Option Pricing under Heston and $\frac{3}{2}$ Stochastic Volatility Models: an Approximation to the Fast Fourier Transform

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Abstract

The purpose of this thesis is to build a fast and accurate technique for computing option prices under stochastic volatility assumption. Currently, the methodology based on the fast Fourier transform is widely used to deal with that issue. Here we derive and suggest a second order approximation, which offers faster, transparent and deeper interpretations in comparison with existing the ones. Thus, a tractable connection between the Black-Scholes prices and the stochastic volatility prices might be established.
Acknowledgements

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Chapter 1

Introduction

With the development of financial markets, the interested parties have the chance of choosing among a variety of financial instruments for selecting those which suit in optimal way their practical needs. Options play a significant role in the market game - they can be used for hedging, for speculative purposes, for building new synthetic instruments and are therefore subject to a profound attention from practitioners and academics.

The first model for option pricing, providing with simple closed-form formulas, was established by Black and Scholes (1973) (BS model hereafter). It became very popular in the following years. Its simplicity is related not only to easy implementation, but also to its drawbacks. Clearly, as every model does, it relies on certain assumptions for building a relevant framework. Some of these simplifications of reality, such as liquidity, no transaction costs or efficient markets are hard to relax, although clearly being unrealistic. Some of the other BS assumptions, however, can be relaxed by introducing stochastic volatility models. A brief justification for the necessity to account for the stochastic nature of volatility will be presented next.

1.1 Justification of the needs of a stochastic volatility model

A further look at the BS assumptions provides an area for elaborating on more complex and thus better performing models, describing important financial instruments, reflecting the reality. With respect to the constant volatility and the log-normal distribution of returns, academics and practitioners have early started seeking for suitable substitute assumptions that would be more realistic. Regarding returns, what is usually observed is excess kurtosis, negative skewness and heavier tails in the density distribution as compared to the familiar normal density. Concerning volatility, a popular observation by Mandelbrot (1963) is that "large changes are being followed by large changes, and small changes are being followed by small changes". That effect is known as volatility clustering which does not exist in
the BS model world, where volatility is assumed to be constant. It is worth mentioning the empirically observed "leverage effect", which means that usually increases in the volatility values result in a decrease in the underlying asset price, i.e. they are negatively correlated (e.g. Bakshi, Cao, and Chen (1997)). For a discussion on stylized facts and analysis of the asset returns’ empirical behaviour, the reader can refer to Cont (2001). A profound study on financial time series properties can be found in Tsay (2005).

In the world of option pricing, it is not just the case that volatility is not constant. It can be shown (e.g. Gatheral (2006)) that in practice volatility takes different values, depending on moneyness (how much in-the-money or out-of-the-money the option is at the current time \(t\)) and term structure. The existence of a "volatility smile" has been widely discussed and studied, to come up with the observation that implied volatility (the nonconstant volatility value to be incorporated in the BS formulae to obtain the observable market prices) takes higher values for options being far in or out of the money, as compared to the ones at the money. From here, plotting implied volatility values against moneyness, a convex function may be observed. And, if implied volatility is charted against time, for a fixed strike, then a time skew is being observed. According to the BS model assumptions, that function should be a flat line or \(y(\sigma_t) = \sigma\). For a visual representation of both effects mentioned, a volatility surface can be produced. In Figure 1.1 a graph representing the relationships between implied volatility, moneyness and time to maturity is plotted (of September 15th 2005, see Gatheral (2006)).

![Figure 1.1: SPX implied volatility surface.](image)

As empirically confirmed by the numerous studies on stock, index or interest rate returns, the volatility is not a constant. Not only there is a scientific evidence in favour of the
nonconstant volatility, but also there is even a traded index on the implied volatility of the S&P500 options, the Chicago Board Options Exchange Market Volatility Index, or VIX, see Figure 1.2.

Hence, a more realistic model would account for the above mentioned observed properties. In practice, the academic branches that take into consideration the volatility smile effect include local volatility models, stochastic volatility models and jump-diffusion models with different specifications.

When focusing on option pricing under a stochastic volatility model, a broadly used pricing method involves characteristic functions inversion. That tool is known as fast Fourier transform (referred to as FFT hereafter) and was first mentioned in Cooley and Tukey (1965) and later elaborated by Carr and Madan (1999). The approach works by providing with prices based on a certain parameter set input. In that way one obtains just a number and thus fails to reveal much about what this number is actually composed of. In a recent contribution by Drimus (2011a), an approximation to the FFT was developed, derived by a Taylor expansion and it consists of the corresponding BS price and certain Greeks. The methodology is tractable, gives a reasonable forecast about the price structure and its relation to the BS elements.

1.2 Purpose and problem statement

The main reference and inspiration for the present work can be found in Drimus (2011a) and Drimus (2011b). The approximation to the FFT derived in Drimus (2011a) is based
on the Heston stochastic volatility model, see Heston (1993). The purpose of this thesis is to replicate that expansion for a more advanced setting, or namely for the so called 3/2 stochastic volatility model, see for example Drimus (2011b). By comparing the performance of the 3/2-based and the Heston-based approximation, the main aim of this thesis is to try to answer the following question:

*Is the FFT approximation provided by Drimus (2011a) a viable tool for studying option prices and implied volatilities under the 3/2 model?*

For drawing a conclusion, the following related research questions will be addressed:

- How is the FFT methodology structured and what are its drawbacks?
- What is the motivation for going beyond the Heston model?
- How is the approximation from Drimus (2011a) derived in the 3/2 case and how is it different from the Heston counterpart?
- What are the necessary conditions for having a well-behaved FFT approximation under the 3/2 model?
- Are there any drawbacks for the expansion under the 3/2 setting?

### 1.3 Limitations

The financial instruments studied are options written on an underlying asset with stochastic price and variance dynamics. One could further complicate the setting by incorporating jumps, which would bring the model closer to reality. For a further justification for the use and properties of stochastic volatility models with jumps, the reader may refer to Cont and Tankov (2004). Also, there is a significant amount of research on models driven by Lévy processes - Generalized Poisson, Normal Inverse Gaussian, Variance Gamma, say. However, these models are beyond the scope of this work.

All valuations are performed on European call options only. However, the results can be extended to put options in a straightforward way.

Also, in terms of terminology, notation and derivations, it is assumed that the reader is familiar with the basics of stochastic calculus and the BS general setting.

Finally, the parameters used for testing the models are from calibrations performed using index options, or namely S&P500 options.
1.4 Thesis structure

The thesis is structured as follows. Chapter 2 gives a theoretical basis for the results and analyses to be further developed. It provides with a thorough discussion on general option pricing theory. Chapter 3 puts an emphasis on Fourier transforms and the FFT option pricing method, in particular. The necessary theoretical background for pricing with the latter method will be elaborated as a build-up for the further analysis. Chapter 4 gives details about the Heston stochastic volatility model, provides with an insight about the contribution of Drimus (2011a) and elaborates on the price approximation derivation and properties. Analysis and applications are discussed and a comparison with the FFT is conducted. Chapter 5 applies the same methodology to the 3/2 stochastic volatility model and again compares its performance with the FFT results. Chapter 6 provides a discussion on the findings and the properties with respect to the expansion and its performance and also gives an interpretation about the parameters’ behaviour. Finally, a conclusion and ideas for further research are to be found in Chapter 7.


Chapter 2

Theoretical background

2.1 Stochastic volatility models - general setting

After the introduction of the BS model in 1973, many researchers have tried to arrive at models fitting reality better. A wide area of research is dedicated to the stochastic volatility models and we will give a general description.

Let \( f_S t \geq 0 \) denote the underlying asset process and \( f_v t \geq 0 \) be the variance process. The general form of stochastic volatility models is characterized by \( \mu \) being the expected rate of return, \( \delta \) denoting the dividend yield, \( \epsilon \) being the volatility of variance and \( \alpha \) and \( \beta \) representing two arbitrary functions, depending on the underlying asset, the variance itself and time. Also, \( \{B_t\}_{t \geq 0} \) and \( \{W_t\}_{t \geq 0} \) represent two Wiener processes, possibly correlated with \( \mathbb{E}[dB_t dW_t] = \rho dt \). The general form of a stochastic volatility model is

\[
dS_t = (\mu - \delta)S_t dt + \sqrt{\nu_t}S_t dB_t,
\]

\[
dv_t = \alpha(S_t, v_t, t) dt + \epsilon \beta(S_t, v_t, t) \sqrt{\nu_t} dW_t. \tag{2.1.1}
\]

As one can easily notice, the model generalizes the BS world setting. It is important to note that the model (2.1.1) is defined under the physical measure \( \mathbb{P} \). As shown in Gatheral (2006), one can derive a general partial differential equation (PDE) that every contract written on an underlying with a variance having the dynamics (2.1.1) should satisfy in order for no arbitrage opportunities to exist on the market. The equation is obtained by forming a risk-free portfolio, this time involving a position in an extra derivative as compared to the constant volatility case, due to the second source of randomness, or namely the variance stochastic dynamics. The price of a derivative contract \( V \) should satisfy:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \epsilon \beta v S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \epsilon^2 \beta^2 v \frac{\partial^2 V}{\partial v^2} + r S \frac{\partial V}{\partial S} - r V = -\alpha' \frac{\partial V}{\partial v}, \tag{2.1.2}
\]

where \( \alpha' = \alpha(S_t, v_t, t) - \phi \beta(S_t, v_t, t) \sqrt{v} \) with \( \phi(S, v, t) \) being the market price of volatility risk.
and \( r \) is the risk-free rate. Similarly to the Capital Asset Pricing Model (CAPM) concept, for bearing the additional volatility risk, investors require an extra return of the amount \( \phi(S,v,t) \).

According to the Feynmann-Kač theorem (Karatzas and Shreve (1991)), the Cauchy problem consisting of the general PDE above and a terminal condition \( \text{Payoff}(S_T^*,v_T^*,T) \) has a solution

\[
P(S_t,v_t,t) = \mathbb{E}_t^Q[e^{-rT}\text{Payoff}(S_T^*,v_T^*,T)\bigl|\mathcal{F}_t]
\]  

(2.1.3)

with \( r \) being the short interest rate for a contract maturing at time \( T \) and \( S_t^* \) and \( v_t^* \) are the stock price and variance processes, satisfying

\[
dS_t^* = (r - \delta)S_t^*dt + \sqrt{v_t^*}S_t^*dB_t,
\]

\[
dv_t^* = \alpha'(S_t^*,v_t^*,t)dt + \epsilon\beta(S_t^*,v_t^*,t)\sqrt{v_t^*}dW_t.
\]

(2.1.4)

The expected value in (2.1.3) is taken conditional on the information flow \( \mathcal{F}_t \) available at the present time \( t \) and under a probability measure \( Q \). The latter is an equivalent martingale measure, that is, it has the same null set as the physical measure \( P \) and it assures that the new discounted processes (2.1.4) are martingales. Relation (2.1.3) states that to calculate an arbitrage-free price of a derivative, one needs the expected payoff, discounted at the risk-free rate, under \( Q \). In other words, under the risk-neutral setting, one is employing the dynamics (2.1.4) instead of (2.1.1)

Remark 2.1.1. Further, all calculations and derivations will be performed under the dynamics (2.1.4). When restricted to option pricing, and with parameters obtained from direct calibration to market prices, one is already in the risk-neutral setting. As explained in Gatheral (2006), that allows us to set the market price of volatility risk \( \phi(S,v,t) \) equal to zero. Also, the Wiener process driving the spot dynamics can be represented as \( dB_t = \rho dW_t + \sqrt{1 - \rho^2}dZ_t \), where \( \{Z_t\}_{t \geq 0} \) is a Wiener process, independent of \( \{W_t\}_{t \geq 0} \). In terms of notation, everything that follows will be according to the dynamics (2.1.4), with \( S_T^* = S_T \) and \( \alpha' = \alpha \) or

\[
dS_t = (r - \delta)S_tdt + \sqrt{v_t}S_t(\rho dW_t + \sqrt{1 - \rho^2}dZ_t),
\]

\[
dv_t = \alpha(S_t,v_t,t)dt + \epsilon\beta(S_t,v_t,t)\sqrt{v_t}dW_t.
\]

(2.1.5)
In contrast to models, characterized by a single stochastic process, such as the BS model, in a stochastic volatility model markets are incomplete. As volatility is not a traded asset, it is not possible to achieve a perfect replicating strategy. Having an incomplete market leads to having a set of *equivalent martingale measures* which is not a singleton. However, as noted in Remark 2.1.1, when dealing with parameters obtained by a calibration, one is already being transferred to the risk-neutral setting. Having a non-empty set of equivalent martingale measures ensures that there are still no arbitrage opportunities. More details on these general model settings can be found at the paper of Harrison and Pliska (1981).

Having the above representations in hand, for a stochastic volatility model, one can think of each value taken by the volatility as a BS parameter and thus a multitude of BS call prices can be calculated. Then, by taking an average for all possible paths of $v_t$, the desired price is obtained. According to the theorem of Romano and Touzi (1997) a call option price under a stochastic volatility model with the general characteristics (2.1.5), can be expressed as an expected value of BS prices for given initial value for the underlying asset $S_0$ and for the variance $v_0$

$$C_{SV}(S_0, v_0, T) = \mathbb{E}^{Q}[(C_{BS}(S_{eff}^T, \sqrt{v_{eff}^T}, T)],$$

with an *effective spot* $S_{eff}^T$ and *effective variance*, $v_{eff}^T$ at time $T$ defined as

$$S_{eff}^T = S_0 \exp \left( -\frac{1}{2} \int_0^T \rho^2 v_s ds + \int_0^T \rho \sqrt{v_s} dW_s \right),$$

$$v_{eff}^T = \frac{1}{T} \int_0^T (1 - \rho^2) v_s ds,$$

with $v_t$ having the model specified dynamics and $\rho$ denoting the correlation coefficient between the Brownian motions characterizing the underlying asset and variance dynamics. In other words, the modeled derivative price under a stochastic volatility model can in fact be represented as a mixture of BS prices with an *effective spot* and *effective variance*, as defined. These values depend solely on the variance process.

### 2.2 First contributions to stochastic volatility models

Below, the models introduced by Hull and White (1987) and Stein and Stein (1991) will be briefly described, being some of the first contributions to stochastic volatility modeling.

One of the first stochastic volatility models, introduced after the market crash of 1987, is developed in Hull and White (1987). At that time, it was already clear that the BS model
The general form of the modeled underlying asset and volatility dynamics is:

\[
\begin{align*}
\text{d}S_t &= (r - \delta) S_t dt + v_t S_t dZ_t, \\
\text{d}v_t &= \psi v_t dt + \epsilon v_t dW_t,
\end{align*}
\]

where \( r \) is the risk-free rate of the market, \( \delta \) is the corresponding dividend yield, \( \psi \) is the mean of the volatility process and \( \epsilon \) is the volatility of volatility. The correlation coefficient \( \rho \) is zero, that is, the price and volatility dynamics are driven by independent random processes, as \( dZ_t dW_t = 0 \).

The Hull and White model is a simple form of the stochastic volatility models developed later. Its main drawbacks are the zero correlation assumption and the lack of an incorporated mean-reverting part for the volatility dynamics. The former has been observed not to hold when dealing with real data, as was shown in the research of Bakshi, Cao, and Chen (1997), for example. The latter study shows empirically that a positive impulse within the volatility is related to a negative effect within the corresponding stock price, that is, they are negatively correlated, or \( \rho \leq 0 \). That leads to the previously mentioned ”leverage effect”. The drawback concerning the mean-reverting volatility structure is also relevant, as shown in an early study by Merville and Pieptea (1989). The authors ”document that the mean-reverting hypothesis is correct” and that ”there is a long-term value toward which the instantaneous variance is pulled”.

A couple of years afterwards, Stein and Stein (1991) introduce a model with spot and volatility dynamics described by

\[
\begin{align*}
\text{d}S_t &= (r - \delta) S_t dt + v_t S_t (\rho dW_t + \sqrt{1 - \rho^2} dZ_t) \\
\text{d}v_t &= k(\theta - v_t) dt + \epsilon dW_t
\end{align*}
\]

with the instantaneous volatility being modeled as an Ornstein-Uhlenbeck process and introducing a mean-reverting structure, with a speed-reversion denoted by \( k \) and long-term volatility level denoted by \( \theta \). The parameter \( \epsilon \) is the volatility of the volatility \( v_t \). The authors provide with closed-form option pricing solutions. With that structure, however, volatility is not prevented from going negative, which is not reasonable. Also, the model assumes no correlation between the processes, or \( \rho = 0 \). That problem is resolved by a later study of Schoebel and Zhu (1999), where the authors allow for correlation to take values different from zero and thus solve the leverage effect issue, but the drawback with the possibly
negative volatility remains.

2.3 Back to pricing

For pricing an option by taking the expected payoff, discounted with the risk-free rate as in (2.1.3), the risk-neutral density of the underlying asset is needed and it should be taken conditional on the information set \( \mathcal{F}_t \). Knowing the density in question, one can easily obtain the desired contingent claim’s price. That is the case with the BS model, where just the density of a normal random variable is needed. However, for many other (and more realistic) models, the desired expression is unknown. Luckily, these models usually have the characteristic function of the spot dynamics available. With that tool in hand, the behaviour of the random variable it belongs to is completely tractable. A popular and efficient technique employed in option pricing literature involves using the Fourier transform of the random variables of interest. An analysis of the general application and of different approaches is reviewed by Schmelzl (2010).
Chapter 3

Option pricing and
the Fourier transform

This chapter gives an insight about operations involving the Fourier transform of a random variable and its inversion, applied to option pricing and is based on Schmelzle (2010). One may also find useful the work of Dufresne, Garrido, and Morales (2009).

3.1 General form and notations

First, some general definitions will be presented. They will be a base for the fast Fourier transform (FFT) option pricing methodology, which will be discussed later in this chapter.

Let \( \mathcal{F}[f(x)] \) denote the Fourier transform of a real function \( f(x) \in \mathbb{R} \), i.e.

\[
\mathcal{F}[f(x)] = \mathbb{E}[e^{iux}] = \int_{-\infty}^{\infty} e^{iux} f(x) dx,
\]

(3.1.1)

with \( i \) being the imaginary unit, and \( u \) being a real number \( (u \in \mathbb{R}) \). In the case when \( f(x) = f_X(x) \) is a probability density function of a random variable \( X \), the transform above is the characteristic function of \( X \) to be denoted by \( \phi_X(u) \).

It is important to note that there exists a one-to-one correspondence between the characteristic function and the probability density function (PDF) of a distribution. Random variables are fully described by their characteristic functions, that is, if \( \phi_X(u) \) is known, the distribution of \( X \) is completely defined. Also, by knowing the characteristic function, one can easily recover the PDF by using the inversion formula

\[
f_X(x) = \mathcal{F}^{-1}[\phi_X(u)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \phi_X(u) du.
\]

(3.1.2)
CHAPTER 3. OPTION PRICING AND THE FOURIER TRANSFORM

The Fourier transform (3.1.1) exists when
\[ \int_{-\infty}^{\infty} |f(x)| \, dx < \infty, \]  
(3.1.3)
i.e. the function \( f(x) \) should be absolutely integrable. Respectively, for existence of the inverse transform (3.1.2), \( \varphi_X(u) \) has to fulfill the integrability condition.

For building option prices in the next section, and in particular, for being able to obtain the Fourier transform of an option payoff, one will work with \( u \) being substituted by a complex number \( z = z_r + iz_i \) (\( z \in \mathbb{C} \)). When using the complex counterpart for \( u \), the expectation in (3.1.1) results in a Generalized Fourier transform. Then, the inversion formula takes the form:
\[ f_X(x) = \mathcal{F}^{-1}[\varphi_X(u)] = \frac{1}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} e^{-iux} \varphi_X(u) \, du. \]  
(3.1.4)
Taking into account the above facts, we may build option prices, based on the Fourier transform and its inverse.

The next section we will provide details on two pricing approaches, that are based on the methodology, referred to as FFT.

3.2 Pricing of options

The famous risk-neutral option pricing approach, employing the Feynman-Kač theorem (e.g. see (2.1.6) and Karatzas and Shreve (1991)), involves taking expectation of the contract’s payoff at maturity with respect to a risk neutral measure and thus the relevant PDF of the random log-price is needed to perform the calculation. In practice, a PDF is not always easily obtained. Instead, characteristic functions always exist. The general idea behind using the techniques described in Section 3.1 is that by using characteristic functions, one is being transferred to making computations within the Image space rather than the Real one. It turns out that for a number of functions the former operations are much easier. That is, for more complicated processes, one can employ the relationship between density and characteristic function of a distribution, with the procedure being composed in two steps. First, a mapping of the characteristic function to the payoff of interest is generated and second, the inversion formula is applied to obtain the prices of interest.

A technique involving Fourier transform methods was used by Heston (1993) for obtaining the widely used pricing formulae. However, the approach used there is beyond the scope
of this thesis. For details on the derivation of the latter formulae, the reader is referred to Heston (1993).

In this thesis the focus will be on a methodology, known as the FFT, which is able to produce entire vectors and matrices of prices by performing the characteristic function inversion just once. In this way FFT is computationally and technically more efficient than previous related approaches. The methodology was introduced in Carr and Madan (1999) and then extended by Lewis (2001) and in what follows both contributions will be presented.

The method of Carr and Madan (1999) consists of calculating the Fourier transform of the entire option price without separating the payoff from the risk-neutral density. We will discuss the case of a call option, the put counterpart can be derived in an analogous way. The approach begins with the familiar risk-neutral valuation, which for a European call price with a strike $K$ and a spot at maturity $S_T$ satisfies the relation

$$C_T(k) = e^{-rT}E^Q[(S_T - K)^+] = e^{-rT} \int_{k}^{\infty} (e^x - e^k)q(x)dx,$$

where $x = \log S_T$, $k = \log K$ and $q(x)$ is the risk-neutral density. The main idea is to transform the expression for $C_T(k)$ and then use the inversion formula and thus obtain the desired price. In that way the unknown density function $q(x)$ is avoided.

However, due to the fact that $\lim_{k \to -\infty} C_T(k) = S_0$, the condition for absolute integrability (3.1.3) is not fulfilled and this means that the Fourier transform of $C_T$ does not exist. To avoid this problem, the authors modify the expression for the call price $C_T$, by introducing a damping factor of the form $e^{ak}$ with $a > 0$. Then, the new function

$$c_T(k) = e^{ak}C_T(k)$$

is absolutely integrable for a suitably chosen $a$ and hence both the Fourier transform and the inverse transform exist and as $k$ approaches minus infinity the price of a call goes to zero, as it should. Let $\psi(u)$ denote the Fourier transform of $c_T(k)$. Then it takes the form

$$\psi(u) = \frac{e^{-rT}\phi_T(u - (a + 1)i)}{a^2 + a - a^2 + i(2a + 1)u},$$

with $\phi_T(u)$ being the characteristic function of the price process under the risk-neutral setting (or the Fourier transform of the risk-neutral density $q(x)$). Finally, the call price is obtained
by taking the inverse transform, accounting for the dampening factor:

\[
C_T(k) = \frac{e^{-ak}}{2\pi} \int_{-\infty}^{+\infty} e^{-iku} \psi(u) du = \frac{e^{-ak}}{\pi} \int_{0}^{+\infty} \text{Re}[e^{-iku} \psi(u)] du. \tag{3.2.1}
\]

The option pricing method to be employed in the current work is the one elaborated by Lewis (2001). The main features will be briefly highlighted here, however, for a thorough discussion one has to be familiar with the facts given by Lewis (2001) or Schmelzle (2010). The notation in what follows will correspond to the one used in the latter paper.

The main idea is that instead of transforming the entire option price, as was done in the Carr and Madan (1999) framework, one could use the payoff function’s Fourier transform. Here the generalized Fourier transform will be needed, which for a payoff \( w(x) \) is given by

\[
\hat{w}(z) = \int_{-\infty}^{+\infty} e^{ixz} w(x) dx
\]

and \( z \) is a complex number. The simple Fourier transform, which was defined for a real variable \( u \) in (3.1.1) will not exist for payoff functions. For further details on the justification for using the generalized Fourier transform, the reader is referred to Lewis (2000) and Lewis (2001). In the case of a European call option, the payoff transform is

\[
\hat{w}(z) = \int_{-\infty}^{+\infty} e^{ixz} (e^{x} - K)^+ dx = -\frac{K^{iiz+1}}{z^2 - iz}.
\]

The integration limit at \(+\infty\) only exists if the imaginary part \( z_i \) is greater than one. One can also say that the call payoff transform \( \hat{w}(z) \) is well behaved within a strip of regularity \( S_w \) where \( \Im(z) > 1 \), where \( \Im(z) \) is the imaginary part of \( z \). To generalize, an option payoff Fourier transform can be obtained, but it is only properly defined for some strip \( S_w \) which is bounded, or \( c_1 < \Im(z) < c_2 \) for some real valued \( c_1 \) and \( c_2 \).

In Table 3.1, some payoff transforms are given, along with the relevant strips of regularity, with \( w(x) \) being the corresponding payoff function and \( \hat{w}(z) \) - its generalized Fourier transform.

When applying the generalized Fourier transform to the log-price PDF \( q(x) \), the characteristic function \( \phi_X(z) \) is obtained and it is again well defined only within a certain strip of regularity. Due to the symmetry property of generalized Fourier transforms \( \phi(-z) = \bar{\phi}(z) \), the following is true: if the conjugate \( \bar{z} \) be well defined along a strip of regularity \( S_X \), then \( \phi(-z) \) is well defined in the strip \( \bar{S}_X \).
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<table>
<thead>
<tr>
<th>Claim</th>
<th>Payoff</th>
<th>( \hat{w}(z) )</th>
<th>Strip ( \mathcal{S}_w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call</td>
<td>((e^x - K)^+)</td>
<td>(-\frac{K^{i+1}}{z^2 - iz})</td>
<td>(\Im(z) &gt; 1)</td>
</tr>
<tr>
<td>Put</td>
<td>((K - e^x)^+)</td>
<td>(-\frac{K^{i+1}}{z^2 - iz})</td>
<td>(\Im(z) &lt; 0)</td>
</tr>
<tr>
<td>Covered Call</td>
<td>(\min(e^x, K))</td>
<td>(\frac{K^{i+1}}{z^2 - iz})</td>
<td>(0 &lt; \Im(z) &lt; 1)</td>
</tr>
</tbody>
</table>

Table 3.1: Generalized Fourier transforms and strips of regularity for some claims

To arrive at an option value \(P(S_0, K, T)\), according to that approach, one needs to apply the Plancherel-Parseval identity (see Schmelzle (2010)) and then it will be true that

\[
P(S_0, K, T) = e^{-rT} \mathbb{E}_Q[w(x)] = \frac{e^{-rT}}{2\pi} \mathbb{E}_Q \left[ \int_{iz_1 - \infty}^{iz_1 + \infty} e^{-izx} \hat{w}(z) dz \right]
\]

\[
= \frac{e^{-rT}}{2\pi} \int_{iz_1 - \infty}^{iz_1 + \infty} \mathbb{E}_Q[e^{-izx}] \hat{w}(z) dz
\]

\[
= \frac{e^{-rT}}{2\pi} \int_{iz_1 - \infty}^{iz_1 + \infty} \phi_T(-z) \hat{w}(z) dz.
\]  

(3.2.2)

The integration is shifted to the Fourier space with the corresponding Fourier representations of the payoff function \(w(x)\) and the risk-neutral density \(q(x)\). In (3.2.2) one has to use Fubini’s theorem with the aim of changing the integration order and reversing the expectation and the integral. To that end, the integrand needs to be well defined. As noted, due to the reflection symmetry property, \(\phi_T(-z)\) behaves reasonably when \(z \in \mathcal{S}_X\) and \(\hat{w}(z)\) is well defined when \(z \in \mathcal{S}_w\), so for applying Fubini’s theorem, the integration should be in the contour \(\mathcal{S}_X \cap \mathcal{S}_w\). This is the region where both functions are well defined.

For obtaining prices of particular option types, one may extract the relevant payoff transform from Table 3.1. In the case of a European call the formula derived is

\[
C(S_0, K, T) = -\frac{Ke^{-rT}}{2\pi} \int_{iz_1 - \infty}^{iz_1 + \infty} \frac{e^{-izk} \phi(-z)}{z^2 - iz} dz.
\]

When the denominator \(z^2 - iz\) has a value of zero, the regularity in the strip \(\mathcal{S}_X\) will not be achieved. That is, the points \(z = 0\) and \(z = i\) are critical and have to be taken into consideration separately. Integrating in the range of \(z_i \in (0, 1)\) one only to consider the residue at \(z = i\), which is \(iS_0e^{-rT} / 2\pi\). Referring to Cauchy’s Residue Theorem (see Ahlfors (1979)), the
value of the call will be equal to

\[ C(S_0, K, T) = S_0 e^{-\delta T} - \frac{Ke^{-rT}}{2\pi} \int_{iz_1 - \infty}^{iz_1 + \infty} \frac{e^{-izk \phi_0(z)}}{z^2 - iz} \, dz. \]  

(3.2.3)

By choosing \( z_i = 1/2 \) at (3.2.3) and changing the variable \( z \) to \( z = u + \frac{1}{2} \), one obtains

\[ C(S_0, K, T) = S_0 e^{-\delta T} - \sqrt{S_0 K} e^{-(r-\delta)T} \frac{1}{\pi} \int_0^{+\infty} Re\left[ e^{iuk \phi_0(u - i/2)} \right] \frac{1}{u^2 + 1/4} \, du \]  

(3.2.4)

with \( k = \log(K/S_0) \) and \( \phi_0 \) being the characteristic function of the standardized log-price at maturity, \( X_T - X_0 = \log\frac{S_T}{S_0} \). Formula (3.2.4) as well as other important results regarding the practical implementation of the Fourier transform apparatus can be found in Gatheral (2006).

### 3.3 Practical problems with the fast Fourier transform

When implementing the fast Fourier transform to option pricing, one needs to understand the errors that may arise from that procedure. To start with, a truncation error should be considered. It appears from the fact that when performing the numerical integration (in this work, coded in MATLAB with the `quadv` function), the upper limit is not infinity. Thus, the result gives a small deviation from the theoretical value to be obtained when integrating to infinity. Then, for the integral computation, a discrete number of points are used, the sampling is not performed on a continuous set and that gives rise to the so called sampling error. Finally, in some cases, there might exist a roundoff error, due to the complicated structure of the Fourier transforms of some functions. It is not very severe for shorter maturities. A discussion on the nature and existence of the above-mentioned errors, as well as recommendations for a number of model frameworks, can be found in Lee (2004).

It is worth giving attention to the numerical procedure employed in inverting the Fourier transform and obtaining the desired contingent claims’ prices. The FFT prices and implied volatilities in the following chapters are obtained via a numerical integration in MATLAB. The function used for performing the procedure is `quadv`, which approximates the integral of interest using recursive adaptive Simpson quadrature and allows for vectorized complex functions to be integrated. However, that issue is beyond the scope of this thesis. For further details, implementation strategies and comparisons, the reader can refer to Schmelzle (2010).
Also, a discussion on the MATLAB numerical integration functions `quad` and `quadl` can be found in Gander and Gautschi (1998).

Another problem that practitioners face when implementing the Fourier transform methods in option pricing is related to the calibration issue. As mentioned in the paper of Mikhailov and Noegel (2003), it turns out that the choice of the initial parameter values is crucial for the results. In a scheme that is in nature a minimization problem, one has to deal with a function that possesses a number of minima. In other words, without a good guess about the initial values, one could easily end up at a local, instead of a global minimum or obtain no result at all. A way out, proposed by Moodley (2005), involves creating a boundary condition for the squared errors, involving the bid and ask quotes and it has the following form

$$\sum_{i=1}^{N} b_i |C_i^{\Omega_0} - C_i^M|^2 \leq \sum_{i=1}^{N} b_i [\text{bid}_i - \text{ask}_i]^2,$$

(3.3.1)

where $N$ is the number of options in the portfolio, $C_i^{\Omega_0}$ is the estimated price of the $i$-th option, given the initial parameter set $\Omega_0$, $C_i^M$ is the $i$-th observed market price (average of the bid and ask quotation) and $b_i$ is the weight of the particular option and $\sum_{i=1}^{N} b_i = 1$. Then, if condition (3.3.1) is not satisfied, one should perform the calibration again with a new set of initial values. Moodley (2005) suggests that there is no guarantee that the solution is a global and not just a local minimum, but given that (3.3.1) is fulfilled, the result is satisfactory. For a more detailed discussion on calibration procedures, timing and performance, the reader may consult Moodley (2005). To summarize, calibration of a model which involves the FFT performs satisfactorily well, given that the set of initial parameter values is close to the true one. However, the results are unreasonable when the deviation is larger than expected. That makes the FFT a strong tool, but only given one already has a good clue about the initial parameters to run the optimization with.

Taking into account the problems just mentioned, the FFT is still enjoying considerable support. In the next section an approximation to the FFT, derived by Drimus (2011a) will be presented. It will be applied for the Heston stochastic volatility model. The approximation performance will be then compared to the original FFT results, based on the Lewis (2001) approach, which will serve as a benchmark.
Chapter 4

Heston expansion

In this chapter the stochastic volatility model proposed by Heston (1993) will be presented and the main related results from Drimus (2011a) will be reviewed and analysed. Some of the concepts will be further investigated in details. The results will be illustrated with two different parameter sets, both obtained by a calibration to market prices (i.e. Bakshi, Cao, and Chen (1997) and Drimus (2011b)).

4.1 General Heston framework

Given a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, Q)\), equipped with a measure \(Q\), Heston (1993) introduced the following model

\[
\begin{align*}
    dS_t &= (r - \delta)S_t dt + \sqrt{v_t}S_t(\rho dW_t + \sqrt{1 - \rho^2}dZ_t), \\
    dv_t &= k(\theta - v_t)dt + \epsilon v_t dW_t.
\end{align*}
\]

(4.1.1)

With this formulation, the variance dynamics follow a square-root diffusion, also referred to as a Cox-Ingersoll-Ross (CIR) process, or \(\{v_t\}_{t \geq 0}\) is a CIR\((k, \theta, \epsilon)\). That square-root diffusion process was first developed by Cox, Ingersoll, and Ross (1985) and was used for interest rate modeling. In the Heston stochastic volatility setting, the CIR process defines the variance dynamics. Modeled in this way, the variance has a mean-reverting structure, with \(k\) being the speed of mean-reversion and \(\theta\) being the long term level of \(v_t\). Intuitively, having \(k > 0\) and \(\theta > 0\), the model (4.1.1) makes sure that if the variance is higher than its usual values, then it will be pulled down towards them; and vice versa, if the variance is too low, then the quantity \(\theta - v_t\) will be positive and the process will be dragged up towards its long term level.

For the model (4.1.1), several improvements of the classical BS setting can be achieved. The speed of mean-reversion, or \(k\), is a parameter that takes account of the empirically observed clustered volatility. Also, as noted earlier, log returns are usually leptokurtic and
controlled by $\epsilon$, being the variance volatility. High $\epsilon$ values provide higher peak, and in particular, when $\epsilon = 0$ the variance is deterministic. Another important specification of the Heston stochastic volatility framework is that the Wiener processes involved in the underlying and volatility dynamics are correlated and the correlation coefficient is allowed to take values different from zero. That means that the leverage effect can be incorporated in the model. The value of $\rho$ affects the tails’ behaviour, but not in a symmetric way, as $\epsilon$ does. That is, $\rho$ accounts for the distribution’s skewness. Having $\rho < 0$ would result in a distribution with a fatter left tail, as compared to the right one. Also, when skewness changes, the implied volatility surface will also change. All these parameters allow for the Heston model to reflect some of the previously mentioned empirically observed properties of returns. It is a more flexible setting in the sense that its configuration allows for capturing distribution structures other than the normal.

The PDE to be satisfied by every contract written on an underlying with variance dynamics as in \((4.1.1)\) comes from the general stochastic volatility PDE given by \((2.1.2)\), with the substituted values for $\alpha(S_t, v_t, t)$ and $\beta(S_t, v_t, t)$, as follows

$$
\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \epsilon v S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \epsilon^2 v \frac{\partial^2 V}{\partial v^2} + r S \frac{\partial V}{\partial S} - r V = -\alpha' \frac{\partial V}{\partial v} = k(v_0 - \theta) \frac{\partial V}{\partial v}.
$$

The last equality holds since in the relation $\alpha' = \alpha - \phi \beta \sqrt{\epsilon}$, the market price of volatility risk $\phi(S, v, t)$ is 0, as explained in the Remark 2.1.1.

The Heston model provides with semi-closed form solutions for pricing European options. The technique employed for deriving prices relies on the use of characteristic functions and their inversion. In that way, a fast and easy to implement method for option pricing has been developed, taking into account the stochastic nature of volatility. That makes the Heston model a widely used methodology due to its being closer to reality (as compared to the previously existent models) and its convenience in terms of implementation.

### 4.2 An approximation for the Heston model prices

The Heston closed-form solutions are based on Fourier transform pricing, which acts like a "black box" producing numbers. It would be useful to gain an understanding about the model structure, as well as a relation between the prices produced by the Heston formulae and the ones resulting from the BS model. A thorough attention to that aspect has been paid in a recent contribution of Drimus (2011a) where an approximation to the FFT prices is derived. It is based on a Taylor expansion and relies on defining a set of equivalent martingale
measures which are connected recursively. The process structure remains the same under any of the new measures. An outline of the derived expansion will be presented shortly.

For building the following main results, one should define the total integrated variance, being

\[ V_T = \int_0^T v_t dt. \]

Taking stock price dynamics, given in (4.1.1) and applying Itô’s formula for \( \log(S_T) \), the following expression for \( S_T \) is valid

\[
S_T = S_0 \exp \left[ (r - \delta)T - \frac{1 - \rho^2}{2} \int_0^T v_t dt - \frac{\rho^2}{2} \int_0^T v_t dt + \sqrt{1 - \rho^2} \int_0^T \sqrt{v_t} dZ_t + \rho \int_0^T \sqrt{v_t} dW_t \right].
\]

Let the process \( \{\xi_t\}_{0 \leq t \leq T} \) be the Doléan-Dade’s exponential of \( \int_0^t \rho \sqrt{v_u} dW_u \). Then it satisfies

\[
\xi_t = \mathcal{E} \left( \int_0^t \rho \sqrt{v_u} dW_u \right) = \exp \left( -\frac{\rho^2}{2} \int_0^t v_u du + \rho \int_0^t \sqrt{v_u} dW_u \right). \tag{4.2.1}
\]

Hence, the above expression for \( S_T \) can be represented as

\[
S_T = S_0 \xi_T \exp \left[ (r - \delta)T - \frac{1 - \rho^2}{2} \int_0^T v_t dt + \sqrt{1 - \rho^2} \int_0^T \sqrt{v_t} dZ_t \right]. \tag{4.2.2}
\]

Following its definition, \( \xi \) solves the differential equation \( d\xi_t = \xi_t \rho \sqrt{v_t} dW_t \) and as at \( t = 0, \xi_0 = 1 \), it follows that

\[
\xi_t = 1 + \int_0^t \xi_s \rho^2 v_s dW_s, \tag{4.2.3}
\]

where the integral is a local martingale. It is important to note that an additional condition is needed to assure that the process is also a true martingale. The requirement comes from the fact that a martingale is always a local martingale, but the inverse implication does not hold. That is, not all local martingales are martingales. The true martingale condition will be fulfilled if the Novikov’s condition (see Karatzas and Shreve (1991)) is satisfied, or

\[
\mathbb{E}^Q \left[ \exp \left( \frac{1}{2} \int_0^T \sigma^2(v_t, t) dt \right) \right] < \infty
\]

with \( \sigma^2(v_t, t) \) being the diffusion coefficient of the process \( \{\xi_t\}_{0 \leq t \leq T} \). That is, the condition
for the process to be a true martingale is

$$
E^Q \left[ \exp \left( \frac{1}{2} \rho^2 \int_0^T v_t dt \right) \right] < \infty.
$$

However, that is exactly the Laplace transform of the total integrated variance $V_T = \int_0^T v_t dt$, evaluated at $-\frac{1}{2} \rho^2$, i.e. $L(-\frac{1}{2} \rho^2)$. This condition can be restated as

$$
L\left(-\frac{1}{2} \rho^2\right) < \infty. \quad (4.2.4)
$$

That is, the Laplace transform, which will be given below, should exist at the point $-\frac{1}{2} \rho^2$. This condition, along with other parameter requirements, will be investigated in Section 4.3.

Having that $\{\xi_t\}_{0 \leq t \leq T}$ is a true martingale and $E^Q[\xi_T] = 1$, the Radon-Nikodym derivative (for example, see Pascucci (2011)) can be defined as $\frac{dQ^1}{dQ} = \xi_T$. With that definition, it is true that

$$
Q^1(A) = \int_A \xi dQ,
$$

where $Q^1$ is a probability measure, absolutely continuous with respect to (w.r.t.) $Q$ for an arbitrary $A \in \mathcal{F}_t$. One can also write that

$$
E^{Q^1}[X_T] = E^Q\left[ \frac{dQ^1}{dQ} X_T \right] = E^Q[\xi_T X_T] \quad (4.2.5)
$$

for some random process $X_T$. According to the Girsanov’s theorem (see Karatzas and Shreve (1991)) a new Wiener process $\{W^1_t\}_{(0 \leq t \leq T)}$ can be defined as

$$
W^1_t = W_t - \int_0^t \rho \sqrt{v_u} du
$$

and it will be a standard Wiener process under $Q^1$. Substituting $W^1_t$ in (4.1.1) we get

$$
\begin{align*}
    dv_t &= k(\theta - v_t)dt + \epsilon \sqrt{v_t} (dW^1_t + \rho \sqrt{v_t} dt), \\
    &= (k - \epsilon \rho) \left( \frac{k \theta}{k - \epsilon \rho} - v_t \right) dt + \epsilon \sqrt{v_t} dW^1_t.
\end{align*}
$$

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That is, changing the measure does not affect the model structure. Under $Q^1$ the variance dynamics are exactly the same as in under $Q$, the only difference is in the parameter values of $k$, $\theta$ and $\epsilon$. The model structure is preserved under the new measure $Q^1$ and it is now $CIR\left(k - \epsilon \rho, \frac{k \theta}{k - \epsilon \rho}, \epsilon\right)$. That result can be further generalized. Taking into account the relation

$$\frac{dQ^{(n)}}{dQ^{(n-1)}} = \xi_T^{(n-1)},$$

(4.2.6)

where

$$\xi_T^{(n-1)} = \exp\left(-\frac{\rho^2}{2} \int_0^T v_t dt + \rho \int_0^T \sqrt{v_t} dW_t^{(n-1)}\right)$$

and noting that

$$W_t^{(n)} = W_t^{(n-1)} - \int_0^t \rho \sqrt{v_u} du$$

is a standard Wiener process under $Q^{(n)}$, by recursive calculations one can easily obtain that under $Q^{(n)}$ for $n = 0, 1, 2...$, the variance dynamics in this model of interest behave like a

$$CIR\left(k - n \epsilon \rho, \frac{k \theta}{k - n \epsilon \rho}, \epsilon\right).$$

Another result that will be used in the subsequent analysis involves the relationship between the densities $\xi_T^{(n)}$, $n \geq 1$. Using the recursive connection between the Wiener processes under the different measures mentioned above, or $dW_t^{(n)} = dW_t^{(n-1)} - \rho \sqrt{v_t} dt$ and the definition of $\xi_T$, the following set of equalities hold

$$\xi_T^{(n)} = \exp\left(-\frac{\rho^2}{2} \int_0^T v_t dt + \rho \int_0^T \sqrt{v_t} dW_t^{(n)}\right)$$

$$= \exp\left(-\frac{\rho^2}{2} \int_0^T v_t dt + \rho \int_0^T \sqrt{v_t} (dW_t^{(n-1)} - \rho \sqrt{v_t} dt)\right)$$

$$= \exp\left(-\frac{\rho^2}{2} \int_0^T v_t dt + \rho \int_0^T \sqrt{v_t} dW_t^{(n-1)}\right) \exp\left(-\rho \int_0^T v_t \rho dt\right)$$

$$= \xi_t^{(n-1)} e^{-\rho^2 \int_0^T v_t dt}. \quad (4.2.7)$$

These relations form the base for building the FFT approximation. Also, we will need the the Laplace transform of the total integrated variance for a variance process, following
the dynamics in (4.1.1). It is known from Cox, Ingersoll, and Ross (1985) that

$$L(\lambda; k, \theta, \epsilon) = E[e^{-\lambda V_T}] = A(\lambda; k, \theta, \epsilon)e^{-v_0 B(\lambda; k, \theta, \epsilon)},$$

(4.2.8)

with

$$A(\lambda; k, \theta, \epsilon) = \left(\frac{e^{kT/2}}{\cosh(P(\lambda)T/2) + \frac{k}{P(\lambda)} \sinh(P(\lambda)T/2)}\right)^{\frac{2\epsilon^2}{\sigma^2}},$$

$$B(\lambda; k, \theta, \epsilon) = \frac{\lambda}{P(\lambda)} \left(\frac{2 \sinh(P(\lambda)T/2)}{\cosh(P(\lambda)T/2) + \frac{k}{P(\lambda)} \sinh(P(\lambda)T/2)}\right),$$

and

$$P(\lambda) = \sqrt{k^2 + 2\epsilon^2 \lambda}.$$

Based on the above properties and definitions, an approximation to the FFT, applied to option prices written on an underlying described by the Heston dynamics (4.1.1) will be derived. It will be composed of the BS prices and some of the greeks - gamma, volga and vanna. That is, when applied, the approximation can give a clear distinction of the impact of the derivative’s value convexity, the volatility convexity, the sensitivity of delta towards volatility and the BS price itself. The actual result is produced via a Taylor expansion. To this end, a representation of the BS formulae in terms of the total integrated variance will be preferable. That is, in the case of an European call option, the BS price will be

$$C^{BS}(S_0, V; r, \delta, K, T) = S_0 e^{-\delta T} N(d_1) - K e^{-r T} N(d_2),$$

(4.2.9)

with

$$d_1 = \frac{\log \frac{S_0}{K} + (r - \delta)T + \frac{V}{2}}{\sqrt{V}}$$

and

$$d_2 = d_1 - \sqrt{V}$$

and $N(.)$ denotes the cumulative distribution function (CDF) of a standard normal random variable. By inserting $V = \sigma^2 T$, $\sigma$ being the BS volatility, the familiar BS call formula is obtained. Following the above notation, $C^{BS}(S_0, \sigma^2 T)$ will give the desired BS call price. By the risk-neutral valuation method, discussed in Chapter 2, the price of a call, maturing at $T$ takes the form

$$C(S_0, K, T) = e^{-r T} Q(S_T - K)^+. $$

By using (4.2.2) and applying the theorem by Romano and Touzi (1997) from (2.1.6) and if $\{\mathcal{F}^W_t\}_{0 \leq t \leq T}$ denotes the filtration being generated by the Wiener process within the variance
diffusion, we get that the European call price written on an underlying described by the Heston dynamics can be represented as

$$
C_H(S_0, K, T, v_0, \theta, \epsilon) = e^{-rT}E_Q[E_Q[(S_T - K)^+ | F_T^W]]
$$

$$
= E_Q\left[ C^{BS}\left( S_0 \xi_T, (1 - \rho^2) \int_0^T v_t dt \right) \right]
$$

$$
= E_Q\left[ C^{BS}9S_0 \xi_T, (1 - \rho^2)V_T \right].
$$

That is, the price of interest becomes an expectation of a BS price of a call with initial spot $S_0 \xi_T$ and total variance $(1 - \rho^2)V_T$, with respect to the measure $Q$. By employing the call representation (4.2.10), and building a Taylor expansion around $(S_0, (1 - \rho^2)E_Q[V_T])$, a price approximation by BS prices and BS greeks can be obtained as follows

$$
C^{(n)}_H = C^{BS}(S_0, (1 - \rho^2)E_Q[V_T])
$$

$$
+ \sum_{k=2}^{n} \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \frac{\partial^k C^{BS}}{\partial S^l \partial V^{k-l}}(S_0, (1 - \rho^2)E_Q[V_T])
$$

$$
\times S_0^l(1 - \rho^2)^{k-l}E_Q[(\xi_T - 1)(V_T - E_Q[V_T])^{k-l}],
$$

where $n = 2, 3, ...$

The focus of this thesis will be on the second order expansion of relation (4.2.11). Drimus (2011a) provides details about the third order expansion and shows that its performance as compared to the second order case is not improving dramatically. The third order expansion gives some better results for options that have their strikes below 90% of moneyness, but is slightly outperformed by the second order for those being at the money. That is, the second order approximation is simpler and marginally better for a reasonable range of moneyness. A more thorough discussion can be found in Drimus (2011a).

We will investigate (4.2.11) when $n = 2$, i.e.

$$
C^{(2)}_H = C^{BS} + \frac{1}{2} \frac{\partial^2 C^{BS}}{\partial S^2} \cdot S_0^2 \cdot E_Q[\xi_T - 1]^2 +
$$

$$
\frac{1}{2} \frac{\partial^2 C^{BS}}{\partial V^2} \cdot (1 - \rho^2)^2 \cdot E_Q[V_T - E_Q[V_T]]^2 +
$$

$$
\frac{\partial^2 C^{BS}}{\partial S \partial V} \cdot S_0 \cdot (1 - \rho^2) \cdot E_Q[(\xi_T - 1)(V_T - E_Q[V_T])].
$$

That is, with the use of the Taylor expansion above, one obtains an approximation to the Heston prices in terms of the BS prices, certain greeks and a number of moments which will
be discussed in detail below. The BS greeks involved in the second order expansion (4.2.12) are

\[
\begin{align*}
\text{Gamma} &= \frac{\partial^2 C_{BS}}{\partial S^2}(S, V) = \frac{e^{-\delta T} \varphi(d_1)}{S \sqrt{V}}, \\
\text{Volga} &= \frac{\partial^2 C_{BS}}{\partial V^2}(S, V) = \frac{e^{-\delta T} S \varphi(d_1)}{4V^{3/2}}(d_1d_2 - 1) \quad \text{and} \\
\text{Vanna} &= \frac{\partial^2 C_{BS}}{\partial S \partial V}(S, V) = -\frac{e^{-\delta T} \varphi(d_1)d_2}{2V},
\end{align*}
\]

with \( \varphi(x) \) denoting the PDF of a standard normal distribution and \( d_1 \) and \( d_2 \) being defined by \( (4.2.9) \).

The BS components comprising the approximation, or namely the sensitivities involved, will be briefly discussed. The following details are based on Haug (2003a) and Haug (2003b).

\textit{Delta} is the first derivative of the option value with respect to the underlying, or \( \frac{\partial C_{BS}}{\partial S} \). It is interpreted as the option price change for a small change of the underlying’s value (or equivalently how sensitive the option price is towards the price of the underlying asset). The delta is not a part of the derived approximation above, but it helps understand the behaviour of \textit{gamma}. The main implication of delta is for hedging purposes, or delta hedging. It aims in diminishing the risk exposure from options operations. In the BS world delta hedging involves taking a position in the underlying, such that the portfolio formed by the positions in the option and in the underlying be \textit{delta neutral}, that is, with a 0 delta. For example, if an investor has a short position in some amount of call options, he/she can purchase a number of shares equal to the delta times the shares included in the options contracts. In that way, if the price of the underlying increases, the losses the investor would suffer from the short position in the options will be perfectly offset by the gains from the shares owned. However, as the underlying’s value changes, so does the position required to maintain delta neutrality. In other words, a dynamic re-balancing is required. In the case of a call, the delta takes values between 0 and 1; put options range between -1 and 0, as the put value and the underlying are not moving in the same direction. It should be noted that under a stochastic volatility model, a perfect hedge by taking a position only in the underlying is not possible due to the market being incomplete. A more thorough discussion can be found in Kurpiel and Roncalli (1998).

\textit{Gamma} is the second derivative of the option value with respect to the underlying, see Formula \( (4.2.13) \). It shows how sensitive the delta is towards changes in the underlying’s value. The gamma is an indicator about the intensity of the delta hedge re-balancing. High
gamma indicates that the position needs to be changed more often to avoid potential losses. In terms of the value range for gamma, it peaks when the option is in the money and diminishes as it moves in or out of the money, being positive for long positions and negative for short ones. A portfolio can become gamma neutral by taking a position in an additional option, written on the same underlying. Delta neutrality ensures that if some small changes in the price of the underlying occur, the investor would not suffer losses. Having a gamma neutral portfolio brings for protection from changes of the underlying of a larger size.

Vega is the first derivative of the option value with respect to the volatility of the underlying. It tells how the option price will change for a percentage volatility change. A high vega means that the option is very sensitive to changes in the volatility, which is good news for purchasing the contract in question. If one wants to have a portfolio which is vega neutral as well, then a position in a new option is required (for an investment to be both gamma and vega neutral, positions in two additional options are needed). As some strategies are extremely sensitive towards volatility, the vega is an important value to keep track of, particularly in markets which are characterized by frequent volatility changes.

Volga (referred to as vomma by some authors) is the second derivative of the option value with respect to the volatility of the underlying asset or, in other words, measures the convexity of the option with respect to volatility. It can also be viewed as the first derivative of vega with respect to volatility, which gives its sensitivity towards volatility changes. High values of volga would mean that the option’s sensitivity towards volatility increases even more if a shock occurs, which in turn is a benefit for the holders of the option. A high volga in the terms of a stochastic volatility model can trigger significant price changes. Changes of vega are strongly affected by the existing correlation between the stock and volatility dynamics when working in a stochastic volatility setting. When trading exotic contingent claims, one is strongly interested in the way vega changes - with respect to the volatility (volga) and with respect to the spot (vanna, which will be analyzed below). In working with barrier options, volga plays an important role, as the option behaviour close to the barrier level is of crucial importance for the instrument being exercised or not. Also, it is thoroughly followed in the case of compound options, options on volatility or forward smile options.

Vanna is the derivative of vega with respect to the underlying. As it is a mixed partial derivative, it can be viewed either as the vega sensitivity towards changes in the underlying, or as the delta sensitivity with respect to volatility. If one is interested in constructing a delta or vega hedged portfolio, then vanna is monitored closely. That is, vanna is useful to capture the changes in delta which arise from changes in volatility, as well as changes in the effectiveness of a vega hedge resulting from the movement of the underlying price.
For obtaining option prices from the approximation \( (4.2.12) \) there is a number of expected values to be calculated, or namely \( \mathbb{E}_Q[\xi_T - 1]^2, \mathbb{E}_Q[V_T], \mathbb{E}_Q[V_T - \mathbb{E}_Q[V_T]]^2 \) and \( \mathbb{E}[(\xi_T - 1)(V_T - \mathbb{E}_Q(V_T))] \).

Let us begin by focusing on the calculation of \( \mathbb{E}_Q[\xi_T - 1]^2 \). Using that \( \mathbb{E}_Q[\xi_T] = 1 \) from \( (4.2.3) \), we have

\[
\mathbb{E}_Q[\xi_T - 1]^2 = \mathbb{E}_Q[\xi_T^2] - 1.
\]

As indicated in \( (4.2.6) \), since \( \xi_T^{(n-1)} \) can be used as a measure for transfer from \( Q^{(n-1)} \) to \( Q^{(n)} \), it is true that \( \mathbb{E}_Q[\xi_T^2] = \mathbb{E}_Q[\xi_T] \cdot \mathbb{E}_Q[\xi_T] = \mathbb{E}_Q[\xi_T] \).

From the relation \( (4.2.7) \), it follows that \( \xi_T = \xi_T^{(1)} e^{\rho^2 \int_0^T v_t dt} \). For \( n = 2 \) in \( (4.2.6) \), we have that \( \frac{dQ^{(2)}}{dQ^{(1)}} = \xi_T^{(1)} \). With these results in hand, the final expression for \( \mathbb{E}_Q[\xi_T^2] \) is:

\[
\mathbb{E}_Q[\xi_T - 1]^2 = \mathbb{E}_Q[\xi_T] - 1 = \mathbb{E}_Q[\xi_T^2] e^{\rho^2 \int_0^T v_t dt} - 1 = \mathbb{E}_Q^2[e^{\rho^2 \int_0^T v_t dt}] - 1 = e^{- \rho^2; k - 2\epsilon \rho, \frac{k\theta}{k - 2\epsilon \rho}, \epsilon} - 1. \quad (4.2.14)
\]

That is, the moment of interest can easily be obtained by taking the Laplace transform \( (4.2.8) \) of the total integrated variance \( V_T \) and evaluate it at the point \( -\rho^2 \) under the Heston variance dynamics under \( Q^2 \). Knowing the explicit expression for the Laplace transform, the calculation of the moment above is straightforward.

Turning attention to the expectations involving the total integrated variance \( V_T \), the paper of Dufresne (2001), devoted to the integrated square-root process, offers closed-form expressions for the expected value of the total integrated variance \( \mathbb{E}_Q[V_T] \) and its second central moment \( \mathbb{E}_Q[V_T - \mathbb{E}_Q[V_T]]^2 \) as follows

\[
\mathbb{E}_Q[V_T] = \mathbb{E}_Q^Q \left[ \int_0^T v_t dt \right] = T \theta + \frac{1 - e^{-kT}}{k} (v_0 - \theta) \quad (4.2.15)
\]

and

\[
\mathbb{E}_Q^Q[V_T - \mathbb{E}_Q^Q[V_T]]^2 = -\frac{5\theta \epsilon^2}{2k^3} + \frac{v_0 \epsilon^2}{k^3} + \frac{T \theta \epsilon^2}{k^2} + \frac{2\theta \epsilon^2}{k^3} e^{-Tk} - \frac{v_0 \epsilon^2}{k^3} e^{-2Tk} + \frac{\theta \epsilon^2}{2k^3} e^{-2kT} - \frac{2T v_0 \epsilon^2}{k^2} e^{-Tk} + \frac{2T \theta \epsilon^2}{k^2} e^{-Tk}. \quad (4.2.16)
\]

Keeping the notations by Drimus (2011a), the relations \( (4.2.15) \) and \( (4.2.16) \) will be
denoted by $D_1(v_0, k, \theta, T)$ and $D_2(v_0, k, \theta, \epsilon, T)$, respectively.

What is left is to determine the expectation in the last term of the second-order approximation (4.2.12). Simplifying, we get

$$\mathbb{E}^Q[(\xi_T - 1)(V_T - \mathbb{E}^Q[V_T])] = \mathbb{E}^Q[\xi_T V_T] - \mathbb{E}^Q[V_T].$$

(4.2.17)

Similarly to the calculation of $\mathbb{E}^Q[\xi_T - 1]^2$ in (4.2.14), $\xi_T$ can be used for a change of measure, so $\mathbb{E}^Q[\xi_T V_T] = \mathbb{E}^Q^1[V_T]$. The expectation $\mathbb{E}^Q[V_T]$ is given in (4.2.15), where the parameters correspond to a CIR($k, \theta, \epsilon$). As noted, the process under $\mathbb{E}^Q^1$ will be a CIR($k - \epsilon \rho, k \theta, k \epsilon \rho$). So taking in consideration these relations, the final representation is

$$\mathbb{E}^Q[(\xi_T - 1)(V_T - \mathbb{E}^Q(V_T))] = \mathbb{E}^Q^1[V_T] - \mathbb{E}^Q[V_T] = D_1(v_0, k - \epsilon \rho, k \theta, k \epsilon \rho, T) - D_1(v_0, k, \theta, T).$$

(4.2.18)

Now, substituting the expressions (4.2.14), (4.2.16) and (4.2.18) and with the greeks in hand, the approximation (4.2.12) takes the form

$$C_H^{(2)} = C_{BS} + \frac{1}{2} \frac{\partial^2 C_{BS}}{\partial S^2} \cdot S_0^2 \cdot \left\{ L \left( -\rho^2; k - 2 \epsilon \rho, \frac{k \theta}{k - 2 \epsilon \rho}, \epsilon \right) - 1 \right\}$$

$$+ \frac{1}{2} \frac{\partial^2 C_{BS}}{\partial V^2} \cdot (1 - \rho^2)^2 \cdot D_2(v_0, k, \theta, \epsilon, T)$$

$$+ \frac{\partial^2 C_{BS}}{\partial S \partial V} \cdot S_0 \cdot (1 - \rho^2) \cdot \left\{ D_1(v_0, k - \epsilon \rho, k \theta, k \epsilon \rho, T) - D_1(v_0, k, \theta, T) \right\}.$$  

(4.2.19)

Thus relation (4.2.19) will be our basic formula to be calculated for different parameter values.

### 4.3 Limitations for the model parameters

There are several parameters involved in the Heston model: $k, \theta, \epsilon, v_0, \rho, \lambda$. Here we will give several empirical and theoretical restrictions on their possible values.
4.3.1 Evidence from empirical data

A number of studies was dedicated to the empirical properties of different stochastic volatility models, and in particular to the parameter values allowed. For example, an extensive research was conducted by Bakshi, Cao, and Chen (1997) and provides useful results on different classes of models. In a model following the CIR dynamics, the study shows that $k$, the mean-reversion coefficient, is strictly positive, and in particular $k > 1$. The volatility of the variance is positive, i.e. $\epsilon > 0$. Also, naturally, $\theta$ and $\nu_0$ are nonnegative. Finally, as already mentioned, the correlation between the two Wiener processes characterizing the underlying asset and the variance dynamics should be nonpositive, i.e. $\rho \leq 0$.

4.3.2 Feller condition

The following restriction, referred to as Feller condition (e.g. Feller (1951))

$$2k\theta \geq \epsilon^2 \iff k \geq \frac{\epsilon^2}{2\theta}$$

is a requirement to be fulfilled to prevent a CIR process from reaching the zero line and staying there. Notice that for deriving the second-order approximation, a number of changes of measure are required. Under $Q^1$ and $Q^2$ the Feller condition should be imposed for the corresponding new CIR processes in order to assure their nonnegativity. Generalizing, for the $Q^{(n)}$ measure with $n \geq 1$, one gets

$$k - n\epsilon\rho \geq \frac{\epsilon^2}{\frac{k\theta}{2k - n\epsilon\rho}} \iff k \geq \frac{\epsilon^2}{2\theta}.$$

Thus, the same condition (4.3.1) is valid under any measure $Q^{(n)}$.

4.3.3 Domain for the Laplace transform parameter

By $\lambda$ we denote the point where the Laplace transform of the total integrated variance is evaluated or $L(\lambda) = \mathbb{E}[\exp^{-\lambda V_T}]$. With the Laplace transform definition in (4.2.8), a condition for existence of $P(\lambda) = \sqrt{k^2 + 2\epsilon^2\lambda}$ is $k^2 + 2\epsilon^2\lambda \geq 0$, or

$$\lambda \geq -\frac{k^2}{2\epsilon^2}.$$
4.3.4 Additional conditions for the correlation parameter

As noted in section 4.1, for having a well-behaved process \( \{ \xi_t \}_{0 \leq t \leq T} \), the Novikov’s condition \( L \left( -\frac{1}{2} \rho^2 \right) < \infty \) needs to be fulfilled. Then, from (4.3.2) it follows that

\[
-\frac{1}{2} \rho^2 \geq -\frac{k^2}{2 \epsilon^2},
\]

or

\[
-\frac{k}{\epsilon} \leq \rho \leq \frac{k}{\epsilon}.
\]

(4.3.3)

The right part of that inequality is always fulfilled, as \( \rho \leq 0 \). In the left part, \( -\frac{k}{\epsilon} \leq -1 \), because \( k > 1 \) and \( \epsilon < 1 \) and \( \rho \) cannot take values of less than \( -1 \).

Also, for calculating one of the terms comprising the expansion, as shown in (4.2.14), one needs the Laplace transform of the total integrated variance \( V_T \) under \( Q^2 \), evaluated at \( -\rho^2 \). The restriction is equivalent to

\[
-\rho^2 \geq -\frac{(k - 2\epsilon \rho)^2}{2 \epsilon^2} \iff \rho \leq \frac{k}{\sqrt{2\epsilon(\sqrt{2} - 1)}}.
\]

The last inequality is true for any \( \rho \), as it is always negative and the right hand side is always positive. All in all, there are no severe parameter value requirements to be imposed. The domains needed are consistent with real world data.

4.4 Fast Fourier transform pricing for the Heston model

The characteristic function of the log-price \( x_T = \log S_T \) in the Heston framework is available in closed-form and can be found in Gatheral (2006) and Heston (1993). After some algebra the notation from Hong (2004) is obtained and has the following form

\[
\phi_H(u) = e^{A(u) + B(u) + C(u)},
\]

(4.4.1)
with

\[ A(u) = iu(x_0 + (r - \delta)T), \]
\[ B(u) = \frac{2\zeta(u)(1 - e^{-\psi(u)T})V_0}{2\psi(u) - (\psi(u) - \gamma(u))(1 - e^{-\psi(u)T})}, \]
\[ C(u) = -\frac{k\theta}{\epsilon^2} \left[ 2\log \left( \frac{2\psi(u) - (\psi(u) - \gamma(u))(1 - e^{-\psi(u)T})}{2\psi(u)} \right) + (\psi(u) - \gamma(u))T \right]. \]

In the last formulas we use \( \zeta(u) = \frac{1}{2}(u^2 + iu), \psi(u) = \sqrt{\gamma(u)^2 - 2\epsilon^2\zeta(u)} \) and \( \gamma(u) = k - i\epsilon u. \)

We will utilize representation (4.4.1) due to its easier implementation.

Following the option pricing via the Fourier transform methodology, described in Section 3, obtaining the desired prices and implied volatilities is performed in MATLAB. These results will serve as a benchmark for comparing the performance and behaviour of the output produced by the second-order approximation.

Here is the place to note that besides the initial value problem when calibrating, there exists another practical drawback of the FFT when implementing it to the Heston model. It is referred to as the "Heston trap" and originates from the characteristic function structure, which requires a logarithm of a complex number to be taken. As there exist more than one solution, due to the nature of the trigonometric functions involved, most software packages provide the calculation

\[ \log z = \log |z| + i \arg z \]

with \( \pi < \arg z \leq \pi \), which causes discontinuity when the imaginary part changes its sign. When that happens, the pricing formula, involving the inverse characteristic function may crash. For more details, the reader is referred to Albrecher et al. (2007). A possible solution can be found at Gatheral (2006), where a different formulation is used and the problem is circumvented. There is no theoretical justification behind it, but according to the author, the issue of the Heston trap is not a problem with the other characteristic function formulation.

### 4.5 Heston expansion performance

In this section the performance of the second-order Heston approximation will be compared with the benchmark results from the FFT mechanism described in the previous chapter. The particular characteristic function formulation to be employed was given in Section 4.4. For an illustration, two sets of parameters are used. The one is obtained in the study of Bakshi, Cao, and Chen (1997) and has the following values \( k = 1.15, \theta = \)
0.0348, $\epsilon = 0.39$, $v_0 = 0.0348$, $\rho = -0.64$; the other parameter set is from Drimus (2011b), where $k = 3.8$, $\theta = 0.3095^2$, $\epsilon = 0.9288$, $v_0 = 0.2556^2$, $\rho = -0.7829$. Figures 4.1 and 4.2 show the implied volatilities produced by both parameter sets, with the left panels giving the uncorrelated case ($\rho = 0$) and the right showing the correlation value results from each empirical study. The time to maturity is 6 months and the error in the results is estimated by summing the squared differences between the FFT and the approximation. Moneyness ranges from 80% to 120%, interest rates and dividend yields are assumed to be zero, and the spot price is 1.

As can be seen from Figure 4.1, for the uncorrelated case, the approximation moves closely to the FFT implied volatility and the error amounts to 0.0005%. In the right panel, when $\rho = -0.64$, the curves move together in the in-the-money range, but the approximation starts underestimating the implied volatility values when the calls are about 18% out-of the money. The squared differences are 0.11% in this second case.

![Figure 4.1: Implied volatilities from the approximation and FFT. The parameters are from Bakshi, Cao, and Chen (1997). Left: $\rho = 0$; right: $\rho = -0.64$.](image)

Figure 4.2 is with the parameter set obtained by Drimus (2011a) after a calibration to S&P500 options. The strikes range is again [80%:120%]. The mean squared error is 0.01% for the uncorrelated base case and 0.6% for the $\rho = -0.7829$ at the right panel.

Then Figure 4.3 shows how the Heston approximation price is composed of the BS price, the gamma, volga and vanna term. In this way one can have a good idea about how exactly the Greeks terms add to the simple BS case to arrive at the stochastic volatility counterpart.

These figures and comparisons justify the using of the second order approximation for pricing options. The results are tractable due to the approximation structure. The characteristic function problem is avoided (the "Heston trap" issue) and when calibrating, an acceptable estimate can be obtained even when one has no clue about a reasonable set of initial parameters. Also, the execution time is considerately decreased.
Figure 4.2: Implied volatilities from the approximation and FFT. The parameters are from Drimus (2011a). Left: $\rho = 0$; right: $\rho = -0.7829$.

Figure 4.3: The left panel shows prices for the non-zero correlation parameter set from Bakshi, Cao, and Chen (1997); in the right panel are the corresponding prices from Drimus (2011b).

However, we should note that the Heston model is not the best choice for modeling prices and implied volatilities and that is not just due to implementation problems. A number of research papers has been devoted to studying the connection between particular stochastic volatility models and how close their results are to market implied volatilities. A study by Jones (2003) provides a discussion and a summary of some previous results. Among the findings, it is worth noting the inability of the Heston model to capture fully the excess kurtosis, see Andersen, Benzoni, and Lund (2002). Also, the implied volatility skew is not modeled consistently as compared to the one directly observed from the market, especially for short maturities as noted for example in Mikhailov and Noegel (2003). The exponent power of 0.5 in the variance dynamics characterization is rejected in a number of studies, and that will be discussed in more details in the next chapter. Jones (2003) also notes that the Heston model shows "inability to capture high volatility states dynamics". These considerations bring the question of implementing a more advanced model that would capture at least some of the above presented drawbacks.
Chapter 5

3/2 model for volatility dynamics

Here the so called 3/2 stochastic volatility model will be introduced. It is a non-affine model with a mean-reverting structure. Among the previous implications in the academic literature one can mention its application to short interest rate modeling by Ahn and Gao (1999). Further, Lewis (2000) provides details about the model with respect to equity options, Carr and Sun (2007) use the 3/2 model for pricing variance swaps. In a more recent contribution, Drimus (2011b) use it for pricing options on realized variance.

Both the Heston and the 3/2 models use particular cases of the general variance dynamics form suggested by Bakshi, Ju, and Yang (2006)

\[ dv_t = (\alpha_0 + \alpha_1 v_t + \alpha_2 v_t^2 + \alpha_3 v_t^{-1})dt + \beta_2 v_t^{\beta_3} dW_t. \]

Their study provides with econometric details, based on the squared VIX index as an approximation for the variance. In particular, the authors reject the hypothesis of \( \beta_3 \leq 1 \) and find evidence about the power \( \beta_3 \) being close to 1.3, which favours the 3/2 specification. Furthermore, Bakshi, Ju, and Yang (2006) find evidence that in the drift specification there is more significance on the term \( \alpha_2 v_t^2 \) rather than \( \alpha_1 v_t \), which is again consistent with the 3/2 framework.

The study of Carr and Sun (2007) summarizes some previous works with respect to the power of \( v_t \) at the drift diffusion and concludes that a model with non-affine structure provides with more accurate results as compared to the CIR process. Some of the surveys mentioned in Carr and Sun (2007) are conducted under the physical measure \( \mathbb{P} \) and also reject the square-root specification. Having a model with affine structure allows for a higher tractability, but a non-affine model turns out to be more consistent when looking at real world data.
CHAPTER 5. 3/2 MODEL FOR VOLATILITY DYNAMICS

5.1 General description

Given a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\), the 3/2 model is defined by the following dynamics

\[
\begin{align*}
    dS_t &= (r - \delta)S_t dt + \sqrt{v_t} S_t \left( \rho dW_t + \sqrt{1 - \rho^2} dZ_t \right), \\
    dv_t &= kv_t (\theta - v_t) dt + \epsilon v_t^{3/2} dW_t.
\end{align*}
\]  
(5.1.1)

Compared to the Heston model (4.1.1), the 3/2 model has different dynamics for the variance, which can be looked upon as square-root diffusion process dynamics, multiplied with an extra factor of \(v_t\), or

\[
    dv_t = v_t \left[ k(\theta - v_t) dt + \epsilon \sqrt{v_t} dW_t \right]
\]  
(5.1.2)

and this makes the model structure non-affine.

The speed of mean-reversion is represented by \(kv_t\), that is, the reversion will depend on the variance value itself (for higher values, the process will be pulled towards its long term mean value with higher speed and vice-versa). With reference to real data, that means the model can incorporate and explain fast volatility increases and decreases, that being a point at which Heston fails.

It is worth gaining a better understanding of the parameters’ interpretation and motivation. Here, as in Heston model, the 3/2 model is structured in a way that allows for some of the BS problems to be accounted for. As well as in the Heston model, within the 3/2 framework, the speed of mean-reversion (given by \(kv_t\) now) incorporates the volatility clustering effect. The parameter \(\epsilon\) is again the volatility of the variance, is responsible for the returns distribution kurtosis. Higher values of the parameter \(\epsilon\) result in an increased kurtosis or peakedness and thus heavy tails on the sides. If \(\epsilon\) takes a value of 0, then there will be no random component for the variance. The value of the correlation parameter \(\rho\) captures the so called leverage effect. When it varies, the asymmetry of log returns distribution is reflected and thus the skewness, as was noted above.

Drimus (2011b) provides some interesting insights about the relation and differences between the Heston and the 3/2 models. The author performs a calibration on the same data set (S&P500 options) under both models and the parameters obtained are given in Table 5.1. Then, the relevant volatility paths corresponding to each model are presented in Figure 5.1. For the Heston paths, as noted by Bakshi, Cao, and Chen (1997) and Drimus (2011b),
a violation of the non-zero condition is observed. When constructing the paths, that problem was solved by imposing $v_t = 0$ when $v_t < 0$ and due to the mean-reverting structure it was possible to obtain a graphical representation. As commented in Drimus (2011b), the volatility modeled by the 3/2 dynamics allows for stronger deviations from the long-term average. That is explained by the variance power in the diffusion coefficient and by the speed of mean-reversion, which depends on the variance itself. Also, one should note the behaviour of the Heston paths - they are shifted towards zero. These differences between the two models’ paths favour the 3/2 due to its ability to capture some stronger deviations for the variance, which are not uncommon for many markets. Also, having volatility values close to zero is difficult to justify.

<table>
<thead>
<tr>
<th></th>
<th>$k$</th>
<th>$\theta$</th>
<th>$\epsilon$</th>
<th>$v_0$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heston model</td>
<td>3.8</td>
<td>0.30952</td>
<td>0.9288</td>
<td>0.25562</td>
<td>-0.7829</td>
</tr>
<tr>
<td>3/2 model</td>
<td>22.84</td>
<td>0.46692</td>
<td>8.56</td>
<td>0.24502</td>
<td>-0.99</td>
</tr>
</tbody>
</table>

Table 5.1: Calibration parameter values

![Figure 5.1: Simulated paths for the instantaneous volatility in a model with the Heston dynamics (left panel) and in a model with the 3/2 dynamics (right panel). The parameters are obtained after a calibration of both models to the same volatility surface.](image)

When performing valuation of an option written on an underlying with variance dynamics satisfying (5.1.1), the PDE it must satisfy is:

$$
\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \epsilon v S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \epsilon^2 v^3 \frac{\partial^2 V}{\partial v^2} + r S \frac{\partial V}{\partial S} - r V = -kv(\theta - v) \frac{\partial V}{\partial v} = -\alpha \frac{\partial V}{\partial v},
$$

and it is again derived from the general PDE given in (2.1.2) above, substituting $\alpha = kv(\theta - v)$
and $\beta = v$. As in the Heston case, $\alpha' = \alpha - \beta\phi\sqrt{v}$ with $\phi$ being the market value of volatility risk, which is zero under the risk-neutral setting.

There exists another relation between the 3/2 and the Heston models. If Itô’s lemma would be applied to the inverse of the 3/2 volatility process, one obtains

$$d\left(\frac{1}{\sqrt{v_t}}\right) = k\theta \left(\frac{k + \epsilon^2}{k\theta} - \frac{1}{v_t}\right) dt - \frac{\epsilon}{\sqrt{v_t}} dW_t. \quad (5.1.3)$$

In other words the inverse of the 3/2 model behaves as a Heston model with different set of parameters. That is, the process $\left\{\frac{1}{\sqrt{v_t}}\right\}_{t>0}$ is a CIR($k\theta, \frac{k + \epsilon^2}{k\theta}$, $-\epsilon$).

The characteristic function of the variance process in the 3/2 framework is given in Lewis (2000) and thus it is possible to price options using the Fourier transform methods. Due to the result of Carr and Sun (2007), the Fourier - Laplace transform of the log spot $X_T$ and the total integrated variance is known in closed-form and is given by

$$
\mathbb{E}[e^{iuX_T - \lambda \int_t^T v_s ds}] = e^{iuX_T} \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left(\frac{2}{e^2 y(t, v_t)}\right)^\alpha M(\alpha, \gamma, -\frac{2}{e^2 y(t, v_t)}), \quad (5.1.4)
$$

where $\lambda = a + bi$ ($\lambda \in \mathbb{C}$) and

$$y(t, v_t) = v_t \int_t^T \left\{ \exp \left(\int_u^t k\theta(s) ds\right) \right\} du, \quad (5.1.5)$$

$$\alpha = -\left(\frac{1}{2} - \frac{p}{\epsilon^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{p}{\epsilon^2}\right)^2 + \frac{2q}{\epsilon^2}}, \quad (5.1.6)$$

$$\gamma = 2 \left(\alpha + 1 - \frac{p}{\epsilon^2}\right), \quad (5.1.7)$$

$$p = -k + i\epsilon \rho u, \quad (5.1.8)$$

$$q = \lambda + \frac{iu}{2} + \frac{u^2}{2}. \quad (5.1.9)$$

$M(\alpha, \gamma, z)$ is the confluent hypergeometric function defined as:

$$M(\alpha, \gamma, z) = \sum_0^\infty \frac{(\alpha)_n z^n}{(\gamma)_n n!}. \quad (5.1.10)$$

For the properties of confluent hypergeometric functions, the reader is referred to
Abramowitz and Stegun (1964).

Substituting $t = 0$ and $u = 0$, in (5.1.4) one gets the Laplace transform of the total integrated variance $V_T = \int_0^T v_s ds$ in the 3/2 model. The explicit expression is

$$E[e^{-\lambda \int_0^T v_s ds}] = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left(\frac{2}{\epsilon^2 y(0, v_0)}\right)^{\alpha} M\left(\alpha, \gamma; -\frac{2}{\epsilon^2 y(0, v_0)}\right)$$  \tag{5.1.11}

where again $\lambda = a + bi$. Since $\theta(s) = \theta$ relation (5.1.5) becomes

$$y(0, v_0) = v_0 \int_0^T e^{\int_0^u \theta ds} du = v_0 \int_0^T e^{k\theta u} du = \frac{v_0(e^{k\theta T} - 1)}{k \theta}. \tag{5.1.12}$$

For (5.1.8) and (5.1.9), we have that $p = -k$ and $q = \lambda$ respectively. Inserting these in the equations for $\alpha$ and $\gamma$ (see (5.1.6) and (5.1.7)), the following is obtained:

$$\alpha = -\left(\frac{1}{2} + \frac{k}{\epsilon^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{k}{\epsilon^2}\right)^2 + \frac{2\lambda}{\epsilon^2}} \tag{5.1.13}$$

and

$$\gamma = 2 \left(\alpha + 1 + \frac{k}{\epsilon^2}\right) = 2 \left(\frac{1}{2} - \frac{k}{\epsilon^2} + \sqrt{\left(\frac{1}{2} + \frac{k}{\epsilon^2}\right)^2 + \frac{2\lambda}{\epsilon^2}}\right) + 2 + \frac{2k}{\epsilon^2}$$

$$= 1 + 2\sqrt{\left(\frac{1}{2} + \frac{k}{\epsilon^2}\right)^2 + \frac{2\lambda}{\epsilon^2}}. \tag{5.1.14}$$

We will use these equations in Sections 5.2 and 5.3.

### 5.2 An approximation for the 3/2 model prices

In this section we will apply the main results from Drimus (2011a) to produce an expansion for prices based on the 3/2 stochastic volatility model. In Section 4.2 we showed how the approximation is built under the Heston model. As the reader will see here, despite being different, these two models turn out to share the same properties and relations with respect to the expansion building. This would not be surprising if one takes into account relations (5.1.3) and (5.1.2).
As the stock price dynamics in the 3/2 model are the same as in the Heston framework, here again we define a process \( \{ \xi_t \}_{0 \leq t \leq T} \) as in (4.2.1) and obtain the representation (4.2.2), i.e.

\[
S_T = S_0 \xi_T \exp \left[ (r - \delta) T - \frac{1 - \rho^2}{2} \int_0^T v_t dt + \sqrt{1 - \rho^2} \int_0^T \sqrt{v_t} dZ_t \right].
\] (5.2.1)

For the process \( \{ \xi_t \}_{0 \leq t \leq T} \) to be a true martingale, and thus allow to be used for a change of measure, as in the Heston case above, the Novikov’s condition

\[
L \left( \frac{1}{2} \rho^2 \right) < 1
\]

should be fulfilled. That is, the Laplace transform of the total integrated variance \( V_T \) under the 3/2 dynamics (5.1.11) should exist when evaluated at \( \lambda = -\frac{1}{2} \rho^2 \).

Given that the Novikov’s condition is satisfied, the process \( \{ \xi_t \}_{0 \leq t \leq T} \) is a true martingale. Then the process is used to define a new probability measure \( Q^1 \), that will be equivalent to \( Q \) by definition. Applying Girsanov’s theorem, under \( Q^1 \), we obtain that \( W^1_t = W_t - \int_0^t \rho \sqrt{v_t} du \) will be a standard Brownian motion. Inserting in the variance dynamics equation (5.1.1)

\[
dv_t = kv_t(\theta - v_t) dt + \epsilon v_t^{3/2} (dW^1_t + \rho \sqrt{v_t} dt),
\]

we get

\[
dv_t = (k - \epsilon \rho)v_t \left( \frac{k\theta}{k - \epsilon \rho} - v_t \right) dt + \epsilon v_t^{3/2} dW^1_t.
\]

As in the Heston framework, the model structure is preserved when changing the measure and a model specified as 3/2 \((k, \theta, \epsilon)\) under \( Q \) will be transformed into a 3/2 \( \left( k - \epsilon \rho, \frac{k\theta}{k - \epsilon \rho}, \epsilon \right) \) under the measure \( Q^1 \). Furthermore, that result can be extended to the case of an \( n \)th change of measure where under \( Q^{(n)} \) the variance dynamics will be defined as 3/2 \( \left( k - n\epsilon \rho, \frac{k\theta}{k - n\epsilon \rho}, \epsilon \right) \).

The (4.2.7) relation which was derived in Section 4.2 is also valid in the 3/2 framework, i.e.

\[
\xi_T^{(n)} = \xi_T^{(n-1)} \exp \left( -\rho^2 \int_0^T v_t dt \right).
\]

As can be seen, all properties used in deriving the FFT approximation are shared by the Heston and the 3/2 model with the total integrated variance being the only difference. Once having these tools in hand, it is not hard to develop the same approximation based on a 3/2 stochastic volatility model. As it is a more reasonable and close to reality model, that result is expected to improve further the performance of the approximation and allow for incorporating more of the empirically observed properties of returns and implied volatilities.

Following the same steps as in Section 4.2, the Taylor expansion around \((S_0, (1-\rho)E^Q[V_T])\)
given in (4.2.11) for the case of $n=2$, is

$$
C_{3/2}^{(2)} = C^{BS} + \frac{1}{2} \frac{\partial^2 C^{BS}}{\partial S^2} \cdot S_0^2 \cdot \mathbb{E}_Q[\xi_T - 1]^2
+ \frac{1}{2} \frac{\partial^2 C^{BS}}{\partial V^2} \cdot (1 - \rho^2) \cdot \mathbb{E}_Q[V_T - \mathbb{E}_Q[V_T]]^2
+ \frac{\partial^2 C^{BS}}{\partial S \partial V} \cdot S_0 \cdot (1 - \rho^2) \cdot \mathbb{E}[(\xi_T - 1)(V_T - \mathbb{E}_Q[V_T])].
$$

(5.2.2)

The second-order approximation in the 3/2 case is again composed of the BS price, gamma, volga, vanna and several expected values. The difference in the application of the expansion to the two models appears when calculating the expectations in (5.2.2). Let us start with

$$
\mathbb{E}_Q[\xi_T - 1]^2 = \mathbb{E}_Q[\xi_T] - 1 = \mathbb{E}_Q[\xi_T e^{\rho^2 \int_0^T \nu dt}] - 1
= \mathbb{E}_Q[e^{\rho^2 \int_0^T \nu dt}] - 1 = \mathcal{L}(-\rho^2; k - 2\epsilon\rho, \frac{k\theta}{k - 2\epsilon\rho}, \epsilon) - 1,
$$

(5.2.3)

where the Laplace transform (5.1.11) of the total integrated variance $V_T$ under the 3/2 variance dynamics is used.

Further, the computation of $\mathbb{E}_Q[V_T]$ is to be performed. The expression exists in a closed form and can be found in Itkin and Carr (2010), and it is calculated from of the derivative of the total integrated variance Laplace transform. We have

$$
\mathbb{E}_Q[V_T] = -\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda}\bigg|_{\lambda=0}
= \left[-\log\left(\frac{2}{\epsilon^2 y(0, v_0)}\right) + \frac{\Gamma'\!(2\nu)}{\Gamma(2\nu)} - M^{(1,0,0)}(0; 2\nu; -2\nu; \frac{2}{\epsilon^2 y(0, v_0)})\right] \frac{2}{2k + \epsilon^2}
$$

(5.2.4)

with $y(0, v_0)$ defined as in (5.1.5), $\nu = 1 + \frac{k}{\epsilon^2}$, $M^{(1,0,0)}(a, b, z)$ being the first derivative of the confluent hypergeometric function given in (5.1.10) w.r.t. $a$ and $\Gamma'\!(2\nu)$ being the first derivative of the Gamma function, evaluated at $2\nu$.

For the confluent hypergeometric function derivative, according to Itkin and Carr (2010), it holds that

$$
M^{(1,0,0)}(0, b, -z) = -\left(\frac{z}{b}\right)_2 F_2[(1, 1); (2, 1 + b); -z],
$$

where $_2 F_2$ denotes the generalized hypergeometric function. More details on this family of special functions can be found in Abramowitz and Stegun (1964).
For the second central moment, $E_Q[V_T - E_Q[V_T]]^2$, from basic calculus it follows that

$$E_Q[V_T - E_Q[V_T]]^2 = E_Q[V_T^2] - E_Q[V_T]^2.$$ 

However, there is no readily available expression for $E_Q[V_T^2]$. Here we can go back to the expression for $E_Q[V_T]$, which is given by (5.2.4). It originates from the Laplace transform first derivative. For the calculation of $E_Q[V_T^2]$ the second derivative of the Laplace transform is needed here. It will be obtained numerically, by taking central differences of the Laplace transform given by (5.1.11). The expression to be computed, is

$$\frac{\partial^2 L(\lambda)}{\partial \lambda^2} = \frac{L(\lambda + h) - 2L(\lambda) + L(\lambda - h)}{h^2}.$$ 

Therefore, the final expression for the second central moment is

$$E_Q[V_T - E_Q[V_T]]^2 = \frac{L(\lambda + h) - 2L(\lambda) + L(\lambda - h)}{h^2} - E_Q[V_T]^2$$

for some small time step of size $h$ and evaluated at $\lambda = 0$.

Then, what is left is to evaluate the mixed moment $E_Q[(\xi_T - 1)(V_T - E_Q[V_T])]$. As in the Heston case, it can be represented as

$$E_Q[(\xi_T - 1)(V_T - E_Q[V_T])] = E_Q[\xi_T V_T] - E_Q[V_T] = E_Q^1[V_T] - E_Q[V_T].$$

In fact, $E_Q^1[V_T]$ can be evaluated with the formula for the $E_Q[V_T]$ calculation, but it should be taken into consideration that under the measure $Q^1$ the variance process follows the dynamics $3/2 \left( k - \epsilon \rho, \frac{k \theta}{k - \epsilon \rho}, \epsilon \right)$. $E_Q[V_T]$ is already available from (5.2.4), therefore all moments comprising the second order approximation in the 3/2 case are now known.

Here is the place to give the necessary attention to the numerical method for obtaining the second derivative above, or namely the central differences. That tool allows us to calculate derivatives of functions that would be otherwise impossible to obtain. Since it is an approximation, it gives rise to certain small errors. As the derivative approximation is obtained from a Taylor expansion, the error order can be analyzed. The deviation from the true derivative value consists of two parts - truncation error and rounding error, see Miranda and Fackler (2002).
The truncation error arises because the Taylor expansion is infinite. The computation cannot continue forever, so a residual term has to be defined. Then the calculation is performed by taking the first \(n-1\) terms of the expansion and the last \(n\)th term incorporates the error and is proportional to the chosen time step \(h\). In the case of the second derivative approximation of a function \(f(x)\), we have

\[
f''(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} + O(h^2),
\]

which means that the error order is \(h^2\). In other words the smaller the step size is, the smaller the truncation error.

The other error of interest, arising when employing the numerical differentiation described above, is the rounding error. As the name suggests, it appears from the necessity of rounding the expressions. That is due to the fact that for a very small \(h\), the values of \(f(x + h), f(x)\) and \(f(x - h)\) will be very close. When a subtraction of small numbers is performed, the terms require rounding. It is true that for the rounding error \(R_X\) it holds that

\[
|R_X| \leq \frac{d}{h}
\]

for some small positive number \(d\), see Burden and Faires (2005). That is, the smaller the time step \(h\), the bigger the rounding error.

From these considerations, one should be careful when choosing the size of \(h\). If it is too large, the truncation error will grow and if it is too small, the rounding error will increase. The right size of \(h\) can be estimated with the assistance of the Heston approximation components. As shown in the previous chapter, there exists a closed-form expression for \(\mathbb{E}^Q[V_T^2]\). Thus, knowing also the Laplace transform in the Heston case, one can easily obtain an estimation of the error by comparing the results from the Heston closed form formula and the central difference expression. In this way one can have a reasonable idea about the optimal size of \(h\) to be used in the numerical differentiation. The results are shown in Table 5.2.

After a comparison, the chosen value for derivng the derivative needed for the \(3/2\) approximation construction, is \(h = 0.001\). The choice is justified by the numbers shown and the corresponding error order \(O(h^3)\). Further, a smaller time step would cause some of the codes to run slower and would not provide with a sufficient improvement due to the error characteristics discussed in this section. To summarize, the numerical differentiation gives
Table 5.2: Time step comparison

<table>
<thead>
<tr>
<th>Time step</th>
<th>$E^Q[V_T^2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Formula</td>
<td>0.00123277789</td>
</tr>
<tr>
<td>$h=0.0001$</td>
<td>0.00123277747</td>
</tr>
<tr>
<td>$h=0.001$</td>
<td>0.00123277788</td>
</tr>
<tr>
<td>$h=0.01$</td>
<td>0.00123277969</td>
</tr>
</tbody>
</table>

an accurate estimate of the true value of $E^Q[V_T^2]$ and is a reliable substitute in the second order approximation.

5.3 Limitations for the model parameters

The 3/2 model has the parameters $k, \theta, \epsilon, v_0, \rho$ and $\lambda$ and Gamma function arguments, depending on them. In this section the possible range of these parameters will be briefly discussed.

5.3.1 Model-imposed conditions

As in the Heston model, the speed of mean-reversion has to be positive. In the 3/2 model the speed of mean-reversion is given by $kv$. Since $\{v_t\}_{0 \leq t \leq T}$ is a positive process, then $k$ must also be positive ($k > 0$). Also, as noted earlier, due to empirical evidence, it may be assumed that prices and volatility are negatively correlated, i.e. $\rho < 0$.

5.3.2 Non-explosion condition

Being aware of the fact that the dynamics of $\left\{\frac{1}{v_t}\right\}_{t \geq 0}$ behave as the ones described by the Heston model, a useful constraint can be derived. Within the Heston model, the variance process $\{v_t\}_{0 \leq t \leq T}$ is CIR($k, \theta, \epsilon$). A condition that prevents this process from reaching zero is

$$2k\theta \geq \epsilon^2,$$

see (4.3.1). Thus, the same condition for the reverse process will prevent it from reaching infinity, or exploding. It was shown earlier that, $\left\{\frac{1}{v_t}\right\}_{t \geq 0}$ is a CIR($k\theta, \frac{k+\epsilon^2}{k\theta}, -\epsilon$) process. This
implies that the non-exploding condition for the 3/2 process is:

\[ 2k\theta \left( \frac{k + \epsilon^2}{k\theta} \right) \geq \epsilon^2 \]

or

\[ k \geq -\frac{\epsilon^2}{2} \tag{5.3.1} \]

which is always fulfilled, as \( k > 0 \). That non-explosion condition also should be satisfied under any of the equivalent martingale measures one needs to work with. For the general case regarding the measure \( Q^{(n)} \) for \( n = 0, 1, 2, \ldots \), the requirement is

\[ k - n\epsilon\rho \geq -\frac{\epsilon^2}{2} \tag{5.3.2} \]

which is again always true, as \( k, \epsilon \) and \( n \) are all positive and \( \rho \leq 0 \).

### 5.3.3 Domain for the Laplace transform parameter

Relations (5.1.13) and (5.1.14), with \( k > 0 \), give condition for the possible values of \( \lambda \). We get the following inequalities

\[ \left( \frac{1}{2} + \frac{k}{\epsilon^2} \right)^2 + \frac{2\lambda}{\epsilon^2} \geq 0, \]

equivalent to

\[ \lambda \geq -\left( \frac{\epsilon^2 + 2k}{2\sqrt{2}\epsilon} \right)^2. \tag{5.3.3} \]

### 5.3.4 Gamma function arguments

According to the definition of a Gamma function, its argument should be positive. In what follows, \( \gamma - \alpha \) and \( \gamma \) have to be investigated using (5.1.13) and (5.1.14). We have

\[
\gamma - \alpha = 1 + 2\sqrt{\left( \frac{1}{2} + \frac{k}{\epsilon^2} \right)^2 + \frac{2\lambda}{\epsilon^2}} - \left( -\left( \frac{1}{2} + \frac{k}{\epsilon^2} \right) + \sqrt{\left( \frac{1}{2} + \frac{k}{\epsilon^2} \right)^2 + \frac{2\lambda}{\epsilon^2}} \right) \\
= \frac{3}{2} + \frac{k}{\epsilon^2} + \sqrt{\left( \frac{1}{2} + \frac{k}{\epsilon^2} \right)^2 + \frac{2\lambda}{\epsilon^2}}.
\]
The last expression is always positive since $k > 0$ and inequality (5.3.3).

Considering $\gamma$ itself, it takes positive values only, as can be seen from (5.1.14). These observations result in well-defined Gamma functions within the Laplace transform above. Also, $\gamma$ taking only positive values ensures that the confluent hypergeometric function is well defined.

### 5.3.5 Conditions for the correlation parameter

When computing the expectation in (5.2.3), the Laplace transform of the process, evaluated at $-\rho^2$ under $Q_2$ will have to be calculated, or $L(-\rho^2; k - 2\epsilon\rho, \frac{k\theta}{k - 2\epsilon\rho}, \epsilon)$. Considering condition (5.3.3) for $\lambda$, the following inequality should be satisfied to ensure existence:

$$-\rho^2 \geq - \left(\frac{\epsilon^2 + 2(k - 2\epsilon\rho)}{2\sqrt{2}\epsilon}\right)^2.$$  

Thus

$$-\frac{\epsilon^2 + 2(k - 2\epsilon\rho)}{2\sqrt{2}\epsilon} \leq \rho \leq \frac{\epsilon^2 + 2(k - 2\epsilon\rho)}{2\sqrt{2}\epsilon}.$$

The right hand side is always satisfied, as $k$ and $\epsilon$ are both positive and $\rho \leq 0$. Focusing on the left hand side, we have that

$$(2\sqrt{2} - 4)\epsilon\rho \geq -\epsilon^2 - 2k \iff \rho \leq -\frac{(\epsilon^2 + 2k)}{(2\sqrt{2} - 4)\epsilon},$$

which is always true, as the right side in the last inequality is always positive (because $\epsilon^2 + 2k > 0$ and $2\sqrt{2} - 4 < 0$) and $\rho$ is negative.

Another condition to be satisfied by the correlation parameter is the Novikov's condition

$$L\left(-\frac{1}{2}\rho^2\right) < \infty$$

in order to assure that $\{\xi_t\}_{0 \leq t \leq T}$ is not only a local martingale, but also a true martingale. From (5.3.3) we have

$$-\frac{1}{2}\rho^2 \geq - \left(\frac{\epsilon^2 + 2k}{2\sqrt{2}\epsilon}\right)^2,$$

which leads to

$$-\frac{\epsilon^2 + 2k}{2\epsilon} \leq \rho \leq \frac{\epsilon^2 + 2k}{2\epsilon}.$$
Again the right inequality is always true. After inspecting the left inequality, the restriction for $\rho$ is

$$\rho \geq -\frac{\epsilon}{2},$$

because if it holds for $k = 0$ then it should be true for any $k$.

The Novikov’s condition also needs to be satisfied for a variance process following the specification $3/2 \left( k - n\epsilon\rho, \frac{k\theta}{k-n\epsilon\rho}, \epsilon \right)$ and that brings to

$$-\frac{\epsilon^2 + 2k - 2n\epsilon\rho}{2\epsilon} \leq \rho \leq \frac{\epsilon^2 + 2k - 2n\epsilon\rho}{2\epsilon},$$

where the left inequality holds for $n \geq 1$ and the right inequality is always true. After all, the final restriction for the parameter $\rho$ is

$$\rho \geq -\frac{\epsilon}{2}. \quad (5.3.4)$$

### 5.4 Fast Fourier transform pricing for the 3/2 model

The generalized Fourier-Laplace transform in the 3/2 model is given in (5.1.4). From there, to obtain an expression for the Fourier transform of the log-price ($x_T = \log S_T$), one needs to substitute $\lambda = 0$ and $t = 0$ in (5.1.4). This yields

$$\phi_{3/2}(u) = \mathbb{E}[e^{iuX_T}] = e^{iuX_T} \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left( \frac{2}{\epsilon^2 y(0,v_0)} \right)^\alpha M \left( \alpha, \gamma, -\frac{2}{\epsilon^2 y(0,v_0)} \right),$$

where

$$y(0,v_0) = \frac{v_0(e^{k\theta T} - 1)}{k\theta},$$

$$\alpha = -\left( \frac{1}{2} - \frac{p}{\epsilon^2} \right) + \sqrt{\left( \frac{1}{2} - \frac{p}{\epsilon^2} \right)^2 + \frac{2q}{\epsilon^2}},$$

$$\gamma = 2 \left( \alpha + 1 - \frac{p}{\epsilon^2} \right),$$

$$p = -k + i\epsilon\rho u,$$

$$q = \frac{iu}{2} + \frac{u^2}{2}$$

and $M(\alpha, \gamma, z)$ is the confluent hypergeometric function, defined as in (5.1.10).
The prices and implied volatilities obtained via the FFT here will also serve as a benchmark values for evaluating the performance of the second order approximation. The drawbacks of the FFT methodology were already discussed in Section 3.3.

5.5 3/2 expansion performance

In this section we will compare the results from applying the FFT and the second-order approximation to the 3/2 model. The parameter values used are obtained by Drimus (2011b) after a calibration performed on S&P500 options. The exact estimates are $k = 22.84, \theta = 0.4669^2, \epsilon = 8.56, v_0 = 0.245^2$ and $\rho = -0.99$ and moneyness ranges from 70% to 130%. Because the purpose of this implementation is to compare prices and implied volatilities, we assume that the spot is equal to 1 and zero interest rates and dividend yields.

![Figure 5.2: The left panel shows prices and the right one provides implied volatilities for the approximation and the FFT for $\rho = 0$.](image)

First, the uncorrelated case is investigated, i.e. $\rho = 0$. Prices and implied volatilities for FFT and the approximation move in sync, and this can be seen on Figure 5.2. The mean squared errors are 0.0004% for prices and 0.014% for implied volatilities.

An investigation of the correlated case reveals very different results from the uncorrelated counterpart. As can be seen from Figure 5.3, the approximation is far from giving appropriate results. The mean absolute error when $\rho = -0.99$ is 7.6432% for prices and 218.656% for implied volatilities. In terms of prices, the approximation undervalues the FFT results and behaves unexpectedly at the point at which the option is at-the-money, where a strong overestimation, both for prices and implied volatilities, is observed. As can be seen from the left panel of Figure 5.3, the strange behaviour is driven by the Gamma approximation term. There is some kink occurring at $S/K = 1$, which gives the hint that some kind of
Figure 5.3: The left panel shows prices and the right one provides implied volatilities for the approximation and the FFT for $\rho = -0.99$.

A discontinuity exists. That is, in disagreement with the initial setting, it turns out that the 3/2 volatility model cannot be a basis for an accurate application of the second-order approximation derived above. That is surprising, due to the similar structure between the 3/2 and the Heston and the properties they share. Furthermore, the 3/2 variance dynamics enjoy more empirical support and have a more realistic setting, as previously noted. One of the possible reasons might be the very high negative value ($-0.99$) of the correlation which implies perfect negative dependence between the Wiener processes characterizing the dynamics in (5.1.1). The problems involved will be discussed in the next chapter.
Chapter 6

Discussion

In this section the above results will be compared and discussed. It is interesting to note that despite not having the same interpretation, the parameters obtained by Drimus (2011b) when calibrating the Heston and the 3/2 models differ significantly, although coming from the same data set. Also, the 3/2 approximation shows a good performance in the uncorrelated case and behaves inconsistently when $\rho = -0.99$. As we noted, this almost limiting negative value of $\rho$ implies perfect negative dependence between the processes involved. In a sense, some parameters from the calibration on the 3/2 performed by Drimus (2011b) are quite extreme as compared to their Heston counterparts, or namely $k_{3/2} = 22.84$, $\epsilon_{3/2} = 8.56$, $\rho_{3/2} = -0.99$ as opposed to $k_H = 3.8$, $\epsilon_H = 0.9288$, $\rho_H = -0.7829$. The following question arises - whether the parameter values themselves cause the misbehaviour of the 3/2 approximation results in the previous section.

In order to gain an understanding about the approximation behaviour and potential pitfalls, here we will comment on the sensitivity analysis of the parameter set.

6.1 Model sensitivity to parameter values

As seen from the analysis and application of the 3/2 approximation, it performs good for some parameters and fails for others. Moreover, the problematic values are still coming out of a model calibration to real world prices. Having the model structure itself too sensitive to some extreme parameter values would mean that the approximation is not suitable for each and every case. To gain an understanding of the approximation behaviour, we will show how implied volatilities change when one of the input parameters varies, given all others remain unchanged.

As the Heston calibration parameter set from Drimus (2011b) provides with proper results regarding the expansion, these values will be used as a benchmark. That is, it will be shown how implied volatilities under the Heston approximation setting differ when changing one parameter, all else being equal.
Firstly, the effect of changes in $k$, which is the speed of mean-reversion in Heston model, will be analyzed. The values used are $k = 1.15$ (parameter from Bakshi, Cao, and Chen (1997)), $k = 3.8$, $k = 10$ and $k = 22.84$ (3/2 parameter from Drimus (2011b)).

As can be seen from Figure 6.1, the effect increases with time to maturity. Also, the in-the-money region does not show strong dependence on the parameter change, the effect appearing at the peak and at the right side of the graph. Here, $k = 3.8$ is the value coming out of the calibration and it is thus closest to the FFT implied volatility.

![Figure 6.1: The left panel shows implied volatilities for a maturity of 3 months and at the right maturity is 6 months for different values of $k$.](image)

The long-run variance level $\theta$ and the initial variance $v_0$ affect the implied volatility in exactly the same way - when increased, the entire graph moves up and both out-of-the-money and in-the-money options are affected in an identical way. Again, the effect is higher when time to maturity increases. The results are visualized in Figures 6.2 and 6.3.

For the volatility of variance $\epsilon$, the approximation crashes when the parameter is artificially increased too much. As noted in the study of Jones (2003), higher variance levels result in an increased correlation, and from other hand having increased variance would mean that the volatility of variance is experiencing a shock. So it could be the case that a high value of $\epsilon$ is strongly connected with increases in some of the other parameters. When inspecting the Novikov’s condition for the Heston model, it turns out that high values of the volatility of variance need to be accompanied by increases in the mean-reversion speed $k$, to assure that the inequality $\rho^2 < \frac{k^2}{\epsilon^2}$ holds. Figure 6.4 shows the approximation behaviour for different values of $\epsilon$.

Finally, when studying the effect of the correlation parameter $\rho$, the results are quite interesting. Theoretically $\rho$ has an effect on the skewness of the log returns distribution and therefore change the shape of the implied volatility. However, for the 3/2 model calibrated value of $\rho = -0.99$ the familiar strange peak appears even under the Heston setting. That
CHAPTER 6. DISCUSSION

Figure 6.2: The left panel shows implied volatilities for a maturity of 3 months and at the right maturity is 6 months for different long-term variance levels $\theta$.

Figure 6.3: The left panel shows implied volatilities for a maturity of 3 months and at the right maturity is 6 months for different initial variance levels $\epsilon_0$.

Figure 6.4: The left panel shows implied volatilities for a maturity of 3 months and at the right maturity is 6 months for different volatility of variance values.

is, even with the balanced and well-behaved Heston parameters, there is some flaw with the model originally developed in Drimus (2011a). One can note that as $\rho$ gets closer to $-1$ (see
Figure 6.5, red line at $-0.88$ and light blue line at $-0.99$), there is an overestimation close to the at-the-money region of implied volatilities, and it gets extreme when the parameter approaches $-1$ (or as $|\rho|$ approaches 1).

Figure 6.5: The left panel shows implied volatilities for a maturity of 3 months and at the right maturity is 6 months for different correlation.

In other words, at that limit case, the approximation explodes, which is intuitively reasonable. After reviewing the way the approximation is composed, the Greeks and simulating the different components, the reason for explosive effect was found in the structure of the second term, appearing in the expansion (4.2.12), i.e. in

$$\frac{1}{2} \frac{\partial^2 C^{BS}}{\partial S^2} \cdot S_0^2 \cdot \mathbb{E}_Q[\xi_T - 1]^2.$$ 

As given in (4.2.13), the partial derivative in that expression is the Gamma, or

$$\text{Gamma} = \frac{\partial^2 C^{BS}}{\partial S^2}(S, V) = \frac{e^{-\delta T} \phi(d_1)}{S \sqrt{V}}$$

and it is evaluated at the point $(S_0, (1 - \rho^2)\mathbb{E}_Q[V_T])$, which means that

$$\text{Gamma} = \frac{e^{-\delta T} \phi(d_1)}{S_0 \sqrt{(1 - \rho^2)\mathbb{E}_Q[V_T]}}.$$ 

Having a $|\rho|$ very close to 1 would mean that in the latter expression one has to divide by a very small number, which makes the entire Gamma too large for obtaining reasonable prices and implied volatilities with the second-order approximation. That is independent of the variance process structure and holds for both the Heston and the $3/2$ specifications. In
practice, and especially in the post-crisis period of the last couple of years, having a $\rho$ close to $-1$ is not unreasonable. Such a value would tell that the random element in the variance dynamics moves in the opposite direction to the randomness in the underlying process, and that happens with almost the same magnitude. There is a huge amount of empirical evidence for this fact and the interested reader is referred to the conclusions of Jones (2003). According to the author "the so-called leverage effect, or the negative correlation between the price and instantaneous variance processes, becomes substantially stronger at higher levels of variance" and in fact these higher levels of variance were actually achieved during and after the last financial crisis. Unfortunately, the second-order approximation developed cannot capture such values, for the above stated reasons. Figure 6.6 illustrates that for non-extreme correlation values, the $3/2$ approximation performs good enough.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6_6.png}
\caption{The left panel shows implied volatilities for $\rho$ taking a value of $-0.64$ and at the right is for $\rho = -0.79$, both in the $3/2$ approximation}
\end{figure}

### 6.2 Final remarks

Considering the presented results, a few conclusions emerge. First of all, the second order approximation to the FFT provides with a fast and tractable way to model option prices and implied volatilities. Speed is not an unimportant issue, especially in the world of financial instruments valuation. The expansion mechanism is developed for options written on an underlying whose price and variance dynamics follow either the Heston or the $3/2$ stochastic volatility model. However, it turns out that the approximation structure itself doesn’t allow some input parameters and either crashes or gives peculiar results. Some parameter values, despite having realistic interpretation, are unacceptable for the given model structure and thus the results are unreasonable. These facts suggest that calibration results might not be
reliable in all cases but only under an economically stable setting. At the end we are using a model with an idea to fit the dynamics of financial instrument and nobody can confirm that these models are the right ones. However, considering the initial value problem with the FFT, and given that the calibration of the approximation does not provide with peculiar numbers, calibrating the expansion and then using the obtained parameter set as initial values for the FFT calibration can be a good way out.
Chapter 7

Conclusion

The main goal of this thesis is to derive an option pricing methodology under a 3/2 stochastic volatility model, based on the contribution of Drimus (2011a), who suggested a tractable approximation for producing option prices in the model proposed by Heston (1993). The interest in deriving such an approximation emerged from the fact that the 3/2 model is characterized by a non-affine structure and has a number of properties that make it more flexible as compared to a model with CIR variance dynamics. Despite being more complex, the 3/2 model is still tractable enough, with reasonable parameter interpretations and in terms of the second-order approximation, it can be derived in a similar way to the Heston model. The relation between the two models make them share a number of properties. The interest in pricing methods alternative to FFT arise from the fact that it is working as a "black box" producing numbers in the absence of additional information about how these numbers are related to any other concepts, such as the BS elements.

We begin this thesis with a review and the reasoning behind the use of general stochastic volatility models offering some examples. That is necessary in order to capture the oversimplifications of the BS world, such as the assumptions for normally distributed log-returns and constant volatility, and the model improvements when variance is a stochastic process instead. There is a vast empirical support for the negative skewness and excess kurtosis (and thus fatter tails) of log-returns. Also, volatility varies for different time to maturity and moneyness. Further, theoretical background from the basic option pricing theory is presented. This is made for gaining an understanding about the necessary model-building elements which are later provided.

Then, some main properties and results involving the fast Fourier transform methods are presented. This is necessary for obtaining a theoretical basis for what later serves as a benchmark when building the approximated prices and implied volatilities for the Heston and 3/2 models. The approaches from Carr and Madan (1999) and Lewis (2001) are discussed. The motivation for going beyond the FFT pricing methodology is that it provides with no tractability and just with numbers with no intuition about how they are related to any other concept. Choosing the right initial values and different software approaches for working in
the complex domain are some of the practical problems occurring when implementing the FFT.

Further, the theoretical results from Drimus (2011a) are reviewed, based on the Heston stochastic volatility model. The model is based on a Taylor expansion and relies on defining a set of equivalent martingale measures that are recursively connected. The option prices produced are in terms of BS prices and certain BS sensitivities. This new model is characterized by high tractability as well as by an impressive ability for building accurate prices and implied volatilities, when compared to the FFT results. The parameters used to visualize the second order approximation efficiency are the ones obtained by a study of Bakshi, Cao, and Chen (1997) and those from a calibration to S&P500 options, performed by Drimus (2011b).

Based on the derivations and results for the Heston model, the 3/2 approximation was derived. As there are no available closed-form expressions for some of the second-order approximation components, which is the case for Heston, central differences are used to perform numerical differentiation. We discuss the accuracy and errors due to the numerical procedures and it was concluded that the results are reliable, as compared to the closed-form counterpart. Then again as in the Heston case, a comparison with the FFT results is given. The parameters used are the ones from Drimus (2011b), which are again obtained after calibrating S&P500. However, it turns out that in this case the expansion does not provide with sensible results. Both prices and implied volatilities show a strange behaviour at the point where options are at-the-money. At that region a kink is observed and there is an immense overestimation to prices and implied volatilities, as if numbers explode.

Finally, a discussion on results obtained, our difficulties and experience is presented. The reason behind the strange behaviour in the 3/2 model is found - the high correlation value of $\rho = -0.99$ and the approximation structure itself. The latter cannot handle such values and explodes. However, when parameter values are not that extreme, it behaves similarly to the Heston counterpart. That is to say that the second-order approximation is in fact too sensitive towards different parameter values and cannot provide with a reliable estimation when some ingredients are too extreme. To gain a better understanding on how the approximation structure reacts to different parameters, the behaviour is graphically shown.

The main conclusion is that the second-order approximation is not a reliable tool for extracting option prices and implied volatilities, at least when modeling under a setting which allows for higher volatility deviations and thus strong correlation between the price and variance process.
An idea for further research is to build a similar pricing methodology, based on a model with jump-diffusion process or on a more advanced Lévy process. It would be useful to obtain a relation between prices corresponding to such a model setting and the BS elements.
Bibliography


Bibliography


