Matchings with Externalities and Attitudes

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Intensely studied class of combinatorial problems:

**One-to-One**: The stable marriage problem

**One-to-Many**: House allocation problems, assigning medical interns to hospitals

**Many-to-Many**: Most labor markets, friendships
Externalities

Also known as transaction spillovers

Third parties are influenced by transactions they did not agree to

**Positive externalities:** Education, immunization, environmental cleanup, research

**Negative externalities:** Environmental pollution, smoking, drinking and driving
Externalities in Matchings

Matchings are a natural model for studying externalities

Agents influenced not only by their own choices (matches), but also by the choices that other agents make

Existing work in economics assumes agents have a different utility for every state of the world

Can bounded rational agents reason about such games?

➢ Succinct model of externalities in matchings (polynomial-size preferences in the number of agents)
Let $G = (M, W, \Pi)$ be a matching game, where $M$ and $W$ are agents on the two sides of the market.

Denote by $\Pi(m, w \mid z)$ the influence of match $(m, w)$ on agent $z$ (if the match forms).

The utility of an agent $z$ in matching $A$ is:

$$u(z, A) = \sum_{(m, w) \in A} \Pi(m, w \mid z)$$
Stability is a central question in game theoretic analyses of matchings.

Given a game, which matchings are such that the agents don't have incentives to (i) cut existing matches or (ii) form new matches?

The stable outcomes depend on the solution concept used:

➢ *This work*: pairwise stability and the core
Core Stability

Given a matching game $G = (M, W, \Pi)$, a matching $A$ of $G$ is core-stable if there does not exist a set of agents $B \subseteq N$, which can deviate and improve the utility of at least one member of $B$ while not degrading the others.
Deviation

Each member of a deviating coalition $B$ must perform some action: either sever a match with an agent in $N$, or form a new match with an agent in $B$.

Response

Given matching $A$ and deviation $A'$ of coalition $B$, the response $\Gamma(B, A, A')$ defines the reaction of the agents outside $B$ upon the deviation.
Stability

A matching is stable if no coalition can deviate and improve the utility of at least one member while not degrading the other members in the response of $N \setminus B$

How will society respond to a deviation?

- The deviators need to estimate the response of the residual agents (which may be intractable)
**Attitudes**

**Optimism:** Deviators assume the best case reaction from the rest of the agents; hoping for the formation of matches good for the deviators and removal of all bad matches (attitude à la “All is for the best in the best of all the possible worlds”)

**Neutrality:** No reaction (the deviators behave as if the others are not going to do anything about the deviation)

**Pessimism:** Worst case reaction (deviators assume the remaining agents will retaliate in the worst possible way)
Many other definitions possible:

**Contractual**: Assume retaliation from agents hurt by the deviation, and no reaction from the rest

**Recursive core (Koczy)**: when a coalition deviates, the residual agents react rationally (maximize their own payoff in the response)
Empty Neutral Core

- The complete matching is Pareto optimal, but unstable
- The empty matching may be stable depending on $\varepsilon$, $\Delta$
The complete matching is a tragic outcome for everyone; may be stable depending on $\varepsilon$, $\Delta$
Many-to-Many Matchings

The cores are included in each other
### Many-to-Many Matchings

<table>
<thead>
<tr>
<th>Core</th>
<th>Optimism</th>
<th>Neutrality</th>
<th>Pessimism</th>
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<tbody>
<tr>
<td>Membership</td>
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<td>coNP-complete</td>
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<td>Nonemptiness</td>
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**Theorem:** Checking membership to the neutral core is \(\text{coNP}\)-complete.

**Proof (sketch):**

- Show the complementary problem is \(\text{NP}\)-complete

- Given \(I = (U, s, v, B, K)\), construct game \(G = (M, W, \Pi)\) and matching \(A\) such that \(A\) has a blocking coalition if and only if \(I\) has a solution
Many-to-Many Matchings

\[ A = \{(m_2, w_2), (m_1), (w_1), (x_1), \ldots, (x_n), (y_1), \ldots, (y_n)\} \]

has a blocking coalition \( I \) has a solution
Known as the stable marriage problem

- the Gale-Shapley algorithm used to compute stable outcomes

**The Core with Externalities:**

- Without externalities, the core is equivalent to the pairwise stable set
- The equivalence between pairwise stability and the core no longer holds with externalities
Moreover, under arbitrary $\Pi$ values, even a pairwise stable solution does not always exist.
However, a pairwise stable matching under neutrality and pessimism always exists when $\Pi$ is non-negative.

- Run Gale-Shapley by ignoring externalities and breaking ties arbitrarily
# One-to-One Matchings with Externalities

<table>
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More refined solution concepts - interesting line of work in economics (e.g. the recursive core)

Externalities in social networks

➢ On platforms such as Facebook, agents are influenced by the matchings of others (friendships, subscriptions)

➢ Such cumulative effects can be expressed with additive models, but what is the right solution concept for bounded rational agents in such settings?