}_g$ such that for a 2-form, α, the isomorphism above is given by $\int_{X_g}(\alpha)$ times the symplectic form. (In [Kodaira, 1967] a family of Riemann surfaces of genus $m(2n-1)$ for $m, n \geq 1$ over a compact Riemann surface S. By the universal property this gives a map from S to $\mathcal{M}_{m(2n-1)}$. The image of this map is the manifold we called X_g for $m = g$ and $n = 1$.)

I am still looking for references on the first assumption.

These assumptions will now allow us to conclude that $\partial_{\mathcal{T}} c$ is exact. To conclude this we use that it is invariant under the action of the mapping class group, and therefore it descends from $\mathcal{T}$ to the moduli space of genus g curves, $\mathcal{M}_g$. We know that $H^2(\mathcal{M}_g, \mathbb{C}) \cong \mathbb{C}$. The isomorphism is described by integrating our form over a special Riemann surface embedded in $\mathcal{M}_g$, since $\partial_{\mathcal{T}} c$ is of type $(2,0)$ the integral over a Riemann surface is zero, and so it must be exact. Exactness allows us to find a 1-form $\tilde{c}$ on $\mathcal{M}_g$, such that $\partial_{\mathcal{M}_g} c = -d_{\mathcal{M}_g}\tilde{c}$. We can pull this back to $\mathcal{T}$, where we will still denote the form $\tilde{c}$.

While we cannot yet say the curvature of the Hitchin connection constructed in [Andersen, 2012] is of type $(1,1)$, we can modify it to get a new Hitchin connection for which this is true, by replacing $u(V)$ in the definition with $u(V) + \frac{k}{(2n+k)^2}\tilde{c}(V)$. We expect that $\partial_{\mathcal{T}} c = 0$, but have not yet verified this.

Theorem 6.3.3
The connection $\hat{\nabla} = \nabla^t + u + \frac{k}{(2n+k)^2}\tilde{c}$ *on* $H^k \to \mathcal{T}$ *is a Hitchin connection with curvature* $\frac{ikn}{12\pi(k+n)}\omega_{WP}$.

Proof: First we see that it is a Hitchin connection. By [Andersen, 2012, Lemma 2.2] it is a Hitchin connection, if:

$$\frac{i}{2}V[I]\nabla_\sigma^{1,0}s + \nabla_\sigma^{0,1}((u(V) + \frac{k}{(2n+k)^2}\tilde{c}(V))s) = 0$$

for ∇_σ, the Levi-Civita connection on $\mathcal{M}_{n,k}$. However, since $u(V)$ gives a Hitchin connection this reduces to:

$$\frac{i}{2}V[I]\nabla^{1,0}s + \nabla^{0,1}((u(V) + \frac{k}{(2n+k)^2}\tilde{c}(V))s) = \frac{k}{(2n+k)^2}(\bar{\partial}_{\mathcal{M}_{n,k}}\tilde{c}(V))s = 0.$$

We can now calculate the curvature of this modified Hitchin connection by modifying [Andersen and Gammelgaard, 2011, Theorem 4.8]. For commuting vector fields V and W on $\mathcal{T}$ we calculate:

$$\begin{aligned}
F_{\hat{\nabla}}(V,W) &= [\hat{\nabla}_V, \hat{\nabla}_W] \\
&= [\nabla_V^t + u(V) + \frac{k}{(2n+k)^2}\tilde{c}(V), \nabla_W^t + u(W) + \frac{k}{(2n+k)^2}\tilde{c}(W)] \\
&= [\nabla_V^t + u(V), \nabla_W^t + u(W)] \\
&\qquad + [\frac{k}{(2n+k)^2}\tilde{c}(V), \nabla_W^t + u(W) + \frac{k}{(2n+k)^2}\tilde{c}(W)] \\
&\qquad + [\nabla_V^t + u(V) + \frac{n}{(2n+k)^2}\tilde{c}(V), \frac{n}{(2n+k)^2}\tilde{c}(W)] \\
&= \frac{k}{(2n+k)^2}\partial_{\mathcal{T}}c(V,W) + \frac{ink}{12\pi(2n+k)}\omega_{WP}(V,W) \\
&\qquad + \frac{k}{(2n+k)^2}(-W[\tilde{c}(V)] + V[\tilde{c}(W)]) \\
&= \frac{ink}{12\pi(2n+k)}\omega_{WP}(V,W). \qquad \blacksquare
\end{aligned}$$

Since the curvature is now of type $(1,1)$ we should be able to conclude directly.

Conjecture 6.3.4
There exists a mapping class group invariant inner product on $H^{(k)}$, and it is preserved by the Hitchin connection of Theorem 6.3.3.

However, I have not been able to find the appropiate reference. We know from previous work of Andersen that there is a projectively compatible inner product.

6.4 The Inner Product on H_σ^k

This section is a presentation of my understanding of a method, explained to me by my advisor Andersen, of getting an equation for an inner product on $H^{(k)}$ (the bundle from Chapter 1).

Let $(\cdot,\cdot) : H_\sigma^k \times H_\sigma^k \to \mathbb{C}$ be a Hermitian inner product which is preserved by the Hitchin connection, that is:

$$V[(s_1, s_2)] - (\hat{\nabla}_V s_1, s_2) - (s_1, \hat{\nabla}_V s_2) = 0. \tag{6.7}$$

Where $\hat{\nabla}_V = \nabla_V^t + \hat{u}(V)$, with $\hat{u}(V) = -u(V) + \frac{k}{(n+2k)^2}\tilde{c}(V)$, is the Hitchin connection of Theorem 6.3.3. We denote the Hermitian inner product in the prequantum line bundle by $\langle\cdot,\cdot\rangle$ and make the ansatz:

$$(s_1, s_2)_\sigma = \int_X \langle s_1, s_2\rangle B_\sigma^{(k)} \frac{\omega^n}{n!}.$$

In the following we will denote the projection $C^\infty(\mathcal{M}, \mathcal{L}^{\otimes k}) \to H_\sigma^{(k)}$ by $\pi^{(k)}$. For (6.7) to be fulfilled we must have:

$$\pi^{(k)} \left(V[B_\sigma^{(k)}] + B^{(k)}\hat{u}(V) + \hat{u}(V)^* B^{(k)} \right) \pi^{(k)} = 0. \tag{6.8}$$

To reformulate this we need the following formula

$$\pi_\sigma^{(k)}(\nabla_X s_1) = \pi_\sigma^{(k)}(f_X s_1), \tag{6.9}$$

where $f_X = \Lambda d(\iota_X \omega)$ and Λ is the contraction with ω. Now we can rewrite what it means for (6.7) to be fulfilled.

$$\begin{aligned}
0 =& V[(s_1, s_2)] - (\nabla_V s_1, s_2) - (s_1, \nabla_{\bar{V}} s_2) \\
=& \int_{\mathcal{M}} V[\langle s_1, s_2\rangle B^{(k)}] \frac{\omega^n}{n!} - (\nabla_V s_1, s_2) - (s_1, \nabla_{\bar{V}} s_2) \\
=& \int_{\mathcal{M}} (\langle \hat{\nabla}_V^t s_1, s_2\rangle B^{(k)} + \langle s_1, \hat{\nabla}_{\bar{V}}^t s_2\rangle B^{(k)} + \langle s_1, s_2\rangle V[B^{(k)}]) \frac{\omega^n}{n!} \\
& - ((\nabla_V^t + \hat{u}(V)) s_1, s_2) - (s_1, (\nabla_{\bar{V}}^t + \hat{u}(\bar{V})) s_2) \\
=& \int_X \langle (-B^{(k)}\hat{u}(V) - \hat{u}^*(\bar{(V)}) B^{(k)} + V[B^{(k)}]) s_1, s_2\rangle \frac{\omega^n}{n!}.
\end{aligned}$$

Since s_1 and s_2 are holomorphic sections, this means we have the operator identity:

$$0 = \pi_\sigma^{(k)}(-B^{(k)}\hat{u}(V) - \hat{u}^*(\bar{V}) B^{(k)} + V[B^{(k)}]))\pi_\sigma^{(k)}.$$

Using (6.9) we calculate each term remembering that

$$u(V) = \frac{1}{4k + 2n} \left(\Delta_{G(V)} s + 2\nabla_{G(V)dF_\sigma} s - 4kV'[F_\sigma] s \right).$$

Note this differ from Theorem 1.2.4 because in the above expression I used $-F$.

We find:

$$\pi_\sigma^{(k)} B^{(k)} \Delta_{G(V)} = \pi_\sigma^{(k)} B^{(k)} \nabla_i G^{ij}(V) \nabla_j$$

$$\begin{aligned}
&= \pi_\sigma^{(k)}\nabla_i B^{(k)} G^{ij}(V)\nabla_j - \pi_\sigma^{(k)}(\nabla_i B^{(k)})G^{ij}(V)\nabla_j \\
&= -\pi_\sigma^{(k)} G^{ij}(V)\nabla_j(\nabla_i B^{(k)}) + \pi_\sigma^{(k)} G^{ij}(V)(\nabla_j\nabla_i B^{(k)}) \\
&= -\pi_\sigma^{(k)}\nabla_j G^{ij}(V)(\nabla_i B^{(k)}) + \pi_\sigma^{(k)}\nabla_j(G^{ij}(V))(\nabla_i B^{(k)}) \\
&+ \pi_\sigma^{(k)} G^{ij}(V)(\nabla_j\nabla_i B^{(k)}) = \pi_\sigma^{(k)}(\Delta_{G(V)} B^{(k)}),
\end{aligned}$$

and

$$\begin{aligned}
\pi_\sigma^{(k)} B^{(k)}\nabla_{G(V)dF_\sigma} &= \pi_\sigma^{(k)}\nabla_{G(V)dF_\sigma} B^{(k)} - \pi_\sigma^{(k)}(\nabla_{G(V)dF_\sigma} B^{(k)}) \\
&= \pi_\sigma^{(k)}\nabla_i G^{ij}(V) dF_\sigma(\nu_j) B^{(k)} - \pi_\sigma^{(k)}(\nabla_i G^{ij}(V) dF_\sigma(\nu_j)) B^{(k)} \\
&\quad - \pi_\sigma^{(k)}(\nabla_{G(V)dF_\sigma} B^{(k)}) \\
&= -\pi_\sigma^{(k)}(\Delta_{G(V)} F_\sigma) B^{(k)} - \pi_\sigma^{(k)}(\nabla_{G(V)dF_\sigma} B^{(k)}).
\end{aligned}$$

This shows that:

$$\begin{aligned}
\pi_\sigma^{(k)} B^{(k)} u(V)\pi_\sigma^{(k)} = \pi_\sigma^{(k)}\frac{1}{4k+2n}\Big(&\Delta_{G(V)} B^{(k)} - 2(\Delta_{G(V)} F_\sigma) B^{(k)} \\
&-2\nabla_{G(V)dF_\sigma} B^{(k)} - 4kV'[F_\sigma] B^{(k)}\Big).
\end{aligned}$$

Now $\pi_\sigma^{(k)} u(\bar{V})^* B^{(k)}\pi_\sigma^{(k)}$ is the adjoint of $\pi_\sigma^{(k)} B^{(k)} u(\bar{V})\pi_\sigma^{(k)}$, since $B^{(k)}$ is real, and therefore

$$\pi_\sigma^{(k)} u(\bar{V})^* B^{(k)}\pi_\sigma^{(k)} = \overline{\pi_\sigma^{(k)} B^{(k)} u(\bar{V})\pi_\sigma^{(k)}}$$

Now $\overline{\Delta_{G(\bar{V})} B^{(k)}} = \Delta_{\bar{G}(V)} B^{(k)}$, and we find that:

$$\begin{aligned}
\pi_\sigma^{(k)} u(\bar{V})^* B^{(k)}\pi_\sigma^{(k)} = \pi_\sigma^{(k)}\frac{1}{4k+2n}\Big(&\Delta_{\bar{G}(V)} B^{(k)} - 2(\Delta_{\bar{G}(V)} F_\sigma) B^{(k)} \\
&-2\nabla_{\bar{G}(V)dF_\sigma} B^{(k)} - 4kV''[F_\sigma] B^{(k)}\Big).
\end{aligned}$$

We rewrite (6.8) to:

$$\begin{aligned}
&V[B_\sigma^{(k)}] - \frac{1}{4k+2n}\Big((\Delta_{\tilde{G}(V)} B^{(k)}) - 2(\nabla_{\tilde{G}(V)dF} B^{(k)}) \\
&-2(\Delta_{\tilde{G}(V)} F) B^{(k)} + 2nV[F] B^{(k)}\Big) - V[F] B^{(k)} + \frac{4k}{(4k+2n)^2}(\tilde{c}(V) + \overline{\tilde{c}(V)}) = 0.
\end{aligned} \tag{6.10}$$

From this follows that up to order $\frac{1}{4k+2n}$ the function e^F is a solution. Now that I have explained how to get an equation for $B^{(k)}$, we proceed to study the equation and try to solve it.

We can expand $B^{(k)}$ in powers of $\frac{1}{4k+2n}$ writing

$$B^{(k)} = e^F\left(1 + \sum_{j=1}^{\infty} c_j\left(\frac{1}{4k+2n}\right)^j\right).$$

For the $\frac{1}{4k+2n}$ term, c_1, we have the following equation.

$$V[c_1 e^F] + \Big((\Delta_{\tilde{G}(V)} e^F) - 2(\nabla_{\tilde{G}(V)dF} e^F) \\ -2(\Delta_{\tilde{G}(V)} F)e^F - 2nV[F]e^F\Big) - V[F]e^F c_1 + \tilde{c}(V) + \overline{\tilde{c}(V)} = 0.$$

Continuing the calculation

$$V[c_1]e^F + \Big(((\Delta_{\tilde{G}(V)} F)e^F + dF\tilde{G}(V)dFe^F) - 2(dF\tilde{G}(V)dFe^F) \\ -2(\Delta_{\tilde{G}(V)} F)e^F - 2nV[F]e^F\Big) + \tilde{c}(V) + \overline{\tilde{c}(V)} = 0.$$

And so the final differential equation is:

$$\begin{aligned} V[c_1] &= -(\Delta_{\tilde{G}(V)} F) - dF\tilde{G}(V)dF - 2nV[F] + \tilde{c}(V) \\ &= c(V) + \bar{c}(V) + \tilde{c}(V) + \overline{\tilde{c}(V)}. \end{aligned} \tag{6.11}$$

We see that the last term is a closed 1-form, since $\partial_{\mathcal{T}} c + d_{\mathcal{T}} \tilde{c} = 0$, and from Corollary 6.3.2 and [Andersen and Gammelgaard, 2011, Prop. 4.4 and Prop 4.3] we have $\bar{\partial}_{\mathcal{T}} c + \partial_{\mathcal{T}} \bar{c} = 0$. We can solve (6.11) on $\mathcal{T}$, and for genus greater than 2 we have that the abelianization of $\pi_1(\mathcal{M}_g)$ is trivial, [Powell, 1978], so the solution will be mapping class group invariant. We see this, since c_1 gives a homomorphism from $\pi_1(\mathcal{M}_g)$ to $\mathbb{C}$ which must be the zero map. Now this allows us to determine c_1 up to a function on $\mathcal{M}^0_{n,k}$.

For higher order coefficients we have, doing similar computations, that:

$$V[c_j] = c_{j-1}(c(V) + \bar{c}(V) + \tilde{c}(V) + \bar{\tilde{c}}(V)) - 2n(\tilde{c}(V) + \bar{\tilde{c}}(V))c_{j-2} \\ +(\Delta_{\tilde{G}(V)} c_{j-1} - dF\tilde{G}(V)dc_{j-1}).$$

In future work I hope also to be able to solve for $c_j, j > 1$, with a proper choice of c_1.

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