Essays on Heterogeneous Beliefs, Public Information, and Asset Pricing
Essays on Heterogeneous Beliefs, Public Information, and Asset Pricing

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Preface

This thesis was written in the period from February 2008 to January 2012 during my PhD studies at the Department of Economics and Business, Aarhus University. I am grateful to the Department of Economics and Business for an excellent research environment and for the financial support given to me for participating in courses, seminars, workshops and conferences.

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Zhenjiang Qin
Aarhus, January 2012

Updated Preface

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Aarhus, April 2012
Summary

This thesis consists of three self-contained chapters that conduct theoretic analysis of the impact of heterogeneous beliefs and public information on asset pricing. The aim is to provide better understanding of the role of public information in general equilibriums with heterogeneous beliefs, in both discrete-time and continuous-time frameworks. Although self-contained, these three chapters are interrelated. Chapter 1 explores the impact of learning mechanism in an incomplete market, in which investors trade only in a zero-coupon bond and a stock. It also investigates an effective complete market which introduces an ideal derivative to facilitate Pareto efficient side-betting, and this gets rid of the need of dynamic trading based on signals. Chapter 2 is a derivative-oriented extension of Chapter 1. It demonstrates that the difference in confidence of investors leads to the fact that investors trade and speculate actively in option markets with Gamma trading strategies. As an extreme case of Chapter 1 and Chapter 2, Chapter 3 investigates the role of continuous trading with heterogeneous beliefs and the information contingent on aggregate consumption. It shows that a continuous trading Radner equilibrium with heterogeneous beliefs can implement the same Arrow-Debreu consumption allocations in Chapter 1.

Chapter 1 addresses the following question: How does public signal precision determine cost of capital, trade volume, and investors’ welfare in a framework of heterogeneous beliefs? In an incomplete market setting with heterogeneous prior beliefs, we find that the public information must be imperfect to be valuable to facilitate improved dynamic trading opportunities, which are based on heterogeneously updated posterior beliefs. Moreover, the Pareto efficient public information system gives rise to the highest efficiency of side-betting, and thus the highest ex ante equilibrium interest rate. Furthermore, it also enjoys the maximum ex ante cost of capital, the maximum expected abnormal trading volume, and the best individual welfare. This result is interesting especially when comparing to the result in partial equilibrium, for example, Debreu (1959): Low ex ante cost of equity capital and, thus, high ex ante stock prices is good. However, financial reporting regulation (and other mandated disclosure requirements) is about choosing information systems for the economy at large, thus we need to study the impact of public information in a general equilibrium setting. In contrast, we find that a high cost of capital is good for investors.

In Chapter 2, I establish an absolute option pricing model to provide answers to the following two questions: What is the condition under which an option is a non-redundant asset? What is the role of public signal in option markets? These questions are worth paying attention to because in the Black-Scholes model, the option is a redundant asset. However,
the high trading volume in option markets in the real world signals that the option is far from redundant. Furthermore, Buraschi and Jiltsov (2006) analyze the impact of heterogeneous beliefs conditional some certain state of the economy, and their model is silent with respect to the impact of information quality. It is interesting to analyze the influence of heterogeneous beliefs and information quality unconditionally. Solving the equilibrium numerically, I find that heterogeneous beliefs provide the economic value to option markets in the sense that investors speculate in option markets and public information improves allocational efficiency of option markets only when there is heterogeneity in prior variance. With heterogeneous prior variance, options are non-redundant assets. They can facilitate side-betting and enable investors to take advantage of the disagreements and the differences in confidence. The increased efficiency of side-betting leads to a higher growth rate in the investors’ certainty equivalents and, a higher equilibrium interest rate. With an intermediate signal precision and the option with intermediate strike price, side-betting among investors is the most efficient, and the equilibrium interest rate reaches the maximum point. Moreover, options make the role of public signal more sophisticated. Since when investors trade in option markets, the public signal tends to affect the ex ante equilibrium risk premium, contrasting to the fact that the risk premium is independent of signal precision when investors trade only in a zero-coupon bond and a stock.

Chapter 3 is a heterogeneous-beliefs extension of Duffie and Huang (1985). I model an information structure, in which the information on the aggregate consumption at the terminal date is revealed by an Ornstein-Uhlenbeck Bridge. This information structure allows investors to speculate on the heterogeneous posterior variance of dividend continuously. The market populated with many time-additive exponential-utility investors is dynamically effectively complete, if investors are allowed to trade in only two long-lived securities continuously. The underlying mechanism is that these assumptions imply that the Pareto efficient individual consumption plans are measurable with respect to the aggregate consumption. Hence, I may not need a dynamically complete market to facilitate a Pareto efficient allocation of consumption, the securities only have to facilitate an allocation which is measurable with respect to the aggregate consumption. With normally distributed dividend, the equilibrium stock price is endogenized in a Radner equilibrium as a precision weighted average of the investors’ posterior mean minus a risk premium determined by the average posterior precision. I demonstrate that there exists a trade strategy contingent on the aggregate consumption to replicate the payoff of the "Dividend Square Security", which effectively completes the market in Chapter 1. This fact indicates that from a welfare perspective, continuous trading can be viewed as a replacement of the convexity in the payoff of the dividend derivative, which can be attained by speculating with Gamma trading strategies.
References


Chapter 1

Information and Heterogeneous Beliefs:
Cost of Capital, Trading Volume, and Investor Welfare
Information and Heterogeneous Beliefs: Cost of Capital, Trading Volume, and Investor Welfare

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Abstract

In an incomplete market setting with heterogeneous prior beliefs, we show that public information can have a substantial impact on the \textit{ex ante} cost of capital, trading volume, and investor welfare. In a model with exponential utility investors and an asset with a normally distributed dividend, the Pareto efficient public information system is the system which enjoys the maximum \textit{ex ante} cost of capital, and the maximum expected abnormal trading volume. The public information system facilitates improved dynamic trading opportunities based on heterogeneously updated posterior beliefs in order to take advantage of the disagreements and the differences in confidence among investors. This leads to a higher growth in the investors’ certainty equivalents and, thus, a higher equilibrium interest rate, whereas the \textit{ex ante} risk premium on the risky asset

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is unaffected by the informativeness of the public information system. In an effectively complete market setting, in which investors do not need to trade dynamically in order to take full advantage of their differences in beliefs, the \textit{ex ante} cost of capital and the investor welfare are both higher than in the incomplete market setting, but they are independent of the informativeness of the public information system, and there is no information-contingent trade.

\textbf{Keywords:} Heterogeneous Beliefs; Public Information Quality; Dynamic Trading; Cost of Capital; Investor Welfare
One of the things that microeconomics teaches you is that individuals are not alike. There is heterogeneity, and probably the most important heterogeneity here is heterogeneity of expectations. If we didn’t have heterogeneity, there would be no trade. But developing an analytic model with heterogeneous agents is difficult.


1 Introduction

Financial markets are not complete, and investors in financial markets are not alike—both in terms of preferences, wealth and beliefs. Acknowledging these facts, we develop a simple analytical model with exponential utility investors, who have heterogeneous beliefs over normally distributed dividends, which shows that the public information system plays a key role for the investors’ welfare, the asset prices, and for the trading volume in the financial market. We show that the Pareto efficient public information system is the system, which enjoys the maximum ex ante cost of capital, and the maximum expected abnormal trading volume. In an incomplete market, imperfect public information facilitates dynamic trading opportunities based on heterogeneously updated posterior beliefs, which allow the investors to better take advantage of their disagreements and their differences in confidence.

The vast majority of prior studies in the accounting and finance literature of the impact of public information system choices, such as financial reporting regulation, on equilibrium asset prices, trading volume, and investor welfare, recognize differences in preferences and/or wealth, but assume that the investors’ prior beliefs are identical, although their posterior beliefs may vary due to differences in the information they have received (see, e.g., Harsanyi 1968). In complete markets, this assumption typically leads to so-called no-trade theorems (see, e.g., Milgrom and Stokey 1982), implying that the theory cannot explain the significant trading volume in actual financial markets, for example, around earnings announcements as first documented by Beaver (1968), unless some unmodeled noise trading is injected into the price system (see, e.g., Grossman and Stiglitz 1980, Hellwig 1980, and Kyle 1985).

But why should all investors have been born equal? Some investors may be more optimistic or more confident in their estimates than others, for example, due to different DNA profiles or past experiences which are completely unrelated to the uncertainty and information in financial markets (see, e.g., Morris 1995, for a critical discussion of the common prior assumption in economic theory). Moreover, despite significant financial innovations over the last four decades, financial markets are probably still incomplete even if we allow for dynamic
trading strategies, for example, due to individual idiosyncratic risks (see, e.g., Krueger and Lustig 2010, and Christensen et al. 2011) or heterogeneous prior beliefs. In this paper, we develop a simple equilibrium model with heterogeneous prior beliefs and incomplete markets allowing us to study (in closed-form) the impact of public information system choices on both equilibrium asset prices, trading volume, and investor welfare.

We compare the equilibrium in the incomplete market setting to the equilibrium in an otherwise identical effectively complete market setting in which there exists a derivative security specifically targeted towards the investors’ incentive to take speculative positions based on their heterogeneity in beliefs. In that economy, the investors do not need to trade dynamically in order to take full advantage of their differences in beliefs. The \textit{ex ante} cost of capital and the investors’ welfare are both higher than in the incomplete market setting, but there is no trade, and the public information system plays no role. More generally, this result suggests that the existence of derivative markets and the public information system have complementary roles in facilitating improved investor welfare in financial markets.

A large literature in accounting and finance studies the impact of information on firms’ cost of equity capital both theoretically and empirically. The general theme in this literature seems to be that more public disclosure of information will reduce firms’ cost of equity capital which, in an exchange economy, is equivalent to higher stock prices. The intuition is simple. A firm’s cost of equity capital is the riskless interest rate plus a risk premium. Releasing more informative public signals reduce the uncertainty about the size and the timing of future cash flows and, therefore, also the risk premium.

This intuition, however, pertains only to the cost of capital when measured after the release of information, i.e., the \textit{ex post} cost of capital. Christensen et al. (2010) show that if the cost of capital is measured before any signals from the information system are realized, i.e., the \textit{ex ante} cost of capital, then the public information system has no impact on the \textit{ex ante} cost of capital and, thus, no impact on the \textit{ex ante} stock prices, in competitive exchange economies with \textit{homogeneous prior beliefs} and both public and private investor information. The public information system only serves to affect the timing of release of information and, thus, to affect the allocation of the total risk premium for future cash flows over time.

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\textsuperscript{1}Theoretical studies include Easley and O’Hara (2004), Hughes et al. (2007), Lambert et al. (2007, 2012), Christensen et al. (2010), Armstrong et al. (2011), and Bloomfield and Fischer (2011), while empirical studies include Botosan (1997), Botosan and Plumlee (2002), Easley et al. (2002), and Francis et al. (2008), among many others.

\textsuperscript{2}Although this intuition may seem simple and straightforward, one has to be careful in interpreting these results in multi-period models in which any interim period has elements of both \textit{ex post} and \textit{ex ante} effects (see the discussion in Christensen et al. 2010). In a standard continuous-time model, Veronesi (2000) shows that more precise public signals about economic growth tend to increase conditional equity premia through a higher equilibrium covariance between current consumption and stock returns.
Is a low \textit{ex ante} cost of equity capital and, thus, high \textit{ex ante} stock prices good or bad? In a partial equilibrium analysis focusing on a single firm and its shareholders, the answer is clearly “good.” This is merely a cousin of the familiar \textit{value maximization principle} for competitive markets, cf. Debreu (1959). However, financial reporting regulation (and other mandated disclosure requirements) is about choosing information systems for the economy at large. In such settings, a general equilibrium analysis is in order and, in general, welfare consequences of policy changes cannot be assessed directly through stock market values.

For example, how is the other component of the cost of equity capital, i.e., the riskless interest rate, affected by changes in the information system in the economy? In competitive exchange economies with homogeneous prior beliefs, time-additive preferences, and public information, the \textit{ex ante} riskless interest rates will not be affected by changes in the information system (see, e.g., Christensen et al. 2010 and the references therein). We show that even for an exchange economy, but with heterogeneous prior beliefs, the \textit{ex ante} equilibrium interest rate is affected by the informativeness of the public information system. In particular, the \textit{ex ante} equilibrium interest rate is a linear increasing function of the growth in the investors’ certainty equivalents. More efficient dynamic trading opportunities based on the heterogeneity in prior beliefs and public information increase the growth in certainty equivalents, while (in our particular model) the \textit{ex ante} risk premium is unaffected by the public information system. In other words, from a general equilibrium perspective, the preferred public information system is the system, which enjoys the highest \textit{ex ante} cost of equity capital and, thus, the lowest \textit{ex ante} stock prices.

Our analysis focuses on a competitive \textit{exchange economy} and, thus, a relevant question is whether the higher \textit{ex ante} cost of capital due to more efficient dynamic trading opportunities based on the heterogeneity in prior beliefs and public information comes with a \textit{negative real effect} due to costlier financing of firms production in a more general \textit{production economy}. Interestingly, introducing a riskless standard convex production technology into the setting of this paper, a higher \textit{ex ante} cost of capital is associated with \textit{positive real effects}. A higher \textit{ex ante} cost of capital is a consequence of a higher growth in certainty equivalents and, thus, the intertemporal trade-off between current and future aggregate consumption changes such that it becomes optimal to invest less in production (and, thus, consume more) now and consume less in the future. Such changes in production choices would then reduce the \textit{ex ante} cost of capital, in equilibrium, but not fully back to the level with less efficient dynamic trading opportunities.

Our model is a two-period extension of the classical single-period capital asset pricing model with heterogeneous beliefs of Lintner (1969). For simplicity, we assume there is a single risky asset in non-zero net-supply paying a known dividend at $t = 0$ and a normally
distributed dividend at \( t = 2 \). The investors have time-additive exponential utility, and we assume, for simplicity, that they have identical time-preference rates and risk aversion parameters. However, their prior beliefs at \( t = 0 \) for the dividend at \( t = 2 \) can differ with regard to both the mean and the precision (i.e., the inverse variance or confidence).

It is well known that Pareto efficient allocations in settings with heterogeneous beliefs require not only an efficient sharing of the risks, but also an efficient side-betting arrangement (see, e.g., Wilson 1968). If the investors’ prior precisions are identical, then the Pareto efficient side-betting (or speculative positions) based on their disagreements about the mean can be achieved by trading in the risky asset and the zero-coupon bond at \( t = 0 \): The optimistic (pessimistic) investors hold more (less) than their efficient risk sharing fraction of the risky asset.

If the investors have different prior precisions, trading in the risky asset and the zero-coupon bond at \( t = 0 \) does not facilitate efficient side-betting: An investor with a low (high) prior precision would like to have a payoff at \( t = 2 \) which is a convex (concave) function of the dividend.\(^3\) The key is that investors with low precisions value a convex payoff more than investors with higher precisions and, thus, trading gains can be achieved with non-linear payoffs. Based on the seminal paper, Wilson (1968), we show that if a derivative security in zero net-supply with a payoff at \( t = 2 \) equal to the square of the dividend on the risky asset is also available for trade at \( t = 0 \), then the market is effectively complete such that both Pareto efficient risk sharing and side-betting are achieved (see also Brennan and Cao 1996). On the other hand, if this dividend derivative specifically targeted towards the heterogeneity in the investors’ prior precisions is not available for trade, then it can be valuable to have public information and another round of trading at the interim date \( t = 1 \).

We consider a simple public information system with a public signal at \( t = 1 \) equal to the \( t = 2 \) dividend on the risky asset plus independent noise. The investors have identical normally distributed beliefs for the noise in the signal, i.e., a zero mean and a common signal precision, such that the investors’ posterior precisions for the dividend are equal to their heterogeneous prior dividend precisions plus the common signal precision. This specification allows us to measure the informativeness of the public information system by the signal precision. Hence, while we assume the investors may disagree about the fundamentals in the economy (i.e., the dividends), we assume the investors have homogeneous beliefs about the noise in the information system, i.e., the investors have so-called concordant beliefs (Milgrom and Stokey 1982) or homogeneous information beliefs (Hakansson, Kunkel,\(^3\) Note that this is similar to so-called Gamma strategies in derivatives pricing and risk management (see, e.g., Hull 2009, Chapter 17). However, while the Black-Scholes model can accommodate differences in expected returns, it does not allow for heterogeneous volatilities among investors on the underlying asset.

\(^3\) Note that this is similar to so-called Gamma strategies in derivatives pricing and risk management (see, e.g., Hull 2009, Chapter 17). However, while the Black-Scholes model can accommodate differences in expected returns, it does not allow for heterogeneous volatilities among investors on the underlying asset.
and Ohlson 1982). This is in contrast to the growing so-called differences-of-opinion literature in which the investors have homogeneous beliefs about the fundamentals in the economy, but disagree on how to interpret common public signals. This literature is mainly targeted towards explaining empirical stylized facts for the relationship between trading volume and stock returns, whereas our model allows us to investigate the relationship between the informativeness of the public information system and the equilibrium asset prices and investor welfare (in addition to trading volume).

If the investors have homogeneous prior dividend precisions, there will be no equilibrium trading at $t = 1$ contingent on the public signal. If they also have an identical prior mean, they hold on to the efficient risk sharing fraction of the risky asset after trading at $t = 0$, while disagreements about the mean and the associated efficient side-betting is facilitated by trading at $t = 0$ (as noted above). However, if the investors have heterogeneous prior dividend precisions, they update their posterior beliefs differently, and this gives the basis for additional trading gains contingent on the public signal. In particular, the equilibrium investor demand for the risky asset at $t = 1$ is an increasing (decreasing) function of the public signal for investors with a lower (higher) prior dividend precision than the investors’ average prior dividend precision. Since the public signal is equal to the dividend plus noise, investors with low (high) prior dividend precisions will, in equilibrium, achieve a payoff at $t = 2$ which is a convex (concave) function of the dividend on the risky asset. Hence, another round of trading in the risky asset (and the zero-coupon bond) contingent on the public information at $t = 1$ partly facilitates the efficient side-betting based on the heterogeneity in prior dividend precisions.

However, the investors’ equilibrium payoffs at $t = 2$ are also affected by the independent noise in the public signal, which implies that the additional side-betting opportunities come with a cost. Moreover, reducing the variance of the noise in the public signal (and, thus, increasing the signal precision) reduces the heterogeneity in the investors’ posterior beliefs as well as the risk premium in the equilibrium price of the risky asset. In the limit with a perfect public signal, there will be no equilibrium trading at $t = 1$, since the risky asset and the zero-coupon bond become perfect substitutes. Consequently, the trading gains decrease if the signal precision becomes too high. We show that the trading gains are maximized with an imperfect public information system with a signal precision equal to the investors’

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4 This assumption ensures that Pareto efficient allocations will only include side-betting on the public signal to the extent that it is informative about the fundamentals and not because it is informative about payoff-irrelevant events (see, e.g., the discussion in Christensen and Feltham 2003, Appendix 4A).

average prior dividend precision. This is also the information system which has the maximum expected abnormal trading volume at $t = 1$.

The trading gains following from an imperfect public signal at $t = 1$ translate directly into higher \textit{ex ante} certainty equivalents of the investors’ $t = 2$ consumption, and this reduces the demand for the zero-coupon bond at $t = 0$ and, thus, increases the equilibrium interest rate from $t = 0$ to $t = 2$.\footnote{We assume, for simplicity, that there is no consumption at the interim date $t = 1$ and, thus, only the equilibrium interest rate from $t = 0$ to $t = 2$ has any substance (and not how that interest rate is divided between the two periods).} Hence, the equilibrium interest rate is also maximized for the public information system with a signal precision equal to the investors’ average prior dividend precision. Since the aggregate consumption at $t = 0$ is equal to the exogenous $t = 0$ dividend on the risky asset, and the investors’ trading gains are maximized for this information system, this is also the \textit{unconstrained} Pareto preferred public information system.

However, the investors may not \textit{unanimously} prefer this system over public information systems with different signal precisions. Of course, it is voluntary for the investors to refrain from trading at $t = 1$, for example, an investor with a prior dividend precision equal to the investors’ average prior dividend precision does not engage in signal-contingent trading at $t = 1$. However, the equilibrium interest rate affects the equilibrium asset prices at $t = 0$ and, therefore, the equilibrium value of the investors’ endowments. A low asset price due to a high equilibrium interest rate is of course good if the investor wants to increase the holding of the asset at $t = 0$, but it is bad if the investor wants to reduce the holding of the asset. Hence, the individual investors’ preferences over public information systems depend on their trading gains (which in turn depend on the absolute difference between their personal prior dividend precision and the investors’ average prior dividend precision), and on their endowments of the zero-coupon bond and the risky asset relative to their equilibrium holdings of these assets at $t = 0$. We show how the investors’ endowments can be re-allocated (for example, due to a prior round of trading) such that all investors unanimously support the unconstrained Pareto efficient public information system.

In this paper, the heterogeneous prior beliefs are specified exogenously, and it is common knowledge that investors have different beliefs. However, our analysis can be extended to certain Hellwig-type noisy rational expectations equilibrium settings in which the heterogeneous beliefs are \textit{equilibrium} posterior beliefs resulting from an initial trading round based on homogeneous prior dividend beliefs, diverse private signals for a continuum of rational investors, and a noisy supply of the risky asset (see, e.g., Grundy and McNichols 1989, Kim and Verrecchia 1991a, Kim and Verrecchia 1991b, and Brennan and Cao 1996). It is well known that these models have a multiplicity of linear equilibria (while our model
has a unique equilibrium). Some of these equilibria are fully revealing following subsequent trading rounds based on independent public signals given the dividend (and, thus, do not involve any trading), while there is one linear equilibrium which is only partially revealing and, thus, involves non-trivial trading among rational investors. Of course, the former type of equilibria are deemed “unappealing” if trading volume is the subject under investigation and, thus, this literature focus on the latter.

The key property of the linear partially revealing rational expectations equilibrium is that the rational investors cannot make better inferences about the private information/noise relationship in the equilibrium price of the risky asset as subsequent public signals are released (since, otherwise, the equilibrium price would be fully revealing). This means that the investors react parametrically on equilibrium prices in subsequent trading rounds. Hence, it makes no difference for the impact of public information whether the heterogeneous prior beliefs are specified exogenously (as in our model) or these beliefs are equilibrium posterior beliefs following an initial trading round based on diverse private signals and a noisy supply. Consequently, the results we obtain for the impact of public information for efficient side-betting on trading volume are very similar to the corresponding results in this noisy rational expectations equilibrium literature.

The noisy rational expectations equilibrium literature relies on the introduction of un-modelled noise/liquidity trading. As pointed out by Cao and Ou-Yang (2009, page 303), a “potential problem with this approach is that the argument to explain trading volume is circular: it essentially requires new exogenous supply shocks to the stock to generate trading volume. In this sense, trading is imposed onto the economy rather than endogenously generated.” Furthermore, since these models are single-date consumption models, public information has no impact on ex ante risk premia and interest rates and, thus, no impact on the ex ante cost of capital and the ex ante stock price.

The rest of the paper is organized as follows. Section 2 presents the model and derives the equilibrium asset prices and asset demands in the incomplete market economy with the zero-coupon bond and the single risky asset as the only marketed securities. Section 3 establishes the relationship between the informativeness of the public information system and the equilibrium asset prices, the ex ante cost of capital, the expected abnormal trading volume, and the investors’ welfare in the incomplete market economy. The effectively complete market is introduced in Section 4. Section 5 concludes with some brief remarks on the empirical

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7This condition requires that the independent noise terms in the subsequent public signals must all be independent of the noise terms in the investors’ diverse private signals. Hence, these models do not allow for subsequent public signals being sufficient statistics for earlier private information with respect to the dividend as in the Grossman and Stiglitz type model of Demski and Feltham (1994), which in turn leads to homogeneous posterior beliefs and an efficient risk sharing following the public signal.
and policy implications of our analysis.

2 The Model

In our basic incomplete market model, we examine the impact of heterogeneity in prior beliefs and signal precision on equilibrium asset prices, trading volume, and investor welfare for a two-period economy in which investors have identical preferences but differ in their prior beliefs about the dividends on a single risky asset. The following two subsections describe the model and the equilibrium, respectively.

2.1 Investor Beliefs and Preferences

There are two consumption dates, $t = 0$ and $t = 2$, and there are $I$ investors who are endowed at $t = 0$ with a portfolio of securities, potentially receive public information at $t = 1$, and receive terminal normally-distributed dividends from their portfolio of securities at $t = 2$. The trading of the marketed securities takes place at $t = 0$ and $t = 1$ based on heterogeneous prior and posterior beliefs, respectively. There are two securities available for trade at $t = 0$ and $t = 1$: a zero-coupon bond that pays one unit of consumption at $t = 2$ and is in zero net-supply, and the shares of a single risky asset that has a fixed non-zero net-supply $Z$ throughout. The investors are endowed with $\gamma_i$ units of the $t = 2$ zero-coupon bond and $\bar{z}_i$ shares of the risky asset, $i = 1, 2, \ldots, I$. In addition, the investors are endowed with $\pi_i$ units of a zero-coupon bond, also in zero net-supply, paying one unit of consumption at $t = 0$. Let $\gamma_{it}$ and $x_{it}$ represent the units held by investor $i$ of the $t = 2$ zero-coupon bond and the risky asset after trading at date $t$, respectively. The market clearing conditions at date $t$ are

$$\sum_{i=1}^{I} \gamma_{it} = 0, \quad \sum_{i=1}^{I} x_{it} = Z, \quad t = 0, 1.$$ 

A share of the risky asset pays a dividend $d_0$ at date $t = 0$ and a dividend $d$ at date $t = 2$. We assume the investors have heterogeneous prior beliefs with respect to the $t = 2$ dividend represented by $\varphi_i(d) \sim N(m_i, \sigma_i^2)$, $i = 1, \ldots, I$, where $m_i$ is the expected dividend per share and $\sigma_i^2$ is the variance of the dividend per share for investor $i$.

At $t = 1$, all investors receive a public signal $y$ from an information system $\eta$, which is jointly normally distributed with the dividend paid by the risky asset at $t = 2$. The public signal is given as the dividend plus noise, i.e., $y = d + \varepsilon$, where $\varepsilon$ and $d$ are independent and $\varphi(\varepsilon) \sim N(0, \sigma_\varepsilon^2)$. We refer to $h_\varepsilon \equiv 1/\sigma_\varepsilon^2$ as the common signal precision, and we use $h_{(\cdot)} \equiv 1/\sigma_i^2$ throughout to denote precisions for the associated variances. Hence, while
the investors may disagree about the fundamentals in the economy (i.e., the dividends), we assume the investors have homogeneous beliefs about the noise in the information system, i.e., the investors have concordant beliefs (Milgrom and Stokey 1982) or homogeneous information beliefs (Hakansson, Kunkel, and Ohlson 1982). As noted in the Introduction, this is in contrast to, for example, Cao and Ou-Yang (2009), Banerjee and Kremer (2010), and Bloomfield and Fischer (2011), who assume that the investors have homogeneous beliefs about the fundamentals, i.e., dividends and earnings, but disagree on how to interpret public disclosures about these fundamentals. Our specification of the heterogeneity in beliefs allows us to ask how the informativeness of the public information, i.e., the signal precision $h$, affects the equilibrium asset prices, the trading volume, and the investors’ welfare.

The prior beliefs of investor $i$ for the public signal and the dividend is $\varphi_i(y, d) \sim N(\mu_i, \Sigma_i)$, where

$$\mu_i = \begin{pmatrix} m_i \\ m_i \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i^2 + \sigma_\varepsilon^2 & \sigma_i^2 \\ \sigma_i^2 & \sigma_i^2 \end{pmatrix}.$$ 

Hence, conditional on the public signal, the posterior beliefs of investor $i$ at $t = 1$ about the dividend is $\varphi_{i1}(d | y) \sim N(m_{i1}, \sigma_{i1}^2)$, where

$$m_{i1} = \omega_i y + (1 - \omega_i) m_i, \quad \omega_i = \frac{\sigma_i^2}{\sigma_i^2 + \sigma_\varepsilon^2}, \\
\sigma_{i1}^2 = \omega_i \sigma_\varepsilon^2, \quad h_{i1} = h_i + h_\varepsilon.$$ 

The posterior mean is a linear function of the investors’ signal, while the posterior variance only depends on the informativeness of the information system and not on the specific signal. Investor $i$’s prior distribution with respect to the posterior mean $m_{i1}$, i.e., the pre-posterior beliefs, is a normal distribution with a mean equal to the prior mean $m_i$ of the dividend and variance $\sigma_{i0}^2 = \sigma_i^2 - \sigma_{i1}^2$, i.e., $\varphi(m_{i1}) \sim N(m_i, \sigma_{i0}^2)$.

The investors trade in the zero-coupon bond with equilibrium price $\beta_0$ at $t = 0$ and $\beta_1$ at $t = 1$. We assume without loss of generality that $\beta_1 = 1$ since there is no consumption at $t = 1$. The equilibrium price of the risky asset at $t = 0$ is denoted $p_0(\eta)$, which reflects the fact that the ex ante price at $t = 0$ may be affected by the public information system $\eta$. The ex post equilibrium price of the risky asset at $t = 1$ given the public signal $y$ is denoted $p_1(y)$.

Investor $i$’s consumption at date $t = 0$ and $t = 2$ is denoted $c_{it}$ and we assume the investors have time-additive utility. The investors have common period-specific exponential utility functions, i.e., $u_{i0}(c_{i0}) = -\exp[-rc_{i0}]$ and $u_{i2}(c_{i2}) = -\exp[-\delta] \exp[-rc_{i2}]$, where $r > 0$ is the investors’ common constant absolute risk aversion parameter, and $\delta$ is the common utility discount rate for date $t = 2$ consumption. Our results are qualitatively
unaffect

d by allowing investors to have different risk aversion parameters and different utility discount rates.

2.2 Equilibrium with Public Information and Heterogeneous Beliefs

In this section, we derive the equilibrium in the economy with heterogeneous beliefs, public information and trading in the zero-coupon bond and the single risky asset. There are two rounds of trading: one round of trading at \( t = 0 \) prior to the release of information, and a second round of trading subsequent to the release of the public signal at \( t = 1 \). We solve for the equilibrium by first deriving the equilibrium prices at \( t = 1 \), and given this equilibrium, we can subsequently derive the equilibrium prices at \( t = 0 \).

2.2.1 Equilibrium prices at date \( t = 1 \)

From the perspective of \( t = 1 \), date \( t = 2 \) consumption for investor \( i \) is

\[
c_{i2} = x_{i1}d + \gamma_{i1},
\]

and is thus normally distributed given the public signal \( y \) at \( t = 1 \). Investor \( i \) maximizes his certainty equivalent of \( t = 2 \) consumption subject to his budget constraint, and given period-specific exponential utility this can be expressed as

\[
\max_{x_{i1}, \gamma_{i1}} \text{CE}_{i2} (x_{i1}, \gamma_{i1} \mid y, \gamma_{i0}, x_{i0})
\]

\[
= \max_{x_{i1}, \gamma_{i1}} \gamma_{i1} + m_{i1}x_{i1} - \frac{1}{2}r\sigma_{i1}^2x_{i1}^2,
\]

subject to \( \gamma_{i1} + p_1(y)x_{i1} \leq \gamma_{i0} + p_1(y)x_{i0} \).

The first-order conditions imply that the optimal portfolio at \( t = 1 \) given investor \( i \)’s posterior beliefs is

\[
x_{i1}(y) = \rho h_{i1} (m_{i1}(y) - p_1(y)) , \quad (2a)
\]

\[
\gamma_{i1}(y) = \gamma_{i0} + p_1(y)x_{i0} - p_1(y)x_{i1}(y) , \quad (2b)
\]

where \( \rho \equiv 1/r \) is the investors’ common risk tolerance and \( h_{i1} = h_i + h_\varepsilon \) is investor \( i \)’s
posterior precision for the terminal dividend. Market clearing at date $t = 1$ implies that

$$
\sum_{i=1}^{I} \rho h_{i1} (m_{i1}(y) - p_1(y)) = Z \Leftrightarrow \quad p_1(y) = \bar{m}_1^h(y) - r\bar{\sigma}_1^2Z/I,
$$

where $\bar{m}_1^h(y)$ is the precision weighted average of the investors’ posterior means, i.e.,

$$
\bar{m}_1^h(y) = \frac{1}{I} \sum_{i=1}^{I} \frac{h_{i1}}{\bar{h}_1} m_{i1}(y), \quad \bar{h}_1 = \frac{1}{I} \sum_{i=1}^{I} h_{i1},
$$

and $\bar{\sigma}_1^2$ is the inverse of the average posterior precision, i.e., $\bar{\sigma}_1^2 = 1/\bar{h}_1$.

Inserting the equilibrium price of the risky asset into investor $i$’s demand function in (2a) yields

$$
x_{i1}^*(y) = \rho h_{i1} (m_{i1}(y) - \left[\bar{m}_1^h(y) - r\bar{\sigma}_1^2Z/I\right]).
$$

The posterior beliefs, i.e., $m_{i1}(y)$ and $h_{i1}$, are functions of the priors and the signal precision. Hence, the equilibrium price of the risky asset and the equilibrium demand functions at date $t = 1$ are affected by both the priors, the signal precision and, moreover, they are linear functions of the public signal (through the posterior means, $m_{i1} = \omega_i y + (1 - \omega_i) m_i$), which implies that, in general, there is non-trivial trading at $t = 1$ in equilibrium. Note, however, that if the investors have homogeneous prior precisions (such that $\omega_i = \omega$ and $h_{i1} = \bar{h}_1$ for all $i$), the equilibrium demand is independent of the public signal.

Consider the two extreme cases for the signal precision separately. If the public signal is a perfect signal of the dividend, i.e., $h_{\varepsilon} \to \infty$ ($\sigma_{\varepsilon}^2 = 0$), the investors get to know the realization of the dividend already at $t = 1$ before any second-round trading can occur. In this case, no arbitrage implies that

$$
p_1(y) = y = d.
$$

That is, the equilibrium asset price at $t = 1$ is equal to the dividend and, thus, independent of the prior beliefs (recall that the equilibrium interest rate from $t = 1$ to $t = 2$ is normalized to zero).

When the signal tends to be uninformative, i.e., $h_{\varepsilon} = 0$ ($\sigma_{\varepsilon}^2 \to \infty$), the posterior beliefs are equal to the prior beliefs and, thus,

$$
p_1(y) = \bar{m}_1^h - r\bar{\sigma}_1^2Z/I,
$$
where
\[ m^h = \frac{1}{I} \sum_{i=1}^{I} \frac{h_i}{\bar{h}} m_i, \quad \bar{h} = \frac{1}{I} \sum_{i=1}^{I} h_i, \quad \sigma^2 = \frac{1}{\bar{h}}. \]

In this case, the ex post asset price is, of course, independent of the signal but is a function of the priors; and it is given as a precision weighted average of the investors’ prior mean minus a risk premium determined by the average prior precision. Moreover, with homogeneous prior beliefs, i.e., \( m_i = m \), and \( \sigma_i^2 = \sigma^2 \), for \( i = 1, 2, ..., I \), we have
\[ p_1(y) = m - r\sigma^2 Z/I, \]
which is the standard no-information exponential-utility/normal-distribution version of the CAPM.

### 2.2.2 Equilibrium prices at date \( t = 0 \)

We now determine the equilibrium ex ante prices and demand functions at \( t = 0 \), taking the equilibrium at \( t = 1 \) characterized by Equations (3) and (4) as given. From the perspective of \( t = 0 \), investor \( i \)’s date \( t = 2 \) consumption is
\[ c_{i2} = [d - p_1(y)] x_{i1}^*(y) + p_1(y) x_{i0} + \gamma_{i0}, \]
and investor \( i \)’s date \( t = 0 \) consumption is
\[ c_{i0} = [p_0(\eta) + d_0] z_i + \beta_0 \bar{y}_i + \bar{\gamma}_i - p_0(\eta) x_{i0} - \beta_0 \gamma_{i0}. \]

Conditional on the public signal at \( t = 1 \), investor \( i \)’s \( t = 1 \) certainty equivalent of \( t = 2 \) consumption is
\[ \text{CE}_{i2} (x_{i0}, \gamma_{i0}, x_{i1}^*(y) \mid y) = \gamma_{i0} + p_1(y) x_{i0} + [m_{i1} - p_1(y)] x_{i1}^*(y) - \frac{1}{2} r \sigma_{i1}^2 (x_{i1}^*(y))^2. \] (5)

Note that from the perspective of \( t = 0 \), the second term in \( \text{CE}_{i2} (x_{i0}, \gamma_{i0}, x_{i1}^*(y) \mid y) \) is a normally-distributed variable, while the last two terms contain products of normally-distributed variables if \( x_{i1}^*(y) \) varies with the public signal at \( t = 1 \) (in which case it is a non-degenerate normally distributed variable). Substituting in the equilibrium demand functions and the equilibrium price of the risky asset at \( t = 1 \), i.e., equations (4) and (3), yields the following result.
Remark 1  Investor $i$’s $t = 0$ certainty equivalent of $t = 2$ consumption is given by

$$CE_{i2}(x_{i0}, \gamma_{i0}) = \gamma_{i0} + U_{1i} + U_{2i} + M_i x_{i0} - \frac{1}{2} r V_i x_{i0}^2, \quad (6)$$

where

$$U_{1i} = \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_\varepsilon)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right], \quad (7a)$$

$$U_{2i} = \frac{1}{2} \frac{h_i [m_i \bar{h} - \bar{h} m_i + r Z/I]^2}{h_i^2 + h_\varepsilon h_i}, \quad (7b)$$

$$M_i = \frac{h_\varepsilon h_i m_i + \bar{h}^2 m_i - r \bar{h} Z/I}{h_i^2 + h_\varepsilon h_i}, \quad (7c)$$

$$V_i = \frac{h_\varepsilon}{h_i^2 + h_\varepsilon h_i}. \quad (7d)$$

The certainty equivalent $CE_{i2}(x_{i0}, \gamma_{i0})$ can be expressed as a constant, i.e., $\gamma_{i0} + U_{1i} + U_{2i}$, plus the certainty equivalent of $x_{i0}$ units of a normally-distributed dividend with mean $M_i$ and variance $V_i$. Since there are no wealth effects with exponential utility, the investor’s demand at $t = 0$ for the risky asset is the same as in a single-period model with this prior mean and variance of a normally distributed dividend. However, note that these priors reflect that there will be a second round of trading at $t = 1$ based on the public signal. The term $U_{1i}$ is a function of the signal precision, but as we shall see below (as part of Proposition 2), in equilibrium, the term $U_{2i} + M_i x_{i0} - \frac{1}{2} r V_i x_{i0}^2$ is independent of the signal precision. Thus, the signal precision affects the equilibrium prices and the equilibrium investor welfare only through the terms $U_{1i}$.

With the investors’ $t = 0$ certainty equivalent of their $t = 2$ consumption determined, investor $i$’s decision problem at $t = 0$ can be stated as follows

$$\max_{\gamma_{i0}, x_{i0}} - \exp(-r CE_{i0}(x_{i0}, \gamma_{i0})) - \exp(-\delta) \exp(-r CE_{i2}(x_{i0}, \gamma_{i0})), \quad (8)$$

where

$$CE_{i0}(x_{i0}, \gamma_{i0}) = [p_0(\eta) + d_0] \bar{z}_i + \beta_0 \bar{\gamma}_i + \bar{\kappa}_i - p_0(\eta) x_{i0} - \beta_0 \gamma_{i0}. \quad (9)$$

The first-order condition for investments in the zero-coupon bond is

$$-r \exp(-r CE_{i0}(x_{i0}, \gamma_{i0})) \beta_0 + r \exp(-\delta) \exp(-r CE_{i2}(x_{i0}, \gamma_{i0})) = 0 \iff \quad (8)$$

$$t = \delta + r (CE_{i2}(x_{i0}, \gamma_{i0}) - CE_{i0}(x_{i0}, \gamma_{i0})), \quad (10)$$
where \( \ell \equiv -\ln \beta_0 \) is the zero-coupon interest rate from \( t = 0 \) to \( t = 2 \). The first-order condition for investments in the risky asset is

\[-r \exp\left(-rCE_{x_0}(x_{i0}, \gamma_{i0})\right)p_0(\eta) + r \exp(-\delta)\exp\left(-rCE_{x_2}(x_{i0}, \gamma_{i0})\right)\left[M_i - rV_i x_{i0}\right] = 0.\]

Hence, the \textit{ex ante} price and the demand for the risky asset at \( t = 0 \) can be expressed as

\[
p_0(\eta) = \beta_0 \left[M_i - rV_i x_{i0}\right],
\]
\[
x_{i0} = \rho \frac{M_i - R_0 p_0(\eta)}{V_i},
\]

where \( R_0 = 1/\beta_0 \). Thus, the market clearing condition for the risky asset implies that its equilibrium price at \( t = 0 \) is

\[
\sum_{i=1}^{I} x_{i0} = Z \iff p_0(\eta) = \beta_0 \left[\overline{M'} - r\overline{V} Z / I\right],
\]

where

\[
\overline{M'} \equiv \frac{1}{I} \sum_{i=1}^{I} \frac{v_i}{\overline{v}} M_i, \quad v_i \equiv V_i^{-1}, \quad \overline{v} \equiv \frac{1}{I} \sum_{i=1}^{I} v_i, \quad \overline{V} \equiv \overline{v}^{-1}.
\]

In other words, the equilibrium price of the risky asset is equal to its discounted “risk-adjusted expected dividend,” where the latter is defined as

\[
\mathbb{E}^Q[d] \equiv \overline{M'} - r\overline{V} Z / I.
\]

The following proposition shows properties of the risk-adjusted expected dividend.

**Proposition 1** The \textit{ex ante} equilibrium price of the risky asset at \( t = 0 \) is equal to the equilibrium riskless discount factor times the risk-adjusted expected dividend, i.e.,

\[
p_0(\eta) = \beta_0 \mathbb{E}^Q[d].
\]

The risk-adjusted expected dividend is independent of the information system, and it can be
expressed as a function of the prior means and variances, i.e.,

\[ E^Q[d] = \bar{m}^h - r\sigma^2 Z/I. \] (12)

Hence, given the priors, the risk-adjusted expected dividend is independent of the information system at \( t = 1 \) and, in particular, it is determined entirely by the prior beliefs as if there would be no second round of trading at \( t = 1 \). In other words, the informativeness of the public signal at \( t = 1 \) affects the \textit{ex ante} equilibrium asset price only through the impact on the equilibrium interest rate.

Substituting the \textit{ex ante} equilibrium price of the risky asset (11) into the demand functions (9b), we obtain the investors’ equilibrium demand for the risky asset at \( t = 0 \):

\[ x^*_i = \rho V^{-1}_i [M_i - E^Q[d]]. \] (13)

Substitution of \( M_i, V_i \) and \( E^Q[d] \), and simplifying yield the following result.

**Remark 2** In equilibrium, investor \( i \)'s \( t = 0 \) equilibrium demand for the risky asset is given by

\[ x^*_i = \rho h_i [m_i - E^Q[d]]. \] (14)

Note that the equilibrium demand for the risky asset is the same as in an otherwise identical economy in which there is no public information at \( t = 1 \). In other words, the investors’ equilibrium demands are \textit{myopic}, independently of the informativeness of the forthcoming public signal. The equilibrium demand is increasing in the investors’ prior mean and in the prior dividend precision such that the more optimistic and confident investors invest more in the risky asset than the more pessimistic and less confident investors. This result is a consequence of the investors’ incentive to take speculative positions based on their heterogeneous prior beliefs and, thus, the equilibrium entails “side-betting.” With homogeneous priors, however, all investors hold the same efficient risk sharing equilibrium positions in the risky asset, i.e., \( x^*_{i0} = Z/I \).

---

\(^8\)This means that we can define the risk-adjusted probability measure \( Q \) explicitly such that under \( Q \), the terminal dividend is normally distributed as \( d \sim N(\bar{m}^h - r\sigma^2 Z/I, \sigma^2) \), and the noise \( \varepsilon \) is normally distributed as \( \varepsilon \sim N(0, \sigma^2) \). Note that while the expected dividend under \( Q \) is uniquely determined in equilibrium, the variance of the dividend under \( Q \) is not uniquely determined due to the market incompleteness and, thus, we just take it to be \( \sigma^2 \). Fortunately, the lack of the uniqueness of the variance has no consequences in the subsequent analysis.
Substituting the equilibrium portfolios into the certainty equivalents, we get

\[
\begin{align*}
CE_{i0}^* &= [p_0(\eta) + d_0] \tilde{z}_i + \beta_0 \gamma_i + \pi_i - p_0(\eta)x_{i0}^* - \beta_0 \gamma_{i0}^*, \\
CE_{i2}^* &= \gamma_{i0}^* + U_{1i} + U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2.
\end{align*}
\tag{15a}
\tag{15b}
\]

Substituting the equilibrium certainty equivalents into the expression for the interest rate (8), we obtain

\[
\iota = \delta + r (CE_{i2}^* - CE_{i0}^*). 
\tag{16}
\]

Using the market clearing conditions for the riskless and risky asset, and simplifying yield the equilibrium interest rate.

**Proposition 2** The equilibrium interest rate is given by

\[
\iota = \delta + r \left( \frac{m_i}{I} + \Phi \left( \{m_i, \sigma_i^2\}_{i=1}^{I} \right) \right),
\tag{17}
\]

where

\[
\frac{m_i}{I} \equiv \frac{1}{I} \sum_{i=1}^{I} U_{1i} = \frac{1}{2} \rho \sum_{i=1}^{I} \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_{\varepsilon}}{(\bar{h} + h_{\varepsilon})^2} \right],
\tag{18}
\]

and \( \Phi (\cdot) \) is a function of the priors but independent of the signal precision,

\[
\Phi \left( \{m_i, \sigma_i^2\}_{i=1}^{I} \right) = r \left[ m - d_0 \right] Z/I - \frac{1}{2} r^2 \sigma^2 (Z/I)^2 + \frac{1}{2} \sum_{i=1}^{I} h_i m_i^2 - \frac{1}{2} (m^h)^2 \bar{h}.
\tag{19}
\]

If the investors have homogeneous prior expected dividends, i.e., \( m_i = m \), then

\[
\Phi \left( \{m_i, \sigma_i^2\}_{i=1}^{I} \right) = r \left[ m - d_0 \right] Z/I - \frac{1}{2} r^2 \sigma^2 (Z/I)^2.
\tag{20}
\]

If the investors have homogeneous prior dividend precisions, the equilibrium interest rate is independent of the signal precision.

The equilibrium interest rate is equal to the utility discount rate plus a function of the signal precision and the priors. The function \( \Phi (\cdot) \) is a function of the priors only and, thus, independent of the information system. Hence, the signal precision only affects the equilibrium interest rate and, thus, the equilibrium price of the risky asset (since \( E^Q [d] \) is independent of \( h_{\varepsilon} \) by Proposition 1), through the logarithmic terms \( \{U_{1i}\}_{i=1}^{I} \).

If the investors hold homogenous prior precisions (i.e., \( h_i = \bar{h} \) for all \( i \)), the logarithmic terms are all equal to zero. Thus, in this case the signal precision does not affect the
equilibrium prices at $t = 0$. Moreover, when $m_i = m$, and $\sigma_i^2 = \sigma^2$, for $i = 1, 2, ..., I$, the equilibrium interest rate can be expressed as

$$\iota = \delta + r (m - d_0) Z/I - \frac{1}{2} r^2 \sigma^2 (Z/I)^2.$$ 

Hence, in a benchmark setting with homogeneous prior beliefs, the equilibrium interest rate is given as the utility discount rate plus a risk-adjusted expected dividend growth minus a risk premium for the uncertainty in the dividend growth. Of course, this is the standard expression for the equilibrium interest rate in effectively complete markets with time-additive HARA utilities and homogeneous prior beliefs (see, e.g., Christensen and Feltham 2009). On the other hand, if the investors have homogeneous prior expected dividends, but heterogeneous prior dividend precisions, then there is an additional component to the equilibrium interest rate, i.e.,

$$\iota = \delta + r \bar{U}_1 + r [m - d_0] Z/I - \frac{1}{2} r^2 \sigma^2 (Z/I)^2.$$ 

This additional component, i.e., $r \bar{U}_1$, depends on the signal precision, and it plays a key role in the following analysis.

3 The Impact of Signal Precision on Ex Ante Asset Prices, Trading Volume, and Investor Welfare

We are interested in how the informativeness of the public information system, i.e., the signal precision, affects the ex ante equilibrium prices, the trading volume, and the investors’ ex ante expected utilities at $t = 0$ when the investors hold heterogeneous beliefs including heterogeneous prior means and/or heterogeneous prior dividend precisions. We first examine the impact on the ex ante equilibrium prices and the trading volume.

3.1 Ex ante Equilibrium Prices and Trading Volume

Proposition 1 establishes that the equilibrium asset prices at $t = 0$ are only affected by the signal precision through the equilibrium interest rate. Furthermore, Proposition 2 establishes that the equilibrium interest rate is also independent of the signal precision if the investors hold homogeneous prior dividend precisions. This is due to the fact that in this case there is no equilibrium trading at $t = 1$ based on the public signal (see, e.g., Grundy and McNichols 1989 for a similar result).

**Proposition 3** When the investors hold identical prior dividend precisions, i.e., $h_i = h, i = 1, ..., I$, the date $t = 1$ equilibrium portfolios are independent of the information system and
equal to the date $t = 0$ equilibrium portfolios, i.e.,

$$x_{11}^*(y) = x_{i0}^*, \quad \gamma_{11}^* = \gamma_{i0}^*.$$ 

With heterogeneous prior dividend precisions, however, the signal precision plays a key role in determining the equilibrium interest rate and, thus, the equilibrium price of the risky asset at $t = 0$. As noted above, the impact of the signal precision on the equilibrium interest rate is only through the logarithmic terms in (17). The following proposition characterizes the equilibrium interest rate as a function of the signal precision.

**Proposition 4** Assume the investors have heterogeneous prior dividend precisions. The equilibrium interest rate is bell-shaped with respect to the signal precision $h_\epsilon$. The unique maximum for the equilibrium interest rate is attained when $h_\epsilon = \bar{h}$, and its minimum is attained for uninformative information ($h_\epsilon = 0$) and for perfect information ($h_\epsilon \to \infty$).

The intuition for the result in Proposition 4 can be obtained from equation (16), in which the interest rate is expressed as a linear increasing function of the growth in the investors’ certainty equivalents, $CE_{i2}^* - CE_{i0}^*$. In equilibrium, all investors have the same growth in certainty equivalents. For the two extreme values of the signal precision ($h_\epsilon = 0$ and $h_\epsilon \to \infty$) there is no trading at $t = 1$ based on the public signal: (a) for $h_\epsilon = 0$, no new information is released at $t = 1$ and, thus, the equilibrium portfolios after trading at $t = 0$ remain equilibrium portfolios; and (b) when the signal precision increases, the investors’ posterior beliefs converge and the risk premium in the equilibrium price of the risky asset decreases, and in the limit for $h_\epsilon \to \infty$ all uncertainty is resolved at $t = 1$ and, thus, there is no basis for additional trading. On the other hand, for intermediate values of the signal precision ($h_\epsilon \in (0, \infty)$) there is non-trivial trading based on the public signal at $t = 1$ if the investors have heterogeneous prior dividend precisions. The source of this trading is that the investors can achieve improved side-betting based on their heterogeneously updated posterior beliefs. These gains to trade translate directly into increased certainty equivalents of $t = 2$ consumption and, thus, a higher growth in their certainty equivalents, $ctertis paribus$. A highly informative or an almost uninformative public signal at $t = 1$ yields only limited side-betting benefits and, thus, the highest growth in certainty equivalents is obtained for a unique interior signal precision $h_\epsilon = \bar{h}$. The equilibrium price of the risky asset is the

Note the equilibrium interest rate looks bell-shaped after the transformation $x = \ln(1 + h_\epsilon \cdot 1.5E+07)$. However, the equilibrium interest rate has actually only one inflection point with respect to the signal precision, i.e., $h_{ip} = 2\bar{h}$. When $h_\epsilon \leq h_{ip}$ ($h_\epsilon \geq h_{ip}$), the second derivative of the equilibrium interest rate with respect to the signal precision is negative (positive) and, thus, the equilibrium interest rate is concave (convex) with respect to the signal precision as the signal precision increases.
product of the equilibrium riskless discount factor and the risk-adjusted expected dividend (which is independent of $h_\varepsilon$ by Proposition 1) and, thus, the equilibrium price of the risky asset is inverted bell-shaped as a function of $h_\varepsilon$ with a minimum point at $h_\varepsilon = \bar{h}$.

**Ex ante cost of capital**

The *ex ante* cost of capital defined as the expected rate of return on the risky asset is an ambiguous concept in a setting in which the investors have heterogeneous prior means for the dividend on the risky asset. However, we can define the *ex ante* cost of capital as the (continuously compounded) expected rate of return $\mu^{xa}(\eta)$ using the beliefs implicit in the unambiguous *ex ante* equilibrium price of the risky asset, i.e., $\phi^h(d) \sim N(\bar{m}^h, \bar{\sigma}^2)$,

$$\exp(\mu^{xa}(\eta)) = \frac{\bar{m}^h}{p_0(\eta)}.$$  \hspace{1cm} (21)

Inserting the *ex ante* equilibrium price of the risky asset (11), and using Proposition 1 we get that

$$\mu^{xa}(\eta) = \iota + \omega^{xa},$$

where the risk premium $\omega^{xa}$ is given by

$$\omega^{xa} = \ln \left( 1 + \frac{r\bar{\sigma}^2 Z/I}{\bar{m}^h - r\bar{\sigma}^2 Z/I} \right).$$

Hence, the *ex ante* cost of capital for the risky asset $\mu^{xa}(\eta)$ is equal to the equilibrium interest rate plus a risk premium $\omega^{xa}$, which is independent of the informativeness of the public signal.\(^{10}\) Propositions 2 and 4 then imply that the *ex ante* cost of capital is minimized for no public information ($h_\varepsilon = 0$) and for perfect public information ($h_\varepsilon \to \infty$), while it is maximized for a unique interior signal precision $h_\varepsilon = \bar{h}$. Is a low *ex ante* cost of capital good or bad for the investors? We address this question in the following subsection.

In order to illustrate our results we use the following three-investor example throughout with the parameters given in Table 1.

\(^{10}\)Similarly, if we define investor-specific expected rates of returns based on their prior dividend beliefs, i.e., $\exp(\mu_i^{xa}(\eta)) \equiv m_i/p_0(\eta)$, then the investor-specific “equity premiums,” i.e., $\omega_i^{xa} \equiv \mu_i^{xa}(\eta) - \iota = \ln \left( m_i / \left[ \bar{m}^h - r\bar{\sigma}^2 Z/I \right] \right)$, are also independent of the informativeness of the public signal.
Figure 1: Equilibrium interest rate, risk premium, and ex ante cost of capital as functions of the signal precision $h_c$ given the parameters in Table 1. The scale on the horizontal axis is $x = \ln (1 + h_c \cdot 1.5E+07)$.

<table>
<thead>
<tr>
<th></th>
<th>Investor 1</th>
<th>Investor 2</th>
<th>Investor 3</th>
<th>Aggregate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk aversion ($r$)</td>
<td>0.01%</td>
<td>0.01%</td>
<td>0.01%</td>
<td></td>
</tr>
<tr>
<td>Utility discount rate ($\delta$)</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td></td>
</tr>
<tr>
<td>Prior mean ($m_i$)</td>
<td>800</td>
<td>1,000</td>
<td>1,200</td>
<td></td>
</tr>
<tr>
<td>Prior variance ($\sigma_i^2$)</td>
<td>25,000</td>
<td>37,500</td>
<td>75,000</td>
<td></td>
</tr>
<tr>
<td>Initial dividend ($d_0$)</td>
<td></td>
<td></td>
<td></td>
<td>950</td>
</tr>
<tr>
<td>Supply ($Z$)</td>
<td></td>
<td></td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>

Table 1: Investor and risky asset parameters of the running example.

Figure 1 illustrates the equilibrium interest rate, the risk premium, and the ex ante cost of capital as functions of the signal precision for the parameters in Table 1. Note that these quantities are independent of the investors’ individual endowments and, thus, even though the investors have heterogeneous beliefs, the equilibrium admits aggregation as with HARA utilities in effectively complete markets with homogeneous beliefs.

Trading Volume

The source of the increased growth in the investors’ certainty equivalents is the trading gains based on the investors’ heterogeneously updated posterior beliefs. In this section we demonstrate that the signal precision, which maximizes the trading gains and, thus, the equilibrium interest rate, also maximizes the expected abnormal trading volume.

Investor $i$’s equilibrium net-trade in the risky asset at $t = 1$ is $\tau_i^*(y) \equiv x_{i1}^*(y) - x_{i0}^*$, where $x_{i1}^*(y)$ is given in (4) and $x_{i0}^*$ is given in (14). Inserting the definitions of the posterior means
and precisions in (4), and simplifying yield the following result (see Kim and Verrecchia 1991a and 1991b for a similar result).

**Remark 3** Investor $i$’s equilibrium net-trade in the risky asset at $t = 1$ is given by

$$
\tau^*_i(y) = \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} \left[ y - E^Q[d] \right],
$$

(22)

and the risk-adjusted expected net-trade is equal to zero, i.e.,

$$
E^Q[\tau^*_i(y)] = 0.
$$

Hence, the sensitivity of the investor’s equilibrium net-trade increases with the difference between the investor’s prior dividend precision $h_i$ and the average prior dividend precision $\bar{h}$. Furthermore, the equilibrium net-trade of the investors with lower (higher) prior dividend precisions than the average is increasing (decreasing) in $y = d + \varepsilon$ and, thus, their $t = 2$ consumption is a convex (concave) function of $d$. This relationship between the public signal and $t = 2$ consumption is the source of the improved side-betting opportunities following from the fact that investors with low prior dividend precisions value convex payoffs more than investors with high prior dividend precisions.

Even though the risk-adjusted expected net-trade is equal to zero, the investors’ expected net-trade is *not* equal to zero, and it depends on their personal prior dividend beliefs. Therefore, in order to investigate the impact of the signal precision on the expected trading volume in the securities market as a whole, we define the *abnormal* net-trade of investor $i$ as the difference between the net-trade and the expected net-trade conditional on the $t = 2$ dividend, i.e.,

$$
a\tau^*_i(y) \equiv \tau^*_i(y) - E[\tau^*_i(y) \mid d] = \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} \varepsilon.
$$

Since the investors have concordant beliefs, they have homogeneous beliefs about their abnormal net-trades and, in particular, the abnormal net-trades are normally distributed with a zero mean. Recognizing that some investors are selling while others are buying, the abnormal trading volume per investor is defined as

$$
T^* \equiv \frac{1}{2} \sum_{i=1}^I \left| a\tau^*_i(y) \right|.
$$

The following proposition characterizes the expected abnormal trading volume.
Proposition 5 The expected abnormal trading volume is

\[ E[T^*] = \frac{\sqrt{h^*_\varepsilon}}{\sqrt{h^*_\varepsilon + h^*_\varepsilon}} \cdot \frac{\rho}{2\pi} \cdot \frac{1}{\sqrt{T}} \sum_{i=1}^{T} \left| \bar{h}_i - h_i \right|. \]

Assume the investors have heterogeneous prior dividend precisions. The expected abnormal trading volume is bell-shaped with respect to the signal precision \( h^*_\varepsilon \). The unique maximum for the expected abnormal trading volume is attained when \( h^*_\varepsilon = \bar{h}_i \), and its minimum is attained for uninformative information (\( h^*_\varepsilon = 0 \)) and for perfect information (\( h^*_\varepsilon \to \infty \)).

The proposition establishes that the expected abnormal trading volume has the same comparative statics as the equilibrium interest rate with respect to the signal precision, cf. Proposition 4.\(^{11}\) Of course, the key empirical implication is that there is a direct positive relationship between the empirically unobservable growth in certainty equivalents and the expected (average) abnormal trading volume.

3.2 Ex ante Expected Utilities

The investors’ ex ante expected utilities are affected in two ways by changes in the signal precision. First, changes in the signal precision affects the gains to trade based on heterogeneously updated posterior beliefs and, thus, the growth in their certainty equivalents as illustrated in the preceding subsection. Secondly, the signal precision affects the ex ante equilibrium asset prices through the equilibrium interest rate and, thus, affects the value of the investors’ individual endowments. The latter may affect the investors in different ways depending on their individual endowments relative to their equilibrium portfolio at \( t = 0 \). The following lemma (partly) characterizes the impact of changing the signal precision on the investors’ ex ante expected utilities.

Lemma 1 The derivative of the investors’ ex ante expected utilities with respect the signal precision \( h^*_\varepsilon \) is given by

\[ \frac{\partial}{\partial h^*_\varepsilon} E[U^*_i] = r \exp \left( -rC_E^*_i \right) \left\{ \beta^*_0 \frac{\partial}{\partial h^*_\varepsilon} U^*_i + \left[ \gamma^*_i - \bar{x}_i + E^Q[d] \left( x^*_i - \bar{z}_i \right) \right] \frac{\partial}{\partial h^*_\varepsilon} \beta^*_0 \right\}, \]

where

\[ \frac{\partial}{\partial h^*_\varepsilon} \beta^*_0 = -r \beta^*_0 \frac{\partial}{\partial h^*_\varepsilon} U^*_1. \]

All the investors’ ex ante expected utilities have a stationary point at \( h^*_\varepsilon = \bar{h}_i \).

\(^{11}\) Consequently, we do not present a figure of the expected abnormal trading volume as a function of the signal precision for the numerical example.
The gains to trade based on the public signal is reflected in the first term in the braces of (23), \( \frac{\partial}{\partial h} U_{1i} \), where \( U_{1i} \) is investor \( i \)'s \textit{ex ante} value of the trading gains at \( t = 1 \) and is given in (7a). Of course, the investors can always refuse to engage in any second-round trading at \( t = 1 \) and, thus, the \textit{ex ante} value of trading gains is always non-negative, and it is maximized for \( h_{\varepsilon} = \bar{h} \). Note that an investor with a prior dividend precision \( h_i \) equal to the average dividend precision \( \bar{h} \) has an \textit{ex ante} value of the trading gains equal to zero, i.e., \( U_{1i} = 0 \). This is due to the fact that the investor does not engage in signal-contingent trading at \( t = 1 \) if \( h_i = \bar{h} \) (see equation (22)). On the other hand, all “more confident” and “less confident” investors than the average have a strictly positive \textit{ex ante} value of trading gains, and these \textit{ex ante} values are maximized for all investors exactly at the signal precision at which the equilibrium interest rate is maximized, \( h_{\varepsilon} = \bar{h} \).

The positive \textit{ex ante} values of trading gains (which will be realized when consuming at \( t = 2 \)) shift the investors’ incentive to consume more at \( t = 0 \) (in order to smooth consumption over time) and, thus, increase the equilibrium interest rate. This reduces the value of the investors’ (positive) endowments of the \( t = 2 \) zero-coupon bond and the risky asset through a reduction in the zero-coupon bond price, \( \beta_0 \) (recall that the risk-adjusted expected dividend of the risky asset is independent of the signal precision). This reduction is maximized when the average \textit{ex ante} value of the trading gains \( U_1 \) is maximized at \( h_{\varepsilon} = \bar{h} \). However, the reduction in the asset prices also makes it cheaper to buy these assets. Hence, the impact of the changed equilibrium prices on the investors’ consumption possibilities at \( t = 0 \) depends on whether the investor wants to increase or decrease the holdings of the assets, i.e., whether \( (\gamma_{i0} - \bar{\gamma}_i) \) and \( (x_{i0} - \bar{x}_i) \) are positive or negative. These effects are reflected in the second term in the braces of (23). Hence, in general, even though the signal precision has a unique impact on the \textit{ex ante} value of trading gains and on the equilibrium interest rate, the investors’ \textit{ex ante} expected utilities may be affected in different directions by changes in the signal precision. In other words, changes in the information system may not result in Pareto improvements or Pareto inferior allocations—some investors may be better off while other investors may be worse off depending on their endowments relative to their equilibrium asset holdings at \( t = 0 \).

In order to illustrate these effects of changes in the signal precision on the \textit{ex ante} expected utilities, we define investor \( i \)'s equilibrium \textit{ex ante} certainty equivalent \( CE_{i}^{xa}(h_{\varepsilon}) \) as

\[
- \exp (-rCE_{i}^{xa}(h_{\varepsilon})) \equiv - \exp (-rCE_{i0}^{*}) - \exp (-\delta) \exp (-rCE_{i2}^{*}).
\]

Of course, there is a positive one-to-one relationship between investor \( i \)'s \textit{ex ante} expected utility and his equilibrium \textit{ex ante} certainty equivalent \( CE_{i}^{xa}(h_{\varepsilon}) \). Hence, the change in the
Figure 2: Changes in equilibrium *ex ante* certainty equivalents $\Delta \text{CE}_{i}^{xa}(h_{\varepsilon})$ as functions of the signal precision $h_{\varepsilon}$ given the parameters in Table 1 and endowments $z_{i} = Z/3$, $\tau_{i} = 0, i = 1, 2, 3$, and $\bar{\pi}_{1} = \bar{\pi}_{3} = 5,000$, $\bar{\pi}_{2} = -10,000$. The scale on the horizontal axis is $x = \ln(1 + h_{\varepsilon} \cdot 1.5E+07)$.

equilibrium *ex ante* certainty equivalent relative to the no public information case $\text{CE}_{i}^{xa}(h_{\varepsilon} = 0)$, i.e.,

$$\Delta \text{CE}_{i}^{xa}(h_{\varepsilon}) \equiv \text{CE}_{i}^{xa}(h_{\varepsilon}) - \text{CE}_{i}^{xa}(h_{\varepsilon} = 0),$$

is a measure of investor $i$’s increased *ex ante* expected utility from changing the informativeness of the public information system from being uninformative to having a signal precision of $h_{\varepsilon}$. Figure 2 illustrates these changes in investor welfare for the three-investor example given in Table 1 assuming that the investors have efficient risk sharing endowments of the risky asset and zero endowments of the $t = 2$ zero-coupon bond, i.e., $z_{i} = Z/3$ and $\tau_{i} = 0, i = 1, 2, 3$, while the endowments of the $t = 0$ zero-coupon bond are $\bar{\pi}_{1} = \bar{\pi}_{3} = 5,000$ and $\bar{\pi}_{2} = -10,000$.

Note that all three investors in Figure 2 have a stationary point for their equilibrium *ex ante* certainty equivalent at $h_{\varepsilon} = \bar{h}$ (see Lemma 1). However, while the equilibrium *ex ante* certainty equivalents of investors 1 and 3 are both maximized at $h_{\varepsilon} = \bar{h}$, investor 2’s equilibrium *ex ante* certainty equivalent is minimized at that point. Hence, investors 1 and 3 are both better off with an interior signal precision, whereas investor 2 is better off with no public information ($h_{\varepsilon} = 0$) or perfect information ($h_{\varepsilon} \to \infty$). The parameters in Table 1 are such that investor 2 has a prior dividend precision equal to the average prior dividend precision, i.e., $h_{2} = \bar{h}$. As noted above, this implies that investor 2 does not engage in signal-contingent trading at $t = 1$ and, thus, his *ex ante* value of trading gains $U_{12}$ is equal to zero. Therefore, his equilibrium *ex ante* certainty equivalent is only affected by the changes in the *ex ante* equilibrium asset prices. With the assumed efficient risk sharing endowments of the
risky asset and his negative endowment of the $t = 0$ zero-coupon bond, investor 2 is a "net seller" of assets (for all levels of $h_x$) at $t = 0$ (i.e., $(\gamma_{i0}^* - \overline{\gamma}_i) + EQ[d] (x_{i0}^* - \bar{z}_i) < 0$) and, thus, he is hurt by the lower equilibrium asset prices with interior signal precisions. On the other hand, both investor 1 and investor 3 have strictly positive ex ante values of trading gains, which exceed any loss from selling assets at the lower equilibrium asset prices for interior signal precisions.

It is well known that even though a Pareto improvement can be achieved by changing the public information system, the change of information system may leave some investors better off and others worse off if the implementation of the equilibrium consumption plans requires trading of assets at equilibrium prices which depend on the information system (see Christensen and Feltham 2003, Chapter 7). The above example in Figure 2 illustrates such a setting. In order to investigate whether there exists a Pareto superior information system, consider a setting in which the investors have "equilibrium endowments." The growth in certainty equivalents is the same for all investors in equilibrium, and is maximized exactly for the signal precision at which the equilibrium interest rate and, thus, the ex ante cost of capital, is maximized, i.e., for $h_x = \overline{h}$. Hence, this level of signal precision is the obvious candidate for a Pareto efficient information system in a setting in which the endowments of the three assets can be re-allocated among the investors, and this is indeed the case.

In order to see why, consider the investors’ equilibrium portfolios at $t = 0$. It follows from (14) that the investors’ equilibrium demand for the risky asset at $t = 0$, i.e., $x_{i0}^* = pf_i [m_i - EQ[d]]$, depends on the prior beliefs but is independent of the informativeness of the public signal at $t = 1$. However, the investors’ equilibrium demand for the $t = 2$ zero-coupon bond at $t = 0$ varies with the signal precision. Consider any given signal precision different from the average prior dividend precision, i.e., $h_x \neq \overline{h}$, and the associated equilibrium certainty equivalents, $\{CE_{i0}^*, CE_{i2}^*\}_{i=1,\ldots,I}$, and equilibrium prices, $\beta_0, p_0(\eta)$. We want to show that this system cannot be a Pareto efficient information system. The equilibrium demand for the $t = 2$ zero-coupon bond $\gamma_{i0}^*$ is determined by

$$CE_{i0}^* = d_0 \bar{z}_i + p_0(\eta) [\bar{z}_i - x_{i0}^*] + \beta_0 [\overline{\gamma}_i - \gamma_{i0}^*] + \overline{\gamma}_i.$$  

As noted above, the equilibrium prices are independent of the investors’ individual endowments. Hence, a re-allocation of the endowments of the three assets defined by

$$\hat{\bar{z}}_i \equiv x_{i0}^*, \quad \hat{\gamma}_i \equiv \gamma_{i0}^*, \quad \hat{\gamma}_i \equiv CE_{i0}^* - d_0 x_{i0}^*,$$  

implies that the investors do not trade at $t = 0$ given these endowments. That is, the investors have equilibrium endowments relative to the signal precision $h_x$, and they achieve...
the same certainty equivalents as with the original endowments. It then follows from Lemma 1 that

\[ \frac{\partial}{\partial h} EU_{i} = r \exp (-rCE_{i}) \left\{ \beta_0 \frac{\partial}{\partial h} U_{1i} + \left[ (\gamma_{i0} - \hat{z}_i) + E^0[d] (x_{i0} - \hat{z}_i) \right] \frac{\partial}{\partial h} \beta_0 \right\} \]

\[ = r \exp (-rCE_{i}) \beta_0 \frac{\partial}{\partial h} U_{1i} \]

\[ = \frac{1}{2} \exp (-rCE_{i}) \beta_0 \frac{\partial}{\partial h} \ln \left[ 1 + \frac{(h - h_i)^2}{h_i (h + h_e)^2} \right]. \]

Since the common term for all investors \( h / (h + h_e)^2 \) is a concave function of the signal precision, which is maximized for \( h = \hat{h} \), all investors are weakly better off by marginally increasing (decreasing) the signal precision for \( h < \hat{h} \) \((h > \hat{h})\). Hence, for any \( h \neq \hat{h} \) and heterogeneous prior dividend precisions (such that there are investors with \( h_i \neq \hat{h} \)), there is an allocation of the endowments such that there exists a Pareto superior equilibrium with a marginal change in the signal precision. If \( h = \hat{h} \), no such Pareto improvements can be obtained, since \( \frac{\partial}{\partial h} U_{1i} = 0 \) for all investors in this case. These arguments establish the following result.

**Proposition 6** Assume the investors have heterogeneous prior dividend precisions.

(a) The information system with signal precision \( h = \hat{h} \) is the unique Pareto efficient public information system, and it enjoys the maximum equilibrium ex ante cost of capital and the maximum expected abnormal trading volume.

(b) For given endowments, some investors may be worse off with information system \( h = \hat{h} \) than with \( h \neq \hat{h} \), but at least one investor is better off with information system \( h = \hat{h} \) than with \( h \neq \hat{h} \).

The trading gains are maximized with an intermediate level of signal precision \( h = \hat{h} \), and this yields a superior Pareto efficient allocation with the maximum growth in certainty equivalents and, thus, the maximum ex ante cost of capital and the maximum expected abnormal trading volume. However, as demonstrated above, this level of signal precision may leave some (but not all) investors worse off depending on their endowments and their incentives to trade at \( t = 0 \). Table 2 illustrates a setting in which the endowments for the example in Figure 2 are re-allocated as in (24) in order to achieve equilibrium endowments relative to signal precision \( h = 0 \).
Figure 3: Changes in equilibrium ex ante certainty equivalents $\Delta CE_{i}^{xo}(h_{\varepsilon})$ as functions of the signal precision $h_{\varepsilon}$ given the parameters in Table 1 and the endowments in Table 2. The scale on the horizontal axis is $x = \ln (1 + h_{\varepsilon} \cdot 1.5E+07)$.

<table>
<thead>
<tr>
<th>Endowments</th>
<th>Investor 1</th>
<th>Investor 2</th>
<th>Investor 3</th>
<th>Aggregate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_{i}$</td>
<td>-3.33</td>
<td>51.11</td>
<td>52.22</td>
<td>100</td>
</tr>
<tr>
<td>$\overline{z}_{i}$</td>
<td>34,736</td>
<td>-19,376</td>
<td>-15,360</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_{i}$</td>
<td>34,898</td>
<td>-22,043</td>
<td>-12,856</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Equilibrium endowments relative to $h_{\varepsilon} = 0$.

With the endowments in Table 2, all three investors do not trade at $t = 0$ if $h_{\varepsilon} = 0$, and they achieve the same certainty equivalents as with the endowments in Figure 2. As the signal precision is increased, the investors continue not to trade at $t = 0$ in the risky asset (since $x_{i0}^{*}$ does not depend on $h_{\varepsilon}$), but investor 2, who has a prior dividend precision $h_{2}$ equal to the average prior dividend precision $\overline{h}$, starts to increase his holdings of the $t = 2$ zero-coupon bond due to its lower equilibrium price. The other two investors reduce their holdings of the $t = 2$ zero-coupon bond, i.e., sell units of the bond, even though its equilibrium price decreases. This is due to the fact that increasing the signal precision increases their future trading gains (which are realized at $t = 2$) and, thus, they also want to consume more at $t = 0$ in order to smooth consumption across the two consumption dates. Hence, all three investors gain from a higher signal precision (up to $h_{\varepsilon} = \overline{h}$) as illustrated in Figure 3 (although the gains to investor 2 are hardly visible in the graph). Of course, a prior round of trading at $t = -1$ based on the belief that the signal precision will be $h_{\varepsilon} = 0$, such that the investors have equilibrium endowments at $t = 0$ relative to $h_{\varepsilon} = 0$, ensures that the investors at $t = 0$ unanimously support a public information system change to a system with $h_{\varepsilon} = \overline{h}$.
4 Effectively Complete Market

The analyses in the preceding sections show that the public signal at $t = 1$ plays a key role in determining the \textit{ex ante} cost of capital, the expected abnormal trading volume, and the investors’ welfare. If the signal is uninformative or a perfect signal about the risky asset’s dividend, there is no trading at $t = 1$. However, if the signal is imperfect and the investors have heterogeneous prior dividend precisions, there will be signal-contingent trading at $t = 1$, and the \textit{ex ante} trading gains will be reflected in both a higher \textit{ex ante} cost of capital, a higher expected abnormal trading volume, and higher investor welfare. Hence, the key role of the public signal is to facilitate improved dynamic trading opportunities based on heterogeneously updated posterior beliefs in order to take advantage of the disagreements and the differences in confidence among the investors. However, does the dynamic trading in the zero-coupon bond and the risky asset based on the public signal allow the investors to take full advantage of the differences in their beliefs? This is the question addressed in this section, and the answer is, in general, no. Additional assets, such as various forms of derivative securities, or more trading rounds based on a sequence of public signals may lead to more efficient side-betting based on the heterogeneous beliefs (see Brennan and Cao 1996 for a similar model with several trading dates). In the extreme, additional assets of the right type may even eliminate the need for dynamic trading based on public signals. We examine only this extreme case in this paper, and we leave the intermediate cases for future research.

The initial question is what would be the right types of derivative assets to facilitate fully efficient side-betting? Wilson (1968) gives almost immediately the answer to this question. Wilson shows that with exponential utility and heterogeneous beliefs, a fully Pareto efficient risk-sharing and side-betting contract is such that each investor gets his risk tolerance fraction (i.e., $v_i = \rho_i/\rho_o$, where $\rho_o \equiv \sum_i \rho_i$) of the aggregate outcome (as with homogeneous beliefs) plus a term, which depends on the state of nature. The fraction $\rho_i/\rho_o$ is a constant independent of the state and, thus, in a sense, the fully efficient contract is still a linear contract in terms of the aggregate outcome as it is in the homogeneous beliefs case. The key is that the efficient risk sharing and the efficient side-betting can be separated into two additive terms. In our setting, the dividend at $t = 2$ on the risky asset, $Zd$, constitutes the aggregate outcome. Moreover, the $t = 2$ dividend is also a unique outcome-adequate representation of the state and, thus, the dividend also enters into the efficient side-betting term. The efficient side-betting term has the following form with homogeneous risk aversion (see Christensen and Feltham 2003, Appendix 4A):

$$f_i(d) = \rho \left( f_o(d) - \ln \left( \frac{\rho}{\lambda_i \varphi_i(d)} \right) \right), \quad i = 1, \ldots, I,$$
where

\[
f_i(d) \equiv \frac{1}{I} \sum_{i=1}^{I} \ln \left( \frac{\rho}{\lambda_i \varphi_i(d)} \right),
\]

and \( \lambda_i \) is the positive expected utility weight in a central planner’s efficient risk-sharing and side-betting program. Note that \( \sum_i f_i(d) = 0 \) and, thus, the side-betting terms constitute a zero-sum game. Furthermore, with a normally distributed dividend, \( \varphi_i(d) \sim N(m_i, \sigma_i^2) \), \( i = 1, \ldots, I \), the efficient side-betting terms \( f_i(d) \) can be expressed as quadratic functions of the dividend \( d \). Hence, any fully efficient risk-sharing and side-betting allocation of the \( t = 2 \) dividend can be expressed as

\[
c_{i2}^* = \alpha_{i0} + \alpha_{id} d + \alpha_{is} d^2, \quad i = 1, \ldots, I, \tag{25}
\]

where \( \alpha_{i0}, \alpha_{id}, \alpha_{is} \) are investor-specific constants with \( \sum_i \alpha_{i0} = \sum_i \alpha_{is} = 0 \) and \( \sum_i \alpha_{id} = Z \).

This suggests that a \( t = 2 \) zero-coupon bond, the risky asset itself, and a derivative security in zero net-supply with a payoff \( d^2 \) would be sufficient to trade at \( t = 0 \) to a fully efficient risk-sharing and side-betting allocation in a decentralized market setting without any need for an additional round of trading at \( t = 1 \) based on the public signal.\(^{12}\) The following analysis shows that this is indeed the case.\(^{13,14}\)

Compared to the market structure in the preceding sections, we now add an additional asset, denoted the *dividend derivative*, in zero net-supply with payoff \( d^2 \) at \( t = 2 \), and prices \( \pi_0(\eta) \) and \( \pi_1(y) \) at \( t = 0 \) and \( t = 1 \), respectively. The investors have endowments \( \bar{\theta}_i \) of this asset at \( t = 0 \), and let \( \theta_{it} \) be the units of the asset held after trading at date \( t \) satisfying the

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\(^{12}\)Note that if the investors have identical prior dividend precisions, then \( \alpha_{is} = 0 \), \( i = 1, \ldots, I \), such that the efficient allocation is linear in \( d \) and, thus, the zero-coupon bond and risky asset are sufficient to implement the efficient allocation without any trading at \( t = 1 \). Hence, in this setting, the market structure examined in the preceding sections yields fully efficient allocations independently of the public information system (compare to Proposition 3).

\(^{13}\)Brennan and Cao (1996) were the first to introduce a quadratic derivative as a means to achieve \((ex \ post)\) Pareto optimality in their noisy rational expectations equilibrium setting with heterogeneous posterior equilibrium beliefs.

\(^{14}\)As noted in the Introduction (footnote 4), our assumption of concordant beliefs ensures that Pareto efficient allocations only include side-betting on the public signal to the extent that it is informative about the fundamentals, i.e., the payoff-relevant events. In their differences-of-opinion model, Cao and Ou-Yang (2009) also introduce the \( d^2 \) derivative security in order to effectively complete the market. In their model, however, there is dynamic information-contingent trading in both the risky asset(s) and the derivative security. This is because the side-betting in their model pertains to *payoff-irrelevant* events determining the differences in how the investors interpret the public signal(s). Recall that “differences of opinion make a horse race,” cf. Harris and Raviv (1993). In fact, in the Cao and Ou-Yang (2009) model with exponential utilities and normally distributed dividends (like ours), a fully Pareto efficient allocation based on the prior homogeneous dividend beliefs can be achieved through trading only at \( t = 0 \) in the zero-coupon bond and the risky asset(s). It is not obvious why the investors should chose subsequently to take speculative positions in the financial markets based on public signals they disagree on how to interpret. Any other zero-sum betting game would achieve the same result.
market clearing conditions

\[ \sum_{i=1}^{I} \theta_i = \sum_{i=1}^{I} \theta_{it} = 0, \quad i = 1, \ldots, I, t = 0, 1. \]

Investor \( i \)'s \( t = 2 \) consumption is now given as

\[ c_{i2} = \theta_{i1} d^2 + x_{i1} d + \gamma_{i1}. \]

Given the public signal at \( t = 1 \), the investor’s certainty equivalent of \( t = 2 \) consumption can be calculated (using Lemma 2 in the appendix) to be\(^{15}\)

\[ \text{CE}_{i2} (\theta_{i1}, x_{i1}, \gamma_{i1} \mid y) = \gamma_{i1} + \frac{1}{2} \rho h_{i1} m_{i1}^2 - \frac{1}{2} \rho \left[ r x_{i1} - h_{i1} m_{i1} \right]^2 + \frac{1}{2} \rho \ln \left[ \frac{2 r \theta_{i1} + h_{i1}}{h_{i1}} \right] \]

and, thus, investor \( i \)'s decision problem at \( t = 1 \) given the public signal \( y \) is

\[ \max_{\theta_{i1}, x_{i1}, \gamma_{i1}} \text{CE}_{i2} (\theta_{i1}, x_{i1}, \gamma_{i1} \mid y) \]

subject to \( \gamma_{i1} + p_1(y) x_{i1} + \pi_1(y) \theta_{i1} \leq \gamma_{i0} + p_1(y) x_{i0} + \pi_1(y) \theta_{i0} \).

The first-order conditions imply that the optimal portfolio at \( t = 1 \) given investor \( i \)'s posterior beliefs is

\[ x_{i1}(y) = \rho h_{i1} \left[ m_{i1} - \frac{p_1(y)}{h_{i1} \left[ \pi_1(y) - p_1(y)^2 \right]} \right], \quad (26a) \]

\[ \theta_{i1}(y) = \frac{1}{2} \rho \left[ \frac{1}{\pi_1(y) - p_1(y)^2} - h_{i1} \right], \quad (26b) \]

\[ \gamma_{i1}(y) = \gamma_{i0} + p_1(y) x_{i0} + \pi_1(y) \theta_{i0} - p_1(y) x_{i1}(y) - \pi_1(y) \theta_{i1}. \quad (26c) \]

The market clearing conditions for the two risky assets then imply that

\[ p_1(y) = \bar{m}_1^t - r \bar{\pi}_1^t Z / I, \quad (27a) \]

\[ \pi_1(y) = \bar{\pi}_1^2 + \left[ \bar{m}_1^t - r \bar{\pi}_1^2 Z / I \right]^2. \quad (27b) \]

Note that the equilibrium price of the risky asset at \( t = 1 \) is not affected by the addition of the dividend derivative to the marketed assets (compare to (3)). Inserting the equilibrium prices of the two risky assets into the investors’ demand functions (26) and simplifying yield

\(^{15}\)Here we assume that \( 2 r \theta_{i1} + h_{i1} > 0 \). We later verify that this is indeed the case, in equilibrium.
the equilibrium demands for the two risky assets,

\[ x_{i1}^+(y) = \rho \left[ h_i m_i - hm \right] + Z/I, \quad hm \equiv \frac{1}{I} \sum_{i=1}^{I} h_i m_i, \quad i = 1, \ldots, I, \]  

\[ \theta_{i1}^+(y) = \frac{1}{2} \rho \left[ \bar{h} - h_i \right], \quad i = 1, \ldots, I. \] 

Of course, the equilibrium prices of the two risky assets at \( t = 1 \) both depend on the public signal at \( t = 1 \) and its informativeness. However, the equilibrium demands for the two risky assets do neither depend on the specific public signal \( y \) at \( t = 1 \) nor on the informativeness of the public information system—equilibrium demands at \( t = 1 \) only depend on the investors’ prior beliefs. Hence, there will be no signal-contingent trading at \( t = 1 \) and, therefore, the equilibrium demands for the two risky assets will be the same at both \( t = 1 \) and \( t = 0 \), i.e.,

\[ x_{i0}^+ = x_{i1}^+(y), \quad \theta_{i0}^+ = \theta_{i1}^+(y), \quad \gamma_{i0}^+ = \gamma_{i1}^+(y), \]  

and the equilibrium prices at \( t = 0 \) are the same as in (27) except that they are based on the prior beliefs rather than the posterior beliefs and that they are discounted by the riskless discount factor, i.e.,

\[ p_0(\eta) = \beta_0 E^Q [d], \]  

\[ \pi_0(\eta) = \beta_0 \left[ \sigma^2 + \left( E^Q [d] \right)^2 \right]. \]  

Inserting these equilibrium demands and equilibrium prices into investor \( i \)'s \( t = 2 \) certainty equivalent and simplifying yield his equilibrium \textit{ex ante} certainty equivalent of \( t = 2 \) consumption,

\[ CE_{i2} = \gamma_{i0}^+ + \frac{1}{2} \rho h_i \left[ m_i^2 - \left( E^Q [d] \right)^2 \right] + \frac{1}{2} \rho \left( E^Q [d] \right)^2 \left( h_i - \bar{h} \right) + \Upsilon_i, \]

where

\[ \Upsilon_i \equiv \frac{1}{2} \rho \ln \left[ \frac{\bar{h}_i}{h_i} \right]. \]

\[ ^{16}\text{Note that } \bar{h} - h_i = \bar{h}_1 - h_{i1} \text{ and, thus, } 2 \rho \theta_{i1}^+(y) + h_{i1} = \bar{h}_1 > 0 \text{ which was assumed in the derivation of the investors’ } t = 2 \text{ certainty equivalent.} \]

\[ ^{17}\text{In equilibrium, less confident investors than the average, i.e., } h_i < \bar{h}, \text{ take long positions in the dividend derivative, while more confident investors than the average take short positions. Note that the dividend derivative resembles a “smooth” straddle and, thus, investors, who think the variance of the dividend is high, like the convexity of its payoff, while investors, who think the variance is low, take a short position to get a concave payoff profile. This result suggests that straddles, i.e., long positions in both a call and a put option with the same strike price, play an important role in incomplete market settings with heterogeneous beliefs about the risks on the underlying assets.} \]
Comparing this certainty equivalent to the equilibrium certainty equivalent in the setting without the dividend derivative (see equation (15)) yields that

\[
\begin{align*}
\text{CE}_{y0}^t &= d_0 z_i + \pi_0(\eta) \left[ \delta_i - \theta_{i0}^t \right] + p_0(\eta) \left[ z_i - x_{i0}^t \right] + \beta_0 \left[ \gamma_i - \gamma_{i0}^t \right], \\
\text{CE}_{i2}^t &= \text{CE}_{i2}^* + \left( \gamma_{i0}^t - \gamma_{i0}^* \right) + \frac{1}{2} \rho \left( E^Q[d] \right)^2 (h_i - \overline{h}) + \Upsilon_i - U_{1i}.
\end{align*}
\] (31a)

The equilibrium interest rate is, as in the setting without the dividend derivative, determined as the utility discount rate plus the risk aversion parameter times the growth in certainty equivalents, i.e.,

\[ t = \delta + r \left( \text{CE}_{i2}^t - \text{CE}_{i0}^t \right). \]

Using the market clearing conditions for the riskless and risky asset, and simplifying yield the equilibrium interest rate.

**Proposition 7** Consider the setting in which the investors can trade in the dividend derivative in addition to the zero-coupon bond and the risky asset.

(a) The equilibrium interest rate is given by

\[ t = \delta + r \overline{\Upsilon} + \Phi \left( \left\{ m_i, \sigma_i^2 \right\}_{i=1,...,I} \right), \]

where

\[ \overline{\Upsilon} = \frac{1}{I} \sum_{i=1}^{I} \Upsilon_i = \frac{1}{2} \rho \sum_{i=1}^{I} \ln \left[ \frac{h_i}{\overline{h}} \right]. \] (32)

(b) The equilibrium interest rate is independent of the signal precision \( h_\varepsilon \).

(c) The equilibrium interest rate is strictly higher than the equilibrium interest rate in the setting without the dividend derivative, irrespectively of the signal precision, i.e.,

\[ \overline{\Upsilon} - U_1 > 0, \forall h_\varepsilon \in (0, \infty), \] if, and only if, the investors have heterogeneous prior dividend precisions.

The higher equilibrium interest rate reflects that the investors can achieve more efficient side-betting based on their heterogeneous beliefs by being able to trade also in the dividend derivative instead of having to rely on dynamic trading in the risky asset alone. Of course, since there is no trading based on the public signal at \( t = 1 \) when the investors can also trade in the dividend derivative, the equilibrium interest rate is independent of the informativeness of the public signal.

The \textit{ex ante} equilibrium prices of the risky asset are in both settings, with and without the dividend derivative, determined as the equilibrium discount factor times the risk-adjusted
Figure 4: Equilibrium interest rate, risk premium, and ex ante cost of capital as functions of the signal precision \( h_\varepsilon \) given the parameters in Table 1 for the settings with the dividend derivative and without the dividend derivative (i.e., the incomplete market (ICM) setting). The scale on the horizontal axis is \( x = \ln (1 + h_\varepsilon \cdot 1.5E+07) \).

expected dividend on the risky asset. Since the latter is the same in both settings, the equilibrium ex ante cost of capital, \( \mu^{x_a}(\eta) = \nu + \infty^{x_a} \), is independent of the informativeness of the public signal in the setting with the dividend derivative, but it is uniformly higher than in the setting without the dividend derivative (as a function of \( h_\varepsilon \)). Figure 4 illustrates these differences for the three-investor example in Table 1.

Inserting the investors’ \( t = 0 \) (and \( t = 1 \)) equilibrium portfolios yields that the investors’ \( t = 2 \) equilibrium consumption is

\[
c_{i2} = \theta_{i0}^d d^2 + x_{i0}^d d + \gamma_{i0}^d
\]

\[
= f_i(d) + v_i Z d,
\]

where

\[
f_i(d) = \frac{1}{2} \rho \left[ \bar{h} - h_i \right] d^2 + \rho \left[ h_i m_i - \bar{m} \right] d + \gamma_{i0}^d, \quad v_i = 1/I.
\]

Note that

\[
\sum_{i=1}^{I} f_i(d) = 0, \quad \sum_{i=1}^{I} v_i = 1,
\]

such that the consumption allocation can be expressed as a linear function of the aggregate
outcome with a fixed slope but a state-dependent intercept. Wilson (1968) shows that this is both a necessary and sufficient condition for a fully efficient risk-sharing and side-betting allocation (see also Amershi and Stoeckennius 1983). Hence, the equilibrium in the setting with the dividend derivative constitutes a fully Pareto efficient risk sharing and side-betting allocation. Since the growth in certainty equivalents is higher in the setting with the dividend derivative than in the setting without it (as reflected by the higher equilibrium interest rate), the addition of the dividend derivative yields a more efficient market structure independently of informativeness of the public signal. Of course, some investors are made better off by the addition of the dividend derivative, but again some investors might be worse off depending on the impact of the higher equilibrium interest rate on the value of their endowments.

**Proposition 8** Assume the investors have heterogeneous prior dividend precisions.

(a) The market structure with the dividend derivative strictly dominates (in a Pareto efficiency sense) the market structure without the dividend derivative, and it enjoys a strictly higher equilibrium ex ante cost of capital, irrespectively of the informativeness of the public information system.

(b) For given endowments, some investors may be worse off with the addition of the dividend derivative to the marketed assets, but at least one investor is made better off by the addition of this asset.

Figure 5 shows the changes in the investors’ ex ante certainty equivalents as functions of the signal precision $h_\varepsilon$ relative to the ex ante certainty equivalents without the dividend derivative and a zero signal precision, $h_\varepsilon = 0$, i.e.,

$$\Delta CE_i^{xa}(h_\varepsilon) \equiv CE_i^{xa}(h_\varepsilon) - CE_i^{xa}(h_\varepsilon = 0),$$

where the ex ante certainty equivalents for $h_\varepsilon > 0$ are defined as

$$- \exp (-rCE_i^{xa}(h_\varepsilon)) \equiv - \exp (-rCE_i^{\uparrow}(h_\varepsilon)) - \exp (-\delta) \exp (-rCE_i^{\downarrow}(h_\varepsilon)),$$

for the setting with the dividend derivative and the setting without this asset (the incomplete market (ICM) setting in the preceding sections), respectively, while the base ex ante certainty

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18 Note that the state-dependent intercept is comprised of (a) a fixed component, $\gamma_i^{*0}$, which depends on

the endowments (as in a homogeneous beliefs setting), (b) a side-betting term due to differences between the

precision-weighted prior means, $\rho [\bar{h}_i m_i - \bar{h} \bar{m}] d$, and (c) a side-betting term due to differences in the prior

dividend precisions, $\frac{1}{2} \rho [\bar{h} - h_i] d^2$. 

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equivalents, $\Delta CE_{i}^{xa}(h_{\varepsilon})$, are defined by

$$- \exp(-rCE_{i}^{xa}(h_{\varepsilon} = 0)) \equiv - \exp(-rCE_{i0}^{*}(h_{\varepsilon} = 0)) - \exp(-\delta) \exp(-rCE_{i2}^{*}(h_{\varepsilon} = 0))$$

for both settings. The investors’ endowments of the zero-coupon bonds and the risky asset are the same as in Figure 3, and it is assumed that the investors have zero endowments of the dividend derivative. Hence, Figure 5 shows the $ex \ ante$ value to the investors of having the public information system in the incomplete market setting versus the $ex \ ante$ value to the investors of the addition of the dividend derivative to the marketed assets at $t = 0$. In this example, all three investors benefit from imperfect public information in the incomplete market setting, but they benefit even more from the introduction of the dividend derivative (again the gains to investor 2 are hardly visible in the graph—note that his equilibrium holdings of the dividend derivative is equal to zero).

5 Concluding Remarks

We have developed a simple analytical model of public information and heterogeneous prior beliefs allowing us to study the relationship between the informativeness of the public information system and the investors’ welfare in an incomplete market setting. The source
of the welfare improvements due to public information is the trading gains following from the investors’ speculative positions based on the differences in the precision of their prior beliefs. These trading gains are reflected in an additional positive component in the equilibrium interest rate (in addition to the usual utility discount rate, the risk-adjusted aggregate consumption growth and the risk premium for aggregate consumption risk). Moreover, the model provides a direct positive relationship between welfare improvements and the expected abnormal trading volume and, thus, it provides an empirical measure for the relationship between public information systems and investor welfare. This result is in contrast to the extant literature on the impact of public information on trading volume based on noisy rational expectations models (see, e.g., Kim and Verrecchia 1991a, McNichols and Trueman 1994, and Demski and Feltham 1994). Due to the unmodelled “noise traders,” these models do not allow welfare comparisons of market structures with different public information systems. Similarly, the differences-of-opinion literature is also to a large extent silent about the relationship between trading volume and investor welfare, mainly because trading volume in these models is generated based on unmodelled heterogeneous beliefs about payoff-irrelevant events as opposed to about the fundamentals of the economy.

Our analysis of the incomplete market setting shows that the public information must be imperfect to be valuable. No information and perfect information rule out valuable dynamic trading strategies to take advantage of the differences in prior dividend precisions. In this sense, our results are related to the so-called information-risk problem due to Hirshleifer (1971), and to the literature on dynamically completing markets by trading long-lived securities (see, e.g., Ohlson and Buckman 1981, Duffie and Huang 1985, and Christensen and Feltham 2003, Chapter 7). Hence, the model provides an argument for the gradual release of information by firms through, for example, earnings forecasts and quarterly earnings announcements, such that the news at the annual audited earnings announcements is limited. In the limit, in which the public information about the normally distributed dividend is generated by a continuous arithmetic Brownian motion and trading in the risky asset and the zero-coupon bond is continuous, a dynamically complete market is achieved and, thus, a fully Pareto efficient Arrow-Debreu equilibrium can be implemented irrespectively of any heterogeneity in prior beliefs, preferences, and wealth distributions (see Duffie and Huang 1985).

A continuous public information flow may be considered an extreme model of information flows in actual financial markets. We show that a dividend derivative specifically targeted towards the investors’ incentive to take advantage of their differences in dividend precisions...

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\[\text{Lev (1989) shows that earnings and earnings-related information only has an explanatory power of about 5\% of the cross-sectional and time-series variability of stock returns for medium-size windows.}\]
completely eliminates the need for dynamic trading based on public signals. This result, however, relies heavily on our assumptions of exponential utilities (such that efficient risk sharing and side-betting arrangements are additively separable) and normally distributed dividends and, thus, it cannot be expected to carry over to more general settings. On the other hand, the result does show that derivative markets (i.e., securities with non-linear payoffs) may play an important welfare enhancing role in incomplete markets with “jumps” in public information flows. Careful modeling and investigations of the intimate relationship between public information flows and the need for derivative markets could be a fruitful topic for future research.

Finally, our model provides a direct positive relationship between welfare improvements and the $ex \ ante$ cost of capital, i.e., the Pareto efficient public information system is the system enjoying the maximum $ex \ ante$ cost of capital and, thus, the lowest equilibrium $ex \ ante$ price of the risky asset. This (maybe at first counterintuitive) result shows the importance of using a general equilibrium analysis in the evaluation of public information systems, which have consequences for the economy at large. In our model, the impact of the public information system on investor welfare is reflected through the equilibrium interest rate, i.e., the expected marginal rate of substitution between current and future consumption.

The $ex \ ante$ risk premium, however, is not affected by the public information system, neither in the incomplete nor in the effectively complete market setting. While the impact on the equilibrium interest can be expected to carry over to more general settings as a measure of changes in investor welfare (through the expected marginal utility of future consumption), the lack of an impact on the $ex \ ante$ risk premium is quite likely specific to our particular model with exponential utilities and normally distributed dividends.\footnote{See, however, Krueger and Lustig (2010), who in a related setting with power utilities gives sufficient conditions for market incompleteness being irrelevant for the equity premium.} Qin (2011) shows numerically that adding a call option to our incomplete market setting yields an $ex \ ante$ risk premium which (as a function of the signal precision) is not aligned with investor welfare. The equilibrium interest rate and the investor welfare, however, are still perfectly aligned and attain their unique maximum for a common interior signal precision. Hence, it is probably wise to be cautious in making policy statements about, for example, financial reporting regulation, based on empirical measures of equity premia (which are hard to measure reliably anyway).

\footnote{See, however, Krueger and Lustig (2010), who in a related setting with power utilities gives sufficient conditions for market incompleteness being irrelevant for the equity premium.}
References


Appendix: Proofs

Proof of Remark 1: Substituting the $t = 1$ demand functions (2a) into investor $i$’s certainty equivalent (5) yields

$$\text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y)) = \gamma_{i0} + x_{i0}p_1(y) + \rho h_{i1} (m_{i1} - p_1(y)) (m_{i1} - p_1(y)) - \frac{1}{2} r (\rho h_{i1} (m_{i1} - p_1(y)))^2 \sigma_{i1}^2$$

$$= \gamma_{i0} + x_{i0}p_1(y) + \frac{1}{2} \rho h_{i1} (m_{i1} - p_1(y))^2,$$

where $p_1(y)$ is given by (3). Investor $i$’s posterior mean in (1a) for the $t = 2$ dividend can be expressed as weighted average of the prior mean and the public signal, i.e., $m_{i1} = \omega_i y + (1 - \omega_i) m_i$, where $\omega_i = \sigma_i^2 / (\sigma_i^2 + \sigma_e^2)$. Thus, the precision weighted average of the investors’ posterior can be written as

$$m_{i1}^h = \frac{1}{I} \sum_{i=1}^{I} \frac{h_{i1}}{h_1} m_{i1}$$

$$= \frac{1}{I} \sum_{i=1}^{I} \frac{1}{h_1} [h_{i} y + h_i m_i]$$

$$= \sigma_1^2 [h_{i} y + \bar{h}\bar{m}^h],$$

and, consequently, the $t = 1$ equilibrium price of the risky asset is

$$p_1(y) = \sigma_1^2 [h_{i} y + \sigma^2 \bar{m}^h - r\bar{Z} / I].$$
Inserting in the above expression for the certainty equivalent yields

\[
\text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y)) = \gamma_{i0} + x_{i0}\sigma_{x}^{2} \left[ h_{\epsilon}y + \overline{h}m - rZ/I \right] + \frac{1}{2}\rho h_{i1} \left( \left[ \sqrt{\sigma_{x}^{2}} + (1 - \omega_{i})m_{i} \right] - \sigma_{x}^{2} \left[ h_{\epsilon}y + \overline{h}m - rZ/I \right] \right)^{2}
\]

\[
= \gamma_{i0} + x_{i0}\sigma_{x}^{2} \left[ h_{\epsilon}y + \overline{h}m - rZ/I \right] + \frac{1}{2}\rho h_{i1} \left( \sigma_{x}^{2} \left[ h_{\epsilon}y + h_{i}m_{i} \right] - \sigma_{x}^{2} \left[ h_{\epsilon}y + \overline{h}m - rZ/I \right] \right)^{2}
\]

\[
= \gamma_{i0} + x_{i0}\sigma_{x}^{2} h_{\epsilon}y + x_{i0}\sigma_{x}^{2} \left[ \overline{h}m - rZ/I \right] + \frac{1}{2}\rho h_{i1} \left( \left[ \sigma_{x}^{2} - \overline{\sigma}_{x}^{2} \right] h_{\epsilon}y + \sigma_{x}^{2}h_{i}m_{i} - \sigma_{x}^{2} \left[ \overline{h}m - rZ/I \right] \right)^{2}.
\]

For notational simplicity, let

\[
E_{1i} \equiv \left[ \sigma_{x}^{2} - \overline{\sigma}_{x}^{2} \right] h_{\epsilon},
\]

\[
E_{2i} \equiv \overline{\sigma}_{x}^{2}h_{\epsilon},
\]

\[
E_{3i} \equiv \sigma_{x}^{2} \left[ \overline{h}m - rZ/I \right],
\]

\[
E_{4i} \equiv \sigma_{x}^{2}h_{i}m_{i} - E_{3i},
\]

and substituting into the certainty equivalent yields

\[
\text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y)) = \gamma_{i0} + x_{i0}E_{2i}y + x_{i0}E_{3i} + \frac{1}{2}\rho h_{i1} \left( E_{1i}y + E_{4i} \right)^{2}
\]

\[
= \gamma_{i0} + x_{i0}E_{3i} + \frac{1}{2}\rho h_{i1} \left[ \sigma_{x}^{2} \left[ h_{\epsilon}y + h_{i}m_{i} \right] - \sigma_{x}^{2} \left[ h_{\epsilon}y + \overline{h}m - rZ/I \right] \right] + \frac{1}{2}\rho h_{i1} E_{1i}y^{2}.
\]

Note that \( \text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y)) \) is a quadratic function of \( y \), and from the perspective of investor \( i \), \( y \sim N(m_{i}, \sigma_{x}^{2} + \sigma_{z}^{2}) \). Hence, we can apply the following lemma (see Christensen and Feltham 2003, Appendix 3A, for proof) to calculate investor \( i \)'s certainty equivalent.

**Lemma 2** Let \( w \) be a normally distributed variable with distribution \( w \sim N(q, \sigma) \) and precision \( d = \sigma^{-1} \), and let a quadratic function of \( w \) be given by \( Q(w) = \sigma + tw + \frac{1}{2}tw^{2} \), where \( \sigma > 0 \). The certainty equivalent \( \text{CE}_{Q(w)} \) of the quadratic function as defined by

\[
- \exp \left[ -r\text{CE}_{Q(w)} \right] = \mathbb{E}[ -\exp[ -rQ(w)] ],
\]

is given by

\[
\text{CE}_{Q(w)} = \sigma + \frac{1}{2}\rho dq^{2} - \frac{1}{2}\rho \left[ \frac{r\overline{d} - d(q)}{r\overline{d} + d} \right]^{2} + \frac{1}{2}\rho \ln|r\overline{d} + d| - \ln|d|.
\]
In our model,

\[ w = y, \quad q = m, \quad \bar{d} = (\sigma_i^2 + \sigma_\varepsilon^2)^{-1}, \]
\[ \bar{a} = \gamma_i + x_iE_{3i} + \frac{1}{2}\rho h_{1i}E_{4i}^2, \]
\[ \bar{b} = x_iE_{2i} + \rho h_{1i}E_{4i}, \]
\[ \bar{c} = \rho h_{1i}E_{4i}^2. \]

Hence, investor \( i \)'s \( t=0 \) certainty equivalent of \( t=2 \) consumption can be expressed as

\[
CE_{i2}(x_{i0}, \gamma_{i0}) = \gamma_{i0} + x_{i0}E_{3i} + \frac{1}{2}\rho h_{1i}E_{4i}^2 + \frac{1}{2}\frac{\rho m_{i}^2}{\sigma_i^2 + \sigma_\varepsilon^2}
- \frac{1}{2}\rho \left[ rx_{i0}E_{2i} + h_{1i}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} \right]^2
\]
\[ \frac{- \frac{1}{2}\rho [\ln[h_{1i}E_{1i}^2 + \frac{1}{\sigma_i^2 + \sigma_\varepsilon^2}] + \ln[\sigma_i^2 + \sigma_\varepsilon^2]]}{1 + h_{1i}E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \]
\[ = \gamma_{i0} + \frac{1}{2}\rho \ln[1 + h_{1i}E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)] + \frac{1}{2}\rho h_{1i}E_{4i}^2 + \frac{1}{2}\rho \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2}
+ x_{i0}E_{3i} - \frac{1}{2}\rho \left[ (\sigma_i^2 + \sigma_\varepsilon^2) [rx_{i0}E_{2i} + h_{1i}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2}] \right]^2
\]
\[ 1 + h_{1i}E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2). \]

Collecting terms yields that

\[
CE_{i2} = \gamma_{i0} + U_{i1} + U_{i2} + M_i x_{i0} - \frac{1}{2}r V_i x_{i0}^2, \]

where

\[
U_{i1} \equiv \frac{1}{2}\rho \ln[1 + h_{1i}E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)], \]
\[
U_{i2} \equiv \frac{1}{2}\rho h_{1i}E_{4i}^2 + \frac{1}{2}\frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - \frac{1}{2}\rho \left( \sigma_i^2 + \sigma_\varepsilon^2 \right) (h_{1i}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2})^2
\]
\[ 1 + h_{1i}E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2), \]
\[
M_i \equiv E_{3i} - \frac{(\sigma_i^2 + \sigma_\varepsilon^2) E_{2i} (h_{1i}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2})}{1 + h_{1i}E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)}, \]
\[
V_i \equiv \frac{(\sigma_i^2 + \sigma_\varepsilon^2) E_{2i}}{1 + h_{1i}E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)}. \]

Using the definition of \( E_{1i} = [\sigma_i^2 - \sigma_{1i}^2] h_\varepsilon \), we get

\[
E_{1i} = \left[ \frac{1}{h_i + h_\varepsilon} - \frac{1}{\bar{h} + h_\varepsilon} \right] h_\varepsilon = h_\varepsilon \frac{\bar{h} - h_i}{(h_\varepsilon + h_i) (\bar{h} + h_\varepsilon)}. \]

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This implies that

\[ A_i \equiv 1 + h_{i1} E_{1i}^2 [\sigma_i^2 + \sigma_\varepsilon^2] \]

\[ = 1 + [\bar{h} + h_i] h_\varepsilon^2 \left[ \frac{\bar{h} - h_i}{(\bar{h} + h_i)(\bar{h} + h_\varepsilon)} \right]^2 \left[ \frac{1}{h_i} + \frac{1}{h_\varepsilon} \right] \]

\[ = 1 + \frac{1}{h_\varepsilon + h_i} h_\varepsilon^2 \left[ \frac{\bar{h} - h_i}{\bar{h} + h_\varepsilon} \right]^2 \frac{h_\varepsilon + h_i}{h_\varepsilon h_i} \]

\[ = 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \]

and, thus,

\[ U_{i1} = \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right], \]

which establishes (7a).

Turning to \( V_i \) and using the definitions of \( E_{2i} = \bar{\sigma}_1^2 h_\varepsilon \) and of \( A_i = 1 + h_{i1} E_{1i}^2 [\sigma_i^2 + \sigma_\varepsilon^2] \), we get that

\[ V_i = \left( \sigma_i^2 + \sigma_\varepsilon^2 \right) \frac{(\bar{\sigma}_i^2 h_\varepsilon)^2}{A_i} \]

\[ = \frac{\left( \frac{1}{h_i} + \frac{1}{h_\varepsilon} \right) \left( \frac{h_\varepsilon}{\bar{h} + h_\varepsilon} \right)^2}{1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2}} \]

\[ = \frac{(h_\varepsilon + h_i) h_\varepsilon}{h_i (\bar{h} + h_\varepsilon)^2 + (\bar{h} - h_i)^2 h_\varepsilon} = \frac{h_\varepsilon}{\bar{h}^2 + h_\varepsilon h_i}, \]

which establishes (7d).

Turning to \( M_i \) and using the definition of \( V_i \), we can express \( M_i \) as

\[ M_i = E_{3i} - \frac{(\sigma_i^2 + \sigma_\varepsilon^2) E_{2i} \left( h_{i1} E_{1i} E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} \right)}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \]

\[ = E_{3i} - \frac{h_{i1} E_{1i} E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} V_i}{E_{2i}} \]

\[ = V_i \left[ \frac{E_{3i}}{V_i} - h_{i1} \frac{E_{1i} E_{4i}}{E_{2i}} + \frac{m_i h_\varepsilon}{E_{2i} (h_\varepsilon + h_i)} \right]. \]
Inserting the expressions for \( E_{ji}, j = 1, 2, 3, 4 \), and simplifying yield

\[
M_i = V_i \left[ \frac{\sigma_i^2 \left[ \overline{m}^h - rZ/I \right]}{V_i} - h_{i1} E_{i1} \left( \sigma_i^2 h_i m_i - \sigma_i^2 \left[ \overline{m}^h - rZ/I \right] \right) + \frac{m_i h_i h_\varepsilon}{\sigma_i^2 h_\varepsilon (h_\varepsilon + h_i)} \right]
\]

\[
= V_i \left[ \frac{\sigma_i^2 \left[ \overline{m}^h - rZ/I \right]}{V_i} - \frac{h_\varepsilon}{h_\varepsilon + h_i} \left( h_i m_i - h_i \sigma_i^2 \left[ \overline{m}^h - rZ/I \right] \right) + \frac{m_i h_i h_\varepsilon}{h_\varepsilon h_\varepsilon + (h_\varepsilon + h_i)} \right]
\]

\[
= V_i \left[ \frac{\sigma_i^2 \left[ \overline{m}^h - rZ/I \right]}{V_i} + h_i m_i + (\overline{h} - h_i) \sigma_i^2 \left[ \overline{m}^h - rZ/I \right] \right]
\]

\[
= V_i \left[ h_i m_i + \left( \frac{1}{V_i} + (\overline{h} - h_i) \right) \sigma_i^2 \left[ \overline{m}^h - rZ/I \right] \right].
\]

Substituting the expression for \( V_i \) back in yields

\[
M_i = h_\varepsilon \left[ h_i m_i + \left( \frac{\overline{r} + h_i}{h_\varepsilon} + (\overline{h} - h_i) \right) \sigma_i^2 \left[ \overline{m}^h - rZ/I \right] \right]
\]

\[
= \frac{h_\varepsilon \left[ h_i m_i + \frac{\overline{r}}{h_\varepsilon} + \frac{1}{h_\varepsilon} \left[ \overline{m}^h - rZ/I \right] \right]}{h_\varepsilon h_\varepsilon + (h_\varepsilon + h_i)}
\]

\[
= \frac{h_\varepsilon h_i m_i + \overline{m}^h - rZ/I}{h_\varepsilon h_\varepsilon + h_i} = \frac{h_\varepsilon h_i m_i + \overline{m}^h - rZ/I}{h_\varepsilon h_\varepsilon + h_i},
\]

which establishes (7c).

Finally, turning to \( U_{i2} \) we get

\[
2rU_{i2} = h_{i1} E_{4i}^2 + \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - \frac{(\sigma_i^2 + \sigma_\varepsilon^2) \left( h_{i1} E_{i1} E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} \right)^2}{1 + h_{i1} E_{i1}^2 (\sigma_i^2 + \sigma_\varepsilon^2)}
\]

\[
= h_{i1} E_{4i}^2 + \frac{1}{2} \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - \frac{h_{i1}^2 E_{i1}^2 E_{4i}^2 (\sigma_i^2 + \sigma_\varepsilon^2) + m_i^2 (\sigma_i^2 + \sigma_\varepsilon^2) - 2h_{i1} E_{i1} E_{4i} m_i}{1 + h_{i1} E_{i1}^2 (\sigma_i^2 + \sigma_\varepsilon^2)}
\]

\[
= h_{i1} E_{4i}^2 + \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} + m_i^2 h_{i1} E_{i1}^2 - \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} + 2h_{i1} E_{i1} E_{4i} m_i
\]

\[
= \frac{h_{i1} E_{4i}^2 + \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} + m_i^2 h_{i1} E_{i1}^2 - \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} + 2h_{i1} E_{i1} E_{4i} m_i}{1 + h_{i1} E_{i1}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} = h_{i1} \frac{(E_{4i} + m_i E_{i1})^2}{1 + h_{i1} E_{i1}^2 (\sigma_i^2 + \sigma_\varepsilon^2)}. \]
Substituting the expressions for $E_{1i}$ and $E_{4i}$ in, we obtain

$$2r_{U_2} = (h_\varepsilon + h_i) \left( \frac{1}{h_\varepsilon + h_i} h_i m_i - \frac{1}{1 + [h_\varepsilon + h_i] [\sigma_1 h_\varepsilon - \sigma_2 h_\varepsilon] h_\varepsilon} \left( \frac{[h_\varepsilon + h_i] [\sigma_1 h_\varepsilon - \sigma_2 h_\varepsilon]}{[\sigma_1 h_\varepsilon - \sigma_2 h_\varepsilon]} h_\varepsilon \right)^2 \right)$$

$$= \frac{1}{h_\varepsilon + h_i} \left( h_i m_i - \frac{1}{h_\varepsilon + h_i} \left( h_\varepsilon m_i - rZ/I \right) + m_i \left( \frac{1}{(h_\varepsilon + h_i) \left( h_\varepsilon + h_i \right)} \right) h_\varepsilon \right)^2$$

$$= \frac{1}{h_\varepsilon + h_i} + \left( \frac{h_i m_i (h_\varepsilon + h_i) + m_i h_\varepsilon (h_\varepsilon + h_i) - \left( h_\varepsilon m_i - rZ/I \right) [h_\varepsilon + h_i]}{h_\varepsilon + h_i} \right)^2 \left( \frac{(h_\varepsilon + h_i) (h_\varepsilon + h_i)}{h_\varepsilon (h_\varepsilon + h_i) + h_i (h_\varepsilon + h_i)} \right)$$

$$= \frac{h_i (h_\varepsilon + h_i) \left( m_i \bar{m} - \left( h_\varepsilon m_i - rZ/I \right) \right)}{h_\varepsilon (h_\varepsilon + h_i) + h_i (h_\varepsilon + h_i)} = \frac{h_i \left( m_i \bar{m} - \left( h_\varepsilon m_i - rZ/I \right) \right)}{h_\varepsilon (h_\varepsilon + h_i)}.$$

Hence, we have that

$$U_{i2} = \frac{\rho h_i \left( m_i \bar{m} - \left( h_\varepsilon m_i - rZ/I \right) \right)}{h_\varepsilon (h_\varepsilon + h_i)},$$

which establishes (7b).

**Proof of Proposition 1:** First we calculate

$$\bar{v} \equiv \frac{1}{\bar{I}} \sum_{i=1}^{I} v_i = \frac{1}{\bar{I}} \sum_{i=1}^{I} \frac{h_\varepsilon^2 + h_i h_i}{h_\varepsilon} = \frac{\bar{h}_e^2}{h_\varepsilon} + \bar{h}_e,$$

and

$$\bar{M}_v \equiv \frac{1}{\bar{I}} \sum_{i=1}^{I} \frac{1}{\bar{I}} \sum_{i=1}^{I} \frac{h_\varepsilon^2 + h_i h_i}{h_\varepsilon} \frac{h_\varepsilon h_i h_i m_i + \bar{h}_e \left( h_\varepsilon m_i - rZ/I \right)}{h_\varepsilon^2 + h_i h_i}$$

$$= \frac{1}{\bar{I}} \frac{h_\varepsilon}{h_\varepsilon^2 + h_i h_i} \sum_{i=1}^{I} \frac{h_\varepsilon h_i m_i + \bar{h}_e \left( h_\varepsilon m_i - rZ/I \right)}{h_\varepsilon^2 + h_i h_i}$$

$$= \frac{1}{\bar{I}} \frac{1}{h_\varepsilon^2 + h_i h_i} \left( I h_\varepsilon \bar{m}^2 m_i^2 + \bar{h}_e \left( h_\varepsilon m_i - rZ/I \right) \right) = \frac{h_\varepsilon \bar{m}^2 m_i - \bar{h}_e \left( h_\varepsilon m_i - rZ/I \right)}{h_\varepsilon^2 + h_i h_i}.$$
Hence, the risk-adjusted expected dividend is

\[ E^Q[d] = \frac{h_{\varepsilon}\overline{m}h + \overline{h}h_{\varepsilon}h - \overline{r}Z/I}{\overline{h}^2 + h_{\varepsilon}\overline{h}} - r \frac{h_{\varepsilon}}{\overline{h}^2 + h_{\varepsilon}\overline{h}} Z/I \]

which shows the claim in the proposition that the risk-adjusted expected dividend is independent of the signal precision.

**Proof of Remark 2:** Substituting the expressions for \( M_i \) and \( V_i \), i.e., (7c) and (7d), into (13) and simplifying yield

\[ x^*_i = \rho \frac{h_{\varepsilon}h_i m_i + \overline{h} [\overline{m}h - rZ/I]}{\overline{h}^2 + h_{\varepsilon}h_i} - E^Q[d] = \rho \frac{1}{h_{\varepsilon}} \left[ h_{\varepsilon}h_i m_i + \overline{h} [\overline{m}h - rZ/I] - \left( \overline{h}^2 + h_{\varepsilon}h_i \right) E^Q[d] \right]. \]

Using the expression for \( E^Q[d] \), i.e., (12), yields

\[ x^*_i = \rho \frac{1}{h_{\varepsilon}} \left[ h_{\varepsilon}h_i m_i + \overline{h}^2 E^Q[d] - \left( \overline{h}^2 + h_{\varepsilon}h_i \right) E^Q[d] \right] = \rho h_i \left\{ m_i - E^Q[d] \right\}. \]

**Proof of Proposition 2:** Summing (16) across investors and using the market clearing conditions yield

\[ \iota = \delta + r \frac{1}{I} \sum_{i=1}^{I} (CE^*_i - CE^*_{i0}) \]

\[ = \delta + r \frac{1}{I} \sum_{i=1}^{I} U_{1i} + r \frac{1}{I} \sum_{i=1}^{I} (U_{2i} + M_i x^*_i - \frac{1}{2} rV_i (x^*_i)^2) - rd_0 Z/I. \]

From (7a) we get

\[ r \frac{1}{I} \sum_{i=1}^{I} U_{1i} = \frac{1}{2} r \frac{1}{I} \sum_{i=1}^{I} \ln \left[ 1 + \frac{(\overline{h} - h_i)^2}{h_i} \frac{h_{\varepsilon}}{\overline{h}^2 + h_{\varepsilon}h_i} \right]. \]
Substituting (14) yields

\[
U_{2i} + M_i x_{i0}^* - \frac{1}{2}rV_i (x_i^*)^2 = U_{2i} + M_i \rho \frac{M_i - E^Q[d]}{V_i} - \frac{1}{2}rV_i \left( \rho \frac{M_i - E^Q[d]}{V_i} \right)^2
\]

\[
= U_{2i} + \rho \frac{M_i^2 - M_i E^Q[d]}{V_i} - \frac{1}{2} \rho \frac{M_i^2 + (E^Q[d])^2 - 2M_i E^Q[d]}{V_i}
\]

\[
= U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (E^Q[d])^2}{V_i}.
\]

Substituting in \(U_{2i}, M_i\) and \(V_i\) yields

\[
2r \left[ U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (E^Q[d])^2}{V_i} \right]
\]

\[
= \frac{h_i \left[ m_i \hbar - \hbar m^h + rZ/I \right]^2}{\hbar^2 + h_i h_\varepsilon} + \left[ \frac{h_i h_i m_i + \hbar m^h - rZ/I}{\hbar^2 + h_i h_\varepsilon} \right]^2 - (E^Q[d])^2.
\]

Let \(B \equiv \hbar m^h - rZ/I\) such that

\[
2r \left[ U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (E^Q[d])^2}{V_i} \right]
\]

\[
= \frac{h_i \left[ m_i \hbar - B \right]^2}{\hbar^2 + h_i h_\varepsilon} + \left[ \frac{h_i h_i m_i + \hbar B}{\hbar^2 + h_i h_\varepsilon} \right]^2 - \frac{\hbar^2 + h_i h_\varepsilon}{\hbar^2 + h_i h_\varepsilon} B^2
\]

\[
= \frac{h_i \left[ m_i \hbar - B \right]^2}{\hbar^2 + h_i h_\varepsilon} + \left[ \frac{h_i h_i m_i + \hbar B}{\hbar^2 + h_i h_\varepsilon} \right]^2 - \frac{\left( \frac{1 + h_i h_\varepsilon}{\hbar^2 + h_i h_\varepsilon} \right) \left( \hbar^2 + h_i h_\varepsilon \right)}{\hbar^2 + h_i h_\varepsilon} B^2
\]

\[
= \frac{1}{\hbar \varepsilon \left( \hbar^2 + h_i h_\varepsilon \right)} \left[ h_i \left[ m_i \hbar - B \right]^2 + \left[ h_i h_i m_i + \hbar B \right]^2 - \frac{1}{\hbar} \left( \hbar^2 + h_i h_\varepsilon \right) \left( \hbar^2 + h_i h_\varepsilon \right) B^2 \right]
\]

\[
= \frac{h_i}{\hbar \varepsilon \left( \hbar^2 + h_i h_\varepsilon \right)} \left[ m_i^2 \hbar^2 - B^2 + h_i h_i m_i^2 \hbar - \frac{1}{\hbar^2} h_i h_i B^2 \right]
\]

\[
= \frac{h_i}{\hbar \varepsilon \left( \hbar^2 + h_i h_\varepsilon \right)} \left[ m_i^2 \left[ \hbar^2 + h_i h_\varepsilon \right] - B^2 \hbar^2 + h_i h_i \right]
\]

\[
= h_i \left[ m_i^2 - \frac{B^2}{\hbar^2} \right].
\]
Thus, we obtain that
\[
U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (E^Q[d])^2}{V_i} = \frac{1}{2} \rho h_i \left[ m^2_i - \frac{\tilde{h}m_i - rZ/I}{h} \right]^2 = \frac{1}{2} \rho h_i \left[ m_i^2 - (E^Q[d])^2 \right].
\]

Hence,
\[
\lambda = \delta + \frac{1}{I} \sum_{i=1}^I U_{1i} + \frac{1}{I} \sum_{i=1}^I (U_{2i} + M_i x_{i0} - \frac{1}{2} rV_i (x_{i0})^2) - rd_0 Z/I
\]
\[
= \delta + \frac{1}{I} \sum_{i=1}^I \log \left[ 1 + \frac{(\tilde{h} - h_i)^2}{h_i} \frac{h_i}{(\tilde{h} + h_i)^2} \right] + \Phi \left( \{m_i, \sigma_i^2\}_{i=1, \ldots, I} \right),
\]
where
\[
\Phi \left( \{m_i, \sigma_i^2\}_{i=1, \ldots, I} \right) \equiv \frac{1}{I} \sum_{i=1}^I \log \left[ m_i^2 - (E^Q[d])^2 - rd_0 Z/I \right].
\]
Inserting the expression for \( E^Q[d] \) in (12) yields
\[
\Phi \left( \{m_i, \sigma_i^2\}_{i=1, \ldots, I} \right) = \frac{1}{I} \sum_{i=1}^I \log \left[ m_i^2 - (\tilde{m}_i^2 - r\tilde{\sigma}^2 Z/I)^2 \right] - rd_0 Z/I
\]
\[
= \frac{1}{I} \sum_{i=1}^I \log \left[ \frac{h_i m_i^2}{(\tilde{h} + h_i)^2} \right] - \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{I} \left( \frac{h_i m_i}{\tilde{h}} \right)^2 \frac{Z/I}{I - \frac{1}{I} \sum_{i=1}^I h_i m_i^2} \left( \frac{h_i}{\tilde{h}} \right)^2 \frac{Z/I}{I - \frac{1}{I} \sum_{i=1}^I h_i m_i^2}.
\]
If the investors have homogeneous prior dividend means, i.e., \( m_i = m \), then
\[
\Phi \left( \{m, \sigma_i^2\}_{i=1, \ldots, I} \right) = \frac{1}{I} \sum_{i=1}^I \log \left[ \frac{h_i m_i^2}{(\tilde{h} + h_i)^2} \right] - \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{I} \left( \frac{h_i m_i}{\tilde{h}} \right)^2 \frac{Z/I}{I - \frac{1}{I} \sum_{i=1}^I h_i m_i^2} \left( \frac{h_i}{\tilde{h}} \right)^2 \frac{Z/I}{I - \frac{1}{I} \sum_{i=1}^I h_i m_i^2}
\]
\[
= \frac{1}{I} \sum_{i=1}^I \log \left[ \frac{h_i m_i^2}{(\tilde{h} + h_i)^2} \right] - \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{I} \left( \frac{h_i m_i}{\tilde{h}} \right)^2 \frac{Z/I}{I - \frac{1}{I} \sum_{i=1}^I h_i m_i^2}
\]
\[
= \frac{1}{I} \sum_{i=1}^I \log \left[ \frac{h_i m_i^2}{(\tilde{h} + h_i)^2} \right] - \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{I} \left( \frac{h_i m_i}{\tilde{h}} \right)^2 \frac{Z/I}{I - \frac{1}{I} \sum_{i=1}^I h_i m_i^2}
\]
If the investors have homogeneous prior dividend precisions, i.e., \( h_i = \tilde{h} \), then
\[
\lambda = \delta + \Phi \left( \{m_i, \sigma_i^2\}_{i=1, \ldots, I} \right).
\]
Hence, the equilibrium interest rate is independent of the signal precision, since the function \( \Phi (\cdot) \) only depends on the prior beliefs.

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Proof of Proposition 3: With identical prior dividend precisions, let \( h_i = h \), for \( i = 1, ..., I \). Hence,

\[ h = h_i, \quad m = \frac{1}{I} \sum_{i=1}^{I} m_i, \]

and

\[ m_{i1} = \frac{h_{i\varepsilon}}{h + h_{i\varepsilon}} y + \left( \frac{h}{h + h_{i\varepsilon}} \right) m_i = \frac{1}{h + h_{i\varepsilon}} (h_{i\varepsilon} y + h m_i). \]

This implies that

\[
x_{i11}^*(y) = \rho h_{i1} \left( m_{i1} - \left[ \bar{m}^h - \sigma_2^2 Z/I \right] \right) \\
= \rho [h_{i\varepsilon} + h] \left( m_{i1} - \frac{1}{\rho [h_{i\varepsilon} + h]} I \left( \sum_{i=1}^{I} \rho [h_{i\varepsilon} + h] m_{i1} - Z \right) \right) \\
= \rho (h_{i\varepsilon} y + h m_i) - \frac{1}{I} \left( \sum_{i=1}^{I} \rho [h_{i\varepsilon} + h] \frac{1}{h + h_{i\varepsilon}} (h_{i\varepsilon} y + h m_i) - Z \right) \\
= \rho (h_{i\varepsilon} y + h m_i) - \frac{1}{I} \left( \sum_{i=1}^{I} \rho (h_{i\varepsilon} y + h m_i) - Z \right) \\
= \rho h m_i - \rho \bar{m}^h + Z/I = \rho h \left\{ m_i - \left[ \bar{m}^h - \sigma^2 Z/I \right] \right\}.
\]

Using the expression for \( E^Q[d] \), i.e., (12), we have that

\[ E^Q[d] = \bar{m}^h - \sigma^2 Z/I, \]

and Remark 2 yields that

\[ x_{i11}^*(y) = \rho h \left\{ m_i - E^Q[d] \right\} = x_{i0}^*. \]

Therefore, the demand for riskless asset is also time- and signal-invariant due to

\[ \gamma_{i1}^* = \gamma_{i0}^* + p_1(y) x_{i0}^* - p_i(y) x_{i1}^*(y) = \gamma_{i0}^*. \]

Proof of Proposition 4: By Proposition 2 the equilibrium interest rate attains its maximum value when the logarithmic term in (17),

\[
\frac{1}{2} \sum_{i=1}^{I} \ln \left[ 1 + \frac{(h_i - h_{i\varepsilon})^2}{h_i} \frac{h_{i\varepsilon}}{(h + h_{i\varepsilon})^2} \right],
\]

is maximized, where

\[ h_i = h_i^* \quad \text{and} \quad m_i = m_i^*. \]
is maximized with respect to $h$. This term is maximized whenever the common term for all investors $h/ (\bar{h} + h)^2$ is maximized. The first-order condition is

$$
\left(\frac{\bar{h} + h}{\bar{h} + h} \right)^2 - 2h^{2} \left(\frac{\bar{h} + h}{\bar{h} + h} \right)^4 = 0 \Leftrightarrow \bar{h}^2 + h^2 + 2\bar{h}h - 2h^2 = 0 \Leftrightarrow \bar{h} - h^2 = 0 \Leftrightarrow (\bar{h} - h) (\bar{h} + h) = 0 \Leftrightarrow \bar{h} = h,
$$

where the last equation follows from the fact that $\bar{h} + h > 0$. Similarly, $h/ (\bar{h} + h)^2$ is minimized (and is equal to zero) for $h = 0$ and $h \to \infty$.

Note the second derivative of the equilibrium interest rate with respect to the signal precision is

$$
-4 \left(\frac{\bar{h} + h}{\bar{h} + h} \right)^{-3} + 6h \left(\frac{\bar{h} + h}{\bar{h} + h} \right)^{-4} = 2 \left(\frac{\bar{h} + h}{\bar{h} + h} \right)^{-3} \left(-2 + 3h \left(\frac{\bar{h} + h}{\bar{h} + h} \right)^{-1}\right).
$$

Hence, the equilibrium interest rate has only one inflection point with respect to the signal precision, i.e., $h_{ip} = 2\bar{h}$. Since $\left(\frac{\bar{h} + h}{\bar{h} + h} \right)^{-3} > 0$, thus, when $h \geq h_{ip}$ ($h \leq h_{ip}$), the second derivative of the equilibrium interest rate with respect to the signal precision is positive (negative), thus, the equilibrium interest rate is convex (concave) with respect to the signal precision.

**Proof of Remark 3:** Using that (see the proof of Remark 1)

$$
\bar{m}^h_i = \sigma^2 \left[ h_{\xi} y + \bar{h} m^h \right] = \frac{1}{\bar{h} + h} \left[ h_{\xi} y + \bar{h} m^h \right],
$$

(4) can be re-written as

$$
x^*_{i1}(y) = \rho h_{i1} \left(m_{i1} - \left[ \bar{m}_i^h - r\sigma^2 Z/I \right] \right)
= \rho \left(h_{\xi} y + \bar{h} m^h - \frac{h_i + h_{\xi}}{\bar{h} + h} \left[ h_{\xi} y + \bar{h} m^h - r Z/I \right] \right)
= \rho \left(h_{\xi} \left(\frac{\bar{h} - h_i}{\bar{h} + h} \right) y + h_i m_i - \frac{\bar{h} (h_i + h_{\xi})}{\bar{h} + h} \left[ m^h - r\sigma^2 Z/I \right] \right)
= \rho \left(h_{\xi} \left(\frac{\bar{h} - h_i}{\bar{h} + h} \right) y + h_i m_i - \frac{\bar{h} (h_i + h_{\xi})}{\bar{h} + h} \text{E}^Q[d] \right).
$$
It then follows from (14) that
\[
\tau_i^*(y) = \rho \left( \frac{h}{h + h_{i}} y + h_{i}m_i - \frac{\bar{h} (h_{i} + h_{e})}{h + h_{e}} E^Q[d] \right) - \rho h_{i} \left[ m_i - E^Q[d] \right]
\]
\[
= \rho \left( \frac{h}{h + h_{i}} y - \left[ \frac{\bar{h} (h_{i} + h_{e})}{h + h_{e}} - h_{i} \right] E^Q[d] \right)
\]
\[
= \rho \left( \frac{h}{h + h_{i}} y - \left[ \bar{h} - h_{i} \right] h_{e} E^Q[d] \right) = \rho \frac{h - h_{i}}{h + h_{e}} \left( y - E^Q[d] \right)
\].

The risk-adjusted expected equilibrium net-trade is
\[
E^Q[\tau_i^*(y)] = \frac{h_{e}}{h + h_{e}} (E^Q[y] - E^Q[d]) = \frac{h_{e}}{h + h_{e}} (E^Q[d + \varepsilon] - E^Q[d])
\]
\[
= \rho \frac{h - h_{i}}{h + h_{e}} E^Q[\varepsilon] = 0.
\]

\[\blacksquare\]

**Proof of Proposition 5:** Using that the expected value of the absolute value of a zero-mean normally distributed variable \(X\) is
\[
E[|X|] = \sqrt{2/\pi} \sqrt{\text{Var}[X]},
\]
the expected abnormal trading volume is
\[
E[T^*] = \frac{1}{2 I} \sum_{i=1}^{I} E[|a \tau_i^*(y)|] = \frac{1}{2 I} \sum_{i=1}^{I} \sqrt{2/\pi} \sqrt{\text{Var}[a \tau_i^*(y)]}
\]
\[
= \frac{1}{2 I} \sum_{i=1}^{I} \sqrt{\rho^2 \frac{h_{e}^2}{(h + h_{e})^2} \sigma_{\varepsilon}^2} = \frac{\sqrt{h_{e}}}{h + h_{e}} \rho \sqrt{2/\pi} \sum_{i=1}^{I} |\bar{h} - h_{i}| .
\]

As a function of the signal precision, the expected abnormal trading volume can be expressed as
\[
E[T^*] = a \sqrt{\frac{h_{e}}{(h + h_{e})^2}}
\]
with \(a\) being a positive constant. The comparative statics stated in the proposition then follows from the proof of Proposition 4, and the fact that the square-root function is a strictly increasing function. \[\blacksquare\]

**Proof of Lemma 1:** Using (8), the equilibrium expected utility of investor \(i\) can be
Expressed as

\[ EU_0^* = -\exp(-rCE_{i0}^*) [1 + \exp(-\delta) \exp(-r \{CE_{i2} - CE_{i0}^*\})] \]

\[ = -\exp(-rCE_{i0}^*) [1 + \exp(-\delta) \exp(-\mu - \delta)] = -\exp(-rCE_{i0}^*) [1 + \beta_0]. \]

Hence,

\[
\frac{\partial}{\partial h_\varepsilon} EU_0^* = r \exp(-rCE_{i0}^*) [1 + \beta_0] \frac{\partial}{\partial h_\varepsilon} CE_{i0}^* - \exp(-rCE_{i0}^*) \frac{\partial}{\partial h_\varepsilon} \beta_0
\]

\[ = r \exp(-rCE_{i0}^*) \left\{ [1 + \beta_0] \frac{\partial}{\partial h_\varepsilon} CE_{i0}^* - \rho \frac{\partial}{\partial h_\varepsilon} \beta_0 \right\}. \]

Investor \(i\)'s \(t = 0\) certainty equivalent is given by

\[ CE_{i0}^* = H(\beta_0) - \beta_0 \gamma_{i0}^*, \]

where

\[ H(\beta_0) \equiv [p_0(\eta) + d_0] \bar{z}_i + \beta_0 \bar{s}_i + \bar{r}_i - p_0(\eta)x_{i0}^* \]

is the value of the endowments minus the investment in the risky asset. The equilibrium investment in the zero-coupon bond is given by its first-order condition (8), i.e.,

\[ \iota = \delta + r \left( \gamma_{i0}^* + U_{1i} + U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 - H(\beta_0) + \beta_0 \gamma_{i0}^* \right). \]

Solving for \(\gamma_{i0}^*\) and using Proposition 2 yield that

\[ \gamma_{i0}^* = \frac{\rho (\iota - \delta) - (U_{1i} + U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 - H(\beta_0))}{1 + \beta_0}. \]

\[ = \frac{U_{1i} - U_{1i} + \rho f \left( \{m_i, \sigma_i^2\}_{i=1,...,I} \right) - (U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 + H(\beta_0))}{1 + \beta_0}. \]

Note by the proof of Proposition 2 that all except for the first two and the last term in the numerator are independent of the signal precision. Hence,

\[
\frac{\partial \gamma_{i0}^*}{\partial h_\varepsilon} = \left[ \frac{\partial}{\partial h_\varepsilon} U_{1i} - \frac{\partial}{\partial h_\varepsilon} U_{1i} + H'(\beta_0) \frac{\partial}{\partial h_\varepsilon} \beta_0 \right] (1 + \beta_0) - \gamma_{i0}^* (1 + \beta_0) \frac{\partial}{\partial h_\varepsilon} \beta_0
\]

\[ = \frac{\partial}{\partial h_\varepsilon} U_{1i} - \frac{\partial}{\partial h_\varepsilon} U_{1i} + [H'(\beta_0) - \gamma_{i0}^*] \frac{\partial}{\partial h_\varepsilon} \beta_0}{1 + \beta_0}. \]
This implies that

$$[1 + \beta_0] \frac{\partial}{\partial h^*_\varepsilon} \text{CE}_{i0} = [1 + \beta_0] \frac{\partial}{\partial h^*_\varepsilon} [H(\beta_0) - \beta_0 \gamma^*_i]$$

$$= [1 + \beta_0] \left[ [H'(\beta_0) - \gamma^*_i] \frac{\partial}{\partial h^*_\varepsilon} \beta_0 - \beta_0 \frac{\partial H}{\partial h^*_\varepsilon} \right]$$

$$= [1 + \beta_0] \left[ [H'(\beta_0) - \gamma^*_i] \frac{\partial}{\partial h^*_\varepsilon} \beta_0 - \beta_0 \left[ \frac{\partial}{\partial h^*_\varepsilon} U_1 - \frac{\partial \gamma^*_i}{\partial h^*_\varepsilon} \beta_0 \right] \right]$$

$$= [H'(\beta_0) - \gamma^*_i] \frac{\partial}{\partial h^*_\varepsilon} \beta_0 - \beta_0 \left[ \frac{\partial}{\partial h^*_\varepsilon} U_1 - \frac{\partial \gamma^*_i}{\partial h^*_\varepsilon} \beta_0 \right].$$

Furthermore, it follows from (8) and Proposition 2 that

$$\frac{\partial}{\partial h^*_\varepsilon} \beta_0 = \frac{\partial}{\partial h^*_\varepsilon} \exp \left( - (\delta + r (\text{CE}_{i2}^* - \text{CE}_{i0}^*)) \right)$$

$$= -r \exp \left( - (\delta + r (\text{CE}_{i2}^* - \text{CE}_{i0}^*)) \right) \frac{\partial}{\partial h^*_\varepsilon} [\text{CE}_{i2}^* - \text{CE}_{i0}^*]$$

$$= -r \exp \left( - (\delta + r (\text{CE}_{i2}^* - \text{CE}_{i0}^*)) \right) \frac{\partial}{\partial h^*_\varepsilon} U_1 = -r \beta_0 \frac{\partial}{\partial h^*_\varepsilon} U_1.$$

Hence,

$$\frac{\partial}{\partial h^*_\varepsilon} \text{EU}_{i0}^* = r \exp (-r \text{CE}_{i0}^*) \left\{ [1 + \beta_0] \frac{\partial}{\partial h^*_\varepsilon} \text{CE}_{i0} - \rho \frac{\partial}{\partial h^*_\varepsilon} \beta_0 \right\}$$

$$= r \exp (-r \text{CE}_{i0}^*) \left\{ [H'(\beta_0) - \gamma^*_i] \frac{\partial}{\partial h^*_\varepsilon} \beta_0 - \beta_0 \left[ \frac{\partial}{\partial h^*_\varepsilon} U_1 - \frac{\partial \gamma^*_i}{\partial h^*_\varepsilon} \beta_0 \right] + \beta_0 \frac{\partial}{\partial h^*_\varepsilon} U_1 \right\}$$

$$= r \exp (-r \text{CE}_{i0}^*) \left\{ \beta_0 \frac{\partial}{\partial h^*_\varepsilon} U_1 + [\gamma^*_i - H'(\beta_0)] \frac{\partial}{\partial h^*_\varepsilon} \beta_0 \right\}$$

$$= r \exp (-r \text{CE}_{i0}^*) \beta_0 \left\{ \frac{\partial}{\partial h^*_\varepsilon} U_1 + r [\gamma^*_i - H'(\beta_0)] \frac{\partial}{\partial h^*_\varepsilon} U_1 \right\}.$$

Using the fact that the risk-adjusted expected dividend of the risky asset is independent of \( h^*_\varepsilon \), it follows that

$$H'(\beta_0) = E^Q[d] [\bar{\varepsilon}_i - x^*_i] + \bar{\gamma}_i.$$

Hence,

$$\frac{\partial}{\partial h^*_\varepsilon} \text{EU}_{i0}^* = r \exp (-r \text{CE}_{i0}^*) \left\{ \beta_0 \frac{\partial}{\partial h^*_\varepsilon} U_1 + [\gamma^*_i - (\beta_0 + H'(\beta_0)] \frac{\partial}{\partial h^*_\varepsilon} \beta_0 \right\}.$$
Furthermore, it follows from (8) and Proposition 2 that

$$\frac{\partial}{\partial h_\varepsilon} \beta_0 = \frac{\partial}{\partial h_\varepsilon} \exp \left( - (\delta + r (C_{i\varepsilon}^* - C_{i0}^*)) \right)$$

$$= -r \exp \left( - (\delta + r (C_{i\varepsilon}^* - C_{i0}^*)) \right) \frac{\partial}{\partial h_\varepsilon} [C_{i\varepsilon}^* - C_{i0}^*]$$

$$= -r \exp \left( - (\delta + r (C_{i\varepsilon}^* - C_{i0}^*)) \right) \frac{\partial}{\partial h_\varepsilon} U_1 = -r \beta_0 \frac{\partial}{\partial h_\varepsilon} U_1.$$

Since both $U_{1i}$ and $U_1$ have a unique maximum for $h_\varepsilon = \bar{h}$, all investors’ expected utilities have a stationary point for $h_\varepsilon = \bar{h}$. ■

Proof of Proposition 7: (a) Inserting (31b) into the expression for the equilibrium interest rate and using the market clearing conditions yield

$$\nu = \delta + r \frac{1}{I} \sum_{i=1}^{I} \left[ C_{i2}^\dagger - C_{i0}^\dagger \right]$$

$$= \delta + r \frac{1}{I} \sum_{i=1}^{I} \left[ C_{i2}^* + (\gamma_i^\dagger - \gamma_i^* \right] + \frac{1}{2} \rho (E^0 [d])^2 (h_i - \bar{h}) + \Upsilon_i - U_{1i} - C_{i0}^*$$

$$= \delta + r \frac{1}{I} \sum_{i=1}^{I} \left[ C_{i2}^* + \Upsilon_i - U_{1i} - C_{i0}^* \right]$$

$$= \delta + r \bar{\Upsilon} - r \bar{U}_1 + r \frac{1}{I} \sum_{i=1}^{I} \left[ C_{i2}^* - C_{i0}^* \right] = \delta + r \bar{\Upsilon} - r \bar{U}_1 + r \left[ C_{i2}^* - C_{i0}^* \right],$$

where the second equality follows from the fact that $\sum_i C_{i0}^\dagger = \sum_i C_{i0}^* = d_0$. It then follows from (16) and (17) that

$$\nu = \delta + r \bar{\Upsilon} - r \bar{U}_1 + r \bar{U}_1 + \phi \left( \left\{ m_i, \sigma_i^2 \right\}_{i=1,...,I} \right) = \delta + r \bar{\Upsilon} + \phi \left( \left\{ m_i, \sigma_i^2 \right\}_{i=1,...,I} \right).$$

(b) follows from the fact that both $\bar{\Upsilon}$ and $\phi (\cdot)$ only depend on the prior dividend beliefs. (c) Comparing (17) and (32), we must show that $\bar{\Upsilon} > \bar{U}_1$ for any signal precision $h_\varepsilon$. Note that

$$\max_{h_\varepsilon} U_{1i} = \max_{h_\varepsilon} \frac{1}{2} \rho \ln \left[ 1 + \left( \frac{\bar{h} - h_i}{h_i} \right)^2 \right] \left[ \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right]$$

$$= \frac{1}{2} \rho \ln \left[ 1 + \left( \frac{\bar{h} - h_i}{4\bar{h}h_i} \right)^2 \right] = \frac{1}{2} \rho \ln \left[ \frac{\bar{h}^2 + h_i^2 + 2\bar{h}h_i}{4\bar{h}h_i} \right].$$

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Hence, for investors $i$ with $h_i \neq \bar{h}$

$$\Upsilon_i - U_{1i} = \frac{1}{2} \rho \left\{ \ln \left[ \frac{\bar{h}}{h_i} \right] - \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i (\bar{h} + h_i)^2} \right] \right\} > \frac{1}{2} \rho \left\{ \ln \left[ \frac{\bar{h}}{h_i} \right] - \ln \left[ \frac{\bar{h}^2 + h_i^2 + 2\bar{h}h_i}{4\bar{h}h_i} \right] \right\} = \frac{1}{2} \rho \ln \left[ \frac{4\bar{h}h_i}{(\bar{h} + h_i)^2} \right] = \rho \ln \left[ \frac{2\bar{h}}{\bar{h} + h_i} \right].$$

This implies that in settings with heterogeneous prior dividend precisions

$$\Upsilon - \bar{U}_1 > \frac{1}{T} \sum_{i=1}^{T} \ln \left[ \frac{2\bar{h}}{\bar{h} + h_i} \right] = \frac{1}{T} \sum_{i=1}^{T} \left[ \ln [2\bar{h}] - \ln [\bar{h} + h_i] \right]$$

$$= \ln [2\bar{h}] - \frac{1}{T} \sum_{i=1}^{T} \ln [\bar{h} + h_i] > \ln [2\bar{h}] - \ln \left[ \frac{1}{T} \sum_{i=1}^{T} [\bar{h} + h_i] \right]$$

$$= \ln [2\bar{h}] - \ln [2\bar{h}] = 0,$$

where the second inequality follows from Jensen’s inequality and the fact that $\ln[\cdot]$ is a concave function. This establishes (c).
Chapter 2

Heterogeneous Beliefs, Public Information, and Option Markets
Heterogeneous Beliefs, Public Information, and Option Markets

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Abstract
In an incomplete market setting with heterogeneous prior beliefs, I show that public information and strike price of option have substantial influence on asset pricing in option markets, by investigating an absolute option pricing model with negative exponential utility investors and normally distributed dividend. I demonstrate that heterogeneous prior variances give rise to the economic value of option markets. Investors speculate in option market and public information improves allocational efficiency of markets only when there is heterogeneity in prior variance. Heterogeneity in mean is neither a necessary nor sufficient condition for generating speculations in option markets. With heterogeneous beliefs, options are non-redundant assets which can facilitate side-betting and enable investors to take advantage of the disagreements and the differences in confidence. This fact leads to a higher growth rate in the investors’ certainty equivalents and, thus, a higher equilibrium interest rate. The public information system facilitates improved dynamic trading opportunities in option markets based on the heterogeneously updated posterior beliefs. With an intermediate signal precision and the option with intermediate strike price, the highest efficiency of side-betting is achieved, reflected by a unique maximum point of the ex ante equilibrium interest rate. The public signal precision affects ex ante equilibrium risk premium only via its relationship with option.

Keywords: Heterogeneous Beliefs; Public Information Quality; Option Market; Dynamic Trading; Bayesian Learning

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1 Introduction

Black and Scholes (1973) develop a relative option pricing model by introducing the concept that the value of an option can be replicated by trading underlying asset continuously, hence in their setting, the option is a redundant asset. However, in the real world, hedgers, speculators, and arbitrageurs trade actively in all the main option exchanges\(^1\). This fact signals options are far from redundant, and are capable of facilitating market efficiency. Otherwise, investors are indifferent in holding them, and option markets vanish due to non-negligible maintenance costs. Although several factors such as stochastic volatility, jumps, trading frictions (e.g., transaction costs, and capital constraints), and heterogeneous information, are found to be responsible for the non-redundancy of options\(^2\), apparently this problem is not yet addressed to a completely satisfying extent. On the other hand, public information or signal such as earnings and dividend announcements, mergers and acquisitions, macroeconomic announcements, accounting reports are long recognized to have substantial impacts on the financial markets\(^3\). However, few studies shed light on the relationship between options and public information quality. Thus, questions arise: Under what condition does the public information quality exert influence on option market? How does public information quality affect the non-redundancy of option markets? Does public signal precision interact with option to play important roles in forming an allocational efficient market? As an attempt to answer those questions, this paper provides information-based models with heterogeneous beliefs which yield explanations for non-redundancy of options, and address the role of public signal precision in the option markets.

I develop a two-period absolute option pricing model in an incomplete market setting with many negative exponential utility investors holding heterogeneous prior beliefs, and where the dividend is normally distributed. I achieve mainly two findings. First, heterogeneous prior variances provide economic value to the option markets in the sense that the investors speculate in the option market, which indicates the option is non-redundant, and imperfect public signal improves the allocational efficiency of option markets only when there is heterogeneity in prior precision. Second, even though the strike price of option affects the ex ante risk premium on the risky underlying asset regardless of the presence of the public

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\(^2\)Related works include Hull and White (1987), Heston (1993), Liu and Pan (2003), and Brennan and Cao (1996) among many others. Note Brennan and Cao (1996) use a quadratic derivative to achieve Pareto efficient consumption allocations, but their model requires new exogenous supply shocks to the stock to generate trading volume.

\(^3\)Recent empirical evidence suggests that macroeconomic announcements and employment figures have pronounced impact on financial markets. Literatures include, e.g., Feltham and Pae (2000), Andersen, Bollerslev, Diebold, and Vega (2003), and Richardson, Sloan, Soliman, and Tuna (2005).
signal, the public signal precision (the invert of variance of noise) affects the \textit{ex ante} risk premium only via its relationship with the option.

The model is a derivative oriented and two-period extension of the classic single-period capital asset pricing model (CAPM) with heterogeneous beliefs of Lintner (1969). Specifically, the investors hold different prior beliefs at $t = 0$ on the normally distributed $t = 2$ dividend, i.e., the prior beliefs of mean and precision (the inverse of variance) differ. These assumptions imply that the investors agree to disagree due to, for instance, difference in their experiences or DNA. The investors update their beliefs according to the Bayes’ rule with a public signal received at $t = 1$ from a simple public information system.\footnote{Note this paper considers the economy-wide impacts of public information and, thus, the public information should be interpreted as, for instance, macroeconomic reports of aggregate consumption. And an information system is a set of potential signals that present conditional (or signal dependent) probabilities that each state at the terminal date occurs.} The public signal is equal to the $t = 2$ dividend on the risky underlying asset plus independent noise. Moreover, the investors have concordant beliefs (Milgrom and Stokey 1982) or homogeneous information beliefs (Hakansson, Kunkel, and Ohlson 1982) on the normally distributed noise in the signal, i.e., a zero mean and a common signal precision. These assumptions allow to measure the informativeness of public information system by the public signal precision. Furthermore, the investors can trade and speculate in the option markets, the underlying asset markets and the zero-coupon bond markets at $t = 0$ and $t = 1$, and consume at $t = 0$ and $t = 2$. Solving for equilibriums in an exchange economy, I investigate the impact of the heterogeneity in beliefs, the strike price, and the public information quality on option pricing and the other asset pricing properties in option markets.

Heterogeneity in the prior variance creates the opportunities for speculation in option markets. With homogeneous prior variance, the investors do not trade in the option markets. The intuition is related to the results in Wilson (1968): Pareto efficient allocations in settings with heterogeneous beliefs require not only an efficient sharing of the risks, but also an efficient side-betting arrangement. With homogeneous prior variance, the Pareto efficient side-betting based on their disagreements about the mean can be achieved by trading only in the risky underlying asset and the zero-coupon bond at $t = 0$. The CAPM-like equilibrium price under heterogeneous beliefs is obtained. However, when the investors have different prior precision, trading only in the underlying asset and the zero-coupon bond at $t = 0$ does not facilitate efficient side-betting. The investors tend to speculate in the option markets. Take a European call option market for example, the investor with a low (high) prior precision takes long (short) position in the call option with convex payoff to achieve a terminal payoff which is a convex (concave) function of the dividend. This speculative strategy is the so-
Speculations in option markets increase the allocational efficiency of the equilibrium. This result can be detected from the change of asset pricing properties when investors' speculative behaviors change. First, conditional on identical average prior precision, the higher heterogeneity in beliefs, the more opportunities in speculations, and the more advantage of the disagreements and the differences in confidence among the investors can be taken. This effect leads to a higher efficiency of side-betting and more gains in trading options. The trading gains translate into increased certainty equivalents of the terminal consumption, and result in a higher equilibrium consumption growth, and thus a higher equilibrium interest rate. Second, investors tend to trade in options with an intermediate strike price. Since this type of options carry the most substantial convexity in their payoff, and thus the investors can effectively and actively speculate in the option markets. Third, the imperfect public signal facilitates speculations. When the investors have heterogeneous prior dividend precision, they update their posterior beliefs differently with imperfect public signal, and this gives the basis for additional trading gains contingent on the imperfect public signal. Another round of trading using Gamma trading strategies at $t = 1$ partly facilitates the efficient side-betting. Eventually, a combination of the option with intermediate strike price and public information system of the intermediate signal precision enables the investors to achieve the highest efficiency of side-betting, reflected by the unique maximum point of the 

Public signal precision affects the \textit{ex ante} equilibrium risk premium on the risky underlying asset via options. The underlying mechanism is that the convexity of the option payoff varies with the public signal precision, through this relationship, the speculative positions in the underlying asset and the option are affected by the public signal precision. This fact gives rise to a signal-precision-dependent covariance between the marginal utility of consumption and the dividend and, thus, a signal-precision-dependent \textit{ex ante} equilibrium risk premium. With an intermediate strike price, the impact of the public signal precision on the \textit{ex ante} equilibrium risk premium is nontrivial (see Figure 9). Compare to the benchmark model in Christensen and Qin (2012), in which the \textit{ex ante} risk premium is independent of the public signal precision, the \textit{ex ante} risk premium in this paper is not aligned with investor welfare (as a function of the signal precision). The equilibrium interest rate and the investor welfare, however, are still perfectly aligned. This fact has an implication that it may be wise to be cautious in making policy statements about, for example, financial reporting regulation, based on empirical measures of equity premia (which are hard to measure reliably anyway).

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\footnote{See more about Gamma trading strategies in e.g., Hull 2009, Chapter 17.}
Review of the Literature

Some studies on the impact of information system and heterogeneity in beliefs on asset pricing are closely related to this work. Li (2008) assumes that investors believe the growth rate of the dividend to be a constant and known perfectly. This assumption enables closed-form solutions for vanilla European option prices and closed-form approximations for barrier options. His model offers a rationale for observed implied volatility patterns in an equilibrium setting and is also easy to implement in practice. However, in his model, the role of the option to facilitate side-betting is not explored, and his model is silent with respect to the influence of information system. Those issues are investigated in this paper.

In a similar effort, Buraschi and Jiltsov (2006) employ a model, based on the work by Detemple and Murthy (1994), to investigate the option markets with heterogeneous beliefs. They show that the heterogeneity in beliefs has significant pricing implications by plotting equilibrium asset pricing properties such as stock price and stock volatility as functions of the difference in beliefs. Their results indicate that the heterogeneous beliefs are strongly related to optimal portfolio holdings, stock volatility, equity premium, stock prices, option prices and skewness in equity returns.

My model differs with Buraschi and Jiltsov (2006) in several aspects. First, the investors’ utility functions and the dividend structure are different. Second, Cuoco and He (1994) demonstrate the equilibrium with a stochastic weight in the representative agent utility is in general not Pareto efficient. Hence markets in Buraschi and Jiltsov (2006) are essentially incomplete, and they do not illustrate how much the option can help to improve the efficiency of side-betting. This paper demonstrates to what degree the option can enhance the market allocational efficiency. Third, the approach to study the impacts of heterogeneity in beliefs is different. They plot the asset pricing properties as functions of the difference in the updated beliefs scaled by the signal volatility, hence their heterogeneous beliefs carry the effect of the information system. In fact, they do not study the impact of the information quality. However, I plot the asset pricing properties as functions of the different priors, thus the analysis in this paper probes the impact of the heterogeneity in beliefs and the public information quality separately.

Also note that Buraschi and Jiltsov (2006) and David (2008) solve for the equilibrium by constructing a representative agent utility through taking weight of the two different individual utilities, and the weight is stochastic and endogenized in the equilibrium as a

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6Recent contributions of asset pricing in economies with incomplete information include, e.g., David (1997), Brennan (1998), Veronesi (1999, 2000), and Brennan and Xia (2001). Efforts to establish the so-called differences-of-opinion models in financial markets include, e.g., Cao and Ou-Yang (2009), and Banerjee and Kremer (2010). These literature assume the investors have homogeneous beliefs about the fundamentals in the economy, but disagree on how to interpret common public signals.
function of the difference in beliefs. Hence the change of heterogeneity in beliefs results in the change of the value of the weight. However, when plotting the asset pricing properties as functions of the heterogeneity in beliefs, they fix the value of the weight at a certain level. Hence, their approach to analyze the impacts of the heterogeneous beliefs can be considered to be conditional. Models of unconditional analyses are provided in this paper.

Christensen and Qin (2012) study a benchmark model in which the investors speculate in a risky asset and a zero-coupon bond. They show that the public information system is able to facilitate side-betting and enable the investors to take advantage of the disagreements in the prior variance. A more efficient market gives rise to a higher equilibrium interest rate. 

Based on the idea of Wilson (1968), Christensen and Qin (2012) introduce a dividend derivative which pays off the square of dividend in the terminal date to facilitate side-betting and achieve a Pareto efficient equilibrium. This extreme case gets rid of the need for dynamic trading based on the public signals. Note that the dividend derivative in their model resembles a “smooth” straddle. This idea gives a motivation to employ a true straddle, i.e., long positions in both a call and a put option with the same strike price, as a replacement of the dividend derivative and explore the ability of the options in facilitating side-betting. The straddle may not be able to effectively complete the market. However, the investors still can employ the Gamma strategy by trading straddles to achieve a more satisfactory convexity of their payoff. This paper investigates this intermediate case to address issues such as the role of options to help establish allocational efficient markets and the role of the public information system in the option markets.

This paper is organized as follows. The primitives of the economy and the learning mechanism are established in Section 2. Section 3 establishes a single-period model in which investors receive no signal or perfect signal, and investigate the effects of the heterogeneous beliefs and the strike price on asset pricing properties such as the equilibrium interest rate and the equilibrium risk premium with numerical examples. Section 4 investigates the equilibrium with imperfect public signal and discusses the effects of the public signal precision on asset pricing properties. Section 5 concludes the paper and discusses possible extensions. Proofs and algorithms to solve for the equilibriums are presented in Appendix A. Appendix B provides the derivations of models when investors speculate with straddles.

2 The Model

I examine the impacts of the heterogeneity in priors, the strike price and the public signal precision on option pricing properties in an exchange economy in which many types of agents have identical preferences but differ in their prior beliefs about the distribution of forthcoming
2.1 The Investors’ Beliefs and Preferences

There are two consumption dates, \( t = 0 \) and \( t = 2 \), and there are \( I \) investors who are endowed at \( t = 0 \) with a portfolio of underlying asset, potentially receive public information at \( t = 1 \), and receive terminal normally-distributed dividends from their portfolio of underlying asset at \( t = 2 \). There are three marketed securities: a zero-coupon bond which pays one unit of consumption at \( t = 2 \) and is in zero net supply, the shares of a single risky firm which have net supplies \( Z \) at \( t = 0 \) and \( t = 1 \), and a European call option with a strike price at \( K \) which has zero net supplies at \( t = 0 \) and \( t = 1 \), and the underlying asset is the share of the firm. The net supplies are fixed: the \( I \) investors are endowed with \( z_i, i = 1, 2, \ldots, I \). In addition, the investors have endowments \( \tilde{\theta}_i \), units of the European call option in zero net supply at \( t = 0 \), and the investors are endowed with \( \tilde{\gamma}_i \), units of \( t = 2 \) zero-coupon bond in zero net supply also. The trading of the marketed securities takes place at \( t = 0 \) and \( t = 1 \), based on heterogeneous prior beliefs. Let \( \theta_{it}, x_{it} \), and \( \gamma_{it} \) present the investor \( i \)'s portfolio of the option, the share and the units had of the zero-coupon bond after trading date \( t \), respectively. Hence, the market clearing conditions at date \( t \) are

\[
\sum_{i=1}^{I} \theta_{it} = 0, \quad \sum_{i=1}^{I} \gamma_{it} = 0, \quad \sum_{i=1}^{I} x_{it} = Z = \sum_{i=1}^{I} \tilde{z}_i, \quad t = 0, 1.
\]

The dividend paid by the firm’s shares at date \( t = 0 \) is denoted \( d_0 \), and at date \( t = 2 \) is denoted \( d \). The investors have heterogeneous prior beliefs on the distribution of the dividend, and the individual perceived distribution is represented by \( N(m_i, \sigma_i^2) \), where \( m_i \) is the expected dividend per share and \( \sigma_i^2 \) is the variance of firm dividends per share for the investor \( i \). Note that since the investors have common and not asymmetric information, they are aware of each others’ different inferences, arising from their different priors. Under this heterogeneous beliefs formulation, the investors agree to disagree. Morris (1995) proposes a method to endogenize the difference in beliefs and formulations. He shows that it is fully consistent with rationality to have heterogeneous priors.

The investor \( i \) has a public information system \( \eta \) reporting public signal \( y \) at \( t = 1 \), which has impacts on asset prices. The investors can trade in the riskless asset in zero net supply with price \( \beta_0 \) at \( t = 0 \) and price \( \beta_1 \) at \( t = 1 \). The underlying asset price and the call option price at \( t = 0 \) are denoted \( p_0(\eta) \) and \( \pi_0(\eta) \), respectively. The underlying asset price and call option price at \( t = 1 \) given the public signals \( y \) are represented by \( p_1(y) \) and \( \pi_1(y) \), respectively, which reflect the fact that the \textit{ex post} asset prices may be affected by the public
signals available to the investors at \( t = 1 \).

The investor \( i \)'s consumption at date \( t \) is denoted \( c_{it} \) and they have time-additive utility. The common period-specific utility is negative exponential utility with respect to consumption, i.e., \( u_{i0}(c_{i0}) = -\exp[-rc_{i0}] \) and \( u_{i2}(c_{i2}) = -\exp[-\delta\exp[-rc_{i2}]] \), where \( r > 0 \) is the investors’ common constant absolute risk aversion parameter. Moreover, the investors have common utility discount rate, \( \delta \), for date \( t = 2 \) consumption.

### 2.2 Learning Mechanism

Bayesian learning provides a core concept of information processing in financial markets. I assume that the investors update their beliefs in a Bayesian fashion.

Specifically, the public signal \( y \) is generated by \( y = d + \varepsilon, \varepsilon \sim N(0, \sigma_\varepsilon^2) \), where \( h_\varepsilon \equiv 1/\sigma_\varepsilon^2 \) is the signal precision which is common knowledge of all the investors. In other words, the investors have concordant beliefs (Milgrom and Stokey 1982) or homogeneous information beliefs (Hakansson, Kunkel, and Ohlson 1982). I use \( h_\varepsilon \equiv 1/\sigma_\varepsilon^2 \) throughout to denote the precisions for the associated variances. When \( h_\varepsilon \to \infty \), the public signal is the realization of the dividend. Moreover, the terminal dividend \( d \) and the noise \( \varepsilon \) are independent and jointly normally distributed. Thus, the joint distribution of the public signal and dividend from the perspective of investor \( i \) is \( \varphi_i(y, d) \sim N(\mu_i, \Sigma_i) \), where

\[
\mu_i = \begin{pmatrix} m_i \\ m_i \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i^2 + \sigma_\varepsilon^2 & \sigma_i^2 \\ \sigma_i^2 & \sigma_i^2 \end{pmatrix},
\]

I represent investor \( i \)'s posterior beliefs of the terminal dividend \( d \) given his signal as \( N(m_{i1}; \sigma_{i1}^2) \), hence the posterior of the investor \( i \) at \( t = 1 \) is \( \varphi_{i1}(d | y) \sim N(m_{i1}, \sigma_{i1}^2) \), where

\[
m_{i1} = m_i + \sigma_i^2 \left( \sigma_i^2 + \sigma_\varepsilon^2 \right)^{-1} (y - m_i), \quad (1)
\]
\[
\sigma_{i1}^2 = \sigma_i^2 - \sigma_i^2 \left( \sigma_i^2 + \sigma_\varepsilon^2 \right)^{-1} \sigma_i^2. \quad (2)
\]

Therefore, the posterior mean, \( m_{i1} \), is a linear function of the investor’s public signal, while the posterior variance, \( \sigma_{i1}^2 \), only depends on the public information system and not on the specific signal.

Moreover, when the variance of the disturbance term \( \sigma_\varepsilon^2 \to 0 \), the posterior mean \( m_{i1} \to y \), hence the posterior mean tends to be independent of the priors as public signal precision increases. When \( \sigma_\varepsilon^2 \to 0 \), the posterior variance \( \sigma_{i1}^2 \to 0 \), this indicates that with higher public signal precision, investors are more confident on their inferences. When \( \sigma_\varepsilon^2 \to \infty \), the public signal disappears, thus the posterior beliefs equal to the prior beliefs. From
the perspective of the investor $i$, the signal $y$ is normally distributed with $N(m_i, \sigma_i^2 + \sigma_e^2)$, hence the posterior mean $m_{i1} \sim N(m_i, \sigma_i^4 (\sigma_i^2 + \sigma_e^2)^{-1})$. For further information on Bayesian learning model, see e.g., Raiffa and Schlaifer (1961) and DeGroot (1970).

3 Equilibrium with Impact from Heterogeneity in Beliefs and Strike Price: Benchmark Case when Investors Receive No or Perfect Public Signal

In this section, I derive equilibriums in the economy populated with investors with heterogeneous beliefs. The investors can speculate in European call option markets. In order to see the effect of the heterogeneous beliefs and the strike price of European call options clearly without interruptions from the impact of the public information system, I first investigate a benchmark case in which the investors receive no or perfect public signal at $t = 1$, then there is no basis for trading at $t = 1$ based on posterior beliefs. Hence, the model is essentially equivalent to a single period model.

3.1 Equilibrium in a Single Period Economy

The payoff of a European call option with strike price $K$ at the date $t = 2$ is $\max(d - K, 0)$. Hence, from the perspective of $t = 0$, the date $t = 2$ consumption for the investor $i$ is

$$c_{i2} = \theta_{i0} \max(d - K, 0) + x_{i0}d + \gamma_{i0}.$$  

(3)

Given the period-specific negative exponential utility, the investor $i$’s date $t = 0$ certainty equivalent of date $t = 2$ consumption, receiving no or perfect public information at $t = 1$, $CE_{i2}(\theta_{i0}, x_{i0}, \gamma_{i0})$ can be calculated according to Lemma 1.

Lemma 1 Assume the investors receive no or perfect public information at $t = 1$, given the portfolios in the underlying asset markets, the call option markets, and the zero-coupon bond markets at $t = 0$, the investor $i$’s $t = 0$ certainty equivalent of $t = 2$ consumption is

$$CE_{i2}(\theta_{i0}, x_{i0}, \gamma_{i0}) = \gamma_{i0} + m_{i0}x_{i0} - \frac{1}{2} r\sigma_{i0}^2 x_{i0}^2 + f_i(x_{i0}, \theta_{i0}),$$  

(4)
where
\[
f_i(x_{i0}, \theta_{i0}) = -\frac{1}{r} \ln \Phi \left( \frac{K - (m_{i0} - r\sigma^2_{i0}x_{i0})}{\sigma_{i0}} \right) + \exp \left[ -r\theta_{i0} \left[ -K + m_{i0} - \frac{1}{2} r\sigma^2_{i0}\theta_{i0} - r\sigma^2_{i0}x_{i0} \right] \right] \times \left( 1 - \Phi \left( \frac{K - (m_{i0} - r\sigma^2_{i0}(\theta_{i0} + x_{i0}))}{\sigma_{i0}} \right) \right),
\]
and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution.

Proof: See Appendix A.

The investor \( i \)'s \( t = 0 \) certainty equivalent equals to the date \( t = 0 \) consumption, i.e.,
\[
CE_{i0} = c_{i2} = d_0 z_i + (\tilde{\theta}_i - \theta_{i0}) \pi_0(\eta) + (\tilde{\gamma}_i - \gamma_{i0}) \beta_0 + (z_i - x_{i0}) p_0(\eta),
\]
the investor \( i \)'s decision problem at \( t = 0 \) can be stated as follows
\[
\max_{\theta_{i0}, \gamma_{i0}, x_{i0}} \quad -\exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}).
\]

To solve for the equilibrium, I first solve the investor \( i \)'s optimal portfolio choice problem,
\[
\frac{\partial (-\exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}))}{\partial x_{i0}} = 0,
\]
\[
\frac{\partial (-\exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}))}{\partial \theta_{i0}} = 0,
\]
\[
\frac{\partial (-\exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}))}{\partial \gamma_{i0}} = 0.
\]

More details of calculation of the first derivatives in the first-order condition are provided in Appendix A. The equilibrium portfolios and the prices are implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset, and the market clearing condition for each asset, i.e.,
\[
\sum_{i=1}^I \gamma_{i0} = 0, \quad \sum_{i=1}^I \theta_{i0} = 0, \quad \text{and} \quad \sum_{i=1}^I x_{i0} = Z.
\]

The implicit solutions of the equations are functions of the prior beliefs. In the following numerical example with two investors, there are nine equations with nine unknowns, i.e., the equilibrium portfolios in the underlying asset markets \( x_{i0}, x_{j0} \), the equilibrium portfolios
in the option markets, $\theta_{i0}, \theta_{j0}$, the equilibrium portfolios in the bond markets, $\gamma_{i0}, \gamma_{j0}$, the equilibrium price for the zero-coupon bond, $\beta_0$, the equilibrium price for the underlying asset, $p_0$, and the equilibrium price for the European call option, $\pi_0$. Given the value of the parameters such as the strike price, the prior mean and the prior variance, I can solve the equations numerically. The methods to further simplify the equations to nonlinear equations with only three unknowns and the algorithms to solve for the equilibrium numerically are described in Appendix A.

### 3.2 The Impact of Heterogeneous Beliefs and Strike Price on Asset Pricing Properties

In this section, I demonstrate how the heterogeneity in beliefs and the strike price of option affect the equilibrium prices at $t = 0$ when the investors hold heterogeneous beliefs including heterogeneous prior means and/or heterogeneous prior variances.

Solving for the equilibrium, I find that with homogeneous prior variance, the investors do not trade in the option markets. This is because the Pareto efficient side-betting based on their disagreements about the mean can be achieved by trading only in the risky underlying asset and the zero-coupon bond at $t = 0$. The model reduces to a single period benchmark case in which the investors only trade in the stock and the zero-coupon bond. Hence in this section, I only plot the asset pricing properties such as the equilibrium interest rate and the equilibrium asset prices as functions of heterogeneity in prior precision, to study the impact of the heterogeneous beliefs in the option market. Moreover, I plot the asset pricing properties as functions of the strike price to detect the effect of strike price in option markets.

#### 3.2.1 Equilibrium Interest Rate

The equilibrium interest rate in the single period is defined as $\iota \equiv -\ln \beta_0$. By the investor $i$’s decision problem, i.e., Eq. (6), and the investor $i$’s certainty equivalents at $t = 0$ and $t = 2$, i.e., Eq. (5) and Eq. (4), the first-order condition with respect to the portfolio in the zero-coupon bond is given as

$$-r \exp(-rCE_{i0}) \beta_0 + r \exp(-\delta) \exp(-rCE_{i2}) = 0.$$

Solve for the price of zero-coupon bond, I obtain

$$\beta_0 = \exp\left(- (\delta + r (CE_{i2} - CE_{i0}))\right).$$
With the definition of the equilibrium interest rate, I achieve the following proposition.

**Proposition 1** Assume the investors with heterogeneous beliefs can potentially trade in option markets. The equilibrium interest rate is given as the time discount rate plus the risk-adjusted growth in certainty equivalents, i.e.,

$$\iota = \delta + r (CE_i - CE_o).$$  \hspace{1cm} (7)

In Eq. (7), the equilibrium interest rate is expressed as an increasing linear function of the growth in the investors’ certainty equivalents. In equilibrium, due to the assumptions of common constant absolute risk aversion parameter $r$ and common utility discount rate $\delta$, it directly follows from the first-order condition that all the investors have the same growth in certainty equivalents. Note an identical relationship between the equilibrium interest rate and growth in certainty equivalents can be found in the benchmark model in Christensen and Qin (2012).

<table>
<thead>
<tr>
<th>Risk aversion ($r$)</th>
<th>0.8</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Utility discount rate ($\delta$)</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>Prior mean ($m_i$)</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>Initial endowments of call option ($\theta_i$)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Initial endowments of zero-coupon bond ($\gamma_i$)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Initial dividend ($d_0$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Supply ($Z$)</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

*Table 1: Investor and risky asset parameters of the running example.*

The equilibrium interest rate is endogenized in the equilibrium as a function of the prior precision and the strike price. To see the impact of those two pricing factors, and I plot the equilibrium interest rate as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a given level of the heterogeneity in beliefs in Figure 1 for the parameters in Table 1. Note the parameters are selected to guarantee: First, significant variations in both the level of equilibrium interest rate (from 3% to 6% in Figure 1) and equilibrium risk premium; Second, the scale of both the equilibrium interest rate and the equilibrium risk premium should be reasonable, e.g., the interest rate should be around 5%, and the risk premium should be around 10%; Third, considerably high accuracy of the numerical solutions of nonlinear equation within an acceptable computing time. Also note the general equilibrium properties which are analyzed in this paper maintain effective
regardless the change of the parameters, given the parameters meet basic requirements, for instance, the risk aversion \( r > 0 \), and the prior variance \( \sigma_i^2 > 0 \).

![Figure 1. The Equilibrium Interest Rate.](image)

The equilibrium interest rate is plotted as a function of the heterogeneity in prior precision conditional on a given level of the strike price and the strike price conditional on a given level of the heterogeneity in prior precision. The horizon axes of the left panel indicate two investors’ beliefs on the prior precision. I set the strike price \( K = 0.01 \) when plotting the equilibrium interest rate as function of the heterogeneous beliefs. When plotting the equilibrium interest rate as function of the strike price, the prior variances are set to be \( \sigma_i^2 = 0.0002; \sigma_j^2 = 0.0001 \). I use identical value of parameters for plotting other asset pricing properties in section 3.2, and thus do not repeat those parameter values again.

In each panel, when I plot the asset pricing properties as functions of the heterogeneity in beliefs, a diagonal line indicates that the two investors agree with each other, and depart further from the diagonal line indicates higher heterogeneity in beliefs. As we can see from the left panel of Figure 1, the equilibrium interest rate increases with the heterogeneity in prior precision. Since conditional on identical average prior precision, the higher heterogeneity in beliefs, the more opportunity in speculation in option market, and the more advantage of the disagreements and the differences in confidence among the investors can be taken. This effect leads to a higher efficiency of side-betting and more gains in trading. The trading gains translate into increased certainty equivalents of the \( t = 2 \) consumption, and result in a higher equilibrium consumption growth, and thus a higher equilibrium interest rate. Note the increase of the equilibrium interest rate can be viewed as an analog of the side-betting trading gains. This intuition can be gained directly from Eq. (7).
Furthermore, the diagonal line in the left panel shows that when investors hold homogeneous prior precision, the equilibrium interest rate increases with the prior precision. It is a standard result that with homogeneous belief, the equilibrium interest rate is given as the time discount factors plus a risk-adjusted aggregate consumption growth minus a variance of risk-adjusted aggregate consumption. With homogeneous beliefs, the investors do not trade in the option markets, and the variance of risk-adjusted aggregate consumption decreases with the prior precision.

The right panel of Figure 1 shows the impact of the strike price on the equilibrium interest rate. When the strike price is very high, i.e., the option is very deep out of the money, the value of option is almost zero. When the strike price is much lower than the mean, i.e., the option is very deep in the money, the option resembles a stock paying a dividend with a very high mean but with identical prior variance as that of the underlying asset. As a result, from the right panel of Figure 1, we can see, with very high or very low strike price, the equilibrium interest rate converges to that in the benchmark case in which the investors only trade in a stock and a zero-coupon bond as depicted by the bottom line. This result illustrates the lowest efficiency of side-betting and benefit from the disagreement in the prior variance. When the option is slightly in the money with an intermediate strike price, it is substantially different from nothing or a stock. Thus the investors can effectively and actively speculate in the option markets. This kind of call option can facilitate side-betting, and thus enhance the growth in certainty equivalents and the equilibrium interest rate. With the used parameters, the equilibrium interest rate reaches the maximum point of around 0.062 at an intermediate strike price of around 0.01.

Note that Buraschi and Jiltsov (2006) and David (2008) also plot asset pricing properties as functions of heterogeneity in beliefs. However, they solve for the equilibrium by constructing a representative agent utility through taking weight of the two different individual utilities, and the weight is stochastic and endogenized in the equilibrium as a function of the difference in beliefs. Hence the change of heterogeneity in beliefs results in the change of the value of the weight. However, when plotting the asset pricing properties as functions of the heterogeneity in beliefs, they fix the value of the weight at a certain level. Hence their approach to analyze the impacts of the heterogeneous beliefs can be considered to be conditional. Models in this paper study the asset pricing properties as functions of the heterogeneous beliefs, the strike price, and the public information quality unconditionally.

3.2.2 Equilibrium Risk Premium

Christensen and Qin (2012) define the (continuously compounded) expected rate of return $\mu^{x_o}(\eta)$ using the beliefs implicit in the unambiguous *ex ante* equilibrium price of the risky
underlying asset, i.e., \( \varphi^h(d) \sim N(\bar{m}^h, \sigma^2) \)

\[
\exp(\mu^x(\eta)) \equiv \frac{\bar{m}^h}{p_0(\eta)},
\]

where

\[
\bar{m}^h \equiv \frac{1}{I} \sum_{i=1}^{I} \frac{h_i}{\bar{h}} m_i, \quad \bar{h} \equiv \frac{1}{I} \sum_{i=1}^{I} h_i, \quad \sigma^2 \equiv \frac{1}{\bar{h}}.
\]

I define the expected rate of return in the same way, and obtain the equilibrium risk premium on the risky underlying asset \( \varpi^x = \mu^x(\eta) - \iota \). Since the equilibrium interest rate is defined as \( \iota \equiv -\ln \beta_0 \) and the \emph{ex ante} price of the underlying asset is a product of the equilibrium riskless discount factor and the risk-adjusted expected dividend, i.e., \( p_0(\eta) = \beta_0 \mathbb{E}_Q[d] \), hence the equilibrium risk premium \( \varpi^x = \ln \bar{m}^h - \ln \mathbb{E}_Q[d] \). The following subsection provides the expression for the risk-adjusted dividend \( \mathbb{E}_Q[d] \).

In the benchmark model without the option, the equilibrium risk premium on the risky asset is only affected by the prior mean, the prior precision, the risk aversion and the net supplies of the risky asset. However, when the investors invest in the option markets, the equilibrium risk premium on the risky underlying asset is affected by the strike price. I plot the equilibrium risk premium as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a given level of the heterogeneity in beliefs in Figure 2.

![Figure 2. The Equilibrium Risk Premium. The equilibrium risk premium on the risky underlying asset is plotted as a function of the heterogeneous prior precision conditional on a given level of the strike price and the strike price conditional on a given level of the heterogeneity in prior precision.](image)
From the left panel in Figure 2, we can see the equilibrium risk premium is not affected by the heterogeneity in beliefs, but decreases with the average prior precision. With homogeneous prior precision, the investor’s confidence on the forthcoming dividend increases with the prior precision, and thus decreases with the requirement of compensation for risk.

The curve in the right panel of Figure 2 is just like the inverted risk-adjusted expected dividend. With a fixed level of heterogeneous beliefs, when the strike price is much higher or much lower than the mean, the option resembles nothing or a stock. The equilibrium risk premium on the risky underlying asset converges to that in a benchmark case with a stock and a zero-coupon bond as depicted by the middle line. The intuition of the impact of the intermediate strike price can be gained from the fact that the risk-adjusted expected dividend is the expected dividend plus the covariance of the marginal utility of consumption and the dividend scaled by the expected marginal utility of consumption. See more in Chapter 5 in Christensen and Feltham (2003). Note in the right panel, when the strike price $K$ increases from -0.005 to 0.023, the equilibrium risk premium increases. To understand this fact, I plot the marginal utility of consumption as a function of the realizations of forthcoming dividend in Figure 3.

![Figure 3. Marginal Utility of Consumption.](image)

We can see from Figure 3 that when the strike price $K$ increases from -0.005 to 0.023, the slope of the broken line of marginal utility is in general decreasing, this fact signals the covariance of the marginal utility of consumption and the dividend is in general decreasing, thus the value of $E^Q[d]$ decreases, and the equilibrium risk premium increases.
Furthermore, the marginal utility of consumption is determined by the investors’ portfolios. I plot the portfolios in financial markets as functions of the strike price in Figure 4.

Figure 4. The Equilibrium Portfolios for Investor with Lower Prior Precision. The equilibrium portfolios for the investor with lower prior precision in the underlying asset markets, the option markets, the zero-coupon bond markets are plotted as a function of the strike price conditional on a certain level of heterogeneity in prior variance in the left panel, middle panel, and right panel, respectively.

An investor with low prior precision takes a positive position in the option market. Moreover, since with extreme strike price, the option tends to resemble stock or nothing, and thus has a payoff with little convexity. Therefore, from Figure 4 we can see that, in order to accumulate enough convexity, the investor takes a very high position in the option market. When the option is very deep in the money and resembles a stock, high positive demand in the option leads to a large short position in the stock market and bond market. Relatively, when the option is very out of the money and resembles nothing, the portfolio in the stock market and the bond market converges to that in the benchmark model in which investors only trade in a stock and a bond. The equilibrium portfolios give rise to the state-dependent payoff according to Eq. (3), and thus the marginal utility which is a function of the forthcoming dividend.

3.2.3 Equilibrium Price of Risky Underlying Asset

The underlying asset price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the equilibrium underlying asset price as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a
given level of the heterogeneity in beliefs in Figure 5.

Figure 5. The Equilibrium Underlying Asset Price. The equilibrium underlying asset price is plotted as a function of the heterogeneous prior precision conditional on a given level of strike price and the strike price conditional on a given level of the heterogeneity in prior precision.

To analyze the properties of the underlying asset price, I first establish the following proposition by deriving the investors’ first-order condition with respect to the portfolio in the underlying asset.

**Proposition 2** Assume the investors with heterogeneous beliefs can potentially trade in option markets. The ex ante equilibrium price of the risky underlying asset at \( t = 0 \) is equal to the equilibrium riskless discount factor times the risk-adjusted expected dividend, i.e.,

\[
p_0(\eta) = \beta_0 E^Q[d]. \tag{8}
\]

The risk-adjusted expected dividend is expressed as a function of the prior means and variances, i.e.,

\[
E^Q[d] = \left[ \prod_{i=1}^{I} \frac{\partial CE_{i2} \left( \theta_{i0}, x_{i0}, \gamma_{i0} \right)}{\partial x_{i0}} \right]^{\frac{1}{2}} = \left[ \prod_{i=1}^{I} \left( m_{i0} - r \sigma_{i0}^2 x_{i0} + \frac{\partial f_i \left( x_{i0}, \theta_{i0} \right)}{\partial x_{i0}} \right) \right]^{\frac{1}{2}}. \tag{9}
\]

\(^7\)This means that I can define the risk-adjusted probability measure \( Q \) explicitly such that under \( Q \), the terminal dividend is normally distributed as \( d \sim N(\left[ \prod_{i=1}^{I} \frac{\partial CE_{i2} \left( \theta_{i0}, x_{i0}, \gamma_{i0} \right)}{\partial x_{i0}} \right]^{\frac{1}{2}}, \sigma^2 \). Note that while the expected dividend under \( Q \) is uniquely determined in equilibrium, the variance of the dividend under \( Q \) is not uniquely determined due to the market incompleteness and, thus, I just take it to be \( \sigma^2 \). Fortunately, the lack of the uniqueness of the variance has no consequences in the subsequent analysis.
As the strike price varies, the ex ante equilibrium price of the risky underlying asset can either be higher or lower than that in the benchmark model in which investors trade only in a stock and a zero-coupon bond.

Proof. See Appendix A for details of the derivation of the equilibrium underlying asset price.

As indicated by Proposition 2, the left (right) panel in Figure 5 is a balanced result of the left (right) panel in Figure 1 and Figure 2. With the parameters I use, the price of the risky underlying asset decreases with the heterogeneity in beliefs. The diagonal line indicates that with homogeneous belief, the risky asset price increases with the prior precision.

Moreover, the right panel in Figure 5 shows that, with a fixed level of heterogeneous beliefs, when the strike price is much higher or much lower than the mean, the price of the risky underlying asset converges to that in a benchmark case with a risky asset and a zero-coupon bond as depicted by the middle line of around 8.62.

### 3.2.4 Equilibrium Call Option Price

The European call option price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the equilibrium call option price as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a given level of the heterogeneity in beliefs in Figure 6.

---

Figure 6. The Equilibrium Call Option Price. The equilibrium call option price is plotted as a function of the heterogeneity in beliefs conditional on a given level of strike price and the strike price conditional on a given level of the heterogeneity in prior precision.
Similar to the case of the equilibrium underlying asset price, to analyze the option price, I establish the following proposition by deriving the investors’ first-order condition with respect to the portfolio in the call option.

**Proposition 3** Assume the investors with heterogeneous beliefs can potentially trade in option markets. The ex ante equilibrium price of the European call option at \( t = 0 \) is equal to the equilibrium riskless discount factor times the risk-adjusted expected option payment, i.e.,

\[
\pi_0(\eta) = \beta_0 E^Q [\max(d - K, 0)]. \tag{10}
\]

The risk-adjusted expected dividend is expressed as a function of the prior means and variances, i.e.,

\[
E^Q [\max(d - K, 0)] = \left[ \prod_{i=1}^I \frac{\partial CE_i}{\partial \theta_{i0}} \right]^{\frac{1}{2}} = \left[ \prod_{i=1}^I \frac{\partial f_i(x_{i0}, \theta_{i0})}{\partial \theta_{i0}} \right]^{\frac{1}{2}}. \tag{11}
\]

As we can see from the left panel in Figure 6, with the parameters I use, the impact of the heterogeneity in beliefs on the equilibrium call option price is almost invisible. The diagonal line indicates that with homogeneous prior precision, the equilibrium call option price decreases with the prior precision. In this case, the investors do not trade in the option markets, and the option is a redundant asset. The model reduces to a benchmark case that matches the result of the Black-Scholes model: The higher stock volatility, the higher value of the option.

Moreover, the right panel in Figure 6 shows that, of course, the deeper in the money, the more valuable the option is. When the option is very deep out of the money, the equilibrium call option price is nearly zero.

### 3.2.5 Equilibrium Expected Utilities

The impact of the option on individual utility is similar to that of public signal precision in Christensen and Qin (2012), hence I only clarify the underlying mechanism concisely. The investors’ *ex ante* expected utilities are affected in two ways by changes of the strike price. First, changes in the strike price affects the gains to trade based on heterogeneously updated posterior beliefs and, thus, the growth in their certainty equivalents. Secondly, the strike price affects the *ex ante* equilibrium asset prices through the equilibrium interest rate and the equilibrium risk premium and, thus, affects the value of the investors’ individual endowments. The latter may affect the investors in different ways depending on their individual endowments relative to their equilibrium portfolio at \( t = 0 \). Therefore, the investors may not
unanimously prefer the option with an intermediate strike price. Since a low equilibrium asset price is of course good if the investor wants to reduce the holding of the asset at $t = 1$, but it is bad if the investor wants to reduce the holding of the asset. However, with equilibrium endowments of the market security, the equilibrium prices are independent of the investors’ individual endowments, the investors do not trade at $t = 0$ given these endowments, all the investors can benefit from the option with an intermediate strike price.

4 Equilibrium with Impact from Heterogeneity in Beliefs, Strike Price and Public Signal: Case when Investors Update Beliefs with Imperfect Public Signal

In this section, I numerically solve for the ex ante equilibrium in a two-period economy in which investors can update their beliefs according to the Bayes’ rule with imperfect public signal. There are two rounds of trading: one round of trading at $t = 0$ prior to the release of the information, and a second round of trading subsequent to the release of the public signal $y$ at $t = 1$. I solve for the ex ante equilibrium by first deriving the ex post equilibrium prices at $t = 1$, and given this ex post equilibriums, I can subsequently derive the ex ante equilibrium prices at $t = 0$.

4.1 Ex Post Equilibrium at Date $t = 1$

I first derive the ex post equilibrium at $t = 2$ conditional on the posterior beliefs. From the perspective of $t = 1$, date $t = 2$ consumption for the investor $i$ is

$$c_{i2} = \theta_{i1} (y) \max(d - K, 0) + x_{i1} (y) d + \gamma_{i1} (y).$$

Given the period-specific negative exponential utility, the investor $i$ maximizes his certainty equivalent of $t = 2$ consumption conditional on the public information at $t = 1$, subject to his budget constraint, i.e.,

$$\max_{\theta_{i1}(y), x_{i1}(y), \gamma_{i1}(y)} \text{CE}_{i2} (\theta_{i1}(y), x_{i1}(y), \gamma_{i1}(y)| m_{i1}, \sigma_{i1}^2)$$

subject to $\beta'_1 \gamma_{i1} (y) + p_1 (y) x_{i1} (y) + \pi_1 (y) \theta_{i1} (y) \leq \beta_1 \gamma_{i0} + p_1 (y) x_{i0} + \pi_1 (y) \theta_{i0}$,

where $\text{CE}_{i2} (\theta_{i1} (y), x_{i1} (y), \gamma_{i1} (y)| m_{i1}, \sigma_{i1}^2)$ can be calculated by Lemma 1.

The ex post certainty equivalent of $t = 2$ consumption of the investor $i$ conditional on
the public information at \( t = 1 \) is

\[
\text{CE}_{i_2} (\theta_{i_1} (y), x_{i_1} (y), \gamma_{i_1} (y) | m_{i_1}, \sigma^2_{i_1}) = \gamma_{i_1} (y) + m_{i_1} x_{i_1} (y) - \frac{1}{2} r \sigma^2_{i_1} x_{i_1}^2 (y) + f_i (x_{i_1}, \theta_{i_1}),
\]

where

\[
f_i (x_{i_1} (y), \theta_{i_1} (y)) = -\frac{1}{r} \ln \left\{ \Phi \left( \frac{K - (m_{i_1} - r \sigma^2_{i_1} x_{i_1} (y))}{\sigma_{i_1}} \right) \right\} + \exp \left[ -r \theta_{i_1} (y) \left[ -K + m_{i_1} - \frac{1}{2} r \sigma^2_{i_1} \theta_{i_1} (y) - r \sigma^2_{i_1} x_{i_1} (y) \right] \right] \times \left[ 1 - \Phi \left( \frac{K - (m_{i_1} - r \sigma^2_{i_1} (\theta_{i_1} (y) + x_{i_1} (y)))}{\sigma_{i_1}} \right) \right],
\]

and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution.

Assume \( \beta_1 = 1 \) as the numeraire in the model, I solve the investor \( i \)'s optimal portfolio choice problem. The equilibrium portfolios and the prices are the implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset,

\[
m_{i_1} - r \sigma^2_{i_1} x_{i_1} (y) + \frac{\partial f_i (x_{i_1} (y), \theta_{i_1} (y))}{\partial x_{i_1}} - p_i (y) = 0,
\]

\[
\frac{\partial f_i (x_{i_1} (y), \theta_{i_1} (y))}{\partial \theta_{i_1}} - \pi_i (y) = 0,
\]

and the market clearing condition for each asset, i.e.,

\[
\sum_{i=1}^{I} \theta_{i_1} (y) = 0, \text{ and } \sum_{i=1}^{I} x_{i_1} (y) = Z.
\]

Note the implicit solutions are functions of the public signal \( y \), the public signal precision, the strike price, and the posterior beliefs (hence the prior beliefs). In the numerical example with two investors, there are six equations with six unknowns, i.e., the ex post equilibrium portfolios in the underlying asset markets, \( x_{i_1} (y), x_{j_1} (y) \), the ex post equilibrium portfolios in the option markets, \( \theta_{i_1} (y), \theta_{j_1} (y) \), the ex post equilibrium price for the underlying asset, \( p_i (y) \), and the ex post equilibrium price for the option, \( \pi_i (y) \). Given the public signal \( y \), the signal precision and the prior beliefs, I can solve the equations numerically. The methods to further simplify the equations to the nonlinear equations with only two unknowns and the algorithms to solve for the equilibrium numerically are described in Appendix A.
4.2 Ex Ante Equilibrium at Date $t = 0$

I now determine the *ex ante* equilibrium price and the demands for assets functions at $t = 0$, taking the equilibrium at $t = 1$ characterized by the system of equations, i.e., Eq. (12a), Eq. (12b), and Eq. (12c) as given. From the perspective of $t = 0$, the date $t = 2$ consumption for the investor $i$ is

$$c_{i2} = \theta_{i1}(y) \max(d - K, 0) - \pi_1(y) + \theta_{i0}\pi_1(\eta) + x_{i1}(y)(d - p_1(y)) + x_{i0}p_1(y) + \gamma_{i0}$$

$$= \theta_{i1}(y) \max(d - K, 0) + x_{i1}(y)d + (\theta_{i0} - \theta_{i1}(y))\pi_1(y) + (x_{i0} - x_{i1}(y))p_1(y) + \gamma_{i0}.$$

By Lemma 1, conditional on the public information at $t = 1$, the investor $i$’s certainty equivalent of $t = 2$ consumption is

$$CE_{i2}(x_{i0}, \theta_{i0}, \gamma_{i0}, p_0, \beta_0, \pi_0, x_{i1}(y), \theta_{i1}(y), p_1(y), \pi_1(y)|F_1) = (\theta_{i0} - \theta_{i1}(y))\pi_1(y) + (x_{i0} - x_{i1}(y))p_1(y) + \gamma_{i0} + m_{i1}x_{i1}(y) - \frac{1}{2}r\sigma_{i1}^2x_{i1}^2(y)$$

$$- \frac{1}{r}\ln\{\Phi\left(\frac{K - (m_{i1} - r\sigma_{i1}^2x_{i1}(y))}{\sigma_{i1}}\right) + \exp\left[-r\theta_{i1}(y)\left[-K + m_{i1} - \frac{1}{2}r\sigma_{i1}^2\theta_{i1}(y) - r\sigma_{i1}^2x_{i1}(y)\right]\right]$$

$$\times \left(1 - \Phi\left(\frac{K - (m_{i1} - r\sigma_{i1}^2(\theta_{i1}(y) + x_{i1}(y)))}{\sigma_{i1}}\right)\right),$$

where $F_1$ denotes the investors’ public information at date $t = 1$.

Conditional on the information at $t = 1$, the investor $i$’s certainty equivalent of $t = 2$ consumption varies with the investors’ posterior mean $m_{i1}$, and thus by Eq. (1), is a function of the public signal $y$. From the perspective of investor $i$, $y \sim N(m_i, \sigma_i^2 + \sigma_z^2)$. Given each realization of the public signal $y$, I can solve for the *ex post* equilibrium and obtain the *ex post* equilibrium prices and demands at $t = 1$ numerically. Substituting in the equilibrium prices and demands at $t = 1$ conditional on each public signal $y$, the expected utility at $t = 2$ from the perspective of $t = 0$ can be written as

$$-E[\exp[-rCE_{i2}(x_{i0}, \theta_{i0}, \gamma_{i0}, \beta_0, p_0, \pi_0, x_{i1}(y), \theta_{i1}(y), p_1(y), \pi_1(y)|F_1)]]$$

$$= -\frac{1}{\sqrt{2\pi(\sigma_i^2 + \sigma_z^2)}} \int_{-\infty}^{+\infty} \exp[-rCE_{i2}(y)] \exp\left(-\frac{(m_i - y)^2}{2(\sigma_i^2 + \sigma_z^2)}\right) dy.$$
Note the expected utility is a function of the \textit{ex ante} equilibrium portfolios, $\gamma_{i0}, x_{i0}, \theta_{i0}$, the priors $m_i, \sigma_i^2$, and the public signal precision $h_e$. The expected utility is a deterministic integral over the infinite interval, and thus can be computed at any desired level of accuracy using standard numerical integration methods (See more about the numerical integration methods in the Appendix A).

The investor $i$’s $t=0$ certainty equivalent equals to the date $t=0$ consumption, i.e.,

\[
CE_{i0} = c_{i0} = d_0 \bar{z}_i + (\tilde{\theta}_i - \theta_{i0}) \pi_0(\eta) + (\tilde{\gamma}_i - \gamma_{i0}) \beta_0 + (\bar{z}_i - x_{i0}) p_0(\eta),
\]

hence the investor $i$’s decision problem at $t=0$ can be stated as follows

\[
\max_{\theta_{i0}, \gamma_{i0}, x_{i0}} - \exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}).
\]

Solve the investor $i$’s optimal portfolio choice problem, the \textit{ex ante} equilibrium portfolios and the prices are implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset, i.e.,

\[
\frac{\partial (- \exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}))}{\partial x_{i0}} = 0,
\]
\[
\frac{\partial (- \exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}))}{\partial \theta_{i0}} = 0,
\]
\[
\frac{\partial (- \exp (-rCE_{i0}) - \exp (-\delta) \exp (-rCE_{i2}))}{\partial \gamma_{i0}} = 0,
\]

and the market clearing condition for each asset, i.e.,

\[
\sum_{i=1}^{I} \gamma_{i0} = 0, \quad \sum_{i=1}^{I} \theta_{i0} = 0, \quad \text{and} \quad \sum_{i=1}^{I} x_{i0} = Z.
\]

Note the implicit solutions are functions of the public signal precision, the strike price, and the prior beliefs. In a two-investor model, there are nine equations with nine unknowns, as we have seen in benchmark model in section 3.1.1. The \textit{ex ante} equilibrium can be solved for numerically.

### 4.3 The Impact of Heterogeneous Beliefs, Strike Price and Public Signal Precision on Ex Ante Asset Pricing Properties

When the investors update their beliefs with imperfect public signal precision, the public information system shows its influence in facilitating side-betting. Solving for the \textit{ex ante}
equilibrium, I find that when the investors receive no public signal or perfect public signal, the model reduces to the benchmark model in which the investors speculate in the option markets in the previous section. Moreover, with homogeneous prior variance, the investors do not trade in the option markets, and asset pricing is independent of the public information system. The intuition is similar to that in the benchmark model, i.e., trading in the underlying asset and the zero-coupon bond is already able to facilitate efficient side-betting.

Furthermore, conditional on a certain level of the public signal precision and the strike price, the impact of the heterogeneous beliefs on the asset pricing properties is quite similar to that in the previous section. The figures and intuition of the panels are almost the same. Thus I do not repeat those results including pictures and explanations again. Therefore in this section, I only plot the asset pricing properties as functions of the public signal precision and the strike price conditional on a certain level of heterogeneity in beliefs.

### 4.3.1 Ex Ante Equilibrium Interest Rate

The *ex ante* equilibrium interest rate from $t = 0$ to $t = 2$ is defined as $\nu \equiv -\ln \beta_0$. By the investor $i$’s decision problem in Eq. (15), and the expression of the investor $i$’s certainty equivalents at $t = 0$ and $t = 2$, i.e., Eq. (13) and Eq. (14), the relationship between the equilibrium interest rate and the growth in certainty equivalents in Eq. (7) still establishes.

The *ex ante* equilibrium interest rate is endogenized in the equilibrium as a function of the priors, the public signal precision and the strike price. To see the impact of the strike price and the public information quality, and I plot the *ex ante* equilibrium interest rate as a function of the strike price and the public signal precision conditional on a certain level of heterogeneity in the prior variance in Figure 7.
Figure 7. The Ex Ante Equilibrium Interest Rate. The ex ante equilibrium interest rate is plotted as a function of the strike price and the public signal precision conditional on a certain level of heterogeneity in the prior variance. The scale on the axis of the public signal precision is $x = \ln (1 + 0.005h_s)$. The parameters used when plotting are presented in Table 1, in addition to the prior variances $\sigma_i^2 = 0.0002; \sigma_j^2 = 0.0001$. I use identical value of the parameters for plotting other asset pricing properties throughout this section, and do not repeat the parameter values again.

The above panels in Figure 7 indicates that, the option and the public signal precision enable the investors to achieve improved side-betting based on their heterogeneously updated posterior beliefs. These gains to trade translate into increased certainty equivalents of $t = 2$ equilibrium consumption and, thus, a higher growth in their certainty equivalents and the ex ante equilibrium interest rate. With an intermediate public signal precision and the option with an intermediate strike price, the highest efficiency of side-betting is achieved, reflected by the unique maximum point of the ex ante equilibrium interest rate.

The left below panel in Figure 7 shows the ex ante equilibrium interest rate surface from a special angle by which we can see, conditional on a certain level of the public signal precision, the ex ante equilibrium interest rate is bell-shaped with respect to the strike price. Since with very low or very high strike price, the option payment is similar to a stock or nothing, thus the option’s ability to facilitate side-betting is limited. However, with an intermediate strike price, the option distinguishes from a fixed income asset and a stock. This kind of option can enhance the market allocational efficiency to the best degree and, thus, leads to the highest growth in certainty equivalents and the highest equilibrium interest rate. Moreover, we can see that when the public signal precision changes from 0 to 10, conditional on each fixed public signal precision, the extreme point of the strike price is always at around 0.009. This result indicates that the extreme point of the ex ante equilibrium interest rate surface is
independent of the public signal precision. This finding matches the result in the benchmark model in Christensen and Qin (2012).

The right below panel in Figure 7 shows the \textit{ex ante} equilibrium interest rate surface from a special angle by which we can see, conditional on a certain level of the strike price, the \textit{ex ante} equilibrium interest rate is bell-shaped with respect to the public signal precision.\footnote{Note the equilibrium interest rate looks bell-shaped after the transformation \( x = \ln (1 + 0.005h) \). The actual equilibrium is not exact bell-shaped, but still keep the property that the interest rate is first increasing, then decreasing, i.e., the equilibrium interest rate has only one maximum point.} Furthermore, when the strike price changes from -0.04 to 0.01, conditional on each fixed level of the strike price, the extreme point of the public signal precision moves from around 3.5 to around 4.5. This result indicates that the extreme point of the \textit{ex ante} equilibrium interest rate surface is a function of the strike price.

The intuition for the fact that the \textit{ex ante} equilibrium interest rate is bell-shaped with respect to public signal precision conditional on a certain level of the strike price can be gained from Eq. (7). Actually, the equilibrium consumption growth is bell-shaped with respect to the public signal precision.

With a given level of heterogeneity in priors, the public signal precision has two countervailing effects on equilibrium interest rate. The first type effect is that when the public signal precision is very low, a better public information quality tends to give rise to more efficient side-betting based on the updated beliefs, and results in a higher equilibrium interest rate. However, when the public signal precision is very high, the investors tend to have very similar inference at \( t = 1 \), therefore, lowers the efficiency of side-betting and the equilibrium consumption growth, and thus lowers the \textit{ex ante} equilibrium interest rate. This is the second type effect of the public signal precision. When the public signal precision is extremely low, the first effect dominates the second type effect, vice versa. Eventually, the counterbalancing effects yield the bell-shaped \textit{ex ante} equilibrium interest rate.

In order to see clearly the impact of the option and the public signal precision on the equilibrium interest rate, I compare the equilibrium interest rate in this model with the equilibrium interest rates in some benchmark models in Christensen and Qin (2012) in Figure 8.
Figure 8. The Equilibrium Interest Rates in Different Models. The equilibrium interest rates in different models are plotted as functions of the public signal precision conditional on a certain level of the strike price. Note the bottom line of around 0.036 depicts the equilibrium interest rate in the benchmark model with only a zero-coupon bond and a stock. The above curve with a maximum point of around 0.052 indicates the equilibrium interest rate in the benchmark model in which the investors update their beliefs with imperfect public signal precision. The horizontal line of round 0.059 depicts the equilibrium interest rate in the benchmark model in which the investors speculate in the option markets and receiving no public signal or perfect public signal. The above curve with a maximum point of around 0.063 indicates the equilibrium interest rate in the model in which the investors update their beliefs with imperfect public signal precision and speculate in both the option markets and the underlying asset markets. The highest horizontal line depicts the equilibrium interest rate in the effectively complete market. The scale on the axis of the public signal precision is \( x = \ln (1 + 0.005 h_c) \). The parameters used when plotting the equilibrium interest rate as function of the public signal precision in Figure 8 are presented in Table 1, in addition to the prior variances \( \sigma_i^2 = 0.0002, \sigma_j^2 = 0.0001 \) and the strike price \( K = 0.01 \).

As we can see from Figure 8, the bottom line is the equilibrium interest rate around 0.036 in the benchmark model in which the investors only trade in a zero-coupon bond and a stock. Note the increase of the equilibrium interest rate can be viewed as an analog of the gain from side-betting. Thus, this result illustrates the lowest efficiency of side-betting and benefit from the disagreement in the prior variance. When the investors can trade two rounds and update their beliefs with the imperfect public signal, the equilibrium interest rate is higher showing by the curve with a maximum point of around 0.052. This fact indicates that the imperfect public signal does facilitate side-betting when the investors heterogeneously
update their beliefs. When the investors speculate with an option and receiving no public signal or perfect public signal, the equilibrium interest rate is even higher and increases to around 0.059, and the equilibrium interest rate increases again when the investors update their beliefs with imperfect public signal. This result indicates that the option and the public signal are capable of facilitating side-betting, leading to a higher growth in certainty equivalent and thus a higher equilibrium interest rate. Finally, the highest equilibrium interest rate is obtained when the investors trade with the derivative which pays a square of the dividend in the effectively complete market. This fact illustrates the option and the imperfect public signal still cannot make the investors take full advantage of the difference in beliefs, and the efficiency of side-betting is not as high as that in the effectively complete market.

Note first, Cuoco and He (1994) demonstrate the equilibrium with a stochastic weight in the representative agent utility is in general not Pareto efficient. Hence markets in Buraschi and Jiltsov (2006) are essentially incomplete, and they do not illustrate how much the option can help to improve the efficiency of side-betting. While, the model in this section shows that, the option and the imperfect public signal can facilitate side-betting, and increase the allocational efficiency of the markets to a better degree than that in some benchmark cases. Second, Buraschi and Jiltsov (2006) plot the asset pricing properties such as stock price and stock volatility as functions of the difference in the updated beliefs scaled by the signal volatility, hence the heterogeneous beliefs in their model carry the effect of the information system. However, I plot the asset pricing properties as functions of the different priors, thus the analysis in this paper probes the impact of the heterogeneity in beliefs and the public information quality separately.

4.3.2 Ex Ante Equilibrium Risk Premium

As in the previous section, I define the \( \text{ex ante} \) equilibrium risk premium on the risky underlying asset as \( \varpi^x = \ln \bar{m}^h - \ln E^Q[d] \), where the weighted mean \( \bar{m}^h \) and the risk-adjusted expected dividend, \( E^Q[d] \), have the same definitions as in the previous section. To see the impact of the strike price and the public information quality, I plot the \( \text{ex ante} \) equilibrium risk premium as a function of the strike price and the public signal precision.
conditional on a certain level of heterogeneity in prior variance in Figure 9.

Figure 9. The Ex Ante Equilibrium Risk Premium. The ex ante equilibrium risk premium on the risky underlying asset is plotted as a function of the strike price and the public signal precision conditional on a certain level of heterogeneity in prior variance. The scale on the axis of public signal precision is \( x = \ln (1 + 0.005h_e) \).

The ex ante risk premium surface in the panel is just like the inverted risk-adjusted expected dividend. With a fixed level of heterogeneous beliefs, when the strike price is much higher or much lower than the mean, the ex ante risk premium on the risky underlying asset converges to that in a benchmark case in which the investors trade only in a stock and a zero-coupon bond, which is independent of the public signal precision. However, with an intermediate strike price, the impact of the public signal precision on the ex ante equilibrium risk premium is nontrivial. This fact suggests that even though the strike price of option affects the ex ante risk premium on the risky underlying asset regardless of the presence of the public signal, the informativeness of the public information system affects the ex ante risk premium only via its relationship with the option.

Compare to the benchmark model in Christensen and Qin (2012), in which the ex ante risk premium is independent of the public signal precision, the ex ante risk premium in this paper is not aligned with investor welfare (as a function of the signal precision). The equilibrium interest rate and the investor welfare, however, are still perfectly aligned. This fact has an implication that it may be wise to be cautious in making policy statements about, for example, financial reporting regulation, based on empirical measures of equity premia (which are hard to measure reliably anyway).

Similar to the benchmark model in the previous sections, the intuition can be gained from the fact that the risk-adjusted expected dividend is the expected dividend plus the
covariance of the marginal utility of consumption and the dividend scaled by the expected marginal utility of consumption. Actually, the surface of the covariance of the marginal utility of consumption and the dividend is similar to the inverted *ex ante* risk premium. Furthermore, the marginal utility of consumption is determined by the investors’ portfolios. I plot the portfolios in financial markets as functions of the strike price and the public signal precision conditional on a certain level of heterogeneity in prior variance in Figure 10.

![Figure 10](image)

Figure 10. The Ex Ante Equilibrium Portfolios for Investor with Lower Prior Precision. The *ex ante* equilibrium portfolios for the investor with lower prior precision in the underlying asset markets, the option markets, the zero-coupon bond markets are plotted as functions of the strike price and the public signal precision conditional on a certain level of heterogeneity in prior variance in the left panel, middle panel, and right panel, respectively. The scale on the axis of public signal precision is $x = \ln (1 + 0.005h_e)$.

Conditional on a certain level of public signal precision, for the investor with lower prior precision, the portfolio in the option market is U-shaped. The intuition is similar to that in the benchmark models. Conditional on a certain level of strike price, the *ex ante* equilibrium demand of the option increases with the public signal precision. The intuition is that when the public signal precision increases, the variance of the posterior beliefs of the dividend decreases and, thus, the convexity of the option payoff at $t = 1$ decreases. To gain more payoff convexity, the investors need to trade more in the option market at $t = 0$. This result correspondingly increases the short positions in the underlying asset and the zero-coupon bond.

### 4.3.3 Ex Ante Equilibrium Price of Risky Underlying Asset

The underlying asset price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the *ex ante* equilibrium underlying asset price as a function of the
strike price and public signal precision conditional on a certain level of heterogeneity in prior variance in Figure 11.

Figure 11. The Ex Ante Equilibrium Underlying Asset Price. The ex ante equilibrium underlying asset price is plotted as a function of the strike price and the public signal precision conditional on certain level of the heterogeneity in prior variance. The scale on the axis of the public signal precision is $x = \ln (1 + 0.005h_e)$.

The ex ante equilibrium underlying asset price is jointly determined by the public signal precision and the strike price through their effects on the ex ante equilibrium riskless discount factor and the risk-adjusted expected dividend. Similar to the results in the previous section, Figure 11 is a balanced result of Figure 7 and Figure 9. Moreover, in the benchmark model in which the investors update beliefs with imperfect public information and speculate in a stock and a zero-coupon bond, the lowest stock price is attained at an intermediate public signal precision. Differently, after incorporating the option, the lowest price of the underlying asset
is attained when there is no or perfect public signal, and at an intermediate strike price.

4.3.4 **Ex Ante Equilibrium Call Option Price**

The European call option price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the *ex ante* equilibrium call option price as a function of strike price and public signal precision conditional on a certain level of heterogeneity in prior variance in Figure 12.

![Figure 12. The Ex Ante Equilibrium Call Option Price.](image)

Figure 12. The Ex Ante Equilibrium Call Option Price. The *ex ante* equilibrium call option price is plotted as a function of the strike price and the public signal precision conditional on a certain level of the heterogeneity in prior variance. The scale on the axis of the public signal precision is $x = \ln (1 + 0.005 h_e)$.

The *ex ante* equilibrium call option price is jointly determined by the public signal precision and the strike price through their effects on the *ex ante* equilibrium riskless discount factor and the risk-adjusted expected option payment. We can see from Figure 12 that the impact of the public signal precision on the *ex ante* equilibrium call option price is limited and hardly visible from the plot. The impact of the strike price is quite similar to that in the previous section: The value of the option decreases with the strike price, and converges to zero when the strike price increases to a very high level.

4.3.5 **Ex Ante Equilibrium Expected Utilities**

As mentioned before, the impact of the option and the public information system on individual utility depends on their individual endowments relative to their equilibrium portfolio
at $t = 0$. Therefore, the investors may not unanimously prefer this system over public information system. However, with equilibrium endowments of the market security, the \textit{ex ante} equilibrium prices are independent of the investors’ individual endowments, the investors do not trade at $t = 0$ given these endowments, all the investors can benefit from the imperfect public signal and the option with an intermediate strike price.

4.3.6 Case when Investors Update Beliefs and Speculate with Straddles

As mentioned in Christensen and Qin (2012), in an incomplete market setting with heterogeneous beliefs about the risks on the underlying asset, straddles, i.e., long positions in both a call and a put option with the same strike price can play an important role to facilitate side-betting. Since straddle can resemble the payoff profile of the dividend derivative which pays off the square of dividend in the terminal date and effectively complete the market. The investors, who think the variance of the dividend is high, still can use Gamma strategy by trading straddle to achieve the convexity of its payoff while the investors, who think the variance is low, can take a short position to get a concave payoff profile. Thus the straddles have potential to exert an influence on the incomplete market settings with heterogeneous beliefs about the risks on the underlying assets. I derive the models when the investors speculate with straddle, receiving no public signal, imperfect public signal or perfect public signal, and solve for the equilibrium numerically. I find that the put option is able to facilitate side-betting, increase the growth in certainty equivalents and equilibrium interest rate. However, compare to the change of the level of equilibrium interest rate, the increase of the equilibrium interest rate which due to the incorporation of the put option is very small, and is hardly visible when plotting it. The same happens to other asset pricing properties such as the equilibrium underlying asset price, and the equilibrium risk premium. Hence, I do not repeat the figures in this section, but the derivations of the models are provided in Appendix B.

5 Conclusion

Under the assumptions of time-additive negative exponential investors and normally distributed dividend, this paper studies the relationship between the heterogeneous belief, the strike price and the public signal precision in the option markets without conditional on specific state of the economy. I demonstrate that the option market is important to facilitate financial market allocational efficiency when the investors hold heterogeneous beliefs. Moreover, the option market also makes the role of public signal precision more complicated and sophisticated. The public signal precision affects the \textit{ex ante} risk premium on the risky
underlying asset via this relationship with the option. When the investors do not speculate with the option, the public signal precision is independent of the \textit{ex ante} risk premium. Combine with the right intermediate public signal precision and the right intermediate strike price, i.e., the right type of option, the highest allocational efficiency of the market can be attained, reflected by the unique maximum point of the \textit{ex ante} equilibrium interest rate surface.

In the effectively complete market, an additional asset of the right type eliminates the need for dynamic trading based on public signals and enables the investors to take full advantage of the heterogeneity in beliefs. Compare to that extreme case, I use option to facilitate side-betting and show that in this intermediate case, the public signal still has room to show its potential to facilitate improved dynamic trading opportunities and yields a more efficient market structure.

More trading rounds based on a sequence of the public signals may lead to more efficient side-betting based on the heterogeneous beliefs. To solve for such a multi-period \textit{ex ante} equilibrium, the algorithms in Dumas and Lyasooff (2011) which solve for equilibrium recursively on an event tree can be helpful. Furthermore, as another extreme case, continuous trading may dynamically effectively complete the financial market with heterogeneous beliefs under the main assumptions in this paper. In other words, continuous trading may play a role as a compensation of long lived securities, but the market is essentially incomplete. I leave these cases for future research.

Another direction to extend this paper is to consider a CRRA-lognormal specification. The CARA-normal setting in this paper does buy some analytical tractability. However, more realistic preference and dividend distribution may come up with more empirically testable implications. Besides, the model can be generalized to a heterogeneous risk-aversion case. With identical prior precision, the least risk-averse investors are expected to tolerate risk and short options, while opposite speculative positions are taken by the most risk-averse investors.

References


A Appendix: Proofs and Algorithms

A.1 Proof of Lemma 1

Since the investor $i$’s consumption at $t = 2$ is
\[ c_{i2} = \theta_{i0} \max(d - K, 0) + x_{i0}d + \gamma_{i0}, \]
and let the probability density function of a Normal distribution $N(\mu, \sigma^2)$ be
\[ f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(\frac{-(x - \mu)^2}{2\sigma^2}\right). \]

Hence, the $t = 2$ expected utility for the investor $i$ is
\[
\begin{align*}
\mathbb{E} \left[ \exp \left\{ -r \left[ \theta_{i0} \max(d - K, 0) + x_{i0}d + \gamma_{i0} \right] \right\} \right] \\
= & - \int_{-\infty}^{+\infty} \exp\left\{ -r \left[ \theta_{i0} \max(d - K, 0) + x_{i0}d + \gamma_{i0} \right] \right\} f \left( x; m_{i0}, \sigma_{i0}^2 \right) dx \\
= & - \int_{-\infty}^{K} \exp\left\{ -r \left[ x_{i0}d + \gamma_{i0} \right] \right\} f \left( x; m_{i0}, \sigma_{i0}^2 \right) dx \\
& - \int_{K}^{+\infty} \exp\left\{ -r \left[ \left( \theta_{i0} + x_{i0} \right) d - \theta_{i0}K + \gamma_{i0} \right] \right\} f \left( x; m_{i0}, \sigma_{i0}^2 \right) dx \\
= & - \exp\left\{ -r \left[ \gamma_{i0} + m_{i0}x_{i0} - \frac{1}{2}r\sigma_{i0}^2x_{i0}^2 \right] \right\} \times \int_{-\infty}^{K} f \left( x; m_{i0} - r\sigma_{i0}^2x_{i0}, \sigma_{i0}^2 \right) dx \\
& - \exp\left\{ -r \left[ \gamma_{i0} - \theta_{i0}K + m_{i0} \left( \theta_{i0} + x_{i0} \right) - \frac{1}{2}r\sigma_{i0}^2 \left( \theta_{i0} + x_{i0} \right)^2 \right] \right\} \times \int_{K}^{+\infty} f \left( x; m_{i0} - r\sigma_{i0}^2 \left( \theta_{i0} + x_{i0} \right), \sigma_{i0}^2 \right) dx
\end{align*}
\]
\[ \begin{align*}
&= -\exp \left[ -r \left[ \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 \right] \right] \times \int_{-\infty}^{\frac{\frac{\nu_{i0} - r \sigma_{i0}^2 x_{i0}}{\sigma_{i0}}}{x_{i0}}} \phi(x) \, dx \\
&= -\exp \left[ -r \left[ \gamma_{i0} - \theta_{i0} K + m_{i0} \left( \theta_{i0} + x_{i0} \right) - \frac{1}{2} r \sigma_{i0}^2 \left( \theta_{i0} + x_{i0} \right)^2 \right] \right] \\
&\times \int_{\frac{\frac{\nu_{i0} - r \sigma_{i0}^2 \left( \theta_{i0} + x_{i0} \right)}{\sigma_{i0}}}{x_{i0}}}^{\frac{\frac{\nu_{i0} - r \sigma_{i0}^2 x_{i0}}{\sigma_{i0}}}{x_{i0}}} \phi(x) \, dx \\
&= \exp \left[ -r \left[ \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 \right] \right] \times \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 x_{i0})}{\sigma_{i0}} \right) \\
&- \exp \left[ -r \left[ \gamma_{i0} - \theta_{i0} K + m_{i0} \left( \theta_{i0} + x_{i0} \right) - \frac{1}{2} r \sigma_{i0}^2 \left( \theta_{i0} + x_{i0} \right)^2 \right] \right] \\
&\times \left( 1 - \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 \theta_{i0})}{\sigma_{i0}} \right) \right) \\
&= \exp \left[ -r \left[ \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 \right] \right] \times \left\{ \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 x_{i0})}{\sigma_{i0}} \right) \\
&- \exp \left[ -r \left[ -K + m_{i0} - \frac{1}{2} r \sigma_{i0}^2 \theta_{i0} - r \sigma_{i0}^2 x_{i0} \right] \right] \right\} \\
&\times \left( 1 - \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 \theta_{i0})}{\sigma_{i0}} \right) \right). \\
\end{align*} \]

Therefore, the certainty equivalent of \( t = 2 \) consumption for the investor \( i \) is

\[ \begin{align*}
\text{CE}_{i2} &= -\frac{1}{r} \ln(-U_{i2} \left( \theta_{i0}, x_{i0}, \gamma_{i0} | m_{i0}, \sigma_{i0}^2 \right)) \\
&= \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 - \frac{1}{r} \ln \left\{ \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 x_{i0})}{\sigma_{i0}} \right) \\
&+ \exp \left[ -r \theta_{i0} \left[ -K + m_{i0} - \frac{1}{2} r \sigma_{i0}^2 \theta_{i0} - r \sigma_{i0}^2 x_{i0} \right] \right] \\
&\times \left( 1 - \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 \theta_{i0})}{\sigma_{i0}} \right) \right) \right\}. \\
\end{align*} \]

This completes the proof.

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A.2 Calculate the First Derivatives in Model 3.1 and 3.2 and Derivation of Equilibrium Price of Underlying Asset

This section provides the calculation of the first derivatives in first-order condition. Let
\[ a_i = \frac{K - (m_{i0} - r\sigma_{i0}^2x_{i0})}{\sigma_{i0}}, \]
\[ b_i = \frac{K - (m_{i0} - r\sigma_{i0}^2(\theta_{i0} + x_{i0}))}{\sigma_{i0}}, \]
\[ c_i = \exp \left[ -r\theta_{i0} \left[ -K + m_{i0} - \frac{1}{2}r\sigma_{i0}^2\theta_{i0} - r\sigma_{i0}^2x_{i0} \right] \right], \]
therefore,
\[ f_i(x_{i0}, \theta_{i0}) = \frac{1}{r} \ln \{ \psi_i \}, \quad (A1) \]
where \( \psi_i \equiv \Phi(a_i) + c_i \times (1 - \Phi(b_i)) \), and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution. Hence,
\[
\frac{\partial f_i(x_{i0}, \theta_{i0})}{\partial x_{i0}} = -\frac{\phi(a_i) \sigma_{i0} - \phi(b_i) \sigma_{i0} c_i + (1 - \Phi(b_i)) c_i r \sigma_{i0}^2 \theta_{i0}}{\Phi(a_i) + (1 - \Phi(b_i)) c_i},
\]
\[
\frac{\partial f_i(x_{i0}, \theta_{i0})}{\partial \theta_{i0}} = \frac{\phi(b_i) \sigma_{i0} c_i + (1 - \Phi(b_i)) (m_{i0} - K - r\sigma_{i0}^2\theta_{i0} + r\sigma_{i0}^2 \theta_{i0}) c_i}{\Phi(a_i) + (1 - \Phi(b_i)) c_i},
\]
where \( \phi(x) \) denotes the standard normal probability density function, hence
\[
\phi(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right).
\]

Now I derive the price of the underlying asset as the product of equilibrium riskless discount factor and the risk-adjusted expected dividend. The investor \( i \)'s first-order condition gives
\[
\frac{\partial (- \exp(-rCE_{i0}) - \exp(-\delta) \exp(-rCE_{i2}))}{\partial x_{i0}} = 0 \iff
- r \exp(-rCE_{i0}) p_0(\eta) - \frac{\partial (\exp(-\delta) \exp(-rCE_{i2}))}{\partial x_{i0}} = 0 \iff
\]
\[
r p_0(\eta) \exp(-rCE_{i0}) = - \frac{\partial (\exp(-\delta) \exp(-rCE_{i2}))}{\partial x_{i0}} \iff
\ln (r p_0(\eta) - rCE_{i0}) = \ln \left( - \frac{\partial (\exp(-\delta) \exp(-rCE_{i2}))}{\partial x_{i0}} \right)
\]
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and sum over $i$, yields

\[
I \ln (r p_0(\eta)) - r (C_{E_0} + C_{E_i}) = \sum_{i=1}^{I} \ln \left( -\frac{\partial (\exp (-\delta) \exp (-r C_{E_i}))}{\partial x_{i0}} \right) \Rightarrow
\]

\[
I \ln (r p_0(\eta)) - r d_0 Z = \ln \left( \prod_{i=1}^{I} \frac{-\partial (\exp (-\delta) \exp (-r C_{E_i}))}{\partial x_{i0}} \right) \Rightarrow
\]

\[
\ln (r p_0(\eta)) = \frac{1}{I} r d_0 Z + \frac{1}{I} \ln \left( \prod_{i=1}^{I} \frac{-\partial (\exp (-\delta) \exp (-r C_{E_i}))}{\partial x_{i0}} \right) \Rightarrow
\]

\[
p_0(\eta) = \frac{1}{r} \left( \prod_{i=1}^{I} \frac{-\partial (\exp (-\delta) \exp (-r C_{E_i}))}{\partial x_{i0}} \right)^{\frac{1}{r}} \exp \left[ \frac{1}{I} r d_0 Z \right]
\]

\[
= \frac{1}{r} \left( \prod_{i=1}^{I} \frac{-\partial (\exp (-\delta) \exp (-r C_{E_i}))}{\partial x_{i0}} \right)^{\frac{1}{r}} \exp \left[ \frac{1}{I} r \sum_{i=1}^{I} C_{E_i} \right]
\]

\[
= \frac{1}{r} \left( \prod_{i=1}^{I} \left( m_{i0} - r \sigma^2 x_{i0} + \frac{\partial f_i(x_{i0}, \theta_i)}{\partial x_{i0}} \right) \right)^{\frac{1}{r}} \exp \left[ -\delta - r \sum_{i=1}^{I} C_{E_i} \exp \left[ \frac{r}{I} \sum_{i=1}^{I} C_{E_i} \right] \right]
\]

\[
= \frac{1}{r} \left( \prod_{i=1}^{I} \left( m_{i0} - r \sigma^2 x_{i0} + \frac{\partial f_i(x_{i0}, \theta_i)}{\partial x_{i0}} \right) \right)^{\frac{1}{r}} \exp \left[ -\delta - r \sum_{i=1}^{I} C_{E_i} + r \sum_{i=1}^{I} C_{E_i} \right]
\]

\[
= \beta_0 \left( \prod_{i=1}^{I} \left( m_{i0} - r \sigma^2 x_{i0} + \frac{\partial f_i(x_{i0}, \theta_i)}{\partial x_{i0}} \right) \right)^{\frac{1}{r}}.
\]

**A.3 Algorithm to Compute the Ex Ante Expected Utility at $t = 2$**

The expectation of the utility for the investor $i$ at $t = 2$ can be expressed as

\[
- \mathbb{E} \left[ \exp \left[ -r C_{E_{i2}}(x_{i0}, \theta_{i0}, \gamma_{i0}, \beta_0, p_0, \pi_0, x_{i1}(y), \theta_{i1}(y), p_1(y), \pi_1(y) | \mathcal{F}_1) \right] \right]
\]

\[
= - \frac{1}{\sqrt{2\pi(\sigma_i^2 + \sigma_z^2)}} \int_{-\infty}^{+\infty} \exp \left[ -r C_{E_{i2}}(y) \right] \exp \left[ -\frac{(m_i - y)^2}{2(\sigma_i^2 + \sigma_z^2)} \right] dy.
\]

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The integral over infinite interval can then be evaluated by ordinary integration methods:

\[
\int_{-H}^{H} f(t) \, dt \approx \frac{2H}{n} \left( \frac{f(-H) + f(H)}{2} + \sum_{k=1}^{n-1} f \left( a + k \frac{2H}{n} \right) \right),
\]

when \( H \to \infty \) and \( n \to \infty \), \( \int_{-H}^{H} f(t) \, dt \to \int_{-\infty}^{\infty} f(t) \, dt \).

Note that for each public signal \( y \), a corresponding \textit{ex post} equilibrium can be solved for and yield the \textit{ex post} equilibrium portfolios \( \theta_{i1}, \theta_{j1}, x_{i1}, x_{j1} \), and the \textit{ex post} prices, \( p_i(y) \) and \( \pi_i(y) \). Sum up \(-\frac{1}{\sqrt{2\pi(\sigma^2_{\theta} + \sigma^2_{x})}} \exp \left[ -rCE_{i2}(y) \right] \exp \left( -\frac{(m_i - y)^2}{2(\sigma^2_{\theta} + \sigma^2_{x})} \right) \)
over all the public signal \( y \) yields the expectation.

Also note that the public signal \( y \sim N(m_i, \sigma^2_{\theta} + \sigma^2_{x}) \), hence, let \( u = \frac{y - m_i}{\sqrt{\sigma^2_{\theta} + \sigma^2_{x}}} \), then \( u \sim N(0, 1) \). This transformation can make the probability density of the normal distribution be independent of the public signal precision, and bring some conveniences when programming.

### A.4 Algorithm to Solve Nonlinear Equations in Ex Post and Ex Ante Equilibriums

Denote the system of the nonlinear equations as \( G(x) \), where \( x \) is a vector and \( G(x) \) is a function that returns a vector value. I use nonlinear least-squares algorithms to find \( x \) that is a local minimizer to a function that is a sum of squares, i.e.,

\[
\min_x \| G(x) \|^2_2 = \min_x \sum_i \alpha_i G^2_i(x).
\]

Specifically, in this paper, I have some manipulations corresponding to the properties of the equations. First, I set a larger weight \( \alpha_i \) to the first-order condition equations than the market clearing equations, and the weights depend on the value of the parameters. This is because the first-order condition equations are more nonlinear and difficult to converge properly when iterating. Second, since the coefficients of the portfolio unknowns are very small when the public signal precision is comparatively low, the value of the equations are not sensitive to the change of the value of the portfolios and thus the portfolios unknowns are more difficult to converge. To solve this problem, first time the portfolio unknowns by a large constant, \( N \), to make the value of the portfolios more influential. After solving the equations, I time the same constant, \( N \), again to the solutions of the portfolios and obtain better converged value of portfolios. The value of the constant, \( N \), depends on the value of parameters. With the parameters used in this paper, \( N \in [1, 10000] \), can be chosen regarding...
difference equilibriums.

Regarding the nonlinear least-squares algorithms, I first use the Gauss-Newton method. The Gauss-Newton method is more efficient when searching for a good starting point, but not robust to converge to a solution. If the solution is not well converged, then I turn to the Levenberg-Maquardt method. The robustness of the Levenberg-Marquardt method compensates for its occasional poor efficiency. Above algorithms can be implemented by employing the function \textit{fsolve} in MATLAB. Set parameters 'LargeScale','off','NonlEqnAlgorithm', 'gn' when using the Levenberg-Marquardt method. Robustness measures are included in the method.

For the iteration formulas refer to, e.g.,

\texttt{http://www.mathworks.com/help/toolbox/optim/ug/brnoybu.html\#brnoyco}

The residuals can be very small when solving the equations with the parameters used in this paper. It is possible to obtain the value of \(|G(x)|^2 \leq 10^{-20}, |G_i(x)| < 10^{-10}\), for almost all the \textit{ex post} and \textit{ex ante} equilibriums. Hence the solutions converge properly.

To reduce the computing time, in a two-investor model, I rearrange the equilibrium equations at \(t = 1\) and \(t = 0\). Note the equations are partly nonlinear, for instance, the market clearing condition is linear, and the asset price in the first order condition is also linear. Hence in the \textit{ex post} equilibriums (see section 3.2.1), I can rewrite the equilibrium equations at \(t = 1\) into nonlinear equations with only two unknowns, i.e., the portfolios in the risky asset markets of one investor, \(x_{i1}, \theta_{i1}\),

\[
\begin{align*}
m_{i1} - r\sigma_{i1}^2 x_{i1} + \frac{\partial f_i(x_{i1}, \theta_{i1})}{\partial x_{i1}} &= m_j - r\sigma_{j1}^2 (Z - x_{i1}) + \frac{\partial f_j((Z - x_{i1}), -\theta_{i1})}{\partial x_{j1}}, \\
\frac{\partial f_i(x_{i1}, \theta_{i1})}{\partial x_{i1}} &= \frac{\partial f_j((Z - x_{i1}), -\theta_{i1})}{\partial x_{j1}}.
\end{align*}
\]

After solving the two-unknown equations and obtaining the value of the portfolios, one can use the following linear relationships to calculate the values of the other unknowns:

\[
\begin{align*}
x_{j1} &= Z - x_{i1}, \\
\theta_{j1} &= -\theta_{i1}, \\
p_1 &= m_{i1} - r\sigma_{i1}^2 x_{i1} + \frac{\partial f_i(x_{i1}, \theta_{i1})}{\partial x_{i1}}, \\
\pi_1 &= \frac{\partial f_i(x_{i1}, \theta_{i1})}{\partial x_{i1}}.
\end{align*}
\]

Similarly, the nonlinear equations at \(t = 0\) can be written into nonlinear equations with only three unknowns, i.e., the portfolios in the risky and riskless asset markets of one investor, \(x_{i1}, \theta_{i1}, \theta_{i0}\),

\[
\begin{align*}
m_{i1} - r\sigma_{i1}^2 x_{i1} + \frac{\partial f_i(x_{i1}, \theta_{i1})}{\partial x_{i1}} &= m_j - r\sigma_{j1}^2 (Z - x_{i1}) + \frac{\partial f_j((Z - x_{i1}), -\theta_{i1})}{\partial x_{j1}}, \\
\frac{\partial f_i(x_{i1}, \theta_{i1})}{\partial x_{i1}} &= \frac{\partial f_j((Z - x_{i1}), -\theta_{i1})}{\partial x_{j1}}, \\
\frac{\partial f_i(x_{i1}, \theta_{i1})}{\partial \theta_{i1}} &= \frac{\partial f_j((Z - x_{i1}), -\theta_{i1})}{\partial \theta_{j1}}.
\end{align*}
\]
investor. With fewer unknowns in the nonlinear equations, the computing time reduces greatly, however, the accuracy of the solutions may also be affected slightly.

There are some other practical issues worth mentioning. When I solve for the equilibrium at $t = 1$, I set the value of the public signal $y$ vary widely, since $y$ is normally distributed. For some value of the public signal $y$ and the public signal precision $h$, the value of the cumulative distribution function of the standard normal distribution, $\Phi(\cdot)$ can be very close to 1 or 0 during the iteration. The computer may have difficulty to tell if it is 1/0 or a number very close to 1/0 due to the limited computing precision. If this situation happens, the solutions may not converge properly. To solve this problem, I set a threshold $q = 10^{-10}$, when $\Phi(\cdot) < q$, let $\Phi(\cdot) = 0$, then according to the expression of $\partial f_i (x_{i1} (y), \theta_{i1} (y))$ in Eq. (A1), yields

$$f_i (x_{i1} (y), \theta_{i1} (y)) = -\frac{1}{r} \ln c_i = \theta_{i1} (y) \left[ -K + m_{i1} - \frac{1}{2} r \sigma_{i1}^2 \theta_{i1} (y) - r \sigma_{i1}^2 x_{i1} (y) \right].$$

When $1 - \Phi(\cdot) < q$, let $\Phi(\cdot) = 1$, and thus $f_i (x_{i1} (y), \theta_{i1} (y)) = 0$. Hence, at $t = 1$, the function $f_i (x_{i1} (y), \theta_{i1} (y))$ follows different kinds of form conditional on the value of $\Phi(\cdot)$ in the iteration. Regarding each form, new equations need to be solved. This classification can enhance the accuracy of the solution and reduce computing time considerably.

B Derivation of the Model when Investors Update Beliefs and Speculate with Straddles

As indicated in the introduction, in incomplete market settings with heterogeneous beliefs about the risks on the underlying asset, the straddles, i.e., long positions in both a call and a put option with the same strike price can play an important role to facilitate side-betting.

Based on the model in sections 3.1 and 3.2, I now add an additional European put option in zero net supply with payoff, $\max(K - d, 0)$, at $t = 2$, and the prices $v_0 (\eta)$ and $v_1 (y)$ at $t = 0$ and $t = 1$, respectively. The investors have endowments $\bar{\xi}_i$ of this asset at $t = 0$, and let $\xi_{i0}$ it be the units of the put option held after trading at date $t$ satisfying the market clearing conditions

$$\sum_{i=1}^{I} \bar{\xi}_{it} = \sum_{i=1}^{I} \xi_{it} = 0, t = 0, 1.$$
B.1 Benchmark Case when Investors Speculate with Straddles Receiving No or Perfect Public Signal

I first derive a benchmark model in which the investors receive no public signal or perfect public signal at $t = 1$, hence the model is equivalent to a single period model.

B.1.1 Equilibrium in a Single Period Economy

From the perspective of $t = 0$, the date $t = 2$ consumption for the investor $i$ is

$$c_{i2} = \theta_{i0} \max(d - K, 0) + \xi_{i0} \max(K - d, 0) + x_{i0} d + \gamma_{i0}.$$  

Given the period-specific negative exponential utility, the investor $i$’s $t = 0$ certainty equivalent of $t = 2$ consumption, receiving no or perfect public information at $t = 1$, $CE_{i2} (\xi_{i0}; \theta_{i0}; x_{i0}; \gamma_{i0})$ can be calculated according to Lemma 2.

**Lemma 2** Assume the investors receive no or perfect public information at $t = 1$; given the portfolios in the underlying asset markets, the call option markets, the put option markets, and the zero-coupon bond markets at $t = 0$, the investor $i$’s certainty equivalent of $t = 2$ consumption is

$$CE_{i2} (\xi_{i0}; \theta_{i0}; x_{i0}; \gamma_{i0}) = \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 + g_i (x_{i0}; \theta_{i0}, \xi_{i0}),$$

where

$$g_i (x_{i0}; \theta_{i0}, \xi_{i0}) = -\frac{1}{r} \ln\{\exp \left[ -r \xi_{i0} \left[ K - m_{i0} - \frac{1}{2} r \sigma_{i0}^2 \xi_{i0} + r \sigma_{i0}^2 x_{i0} \right] \right]$$

$$\times \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 \theta_{i0} - r \sigma_{i0}^2 x_{i0})}{\sigma_{i0}} \right)$$

$$+ \exp \left[ -r \theta_{i0} \left[ -K + m_{i0} - \frac{1}{2} r \sigma_{i0}^2 \theta_{i0} - r \sigma_{i0}^2 x_{i0} \right] \right]$$

$$\times \left( 1 - \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 (\theta_{i0} + x_{i0}))}{\sigma_{i0}} \right) \right) \},$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Proof: See Appendix B3.

The investors’ $t = 0$ certainty equivalent equals to the date $t = 0$ consumption, i.e.,

$$CE_{i0} = c_{i2} = d_{i0} \bar{z}_i + (\bar{\xi}_{i0} - \xi_{i0}) \nu_0 (\eta) + (\bar{\theta}_{i0} - \theta_{i0}) \pi_0 (\eta) + (\bar{\gamma}_{i0} - \gamma_{i0}) \beta_0 + (\bar{z}_i - x_{i0}) p_0 (\eta),$$

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the investor $i$’s decision problem at $t = 0$ can be stated as follows

$$\max_{\xi_{i0}, \theta_{i0}, \gamma_{i0}, x_{i0}} - \exp (-r \text{CE}_{i0}) - \exp (-\delta) \exp (-r \text{CE}_{i2}).$$

To solve for the equilibrium, I first solve the investor $i$’s optimal portfolio choice problem,

$$\frac{\partial (- \exp (-r \text{CE}_{i0}) - \exp (-\delta) \exp (-r \text{CE}_{i2}))}{\partial \xi_{i0}} = 0,$$

$$\frac{\partial (- \exp (-r \text{CE}_{i0}) - \exp (-\delta) \exp (-r \text{CE}_{i2}))}{\partial x_{i0}} = 0,$$

$$\frac{\partial (- \exp (-r \text{CE}_{i0}) - \exp (-\delta) \exp (-r \text{CE}_{i2}))}{\partial \theta_{i0}} = 0,$$

$$\frac{\partial (- \exp (-r \text{CE}_{i0}) - \exp (-\delta) \exp (-r \text{CE}_{i2}))}{\partial \gamma_{i0}} = 0.$$

Note more details of the calculation of the first derivatives in the first-order condition are provided in Appendix B4. Then the equilibrium portfolios and the prices are the implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset, and the market clearing condition for each asset, i.e.,

$$\sum_{i=1}^{I} \gamma_{i0} = 0, \quad \sum_{i=1}^{I} \xi_{i0} = 0, \quad \sum_{i=1}^{I} \theta_{i0} = 0, \quad \text{and} \quad \sum_{i=1}^{I} x_{i0} = Z.$$

### B.2 Case when Investors Update Beliefs with Imperfect Public Signal and Speculate with Straddles

I now derive a dynamic trading model in which the investors trade in both call and put option markets and update their beliefs at $t = 1$ with imperfect public information.

#### B.2.1 Equilibrium Prices at $t = 1$

I first derive the ex post equilibrium at $t = 2$ conditional on the posterior beliefs. From the perspective of $t = 1$, date $t = 2$ consumption for the investor $i$ is

$$c_{i2} = \theta_{i1} (y) \max(d - K, 0) + \xi_{i1} (y) \max(K - d, 0) + x_{i1} (y) d + \gamma_{i1} (y).$$

Given the period-specific negative exponential utility, the investor $i$ maximizes his certainty equivalent of $t = 2$ consumption (conditional on the public information at $t = 1$),
subject to his budget constraint, i.e.,

\[
\max_{\xi_{i1}(y), \theta_{i1}(y), x_{i1}(y), \gamma_{i1}(y)} \text{CE}_{i2} \left( \xi_{i1}(y), \theta_{i1}(y), x_{i1}(y), \gamma_{i1}(y) \right| m_{i1}, \sigma_{i1}^2 \right)
\]

subject to

\[
\beta_{1} \gamma_{i1}(y) + p_{1}(y) x_{i1}(y) + \pi_{1}(y) \theta_{i1}(y) + v_{1}(y) \xi_{i1}(y) \leq \beta_{1} \gamma_{i0} + p_{1}(y) x_{i0} + \pi_{1}(y) \theta_{i0} + v_{1}(y) \xi_{i0},
\]

where \( \text{CE}_{i2} \left( \xi_{i1}(y), \theta_{i1}(y), x_{i1}(y), \gamma_{i1}(y) \right| m_{i1}, \sigma_{i1}^2 \right) \) can be obtained by Lemma 2.

The \textit{ex post} certainty equivalent of \( t = 2 \) consumption of the investor \( i \) (conditional on the public information at \( t = 1 \)) is

\[
\text{CE}_{i2} \left( \xi_{i1}(y), \theta_{i1}(y), x_{i1}(y), \gamma_{i1}(y) \right| m_{i1}, \sigma_{i1}^2 \right) = \gamma_{i1}(y) + m_{i1} x_{i1}(y) - \frac{1}{2} r \sigma_{i1}^2 x_{i1}(y) + g_{i} \left( x_{i1}(y), \xi_{i1}(y), \theta_{i1}(y) \right),
\]

where

\[
g_{i} \left( \xi_{i1}, x_{i1}, \theta_{i1} \right) = - \frac{1}{r} \ln \left\{ \exp \left[ -r \xi_{i1}(y) \left[ K - m_{i1} - \frac{1}{2} r \sigma_{i1}^2 \xi_{i1}(y) + r \sigma_{i1}^2 x_{i1}(y) \right] \right] \right. \\
\quad \times \Phi \left( \frac{K - \left( m_{i1} - r \sigma_{i1}^2 \left( x_{i1}(y) - \xi_{i1}(y) \right) \right)}{\sigma_{i1}} \right) \\
\quad + \exp \left[ -r \theta_{i1} \left( -K + m_{i1} - \frac{1}{2} r \sigma_{i1}^2 \theta_{i1}(y) - r \sigma_{i1}^2 x_{i1}(y) \right) \right] \\
\quad \times \left( 1 - \Phi \left( \frac{K - \left( m_{i1} - r \sigma_{i1}^2 \left( \theta_{i1}(y) + x_{i1}(y) \right) \right)}{\sigma_{i1}} \right) \right) \}.
\]

To solve the investor \( i \)'s optimal portfolio choice problem, let the Lagrangian be

\[
\hat{\mathcal{L}}_{i} = \gamma_{i1}(y) + m_{i1} x_{i1}(y) - \frac{1}{2} r \sigma_{i1}^2 x_{i1}(y) + g_{i} \left( \xi_{i1}(y), x_{i1}(y), \theta_{i1}(y) \right) \\
+ \lambda \begin{pmatrix} \\
\beta_{1} \left( \gamma_{i1}(y) - \gamma_{i0} \right) + p_{1}(y) \left( x_{i1}(y) - x_{i0} \right) \\
+ \pi_{1}(y) \left( \theta_{i1}(y) - \theta_{i0} \right) + v_{1}(y) \left( \xi_{i1}(y) - \xi_{i0} \right) \\
\end{pmatrix}.
\]

Assume \( \beta_{1} = 1 \) as the numeraire in the model, thus from

\[
\frac{\partial \hat{\mathcal{L}}_{i}}{\partial \gamma_{i1}} = 1 + \lambda \beta_{1} = 0,
\]

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yields \( \lambda = -1 \), hence

\[
\mathcal{L}_i = \gamma_{i1} (y) + m_{i1} x_{i1} (y) - \frac{1}{2} r \sigma_{i1}^2 x_{i1}^2 (y) + g_i (x_{i1} (y), \theta_{i1} (y), \xi_{i1} (y)) - \left( \gamma_{i1} (y) - \gamma_{i0} + p_1 (y) (x_{i1} (y) - x_{i0}) + \pi_1 (y) (\theta_{i1} (y) - \theta_{i0}) + v_1 (y) (\xi_{i1} (y) - \xi_{i0}) \right).
\]

The equilibrium portfolios and the equilibrium prices are the implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset,

\[
\frac{\partial \mathcal{L}_i}{\partial x_{i1}} = m_{i1} - r \sigma_{i1}^2 x_{i1} (y) + \frac{\partial g_i (x_{i1} (y), \theta_{i1} (y), \xi_{i1} (y))}{\partial x_{i1}} - p_1 (y) = 0, \quad (B1)
\]
\[
\frac{\partial \mathcal{L}_i}{\partial \theta_{i1}} = \frac{\partial g_i (x_{i1} (y), \theta_{i1} (y), \xi_{i1} (y))}{\partial \theta_{i1}} - \pi_1 (y) = 0, \quad (B2)
\]
\[
\frac{\partial \mathcal{L}_i}{\partial \xi_{i1}} = \frac{\partial g_i (x_{i1} (y), \theta_{i1} (y), \xi_{i1} (y))}{\partial \xi_{i1}} - v_1 (y) = 0, \quad (B3)
\]

and the market clearing condition for each asset, i.e.,

\[
\sum_{i=1}^l \xi_{i1} (y) = 0, \quad \sum_{i=1}^l \theta_{i1} (y) = 0, \quad \text{and} \quad \sum_{i=1}^l x_{i1} (y) = Z. \quad (B4)
\]

**B.2.2 Equilibrium Prices at \( t = 0 \)**

I now determine the equilibrium prices and the equilibrium demands at \( t = 0 \), taking the equilibrium at \( t = 1 \) characterized by the system of equations, i.e., Eq. (B1) to Eq. (B4) as given. From the perspective of \( t = 0 \), the date \( t = 2 \) consumption for the investor \( i \) is

\[
c_{i2} = \xi_{i1} (y) (\max(K - d, 0) - v_1 (y)) + \xi_{i0} v_1 (y) + \theta_{i1} (y) (\max(d - K, 0) - \pi_1 (y)) + \theta_{i0} \pi_1 (y) + x_{i1} (y) (d - p_1 (y)) + x_{i0} p_1 (y) + \gamma_{i0} \]
\[
= \xi_{i1} (y) \max(K - d, 0) + \theta_{i1} (y) \max(d - K, 0) + x_{i1} (y) d + (\xi_{i0} - \xi_{i1} (y)) v_1 (y) + (\theta_{i0} - \theta_{i1} (y)) \pi_1 (y) + (x_{i0} - x_{i1} (y)) p_1 (y) + \gamma_{i0}.
\]

By Lemma 2, conditional on the information at \( t = 1 \), the investor \( i \)'s certainty equivalent
of \( t = 2 \) consumption is

\[
\text{CE}_{i2} (\bar{z}_i, x_{i0}, \theta_{i0}, x_{i1} (y), \theta_{i1} (y), p_0(\eta), p_1(y), \pi_0(\eta), \pi_1(y), \gamma_{i0} | F_1)
\]

\[
= (\xi_{i0} - \xi_{i1} (y)) v_1 (y) + (\theta_{i0} - \theta_{i1} (y)) \pi_1 (y)
+ (x_{i0} - x_{i1} (y)) p_1 (y) + \gamma_{i0} + m_{i1} x_{i1} (y) - \frac{1}{2} r \sigma_{i1}^2 x_{i1}^2 (y)
\]

\[-\frac{1}{r} \ln \{ \exp \left[ -r \xi_{i1} (y) \left[ K - m_{i1} - \frac{1}{2} r \sigma_{i1}^2 \xi_{i1} (y) + r \sigma_{i1}^2 x_{i1} (y) \right] \right] \]

\[\times \Phi \left( \frac{K - (m_{i1} - r \sigma_{i1}^2 (x_{i1} (y) - \xi_{i1} (y))))}{\sigma_{i1}} \right) \]

\[+ \exp \left[ -r \theta_{i1} (y) \left[ -K + m_{i1} - \frac{1}{2} r \sigma_{i1}^2 \theta_{i1} (y) - r \sigma_{i1}^2 x_{i1} (y) \right] \right] \]

\[\times \left( 1 - \Phi \left( \frac{K - (m_{i1} - r \sigma_{i1}^2 (\theta_{i1} (y) + x_{i1} (y))))}{\sigma_{i1}} \right) \right) \},
\]

where \( F_1 \) denotes the investors’ public information at date \( t = 1 \).

Conditional on the information at \( t = 1 \), the investor \( i \)'s certainty equivalent of \( t = 2 \) consumption varies with the investors’ posterior beliefs \( m_{i1} \) and \( \sigma_{i1}^2 \), and thus is a function of the public signal \( y \). Note from the perspective of the investor \( i \), \( y \sim N(m_i, \sigma_i^2 + \sigma_z^2) \).

Moreover, the investor \( i \)'s \( t = 0 \) certainty equivalent equals to the date \( t = 0 \) consumption, i.e.,

\[
\text{CE}_{i0} = c_{i0} = d_0 \bar{z}_i + (\bar{\xi}_i - \xi_{i0}) v_0(\eta) + (\bar{\theta}_i - \theta_{i0}) \pi_0(\eta) + (\bar{\gamma}_i - \gamma_{i0}) \beta_0 + (\bar{z}_i - \bar{x}_{i0}) p_0(\eta),
\]

hence the investor \( i \)'s decision problem at \( t = 0 \) can be stated as follows

\[
\max_{\xi_{i0}, \theta_{i0}, \gamma_{i0}, x_{i0}} - \exp (-r \text{CE}_{i0}) - \exp (-\delta) \exp (-r \text{CE}_{i2}).
\]

Solve the investor \( i \)'s optimal portfolio choice problem, the equilibrium portfolios and the prices are the implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset, i.e.,
\[
\frac{\partial (- \exp(-rCE_i0) - \exp(-\delta) \exp(-rCE_i2))}{\partial \xi_{i0}} = 0, \\
\frac{\partial (- \exp(-rCE_i0) - \exp(-\delta) \exp(-rCE_i2))}{\partial x_{i0}} = 0, \\
\frac{\partial (- \exp(-rCE_i0) - \exp(-\delta) \exp(-rCE_i2))}{\partial \theta_{i0}} = 0, \\
\frac{\partial (- \exp(-rCE_i0) - \exp(-\delta) \exp(-rCE_i2))}{\partial \gamma_{i0}} = 0,
\]

and the market clearing condition for each asset, i.e.,

\[
\sum_{i=1}^{I} \gamma_{i0} = 0, \quad \sum_{i=1}^{I} \xi_{i0} = 0, \quad \sum_{i=1}^{I} \theta_{i0} = 0, \quad \text{and} \quad \sum_{i=1}^{I} x_{i0} = Z.
\]

### B.3 Proof of Lemma 2

Since the investor \(i\)'s consumption at \(t = 2\)

\[
c_{i2} = \theta_{i0} \max(d-K,0) + \xi_{i0} \max(K-d,0) + x_{i0}d + \gamma_{i0},
\]

hence the \(t = 2\) expected utility for the investor \(i\) is

\[
\begin{align*}
&= -E \left[ \exp \left[ -r \left[ \theta_{i0} \max(d-K,0) + \xi_{i0} \max(K-d,0) + x_{i0}d + \gamma_{i0} \right] \right] \right] \\
&= - \int_{-\infty}^{+\infty} \exp \left[ -r \left[ (x_{i0} - \xi_{i0})d + \xi_{i0}K + \gamma_{i0} \right] \right] f \left( x; m_{i0}, \sigma_{i0}^2 \right) dx \\
&= - \int_{-\infty}^{K} \exp \left[ -r \left[ (x_{i0} - \xi_{i0})d + \xi_{i0}K + \gamma_{i0} \right] \right] f \left( x; m_{i0}, \sigma_{i0}^2 \right) dx \\
&\quad - \int_{K}^{+\infty} \exp \left[ -r \left[ (\theta_{i0} + x_{i0})d - \theta_{i0}K + \gamma_{i0} \right] \right] f \left( x; m_{i0}, \sigma_{i0}^2 \right) dx
\end{align*}
\]

\[
= - \exp \left[ -r \left[ \gamma_{i0} + \xi_{i0}K + m_{i0} (x_{i0} - \xi_{i0}) \right] - \frac{1}{2} r \sigma_{i0}^2 (x_{i0} - \xi_{i0})^2 \right] \\
\times \int_{-\infty}^{K} f \left( x; m_{i0} - r \sigma_{i0}^2 (x_{i0} - \xi_{i0}), \sigma_{i0}^2 \right) dx \\
- \exp \left[ -r \left[ \gamma_{i0} - \theta_{i0}K + m_{i0} (\theta_{i0} + x_{i0}) \right] - \frac{1}{2} r \sigma_{i0}^2 (\theta_{i0} + x_{i0})^2 \right] \\
\times \int_{K}^{+\infty} f \left( x; m_{i0} - r \sigma_{i0}^2 (\theta_{i0} + x_{i0}), \sigma_{i0}^2 \right) dx
\]

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\[
\begin{align*}
&= - \exp \left[ -r \left( \gamma_{i0} + \xi_{i0} K + m_{i0} (x_{i0} - \xi_{i0}) - \frac{1}{2} r \sigma_{i0}^2 (x_{i0} - \xi_{i0})^2 \right) \right] \\
&\quad \times \int_{\sigma_{i0}}^{x_{i0} - r \sigma_{i0}^2 (x_{i0} - \xi_{i0})} \phi(x) \, dx \\
&= - \exp \left[ -r \left( \gamma_{i0} - \theta_{i0} K + m_{i0} (\theta_{i0} + x_{i0}) - \frac{1}{2} r \sigma_{i0}^2 (\theta_{i0} + x_{i0})^2 \right) \right] \\
&\quad \times \int_{\sigma_{i0}}^{+\infty} \phi(x) \, dx \\
&= - \exp \left[ -r \left( \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 + \xi_{i0} K - m_{i0} \xi_{i0} \right) \right] \\
&\quad \times \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 (x_{i0} - \xi_{i0}))}{\sigma_{i0}} \right) \\
&= - \exp \left[ -r \left( \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 - \frac{1}{2} r \sigma_{i0}^2 \xi_{i0}^2 + \frac{1}{2} r \sigma_{i0}^2 x_{i0} \xi_{i0} ight) \right] \\
&\quad \times \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 (x_{i0} - \xi_{i0}))}{\sigma_{i0}} \right) \\
&= - \exp \left[ -r \left( \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 - \frac{1}{4} r \sigma_{i0}^2 \theta_{i0}^2 - \frac{1}{2} r \sigma_{i0}^2 \theta_{i0} x_{i0} \right) \right] \\
&\quad \times \left( 1 - \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 (\theta_{i0} + x_{i0}))}{\sigma_{i0}} \right) \right)
\end{align*}
\]
Therefore, the certainty equivalent of \( t = 2 \) consumption for the investor \( i \) is

\[
\text{CE}_{i2} = -\frac{1}{r} \ln(-U_{i2} (\theta_{i0}, x_{i0}, \gamma_{i0} | m_{i0}, \sigma_{i0}^2))
\]

\[
= \gamma_{i0} + m_{i0} x_{i0} - \frac{1}{2} r \sigma_{i0}^2 x_{i0}^2 \\
- \frac{1}{r} \ln\{\exp \left[ -r \xi_{i0} \left[ K - m_{i0} - \frac{1}{2} r \sigma_{i0}^2 \xi_{i0} + r \sigma_{i0}^2 x_{i0} \right] \right] \\
\times \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 (x_{i0} - \xi_{i0}))}{\sigma_{i0}} \right) \\
+ \exp \left[ -r \theta_{i0} \left[ -K + m_{i0} - \frac{1}{2} r \sigma_{i0}^2 \theta_{i0} - r \sigma_{i0}^2 x_{i0} \right] \right] \\
\times \left( 1 - \Phi \left( \frac{K - (m_{i0} - r \sigma_{i0}^2 (\theta_{i0} + x_{i0}))}{\sigma_{i0}} \right) \right) \}. 
\]

This completes the proof.
B.4 Calculate the First Derivatives in Model B.1 and B.2

This section provides the calculation of the first derivatives in the first-order condition in model B.1 and B.2. Let

\[
\begin{align*}
\tilde{a}_i &= \frac{K - (m_{\alpha} - r\sigma_{\alpha}^2 (x_{i0} - \xi_{i0}))}{\sigma_{\nu}}, \\
\tilde{b}_i &= \frac{K - (m_{\alpha} - r\sigma_{\alpha}^2 (\theta_{i0} + x_{i0}))}{\sigma_{\nu}}, \\
\tilde{c}_i &= \exp \left[ -r \theta_{i0} \left[ -K + m_{\alpha} - \frac{1}{2} r\sigma_{\alpha}^2 \theta_{i0} - r\sigma_{\alpha}^2 x_{i0} \right] \right], \\
\tilde{d}_i &= \exp \left[ -r \xi_{i0} \left[ K - m_{\alpha} - \frac{1}{2} r\sigma_{\alpha}^2 \xi_{i0} + r\sigma_{\alpha}^2 x_{i0} \right] \right],
\end{align*}
\]

therefore,

\[
g_i (x_{i0}, \theta_{i0}, \xi_{i0}) = -\frac{1}{r} \ln \{ \tilde{d}_i \times \Phi (\tilde{a}_i) + \tilde{c}_i \times (1 - \Phi (\tilde{b}_i)) \},
\]

and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution. Hence

\[
\begin{align*}
\frac{\partial g_i (x_{i0}, \theta_{i0}, \xi_{i0})}{\partial x_{i0}} &= \frac{\tilde{d}_i \phi (\tilde{a}_i) \sigma_{\nu} - \tilde{d}_i \Phi (\tilde{a}_i) r\sigma_{\alpha}^2 \xi_{i0}}{d_i \Phi (\tilde{a}_i) + (1 - \Phi (\tilde{b}_i)) \tilde{c}_i}, \\
\frac{\partial g_i (x_{i0}, \theta_{i0}, \xi_{i0})}{\partial \theta_{i0}} &= \frac{\phi (\tilde{b}_i) \sigma_{\nu} \tilde{c}_i + (1 - \Phi (\tilde{b}_i)) (m_{\alpha} - \tilde{d}_i \Phi (\tilde{a}_i) + (1 - \Phi (\tilde{b}_i)) \tilde{c}_i)}{d_i \Phi (\tilde{a}_i) + (1 - \Phi (\tilde{b}_i)) \tilde{c}_i}, \\
\frac{\partial g_i (x_{i0}, \theta_{i0}, \xi_{i0})}{\partial \xi_{i0}} &= \frac{\phi (\tilde{a}_i) \sigma_{\nu} \tilde{d}_i + \Phi (\tilde{a}_i) (m_{\alpha} - r\sigma_{\alpha}^2 \xi_{i0} + r\sigma_{\alpha}^2 x_{i0}) \tilde{d}_i}{d_i \Phi (\tilde{a}_i) + (1 - \Phi (\tilde{b}_i)) \tilde{c}_i},
\end{align*}
\]

where \( \phi (x) \) denotes the standard normal probability density function, hence

\[
\phi (x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right).
\]
Chapter 3

Continuous Trading Dynamically Effectively Complete Market
with Heterogeneous Beliefs
Continuous Trading Dynamically Effectively Complete Market with Heterogeneous Beliefs

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Abstract

In a framework of heterogeneous beliefs, I investigate a two-date consumption model with continuous trading over the interval $[0, T]$, in which information on the aggregate consumption at time $T$ is revealed by an Ornstein-Uhlenbeck Bridge. This information structure allows investors to speculate on the heterogeneous posterior variance of dividend throughout $[0, T)$. The market populated with many time-additive exponential-utility investors is dynamically effectively complete, if investors are allowed to trade in only two long-lived securities continuously. The underlying mechanism is that these assumptions imply that the Pareto efficient individual consumption plans are measurable with respect to the aggregate consumption. Hence, I may not need a dynamically complete market to facilitate a Pareto efficient allocation of consumption, the securities only have to facilitate an allocation which is measurable with respect to the aggregate consumption. With normally distributed dividend, the equilibrium stock price is endogenized in a Radner equilibrium as a precision weighted average of the investors’ posterior mean minus a risk premium determined by the average posterior precision. The stock price is also a sufficient statistic for computation of the price of redundant dividend derivative and the equilibrium portfolios. The investors form their Pareto optimal trading strategies as if they intend to dynamically endogenously replicate the value of the dividend derivative.

Keywords: Heterogeneous Beliefs; Continuous Trading; Dynamically Effectively Complete Market; Asset Pricing

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1 Introduction

In a homogeneous belief setting, Duffie and Huang (1985) and Duffie and Zame (1989) demonstrate that continuous trading can play a role as a compensation of long lived securities to dynamically complete the financial market, with finite number of securities. In other words, a continuous trading Radner equilibrium can implement the same Arrow-Debreu consumption allocations. These are two of the very few papers to address the issue of welfare consequences of continuous-trading opportunities in a few long-lived securities (Sundaresan 2000).

However, Duffie and Zame (1989) model the endowment by Arithmetic Brownian Motion, and their information structure cannot be generalized to the heterogeneous beliefs case with heterogeneous volatility. Since it follows from Girsanov’s Theorem that the instantaneous volatility of the endowment is identical across investors under both individual perceived dynamics and risk-neutral dynamics. Therefore, to investigate the effects of speculation on the heterogeneity of perceived dividend variance, another information structure which allows difference in perceived variance is in order. Furthermore, in Duffie and Huang (1985), information structure is modeled by a complete probability space which constitutes all possible states of the world that could exist at a terminal date. The investors can receive information which is not relevant to the Pareto optimal consumption. However, in the real world, investors usually receive only part of the information in the economy, for instance, information on the aggregate consumption. Naturally, questions arise: How to generalize Duffie and Huang (1985) to a case with heterogeneous beliefs and information only relevant to aggregate consumption? Can the Pareto consumption allocations in the Arrow-Debreu equilibrium with heterogeneous beliefs in Christensen and Qin (2012) be implemented by some counterpart continuous trading model? In this paper, I show that, fortunately, dynamically complete market can be substituted by additional assumptions about preferences, in order to yield effectively identical results. I find with more restrictive assumptions of utility, i.e., with time-additive negative-exponential utility, continuous trading can dynamically effectively complete the financial market with heterogeneous beliefs. This result can considered to be a consequence of continuous-time Pareto efficient side-betting based on the heterogeneously perceived variance of dividend.

This work is a heterogeneous-belief extension of Duffie and Huang (1985). I summarize the main results in the following three aspects. First, an information structure is constructed to allow heterogeneity in perceived variance of the terminal dividend. The investors can speculate on the variance of the dividend throughout the interval $[0, T)$. Second, with continuous-time Pareto efficient side-betting on confidence among investors, the market is
dynamically effectively complete, if investors are allowed to trade in only two long-lived securities continuously. Third, I provide an example to endogenously replicate the value of the redundant derivative in a Radner equilibrium.

I investigate a two-date consumption model in which information on the aggregate consumption at the terminal date $T$ is revealed by an Ornstein-Uhlenbeck Bridge, which is driven by a Standard Brownian Motion. The investors update their beliefs on the normally distributed dividend in a Bayesian fashion. The continuous speculative trade motivated by the heterogeneous beliefs can dynamically effectively complete the markets. The underlying mechanism is that the assumptions of heterogeneous beliefs and time-additive exponential utility imply that the Pareto efficient individual consumption plans are measurable with respect to aggregate consumption. Hence, I may not need a dynamically complete market to facilitate the Pareto efficient allocation of consumption, the securities only have to facilitate an allocation which is measurable with respect to the aggregate consumption. The dynamically effectively complete market in the model has the property that any consumption plan, which is measurable with respect to aggregate consumption, can be implemented, despite the fact that investors may not be able to implement any financially feasible consumption plan as they are in a dynamically complete market.

The model in this paper is a counterpart extreme case of that in Christensen and Qin (2012). Based on the idea of Wilson (1968), they introduce a dividend derivative which pays off the square of dividend at the terminal date to facilitate side-betting and achieve a Pareto efficient equilibrium. This extreme case gets rid of the need for dynamic trading based on public signals. In contrast, in this paper, the investors trade dynamically based on continuous signals. The payoff of the "Dividend Square Security" can be endogenously replicated by trading a riskless bond and a risky asset continuously. Specifically, under the assumptions of exponential utility and the normally distributed dividend, I derive analytical expressions of the equilibrium security prices and the equilibrium portfolios in the Radner equilibrium to implement the Arrow-Debreu equilibrium in Christensen and Qin (2012). The equilibrium stock price in the Radner equilibrium is given as a precision weighted average of the investors’ posterior mean minus a risk premium determined by the average posterior precision. The stock price is driven by the heterogeneously updated posterior beliefs and, thus, driven by the prior beliefs and the public signals (as functions of the Brownian motion). Moreover, the stock price is a sufficient statistic for computation of the optimal portfolios, the optimal wealth processes, and the price of the redundant dividend derivative. The investors form their Pareto optimal trading strategies by investing as if they intend to dynamically replicate the value of the dividend derivative. Therefore, continuous trading can be viewed as a replacement of the convexity in the payoff of the derivatives needed for Pareto efficient
side-betting, and implement the Pareto efficient consumption allocations.

Review of the Literature

Current work is closely related to several continuous-time models in different directions. Duffie and Huang (1985) study a dynamically complete market with homogeneous belief. Compare to their model, this paper’s information structure allows me to explicitly study the effects of heterogeneous updated posterior beliefs on asset pricing properties such as equilibrium stock price and equilibrium portfolios. Besides, the dynamically effectively complete market in this paper cannot facilitate all kinds of consumption plans as the dynamically complete market can.

With homogeneous prior belief, Brennan and Cao (1996) assume investors receive signals with different precision. New exogenous supply shocks are needed to generate the trading volume in stock to achieve Pareto efficient consumption allocations, which can also be obtained by trading in a quadratic option. In contract, in this paper, the trading on stock is endogenized in the equilibrium as a results of speculation on heterogeneous variances. In a framework of homogeneous belief, Christensen, Graversen, and Miltersen (2000) show that continuous trading of long-lived contingent claims on aggregate consumption can substitute the need for an infinite number of primitive securities in a dynamically effectively complete market. Zuasti (2008) extends the above literature on continuous trading, by formally including insurance as a non-tradable asset and studying its price and demand. He provides a framework of heterogeneous von Neumann–Morgenstern preferences to study of the interaction between insurance and dynamic financial market which is effectively complete with homogeneous belief. Anderson and Raimondo (2008) provide a non-degeneracy condition on the terminal security dividends to insure completeness in equilibrium with homogeneous belief.

Beyond the classical two-consumption date economy, many other works are contributed to study the infinite-consumption model with heterogeneous beliefs. Buraschi and Jiltsov (2006) and David (2008) investigate continuous-time model in which the power utility investors update beliefs with the dividend process following a geometric Brownian motion, with heterogeneous prior beliefs. The market in their model is effectively incomplete. Moreover, their information structure does not allow heterogeneity in the dividend volatility. Thus, investors conduct no speculative activity with respect to the volatility of dividend and signals. Beker and Espino (2011), in a discrete-time framework, analyze the dynamic properties of portfolios that sustain dynamically complete markets equilibria when investors have heterogeneous priors.

Although both the Black-Scholes model and this paper involve the notion of replicating
the value of redundant derivative, the method in this paper fundamentally differs from that in the Black-Scholes model. First, the Radner equilibrium endogenously replicates the payoff of the redundant asset, in which both the price process of the underlying asset and the replicating strategies are endogenized in the Radner equilibrium. In contrast, in the Black and Scholes (1973) option pricing model, the underlying price process is exogenously given. Furthermore, Black and Scholes (1973) cannot be generalized to the case with heterogeneously perceived volatility, since under individual equivalent probability measure, the volatility has to be identical across investors.

This paper is organized as follows. The primitives of the economy and the learning mechanism in continuous time are established in Section 2. Section 3 establishes a continuous trading Radner equilibrium to implement the same Arrow-Debreu consumption allocations with heterogeneous beliefs in Christensen and Qin (2012). Propositions present the expressions of the equilibrium stock price, the equilibrium portfolios, and the price of the dividend derivative as functions of posterior beliefs, revealing the impacts of public signals. Section 4 concludes the paper. Proofs of lemmas are provided in Appendix A, and the proofs of propositions and the methods to implement the allocation in the Arrow-Debreu equilibrium with heterogeneous beliefs are provided in Appendix B.

2 The Model

I examine an economy with two consumption dates and investors can trade continuously in between with heterogeneous beliefs on the terminal dividend. The model extends the model in Duffie and Huang (1985) to a heterogeneous-belief framework with information only contingent on the terminal aggregate consumption. The investors implement their Pareto optimal consumption plans in a continuous-time and continuous state-space economy.

2.1 The Investors’ Beliefs and Preferences

Uncertainty in the economy is represented by an individual-specific product probability space\(^1\) \((\Omega_{D_T} \times \Omega, \mathcal{F}_{D_T} \times \mathcal{F}, P_{D_T}^i \times P)\), where \(\Omega_{D_T}\) and \(\Omega\) are independent. The sample space \(\Omega_{D_T}\) contains all possible terminal dividend, and \(\Omega\) on which is defined a one-dimensional

\(^1\)For a brief introduction of product probability space, see, e.g. Grigoriu (2002). Consider two probability spaces \((\Omega_1, \mathcal{F}_1, P_1)\), and \((\Omega_2, \mathcal{F}_2, P_2)\), describing two experiments. These two experiments can be characterized jointly by the product probability space \((\Omega_0, \mathcal{F}_0, P_0)\) with the following components.

(i) Product sample space:

\[ \Omega_0 = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_k \in \Omega_k, k = 1, 2\} . \]

(ii) Product \(\sigma\)-field: \(\mathcal{F}_0 = \mathcal{F}_1 \times \mathcal{F}_2 = \sigma(\mathbb{R})\), where measurable rectangles
Brownian motion \( W \). Let \( \{ \mathcal{F}_t \} \) denote the augmented filtration generated by \( W(t) \), and \( \mathcal{F}_{DT} \) is a \( \sigma \)-field independent of \( \mathcal{F} \). The field \( \mathcal{F}_{DT} \), whose role is to allow for heterogeneity in investors’ priors, consists of all possible initial beliefs. The complete information filtration is the augmentation of the filtration \( \mathcal{F}_{DT} \times \{ \mathcal{F}_t \} \).

There are two consumption dates, \( t = 0 \) and \( t = T \), and there are \( I \) investors who are endowed at \( t = 0 \) with a portfolio of marketed securities. The investors potentially receive public information which is revealed by a Standard Brownian Motion continuously at \( t \in [0, T] \), and receive terminal normally-distributed dividends at \( t = T \). The trading of the marketed securities takes place at \( t \in [0, T] \). There are two marketed securities: a zero-coupon bond which pays one unit of consumption at \( t = T \) and is in zero net supply, the shares of a single risky firm which have net supplies \( Z \) at \( t \in [0, T] \). The assumptions of endowment in this paper are identical to that in Christensen and Qin (2012). The investors are endowed with \( \pi_i \) units of the \( t = T \) zero-coupon bond and \( z_i \) shares of the risky asset, \( i = 1, 2, \ldots, I \). In addition, the investors are endowed with \( \pi_i \) units of a zero-coupon bond, also in zero net-supply, paying one unit of consumption at \( t = 0 \). Let \( x_{it} \) and \( \gamma_{it} \) present investor \( i \)'s portfolio of share and units had of the zero-coupon bond after trading at date \( t \), respectively. Hence, the market clearing conditions at date \( t \) are

\[
\sum_{i=1}^{I} \gamma_{it} = 0, \quad \sum_{i=1}^{I} x_{it} = Z \equiv \sum_{i=1}^{I} z_i, \quad t \in [0, T].
\]

The investor \( i \)'s consumption at date \( t \) is denoted \( c_{it} \) and they have time-additive utility. The common period-specific utility is negative exponential utility with respect to consumption, i.e., \( u_{i0}(c_{i0}) = -\exp[-rc_{i0}] \) and \( u_{iT}(c_{iT}) = -\exp[-\delta] \exp[-rc_{iT}] \), where \( r > 0 \) is the investors’ common constant absolute risk aversion parameter. Moreover, the investor \( i \)'s has common utility discount rate, \( \delta \), for date \( t = T \) consumption.

### 2.2 Learning Mechanism

In this section, I construct an information structure which is a continuous-time extension of the two-period learning model in Christensen and Qin (2012). This information structure

\[
\mathcal{R} = \{ A_1 \times A_2 : A_1 \in \mathcal{F}_1, \ A_2 \in \mathcal{F}_2 \}.
\]

(iii) Product probability measure \( P_0 = P_1 \times P_2 \), on the measurable space \( (\Omega, \mathcal{F}) \): The probability \( P_0 \) is unique and has the property

\[
P_0 (A_1 \times A_2) = P_1 (A_1) P_2 (A_2) : A_1 \in \mathcal{F}_1, \ A_2 \in \mathcal{F}_2.
\]
satisfies that: (1) Signals gradually reveal the information about the terminal dividend; (2) Investors hold heterogeneous prior beliefs, and continuously update their beliefs according to the Bayes’ rule. They know the terminal dividend perfectly at \( t = T \); (3) The information structure should allow heterogeneity in perceived variance of the terminal dividend and, thus, the investors can speculate on the variance of the dividend throughout the interval \([0, T]\).

A share of the risky asset pays a dividend \( d_0 \) at date \( t = 0 \) and a dividend \( d_T \) at date \( t = T \). At \( t = 0 \), the investor \( i \) views \( d_T \) as a normally distributed variable, with mean \( m_{i0} \) and variance \( \sigma_{i0}^2 \). The investors observe a continuous signal process, \( y_t \), follows the differential form according to Eq. (3) in the following Lemma.

**Lemma 1 (Ornstein-Uhlenbeck Bridge)** Given deterministic functions \( A(t) > 0 \) and \( B(s) \) satisfying that

\[
\lim_{t \to T} A(t) = 0 \quad \text{and} \quad \lim_{s \to T} B(s)^2 e^\int_0^s A(u)^{-1} du = 0,
\]

then there exists a unique stochastic signal process \( y = (y_t) \) solving the following SDE\(^2\)

\[
dy_t = A(t)^{-1} (d_T - y_t) \, dt + B(t) \, dW_t, \quad t \in (0, T), \quad y_0 \in \mathbb{R},
\]

and \( y_t \to d_T, \text{ P–a.s.}, \text{ as } t \to T \), where \( d_T \) is the terminal dividend at \( t = T \).

Proof. See the Appendix.

Note the terminal dividend \( d_T \) and Brownian motion in the signal process is independent, and the signal \( y_t \) is measurable to \( \Omega_{D_T} \times \Omega \). Particularly, the prior beliefs about \( d_T \) at date \( t = 0 \) is \( \mathcal{F}_{D_T} \)-measurable. This assumption is consistent with the discrete-time information structure in Christensen and Qin (2012)\(^3\).

Given the signal process defined above, the posterior mean and posterior variance of each investor can be derived by employing the standard filtering theorem in Liptser and Shiryayev (1977). To ensure the signal process converges to \( d_T \), the coefficients of the signal process have to meet condition (2). Moreover, according to Liptser and Shiryayev (1977),

\(^2\)Although the investors perceive the terminal dividend \( d_T \) as a random variable, the nature determines the terminal dividend. Thus, the terminal dividend in the SDE of the public signal can be viewed as an exogenous parameter. As a result, the public signal process is adapted to the filtration \( \{\mathcal{F}_t\} \).

\(^3\)The information structure in this paper is different from the Kyle-Back model of "insider trading" (see Kyle 1985 and Back 1992) or the dynamic Markov bridges motivated by models of insider trading (see Campi, Cetin, and Danilova 2011), in which a gradually informed insider observes a signal process (unknown to the market), and the signal process converges to a terminal value which is stochastic and not known in advance.
the following conditions are required to achieve a Bayesian learning:

\[ \int_0^T A(t)^{-2} \, dt < \infty, \quad \int_0^T B(t)^2 \, dt < \infty. \]  

(4)

I specify that

\[ A = \alpha^{-1} (T - t)^{-\frac{k}{2}}, \quad -1 < k < 0, \quad \alpha > 0, \]  

(5)

and

\[ B = \beta (T - t)^{q}, \quad q > 0, \quad \beta \in R, \]  

(6)

and proof that the specified coefficients \( A \) and \( B \) meet all the requirements in (2) and (4). See proofs in the Appendix.

With the specified coefficients, the signal process which converges to the terminal dividend can be stated as

\[ dy_t = \alpha (T - t)^{\frac{k}{2}} (d_T - y_t) \, dt + \beta (T - t)^{q} \, dW_t, \quad t \in (0, T), \quad y_0 \in R. \]  

(7)

Note there is a linear dependence of the observable component \( y_t \) in the drift coefficient of the signal process. Denote the expectation and variance conditional on observed signals up to date \( t \) by \( m_{it} \) and \( \sigma_{it}^2 \). I use \( h_{it} \equiv 1/\sigma_{it}^2 \) throughout to denote precisions for the associated variances. According to Theorem 10.3 in Liptser and Shiryayev (1977), the dynamics of the conditional expectations and conditional variances are given by (also see Lemma 2 in the Appendix, i.e., a benchmark case of Theorem 10.3 in Liptser and Shiryayev (1977))

\[ dm_{it} = \frac{\alpha \sigma_{it}^2}{\beta^2} (T - t)^{\frac{k}{2}-2q} \left[ dy_t - \alpha (T - t)^{\frac{k}{2}} (m_{it} - y_t) \, dt \right], \]  

(8)

and

\[ d\sigma_{it}^2 = -\sigma_{it}^4 \frac{\alpha^2}{\beta^2} (T - t)^{k-2q} \, dt. \]

The posterior variance follows an ODE, solve for the ODE, I obtain,

\[ \sigma_{it}^2 = \frac{(k - 2q + 1) \beta^2 \sigma_{i0}^2}{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} - \alpha^2 \sigma_{i0}^2 (T - t)^{k-2q+1} + \beta^2 (k - 2q + 1)}. \]  

(9)

When \( t \to 0, \sigma_{it}^2 \to \sigma_{i0}^2 \), and when \( t \to T, \text{ if } k - 2q + 1 < 0, \text{ then } (T - t)^{k-2q+1} \to \infty, \text{ thus, } \sigma_{it}^2 \to 0 \). Note \( k < 0 < 2q - 1 \), thus, when \( q > \frac{1}{2} \), the posterior variance decreases from the prior variance to zero, and all the investors know the terminal dividend at \( t = T \) perfectly.

The heterogeneously updated beliefs give basis for side-betting over the interval \([0, T)\).

With normally distributed terminal dividend, Christensen and Qin (2012) employ a
Bayesian learning model in discrete time to model the information structure. As a counterpart information structure in continuous time, the information is revealed and the posterior mean is driven by the Standard Brownian Motion with multiplicity one. This mathematical characteristic has important asset pricing implications, as shown in following sections.

3 Equilibrium with Heterogeneity in Beliefs and Information on Terminal Aggregate Consumption

In this section, I show with heterogeneous beliefs, how the trading strategies and the asset prices in a continuous trading Radner (1972) equilibrium implement the identical Arrow-Debreu consumption allocations in Christensen and Qin (2012), with only two long-lived securities.

3.1 Arrow-Debreu Equilibrium

Under the assumptions of exponential utility investors with heterogeneous beliefs and normally distributed dividend, Christensen and Qin (2012) show that effectively complete market can be achieved if allow the investors trade in only three assets, i.e., a zero coupon bond, a stock, and a derivative which pays off the square of the dividend at the terminal date. In such a Arrow-Debreu equilibrium, the Pareto efficient consumption is a linear function of the aggregate consumption plus a state-dependent term. In other words, the investors share risk (side bet) linearly with heterogeneous beliefs.

3.2 Radner Equilibrium

It is a standard result that the assumptions of heterogeneous beliefs and time-additive preferences represented by exponential utility imply that the Pareto efficient individual consumption plans are measurable with respect to the aggregate consumption (see, e.g., Christensen and Feltham (2003), Chapter 4). Hence, under such a framework, to facilitate the Pareto efficient allocation of consumption, the securities only have to facilitate allocations which are

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4 A brief introduction of martingale multiplicity is given as follows. The space of square-integrable martingales on \((\Omega, \mathcal{F}, P)\) which are null at zero is denoted \(M^2_P\). Two martingales \(\tilde{X}\) and \(\tilde{Y}\) are said to be orthogonal if the product \(\tilde{X}\tilde{Y}\) is a martingale. Defined an orthogonal 2-basis for \(M^2_P\) as a minimal set of mutually orthogonal elements of \(M^2_P\) with the representation property. Then, the number of elements of a 2-basis, whether countably infinite or some positive integer, is called the multiplicity of \(M^2_P\), denoted \(M(M^2_P)\). Refer to the Appendix in Duffie and Huang (1985) for a detailed description of the martingale multiplicity.
measurable with respect to aggregate consumption. The notion of complete market which needs infinite securities is not necessary anymore.

### 3.2.1 Implementing Pareto Efficient Consumption Allocation

Duffie and Huang (1985) provide a procedure to implement the Pareto consumption plan in dynamically complete market. Each consumption allocation including those not measurable to the aggregate consumption can be implemented by the price-contingent portfolio. However, in this paper, investors only receive the information which is about the aggregate consumption, thus, I can use their procedure to implement consumption plan, which is measurable to the aggregate consumption.

Assume $Q$ as the martingale measure, the space of square-integrable martingales under $Q$, denoted $M^2_Q$; its multiplicity, denoted $M(M^2_Q)$. To employ their procedure, I first have to specify an orthogonal 2-basis for $M^2_Q$. Since the multiplicity, $M(M^2_Q)$, determines how many securities are needed to dynamically effectively complete the market.

According the information structure in the previous section, a Standard Brownian Motion $W$ reveals the information on the aggregate consumption. It is a well known result that the underlying Brownian Motion $W$ is a 2-basis for $M^2_P$. Assuming $Q \approx P$, the process

$$\Lambda(t) = E \left[ \frac{dQ}{dP} \bigg| \mathcal{F}_t \right], \quad t \in [0, T],$$

is a square-integrable martingale on $(\Omega, \mathcal{F}, P)$, with $E[\Lambda(T)] = 1$.

Note in this paper, individual Pareto optimal consumption allocation is measurable with respect to the aggregation consumption, hence, applying Theorem 4.1 in Duffie and Huang (1985), there exists a trading strategy contingent on the information of the aggregate consumption to implement the individual Pareto optimal consumption plan. In other words, there exists some $\varrho \in L^2_P[W]$ giving the representation

$$\Lambda(t) = 1 + \int_0^t \varrho(s) \, dW(s),$$

where $\Lambda(t)$ is measurable to the aggregate consumption. It follows from Ito’s Lemma that, defining the process $\eta(t) = \varrho(t)/\Lambda(t)$, yields the alternative representation

$$\Lambda(t) = \exp \left( \int_0^t \eta(s) dW(s) - \frac{1}{2} \int_0^t [\eta(s)]^2 \, ds \right).$$
From this representation, the new process

\[ W^* (t) = W(t) - \int_0^t \eta(s) ds, \]

defines a Standard Brownian Motion on \((\Omega, \mathcal{F}, Q)\) by Girsanov’s Fundamental Theorem (Liptser and Shiryayev 1977, p.232). It remains to show that \(W^*\) is itself a 2-basis for \(M^2_Q\), but this is immediate from Theorem 5.18 of Liptser and Shiryayev (1977), using the uniform absolute continuity of \(P\) and \(Q\).

With the orthogonal 2-basis for \(M^2_Q, W^*\), I can apply the procedure in Duffie and Huang (1985) which includes four steps to implement all the allocations which are measurable with respect to aggregate consumption by trading in only two long-lived securities: (See details in the Appendix)

1. Specify a set of long-lived securities: Since the multiplicity \(M^2_Q\) is 1, according Proposition 5.1 in Duffie and Huang (1985), I only need \(M^2_Q + 1\), i.e., two securities to dynamically effectively complete the markets: A riskless bond and a risky asset.

2. Announce a price for \(t = 0\) consumption and price processes for the long-lived securities: Duffie and Huang (1985) point out that the valid price processes for the riskless bond and stock should be 1 and the orthogonal 2-basis for \(M^2_Q\) on \(Q, W^*(t)\), respectively.

3. Allocate a trading strategy to each investor which generates that investor’s Arrow-Debreu allocation and which, collectively, clears markets.

4. Prove that no investor has any incentive to deviate from the allocated trading strategy.

Essentially, by marketing only two long-lived securities, one paying \(W^*(T)\) in date \(T\) consumption, the other paying one unit of date \(T\) consumption with certainty, and announcing their price processes as \(W^*(t)\) and 1 (for all \(t\)), a Radner equilibrium in dynamically effectively complete markets is achieved. Summarize the above results, I achieve the following proposition.

**Proposition 1** Assume the exponential-utility investors rationally update their posterior beliefs of the terminal dividend according to the optimal filtering equations (8) and (9), given the observed realizations of the signal.

(a) The market in which the investors can trade in one stock in addition to a riskless bond continuously is dynamically effectively complete.

(b) The Radner equilibrium implements the Pareto optimal consumptions in the corresponding Arrow-Debreu equilibrium. Both the stock price and the trading strategies in the Radner equilibrium are contingent on the information of the aggregate consumption.
Proof: See Appendix.

Proposition 1 notes the stock price and the trading strategies are contingent on the information of the aggregate consumption, however, it does not provide the explicit expressions for the stock price and portfolios to show how they exactly depend on the public signals. Next subsection presents concrete expressions of the equilibrium stock price and the equilibrium portfolios in the Radner equilibrium, under specific assumptions of utility and dividend structure.

To sum up, the market in this paper is called dynamically effectively complete markets. This type of market is sufficient to ensure the existence of an equilibrium with Pareto optimal consumption allocations, despite the fact that investors may not be able to implement any financially feasible consumption plan as they are in a dynamically complete market.

3.2.2 Equilibrium Security Prices with Impacts from Posterior Beliefs and Public Signals

In general, it is technically difficult to derive the candidate prices to implement the Arrow-Debreu equilibrium, since these prices are usually given by conditional expectations which cannot be computed explicitly. However, under the assumptions of exponential utility and the normally distributed dividend, in this paper, I can derive the explicit expressions of the equilibrium security prices and equilibrium portfolios in the Radner equilibrium to implement the Arrow-Debreu equilibrium in Christensen and Qin (2012).

First of all, I assume the investors trade in the riskless bond with price 1 all the time. In the dynamically effectively complete market, I can employ the standard theory of martingale approach\(^5\) to solve for the individual-specific state-price deflator, and thus, the security price.

The individual decision problem for the investor \(i\) is

\[
\max u_{iT}(c_{iT}) \quad s.t. \quad E^i [\zeta_{iT}c_{iT} | \mathcal{F}_t] \leq w_{i0},
\]

where \(\zeta_{iT}\) is the individual-specific state-price deflator at the terminal date \(T\), and \(w_{i0}\) is the wealth of investor \(i\) at date 0. Let the Lagrangian multiplier be \(\lambda_{i0}\), thus, the First-Order Condition gives

\[
u'_{iT}(c_{iT}^*) = \lambda_{i0}\zeta_{iT}^*;
\]

thus, the individual-specific state-price deflator at the terminal date \(T\), \(\zeta_{iT}\), is given as a

\(^5\)For an introduction of the standard theory of martingale approach, see, e.g., the Chapter 9 in ?), or a more resent paper, ?).

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function of the optimal terminal consumption,
\[ \zeta_{iT}^* = \frac{u'_T(c_{iT}^*)}{\lambda_{i0}} = \frac{r}{\lambda_{i0}} \exp(-rc_{iT}^*). \]

Note the individual-specific state-price deflator is proportional to the marginal utility of the optimal terminal consumption. Christensen and Qin (2012) derive the Pareto optimal terminal consumption in the Arrow-Debreu equilibrium,
\[ c_{iT}^* = \theta_{i0}^t d_T^t + x_{i0}^t d_T + \gamma_{i0}^t, \]
where \( \theta_{i0}^t, x_{i0}^t \) and \( \gamma_{i0}^t \) are the \( t = 0 \) equilibrium portfolios of the dividend derivative, the stock, and the riskless bond in the effectively complete market, respectively. Specifically,
\[ x_{i0}^t = \rho [h_{i0} m_{i0} - \overline{hm}] + Z/I, \quad \rho \equiv \frac{1}{r}, \quad \overline{hm} \equiv \frac{1}{I} \sum_{i=1}^{I} h_{i0} m_{i0}, \quad i = 1, \ldots, I, \]
\[ \theta_{i0}^t = \frac{1}{2} \rho [\overline{h} - h_{i0}], \quad \overline{h} \equiv \frac{1}{I} \sum_{i=1}^{I} h_{i0}, \quad i = 1, \ldots, I. \]

Given the portfolios \( \theta_{i0}^t, x_{i0}^t \) and \( \gamma_{i0}^t \) all known, and from the perspective of the investor \( i \) at date \( t \), \( d_T \sim N (m_{it}, \sigma_{it}^2) \), the conditional expectation of the individual-specific state-price deflator at the terminal date \( T \), \( \zeta_{iT} \), can be calculated. Also note that due to the fact that the riskless interest rate is constantly zero, the state-price density process, \( \zeta_i^* = (\zeta_{iT}^*) \), for each investor \( i \), is a martingale, i.e.,
\[ \zeta_{ti}^* (y_t) = E^i [\zeta_{iT}^* | \mathcal{F}_t], \]
hence, the stock price at date \( t \) can be computed according to
\[ p_t (y_t) = \frac{E^i [\zeta_{iT}^* d_T | \mathcal{F}_t]}{\zeta_{it}^*}. \]

Calculate the conditional expectation, the stock price at date \( t \) is simplified as a function of the priors and the posterior beliefs,
\[ p_t (y_t) = \frac{\overline{hm} - h_{i0} m_i + h_{it} m_{it} (y_t) - rZ/I}{\overline{h} - h_{i0} + h_{it}}. \]

At the first glimpse, the stock price looks individual specific. However, by substituting for the posterior mean \( m_{it} (y_t) \) and the posterior precision \( h_{it} \) (in terms of the priors and signal \( y_t \)), I show the price is common for all the investors in the following proposition.

**Proposition 2** Assume the economy is populated by exponential-utility investors who per-
ceive normally distributed dividend with heterogeneous beliefs. The equilibrium stock price in the Radner equilibrium is given as a precision weighted average of the investors’ posterior mean minus a risk premium determined by the average posterior precision, i.e.,

\[ p_t(y_t) = \bar{m}_t^h(y_t) - r\bar{\sigma}_t^2 Z/I, \]

where \( \bar{m}_t^h \) is the precision weighted average of the investors’ posterior means, i.e.,

\[ \bar{m}_t^h(y_t) = \frac{1}{I} \sum_{i=1}^{I} h_{it} m_{it}(y_t), \quad \bar{\sigma}_t^2 = \frac{1}{I} \sum_{i=1}^{I} h_{it}, \]

and \( \bar{\sigma}_t^2 \) is the inverse of the average posterior precision, i.e., \( \bar{\sigma}_t^2 \equiv 1/\bar{\sigma}_t^2. \)

Proof. See Appendix.

Note the expression of the stock price is consistent to that in Christensen and Qin (2012), in a sense that the stock price is given as the discounted expected risk-adjusted dividend. In this paper, the riskless discount factor is a constant one, hence, the stock price is immediately equal to the expected risk-adjusted dividend, i.e., \( p_t(y_t) = E^Q[d_T|\mathcal{F}_t] \). Moreover, define a representative investor holding a consensus belief of the terminal dividend as \( d_T \sim N(\bar{m}_t^h, \bar{\sigma}_t^2) \), thus, a homogeneous-belief model with the representative investor generates the same equilibrium prices as in Proposition 2. Obviously, the stock price is driven by the heterogeneously updated posterior beliefs and, thus, driven by the prior beliefs and the public signals (as functions of the Brownian motion). Aggregation of heterogeneous beliefs is also discussed in Chiarella, Dieci, and He (2006) and Jouini and Napp (2007).

3.2.3 Equilibrium Portfolios with Impacts from Posterior Beliefs and Public Signals

With the concrete expression of equilibrium stock price, I can derive the self-financing optimal trading strategies. Note in most stochastic optimization problems, posed in the general financial market models, investors ascertain only the existence of the associated portfolio strategies, since the Martingale Presentation Theorem does not provide explicit relating integrand. However, following the pioneer work on explicit descriptions of the integrand by

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\[^{6}\text{Similar to Christensen and Qin (2012), I define the risk-adjusted probability measure } Q \text{ explicitly such that conditional on the information at date } t \text{ under } Q, \text{ the terminal dividend is normally distributed as } d \sim N(\bar{m}_t^h - r\bar{\sigma}_t^2 Z/I, \bar{\sigma}_t^2). \text{ Note while the expected dividend under } Q \text{ is uniquely determined in equilibrium, the variance of the dividend under } Q \text{ is not uniquely determined due to the market incompleteness and, thus, I just take it be } \bar{\sigma}_t^2. \text{ Fortunately, the lack of the uniqueness of the variance has no consequences in the subsequent analysis.} \]
Clark (1970), Ocone and Karatzast (1991) generalize the Clark-Ocone formula. Using the tool from Malliavin calculus, they derive a general representation formula for the optimal portfolios\(^7\). Their formulae provide very explicit expressions for the optimal portfolios in feedback form on the current level of wealth, in the market with deterministic riskless interest rate and deterministic market price of risk. Note rewriting the form of the stock price, Eq. (B11) shows that the market price of risk implied in stock price under \(P\)-measure is deterministic, and the riskless interest rate is constantly zero. Thus, by using the generalized Clark-Ocone formula in Proposition 2.5 and formula of Eq. (3.11) in Ocone and Karatzast (1991), I calculate the self-financing optimal trading strategies, and obtain the following proposition.

**Proposition 3** Assume the economy is populated by exponential-utility investors who perceive normally distributed dividend with heterogeneous beliefs. In the Radner equilibrium, the equilibrium portfolios in the risky asset and riskless bond are respectively given as

\[
x^*_i(y_t) = 2\theta^\dagger_{i0}p_t(y_t) + x^\dagger_{i0}, \quad \gamma^*_i(y_t) = \theta^\dagger_{i0}\sigma_t^2 + \gamma^\dagger_{i0} - \theta^\ddagger_{i0}p_t^2(y_t).
\]

Proof. See Appendix.

As shown by the expressions of the equilibrium portfolios, intuitively, to achieve the Pareto optimal consumption \(c^*_{iT}\), the investor \(i\) hold constantly \(x^\dagger_{i0}\) share of the stock, and \(\gamma^\dagger_{i0}\) share of the riskless bond, then he achieves the part of \(x^\dagger_{i0}d_T + \gamma^\dagger_{i0}\) in the optimal consumption. In the following subsection, I show that the part of \(\theta^\dagger_{i0}\sigma_t^2\) in the optimal consumption is achieved by holding \(2\theta^\dagger_{i0}p_t(y_t)\) share of the stock, and \(\theta^\ddagger_{i0}(\sigma_t^2 - p_t^2(y_t))\) share of the riskless bond. Furthermore, the stock price \(p_t(y_t)\) is a sufficient statistic for computation of the optimal portfolios and thus, the optimal wealth processes.

### 3.2.4 Replicate Payoff of Derivative Paying off the Square of Terminal Dividend

Christensen and Qin (2012) introduce a dividend derivative which pays off the square of the dividend at the terminal date to effectively complete the market in a Arrow-Debreu Equilibrium. The "Dividend-Square Security" with ideal convexity in its payoff profile facilitates Pareto efficient side-betting. Interestingly, similar to Black and Scholes (1973), in which the value of derivative (option) can be replicated by trading underlying asset continuously, here in my model, the payment of the "Dividend-Square Security" in Christensen and Qin (2012)

\(^7\)Other particularly significant works on explicit descriptions of the integrand are Haussmann (1979), Ocone (1984), and Karatzast, Ocone, and Li (1991), among others. Davis (2005) survey the problems of martingale representation, especially those martingales with finite multiplicity. More recently, Renaud and Remillard (2007) apply the Clark-Ocone formula to option pricing and obtain explicit trading portfolios.
can be replicated by trading two securities continuously. Since any function of the dividend is measurable with respect to aggregate consumption, thus, the payoff of "Dividend-Square Security" is measurable with respect to aggregate consumption. Hence, there exists a trade strategy to replicate the payoff of the "Dividend-Square Security". This fact shows that from a welfare perspective, continuous trading is a replacement of the convexity in the payoff of the derivative, which can be attained by using Gamma trading strategies.

Specifically, the dividend derivative in the Radner equilibrium is a redundant asset. The price of "Dividend-Square Security" and the replicating trading strategies are given by the following proposition.

**Proposition 4** Assume the economy is populated by exponential-utility investors who perceive normally distributed dividend with heterogeneous beliefs. The price of the redundant dividend derivative in the Radner equilibrium is given as

\[
\pi_t (y_t) = \mathbb{E}^i \left[ \tilde{\xi}_t d^2_t | \mathcal{F}_t \right] = \sigma^2_t + p_t^2 (y_t),
\]

and the value of \( \theta_{10}^i \) share of the dividend derivative can be replicated by investing in \( 2\theta_{10}^i p_t (y_t) \) share of the stock, and \( \theta_{10}^i (\sigma^2_t - p_t^2 (y_t)) \) share of the riskless bond continuously.

Proof. See Appendix.

Note the riskless discount factor is a constant one, hence, the price of the dividend derivative is immediately equal to the expected risk-adjusted payment of square of the dividend, i.e., \( \pi_t (y_t) = \mathbb{E}^Q [d^2_t | \mathcal{F}_t] \). Intuitively, the risk-adjusted expectation of the derivative payoff is affected by the posterior beliefs and, thus, affected by the prior beliefs and public signals. Moreover, the expression of the price of the dividend derivative shows that the investors form their Pareto optimal trading strategies by investing as if they intend to dynamically replicate the value of the dividend derivative. This result intuitively demonstrates how the dynamically effectively complete market in this paper is equivalent to the effectively complete market in Christensen and Qin (2012).

Although the interpretation of the equilibrium involves the notion of replication, the method in this paper fundamentally differs from that in the Black-Scholes model. First, the Radner equilibrium endogenously replicate the payoff of the redundant asset, in which both the price process of the underlying asset and the replicating strategies are endogenized in the Radner equilibrium. In contrast, in the Black and Scholes (1973) option pricing model, the underlying price process is exogenously given. Furthermore, Black and Scholes (1973) cannot be generalized to the heterogeneous beliefs case with heterogeneous volatility, since
under individual equivalent probability measure, the volatility has to be identical across investors. However, information structure in this paper allows heterogeneity in perceived variance of the terminal dividend. It is the investors’ speculations on the variance of the dividend throughout the interval \([0, T]\) dynamically effectively complete the market.

4 Conclusion

The assumptions of heterogeneous beliefs and the information on aggregate consumption have substantial influence in continuous-time financial models. Comparing to the benchmark homogeneous belief model, I achieve a less strong but effectively equivalent result, i.e., continuous trading can effectively dynamically complete the financial market with heterogeneous beliefs. The investors in such an economy can deal with all the risk on the aggregate consumption and attain their Pareto optimal allocations by trading in a few securities.

The assumptions of negative exponential utility and normally distributed dividends enable me to achieve explicit expressions of the equilibrium security prices and equilibrium portfolios in the Radner equilibrium to implement the Arrow-Debreu equilibrium. More realistic assumptions of preferences and non-normal distributed dividend may not lead to these analytical equilibrium properties. However, adding jumps into the information structure in this paper may maintain some nice properties, I leave this for future research.

References


A Appendix: Proof of Lemmas

A.1 Proof of Lemma 1

The proof is similar to that in the Appendix of Christensen, Larsen, and Munk (2011). However, the structure of the drift and volatility is different, and I introduce a new condition, i.e., Eq. (2) to ensure the convergence to the terminal dividend.

I define the deterministic function \( b(t) \equiv -A(t)^{-1} \) and note that \( b(t) \to -\infty \) as \( t \to T \). A direct application of Ito’s product rule gives that the stochastic process
\[ \tilde{X}_s = e^{\int_0^s b(u)du} \left( \tilde{X}_0 + \int_0^s e^{-\int_0^t b(u)du} dT A(t)^{-1} dt + B(t) dW_t \right), \quad s \in [0, T), \]

satisfies the SDE (3). Furthermore, L'Hopital’s rule gives

\[
\lim_{s \to T} \frac{\int_0^s e^{-\int_0^t b(u)du} (dT A(t)^{-1}) dt}{e^{\int_0^s b(u)du}} = \lim_{s \to T} \frac{dT A(s)^{-1}}{-b(s)} = d_T.
\]

The proof can therefore be concluded by showing

\[ e^{\int_0^s b(u)du} M_s = e^{\int_0^s b(u)du} \int_0^s e^{-\int_0^t b(u)du} B(s) dW_t \to 0, \text{ P-a.s.}, \]

as \( s \to T \). The quadratic variation of \( M \) is given by

\[ \langle M \rangle_s = \int_0^s e^{-2\int_0^t b(u)du} \frac{1}{B(t)^{-2}} dt, \quad s \in [0, T). \]

If \( \langle M \rangle_T < \infty \), I trivially have that \( M \) is a continuous martingale on the interval \([0, T]\) and, in particular, \( M_T \) is a real valued random variable and the claim follows. If \( \langle M \rangle_T = \infty \), I can use Exercise II.15 in Protter (2004) to see that

\[ \lim_{s \to T} \frac{M_t}{\langle M \rangle_t} = 0, \text{ P-almost surely.} \]

L'Hopital’s rule gives

\[
\lim_{s \to T} \langle M \rangle_s e^{\int_0^s b(u)du} = \lim_{s \to T} \frac{\int_0^s e^{-2\int_0^t b(u)du} \frac{1}{B(t)^{-2}} dt}{e^{\int_0^s b(u)du}} = \lim_{s \to T} \frac{e^{-\int_0^s b(u)du}dt}{-b(s) B(s)^{-2}},
\]

and the condition (2) suffices to ensure the above limit is zero. This completes the proof.

### A.2 Proof the Specified Coefficients Meet the Requirements of Signal Process and Filtering Equation

Now I proof the coefficients

\[ A = \frac{1}{\alpha (T-t)^{\frac{1}{2}}}, \quad -1 < k < 0, \quad \alpha > 0, \]

and

\[ B = \beta (T-t)^{q}, \quad q > 0, \quad \beta \in R, \]
meet following conditions\(^8\)

\[
\lim_{t \to T} A(t) = 0, \quad \lim_{s \to T} B(s)^2 e^{\int_0^s A(u)^{-1}du} = 0, \quad \int_0^T A(t)^{-2} \, dt < \infty, \quad \int_0^T B(t)^2 \, dt < \infty.
\]

Since \(k < 0\), so that \(A \to 0\) as \(t \to T\), and \(B \to 0\) as \(t \to T\). Moreover, to proof

\[
\lim_{s \to T} e^{\int_0^s A(u)^{-1}du} = 0,
\]

I only need to proof

\[
\lim_{s \to T} e^{\int_0^s A(u)^{-1}du} < \infty.
\]

Note let \(v = T - u\),

\[
\lim_{s \to T} e^{\alpha \int_0^s (T-u)^{1/2} du} = \lim_{s \to T} e^{-\alpha \int_0^s (T-u)^{1/2} d(T-u)} = \lim_{s \to T} e^{-\alpha \int_0^{T-s} u^{1/2} du}
\]

\[
= \lim_{s \to T} e^{\alpha \left( -\frac{u^{1/2+1}}{2+1} \right)_{T-s}} = \lim_{s \to T} e^{\alpha \left( -\frac{(T-s)^{1/2+1}}{2+1} \right)}
\]

\[
= \lim_{s \to T} e^{\alpha \left( T^{1/2+1} - (T-s)^{1/2+1} \right)} = \lim_{s \to T} e^{\frac{2\alpha}{k+2} \left( T^{1/2+1} - (T-s)^{1/2+1} \right)}
\]

\[
= \lim_{s \to T} e^{\frac{2\alpha}{k+2} \left( T^{1/2+1} - (T-s)^{1/2+1} \right)} e^{-\frac{2\alpha}{k+2} \left( T^{1/2+1} - (T-s)^{1/2+1} \right)}
\]

since \(k > -2\), thus, \(k^2 + 1 > 0\), and

\[
\lim_{s \to T} e^{-\frac{2\alpha}{k+2} \left( T-s \right)^{1/2+1}} = 1,
\]

---

\(^8\)Note a general Brownian bridge converging to \(d_T\)

\[
dy_t = \frac{d_T - y_t}{T - t} \, dt + dW_t,
\]

does not meet the condition required by the filtering equation in Liptser and Shiryayev (1977), since

\[
\int_0^T \left( \frac{1}{T-t} \right)^2 \, dt = -\int_0^T (T-t)^{-2} \, d(T-t),
\]

let \(T - t = u\),

\[
\int_0^T \left( \frac{1}{T-t} \right)^2 \, dt = -\int_0^u u^{-2} \, du = -\left( \frac{u^{-1}}{-1} \right)_{T}^{0} = \frac{1}{u} \bigg|_0^T = \infty.
\]
thus,
\[
\lim_{s \to T} \beta^2 (T - s)^{2q} e^{\alpha \int_0^s (T-u)^{-2q} du} = 0.
\]

Furthermore, given \(0 > k > -1\),
\[
\int_0^T A(t)^{-2} dt = \alpha^2 \int_0^T (T-t)^k dt,
\]
thus, as \(t \to T\), \((T-t)^k \to \infty\), I now have to check the convergence of improper integrals of the second kind with singularity at \(t = T\). Note
\[
\int_0^T A(t)^{-2} dt = \alpha^2 \int_0^T (T-t)^k dt
= -\alpha^2 \int_0^T (T-t)^k d(-t)^{u=-t} - \alpha^2 \int_0^{-T} (u+T)^k du
= -\alpha^2 \int_0^{-T} (u+(-T))^k du = \alpha^2 \int_0^{-T} (u+(-T))^k du
= \alpha^2 \left[ \frac{1}{1+k} (u+(-T))^{1+k} \right]_0^{-T} = \frac{\alpha^2 T^{1+k}}{1+k},
\]
and
\[
\int_0^T B(t)^2 dt = \beta^2 \int_0^T (T-t)^{2q} dt = -\beta^2 \int_0^T (T-t)^{2q} d(T-t)
= -\beta^2 \int_T^0 u^{2q} du = \beta^2 \int_0^T u^{2q} du = \beta^2 \frac{u^{2q+1}}{2q+1} \bigg|_0^T = \frac{\beta^2 T^{2q+1}}{2q+1}.
\]

thus, the specified coefficients meet the requirements of signal process and filtering equation. With the specified coefficients, I can apply the following lemma, which is a simple specific case of the Theorem 10.3 in Liptser and Shiryayev (1977).

**Lemma 2 (Filtering Equation)** Let \(\xi_t\) be the observable process, and the coefficients of equations in
\[
d\xi_t = (A_1(t) \xi_t + A_2(t) \xi_t) dt + B_1(t) dW_t,
\]
satisfy the conditions of
\[
\int_0^T \left[ A_1(t)^2 + A_2(t)^2 \right] dt < \infty, \text{ and } \int_0^T B_1(t)^2 dt < \infty,
\]

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then the vector \( m_t = M \left( \xi_t \mid \mathcal{F}_t^i \right) \) and the \( \gamma_t = M \left( (\xi_t - m_t)^2 \mid \mathcal{F}_t^i \right) \) are solutions of the system of equations

\[
dm_t = \gamma_t A_1(t) B_1(t)^{-2} \left[ d\xi_t - (A_1(t)m_t + A_2(t)\xi_t) \, dt \right],
\]

and

\[
d\gamma_t = -\gamma_t^2 A_1^2(t) B_1(t)^{-2} \, dt,
\]

with the initial conditions \( m_0 = M(\theta_0|\xi_0) \), and \( \gamma_0 = M((\theta_0 - m_0)^2) \).

### A.3 Learning Model with Specific Coefficients

With the specified coefficients (5) and (6), the signal process can be stated as

\[
dy_t = \alpha (T - t)^{\frac{k}{2}} (d_T - y_t) \, dt + \beta (T - t)^q \, dW_t, \quad t \in (0, T), \ y_0 \in R.
\]

According to Lemma 2, the updating equations of the posterior mean and posterior variance for investor \( i \) is

\[
dm_{it} = \frac{\alpha \sigma_{it}^2}{\beta^2} \left( T - t \right)^{\frac{k}{2} - 2q} \left[ dy_t - \alpha (T - t)^{\frac{k}{2}} (m_{it} - y_t) \, dt \right],
\]

and

\[
d\sigma_{it}^2 = -\frac{\alpha^2}{\beta^2} (T - t)^{k - 2q} \, dt.
\]

The posterior variance follows an ODE, solve for the ODE,

\[
-\frac{1}{\sigma_{it}^2} d\sigma_{it}^2 = \frac{\alpha^2}{\beta^2} (T - t)^{k - 2q} \, dt
\]

\[
\rightarrow \int -\frac{1}{\sigma_{it}^2} d\sigma_{it}^2 = \frac{\alpha^2}{\beta^2} \int (T - t)^{k - 2q} \, dt + C_0
\]

\[
\rightarrow \int -\frac{1}{\sigma_{it}^2} d\sigma_{it}^2 = -\frac{\alpha^2}{\beta^2} \int (T - t)^{k - 2q} \, d(T - t) + C_0
\]

\[
\rightarrow \frac{1}{\sigma_{it}^2} = -\frac{\alpha^2 (T - t)^{k-2q+1}}{\beta^2 (k - 2q + 1)} + C_0,
\]
when $t = 0$, the constant $C_0 = \frac{1}{\sigma_{t_0}^2} + \frac{\alpha_2^2 k^{k-2q+1}}{\beta^2 (k-2q+1)}$, thus,

$$\frac{1}{\sigma_{it}^2} = -\frac{\alpha^2 (T - t)^{k-2q+1}}{\beta^2 (k-2q+1)} + \frac{1}{\sigma_{i0}^2} + \frac{\alpha^2 T^{k-2q+1}}{\beta^2 (k-2q+1)}$$

$$\rightarrow \frac{1}{\sigma_{it}^2} = \frac{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} - \alpha^2 \sigma_{i0}^2 (T - t)^{k-2q+1} + \beta^2 (k-2q+1)}{(k-2q+1) \beta^2 \sigma_{i0}^2}$$

$$\rightarrow \sigma_{it}^2 = \frac{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} - \alpha^2 \sigma_{i0}^2 (T - t)^{k-2q+1} + \beta^2 (k-2q+1)}{(k-2q+1) \beta^2 \sigma_{i0}^2}.$$  

This completes the derivation of the dynamics of the posterior mean and posterior variance.

### A.4 Proof of Lemma 2

This proof is a special case when setting $a_0 = a_1 = a_2 = b_1 = b_2 = A_0 = B_2 = 0$ in Theorem 10.3 of Liptser and Shiryaev (1977). The basic idea of the proof is that using a transformation to get rid of the linear dependence of the observable component $\xi_t$, and thus, I can obtain the filtering equation by applying the Theorem 10.1. In other words, the model reduces to the benchmark case in Brennan (1998).

Linear dependence of the observable component $\xi_t$ is introduced into the coefficients of transfer in (A2). To prove Lemma 2, I shall need the following Lemma 3.

**Lemma 3** Let the matrix process $D = (D_t, F_t)$, be such that for almost all $t$, $0 \leq t \leq T$, ($P - a.s.$)

$$B_{iT}^2 = D_t^2,$$

then there is a Wiener process $\tilde{W}_s$, such that for each $t$, $0 \leq t \leq T$, ($P - a.s.$)

$$\int_0^t B_{1s} dW_s = \int_0^t D_s d\tilde{W}_s.$$

Note Lemma 3 is an one-dimension benchmark case of Lemma 10.4 in Liptser and Shiryaev (1977), which is on a multidimensional Wiener process. By Lemma 3, for the system of equations in (A2), there is also the representation

$$d\xi_t = (A_1 (t) \xi_t + A_2 (t) \xi_t) dt + D (t) d\tilde{W}_t,$$  

let

$$\zeta_t = \xi_t, \bar{\xi}_t = \xi_t - \int_0^t A_2 (s) \xi_s ds,$$  

(A4)
by (A3) I have
\[ W_2(t) = \int_0^t D_2^{-1}(s) \left[ d\xi_s - (A_1(s)\xi_s + A_2(s)\xi_t) \right] ds. \quad (A5) \]

From (A3), (A4), and (A5), I have
\[ d\bar{\xi}_t = A_1(t)\bar{\theta}_tdt + D_2(t)dW_2(t). \]

From the construction of the process \( \bar{\xi} = (\bar{\xi}_t), 0 \leq t \leq T, \) it follows that \( \mathcal{F}_t^\xi \supseteq \mathcal{F}_t^{\bar{\xi}}. \) It will be shown that actually the \( \sigma \)-algebras \( \mathcal{F}_t^\xi \) and \( \mathcal{F}_t^{\bar{\xi}} \) coincide for all \( t, 0 \leq t \leq T. \)

For the proof I shall consider the linear system of equations
\[ d\xi_t = A_2(t)\xi_t dt + d\bar{\xi}_t, \quad \xi_0 = \bar{\xi}_0, \]

obtained from (A4).

This linear system of equations has a unique, strong solution (see Theorem 4.10 in Liptser and Shiryayev (1977) and the note to it) which implies \( \mathcal{F}_t^{\bar{\xi}} \supseteq \mathcal{F}_t^\xi, \) \( 0 \leq t \leq T, \) i.e., \( \mathcal{F}_t^{\bar{\xi}} = \mathcal{F}_t^\xi \) and
\[ \bar{m}_t = M \left( \bar{\xi}_t | \mathcal{F}_t^{\bar{\xi}} \right) = M \left( \bar{\xi}_t | \mathcal{F}_t^\xi \right), \]

hence,
\[ m_t = M \left( \xi_t | \mathcal{F}_t^\xi \right) = M \left( \bar{\xi}_t | \mathcal{F}_t^\xi \right) = \bar{m}_t, \]

and
\[ \bar{\xi}_t - \bar{m}_t = \xi_t - m_t. \]

From this,
\[ \gamma_t = \bar{\gamma}_t. \]

According to Theorem 10.1 in Liptser and Shiryayev (1977), I have
\[ d\bar{m}_t = \bar{\gamma}_t A_1(t) D(t)^{-2} \left[ d\bar{\xi}_t - A_1(t)\bar{m}_t dt \right], \]

and
\[ d\bar{\gamma}_t = -\bar{\gamma}_t^2 A_1^2(t) D(t)^{-2} dt. \]

From this, taking into account that \( m_t = \bar{m}_t \) and \( \gamma_t = \bar{\gamma}_t, \) after some simple transformations I arrive at for \( m_t \) and \( \gamma_t. \)
B  Proof of Propositions

B.1  Proof of Proposition 1 and Algorithm to Implement Pareto Efficient Allocation

The proof and the algorithm is a special case of that in Duffie and Huang (1985). Specifically in this paper, there are only two securities that are traded continuously, and the signals only reveal the information contingent on the aggregate consumption.

Christensen and Qin (2012) provide a specific Arrow-Debreu Equilibrium with specific securities and dividend structure. In general, the economy can be summarized in the usual way by the collection

\[ E = (V_i, v_i, \succeq_i; i = 1, ..., I), \]

where the individual-specific consumption space is thus formalized as \( V_i \subseteq V = R \times L^2(P) \), where \( L^2(P) \) is the space of (equivalence classes) of square-integrable random-variables on \((\Omega, \mathcal{F}, P)\). The consumption pair \( \bar{v}_i = (\bar{\omega}_{i0}, \bar{\omega}_{iT}) \in V_i \) is the ith investor’s endowment. The Arrow-Debreu equilibrium for an economy \( E \) can be defined as a nonzero linear (price) functional \( \Psi : V \rightarrow R \) and a set of allocations \((v^*_i = (\omega_{i0}, \omega_{iT}) \in V_i; i = 1, ..., I)\) satisfying, for all \( i \),

\[ \Psi (v^*_i) \leq \Psi (\bar{v}_i), \]

\[ v \succ_i v^*_i \rightarrow \Psi (v) > \Psi (v^*_i), \ \forall v \in V_i, \]

\[ \sum_{i=1}^{I} v^*_i = \sum_{i=1}^{I} \bar{v}_i. \]

Duffie and Huang (1985) proof that \( \Psi (\omega_{i0}, \omega_{iT}) = \bar{a} \omega_{i0} + E^Q (\omega_{iT}), \ \forall (\omega_{i0}, \omega_{iT}) \in V_i \), where \( \bar{a} \in R_+ \) is a price for date zero consumption. They note this relationship hold for both homogeneous belief case and heterogeneous beliefs case.

The Radner equilibrium to implement the Pareto optimal consumption allocations is comprised of: (1) a set of long-lived securities claiming, i.e., the riskless bond and the stock, with corresponding price processes 1 and \( W^* \), and \( S \equiv [1, W^*]' \); (2) a set of trading strategies \( \theta_i = [\gamma_i, x_i]' \), (omit the subscription \( t \)) for each investor \( i = 1, ..., I \); and (3) a price \( \bar{a} \in R_+ \) for date zero consumption. All of these satisfy budget constrained optimality: for each investor \( i \),

\[ \bar{\omega}_{i0} - \frac{\theta_i'(0) S (0)}{\bar{a}}, \ \bar{\omega}_{iT} + \theta_i'(T) S (T), \]

is \( \succeq_i \) - maximal in the budget set, and market clearing.
With the orthogonal 2-basis for $M^2_Q$, I can apply the procedure in Duffie and Huang (1985) to implement all the allocations which are measurable with respect to the aggregate consumption by trading in only two long-lived securities as follows.

For any investor $i$, for $1 \leq i \leq I - 1$, let $e_i = c_i^T - \bar{\omega}_iT$, and $c_i^*$ is the Pareto optimal consumption at $t = T$, then the process

$$
\bar{X}_i(t) = E_Q(e_i | \mathcal{F}_t) - E_Q(e_i), \ t \in [0, T],
$$

is an element of $M^2_Q$, given $Q \approx P$, which can be reconstructed via Theorem 4.1 in Duffie and Huang (1985) as

$$
\bar{X}_i(t) = \int_0^t x_i(s)dW^*(t), \ \forall t \in [0, T] \ a.s., \quad (B1)
$$

for some $x_i(s) \in L^2_P[W^*].$

In order to meet the accounting restriction

$$
\theta'_i(t)S(t) = \theta'_i(0)S(0) + \int_0^t \theta'_i(s)dS(s), \ \forall t \in [0, T] \ a.s., \quad (B2)
$$

I set the following trading process for the "store-of-value" security:

$$
\gamma'_i(t) = E_Q(e_i) + \int_0^t x_i(s)dW^*(t) - x_i(s)W^*(t), \ t \in [0, T]. \quad (B3)
$$

Substituting Eq. (B3) into Eq. (B1), noting that $W^*(0) = 0, \forall n$, yields and confirms the accounting restriction, Eq. (B2). This yields the final requirement for claiming the trading strategy $\theta_i = (\gamma_i, x_i)'$ is admissible. Evaluating Eq. (B2) at times $T$ and 0, using the definitions of $\bar{X}_i$ yields:

$$
\theta'_i(T)S(T) + \bar{\omega}_iT = c^*_i \ a.s.,
$$

and

$$
\theta'_i(0)S(0) = E_Q(c^*_i - \bar{\omega}_iT) = \Psi(0, c^*_iT) - \Psi(0, \bar{\omega}_iT) = (\bar{\omega}_i0 - c^*_i0) \bar{a},
$$

where $c^*_i0$ is the Pareto optimal consumption at $t = 0$, and the last line making use of the budget constraint on the Arrow-Debreu allocation for investor $i$. Thus by adopting the trading strategy $\theta_i$, and faced with the date-zero consumption price of $\bar{a}$, investor $i$ can consume precisely $(c^*_i0, c^*_iT) = v^*_i$.

The above construction applies for investors 1 through $I - 1$. For the last investor, investor $I$, let $\theta_I = -\sum_{i=1}^{I-1} \theta_i$, thus market clearing is obviously met by construction. To complete
this step it remains to show that $\theta_1$ generates the consumption allocation $(c^*_{i0}, c^*_{iT}) = v^*_i$, but this is immediate from the linearity of stochastic integrals and market clearing in the Arrow-Debreu equilibrium.

Now prove that no investor has any incentive to deviate from the allocated trading strategy by contradiction. Suppose some investor $j$ can obtain a strictly preferred allocation $(c_0, c_T) \succ_j (c^*_j, c^*_j)$ by adopting a different trading strategy $\theta$. Then the Arrow-Debreu price of $(c_0, c_T)$ must be strictly higher than that of $(c^*_j, c^*_T)$, or

$$\tilde{a}c_0 + E^Q(c_T) > \tilde{a}c^*_j + E^Q(c^*_T).$$

Substituting the Radner budget constraint for $c_0$ and $c_T$,

$$\tilde{a}\omega_{j0} - \theta'(0) S(0) + E^Q \left[ \omega_{jT} + \theta'(0) S(0) + \int_0^T \theta'(t) dS(t) \right] > \tilde{a}c^*_j + E^Q(c^*_j),$$

or

$$\tilde{a}\omega_{j0} + E^Q(\omega_{jT}) > \tilde{a}c^*_j + E^Q(c^*_T).$$

The last line uses the fact that $E^Q[\int_0^t \theta(s) dS(t)] = 0$ since $\int \theta dS$ is a $Q-$ martingale. But Eq. (B4) contradicts the Arrow-Debreu budget-constrained optimality of $(c^*_j, c^*_j)$. This establishes the whole implement process. Moreover, the market is dynamically effectively complete, since no investor has any incentive to deviate from the self-financing trading strategy, and all the markets clear. This completes the proof.

### B.2 Proof of Proposition 2

To obtain the price of the risky asset, I first derive the state-price deflator and the expected risk-adjusted dividend, and then calculate the price of the risky asset according to Eq. (10).

#### B.2.1 State-Price Deflator at the Terminal Date

Note the state-price deflator at the terminal date $T$ is given as

$$c^*_T = \frac{u'_i(c^*_i)}{\lambda_{i0}} = \frac{r \exp \left( -r c^*_T \right)}{\lambda_{i0}},$$

where the Pareto optimal consumption at the terminal date

$$c^*_T = \frac{1}{2} \rho \left[ \bar{h} - h_{i0} \right] d_T^2 + \rho \left[ h_{i0} m_i - \bar{m} \right] d_T + \left( \frac{Z}{I} \right) d_T + \gamma_{i0},$$
thus, the state-price deflator is

\[
\zeta_{it}^* = r \exp \left( -r \left[ \frac{1}{2} \rho \left[ \tilde{h} - h_{i0} \right] d_T^2 + \rho \left[ h_{i0} m_i - \bar{m} \right] d_T + (Z/I) d_T + \gamma_{i0}^\dagger \right] \right)
\]

\[
= \exp \left( -\frac{1}{2} \left[ \tilde{h} - h_{i0} \right] d_T^2 - \left[ h_{i0} m_i - \bar{m} \right] d_T - r \left( Z/I \right) d_T \right),
\]

where \( \psi_{i0} = \frac{\lambda_{i0}}{r \exp(-r \gamma_{i0})} \).

### B.2.2 Expected Risk-Adjusted Dividend

Note let \( \bar{\psi}_{i0} = 2 \sqrt{2\pi} \sigma_{it}^2 \psi_{i0} \), thus the expected risk-adjusted dividend follows

\[
\mathbb{E}^i \left[ \zeta_{it}^* d_T | \mathcal{F}_i \right] = \frac{1}{\bar{\psi}_{i0}} \mathbb{E}^i \left[ \exp \left( -\frac{1}{2} \left[ \tilde{h} - h_{i0} \right] d_T^2 - \left[ h_{i0} m_i - \bar{m} \right] d_T - r \left( Z/I \right) d_T \right) d_T | \mathcal{F}_i \right]
\]

\[
= \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \left[ \tilde{h} - h_{i0} \right] d_T^2 - \left[ h_{i0} m_i - \bar{m} \right] d_T - r \left( Z/I \right) d_T - \frac{1}{2} \left( \frac{d_T - m_{it}}{\sigma_{it}} \right)^2 \right) d d_T.
\]

To simplify, let

\[
\alpha_0 \equiv \frac{1}{2} \left[ \tilde{h} - h_{i0} \right] + \frac{1}{2} h_{it},
\]

\[
\alpha_1 \equiv \frac{h_{i0} m_i - \bar{m} - h_{it} m_{it} + r \left( Z/I \right)}{\tilde{h} - h_{i0} + h_{it}},
\]

\[
\alpha_2 \equiv \frac{\left( h_{i0} m_i - \bar{m} - h_{it} m_{it} + r \left( Z/I \right) \right)^2}{\tilde{h} - h_{i0} + h_{it}} - \frac{1}{2} h_{it} m_{it}^2,
\]

hence,

\[
\mathbb{E}^i \left[ \zeta_{it}^* d_T | \mathcal{F}_i \right] = \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp \left( -\alpha_0 \left[ d_T^2 + 2 \alpha_1 d_T \right] - \frac{1}{2} h_{it} m_{it}^2 \right) d d_T
\]

\[
= \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp \left( -\alpha_0 \left[ (d_T + \alpha_1)^2 - \alpha_1^2 \right] - \frac{1}{2} h_{it} m_{it}^2 \right) d d_T
\]

\[
= \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp \left( -\alpha_0 \left( d_T + \alpha_1 \right)^2 + \alpha_2 \right) d d_T.
\]
Since \( h_{it} \geq h_{i0} \), for \( 0 < t \leq T \), hence, \( \alpha_0 > 0 \). Let \( a_1 = \sqrt{\alpha_0} \), and \( b_1 = a_1 \alpha_1 \), thus,

\[
\int_{-\infty}^{+\infty} \exp \left[ \alpha_0 \left[ d_T + \alpha_1 \right]^2 + \alpha_2 \right] d d_T^2 = \int_{-\infty}^{+\infty} \exp \left[ \left( a_1 d_T + b_1 \right)^2 + \alpha_2 \right] d d_T
\]

\[
\begin{align*}
&= \exp \left[ -\alpha_2 \right] \int_{-\infty}^{+\infty} \exp \left[ \left( a_1 d_T + b_1 \right)^2 \right] d d_T^2 \\
&= \frac{\exp \left[ -\alpha_2 \right]}{a_1^2} \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] d \left[ u^2 - 2b_1 u \right] \\
&= \frac{\exp \left[ -\alpha_2 \right]}{a_1^2} \left[ \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] du^2 - 2b_1 \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] du \right],
\end{align*}
\]

let \( u = a_1 d_T - b_1 \), note \( u \to +\infty \), as \( d_T \to +\infty \), \( u \to -\infty \), as \( d_T \to -\infty \), thus,

\[
\begin{align*}
\int_{-\infty}^{+\infty} \exp \left[ \alpha_0 \left[ d_T + \alpha_1 \right]^2 + \alpha_2 \right] d d_T^2 \\
&= \frac{\exp \left[ -\alpha_2 \right]}{a_1^2} \left[ \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] du^2 - 2b_1 \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] du \right] \\
&= \frac{\exp \left[ -\alpha_2 \right]}{a_1^2} \left[ 0 - 2b_1 \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] du \right] \\
&= \frac{\exp \left[ -\alpha_2 \right]}{a_1^2} \left[ -2b_1 \sqrt{\pi} \right] = -\frac{2b_1 \sqrt{\pi}}{a_1^2} \exp \left[ -\alpha_2 \right].
\end{align*}
\]

Thus, the expected risk-adjusted dividend

\[
E^i \left[ \zeta_{iT}^* d_T | \mathcal{F}_t \right] = -\frac{2b_1 \sqrt{\pi}}{a_1^2 \sqrt{\delta_0}} \exp \left[ -\alpha_2 \right].
\]

### B.2.3 State-Price Deflator at Date \( t \)

The state-price deflator at date \( t \) follows
\begin{align*}
\zeta_{it}^* &= \mathbb{E}^{i} [\zeta_{iT}^* | \mathcal{F}_t] \\
&= \frac{1}{\psi_{i0}} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} [\bar{h} - h_{i0}] d_T^2 - [h_{i0}m_i - \bar{h}m] d_T - r (Z/I) d_T - \frac{1}{2} \left( \frac{d_T - m_{it}}{\sigma_{it}} \right)^2 \right) d d_T \\
&= \frac{1}{\psi_{i0}} \int_{-\infty}^{+\infty} \exp \left( -\frac{\alpha_0}{2} [d_T + \alpha_1]^2 + \alpha_2 \right) d d_T \\
&= \frac{1}{\psi_{i0}} \sqrt{2\pi \frac{1}{2\alpha_0}} \exp \left( -\alpha_2 \right) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi \frac{1}{2\alpha_0}}} \exp \left( -\frac{1}{2} \frac{d_T - (-\alpha_1)^2}{2 \frac{1}{2\alpha_0}} \right) d d_T \\
&= \frac{1}{\psi_{i0}} \sqrt{\frac{\pi}{\alpha_0}} \exp \left( -\alpha_2 \right).
\end{align*}

**B.2.4 Stock Price**

Therefore, the stock price at date \( t \) is given as

\[
p_t = \mathbb{E}^{i} [\zeta_{iT}^* d_T | \mathcal{F}_t] = \frac{-2b_1 \sqrt{\pi}}{a_1 \psi_{i0}} \exp \left( -\alpha_2 \right) = \frac{-b_1}{a_1} = \frac{-b_1}{a_1}
\]

\[
= -\alpha_1 = \frac{\frac{\bar{h}m - h_{i0}m_i + h_{it}m_{it} - rZ}{\bar{h} - h_{i0} + h_{it}}}{h_{i0} + J_0(t)}.
\]

At the first glimpse, the stock price looks individual specific. However, by substituting for the posterior mean \( m_{it} \) and the posterior precision \( h_{it} \) (in terms of the priors and signal \( y_t \)), I show the price is common for all the investors in the following subsection.

**B.2.5 Express the Stock Price as a Function of Priors**

I first express the posterior beliefs as functions of priors. By Eq. (9), I obtain the posterior precision

\[
h_{it} = h_{i0} + \frac{\alpha^2 T^{k-2q+1} - \alpha^2 (T-t)^{k-2q+1}}{(k-2q+1) \beta^2} = h_{i0} + J_0(t),
\]

where

\[
J_0(t) \equiv \frac{\alpha^2 T^{k-2q+1} - \alpha^2 (T-t)^{k-2q+1}}{(k-2q+1) \beta^2},
\]

is common for all the investors.

Moreover, note the posterior mean is driven by the signal \( y \), substitute Eq. (7) into Eq.
(8), I have

\[
dm_{it} = \frac{\alpha \sigma_{it}^2}{\beta^2} (T - t)^{\frac{k}{2} - 2q} \left[ \alpha (T - t)^{\frac{k}{2}} dT dt + \beta (T - t)^q dW_t - \alpha (T - t)^{\frac{k}{2}} m_{it} dt \right]
\]

\[
= \frac{\alpha \sigma_{it}^2}{\beta^2} (T - t)^{\frac{k}{2} - 2q} \left[ \alpha (T - t)^{\frac{k}{2}} (dT - m_{it}) dt + \beta (T - t)^q dW_t \right],
\]

thus, posterior mean \( m_{it} \) follows a SDE, solve for the SDE I obtain that

\[
m_{it} = \Phi(t) \left( m_{i0} + \int_0^t \Phi^{-1}(s) \Theta(s) ds + \int_0^t \Phi^{-1}(s) G(s) dW_s \right)
\]

\[
= \Phi(t) m_{i0} + \Phi(t) \int_0^t \Phi^{-1}(s) \Theta(s) ds + \Phi(t) \int_0^t \Phi^{-1}(s) G(s) dW_s,
\]

(B7)

where

\[
\Phi(t) = \frac{\beta^2 (k - 2q + 1)}{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} + \beta^2 (k - 2q + 1) - \alpha^2 \sigma_{i0}^2 (T - t)^{k-2q+1},}
\]

follows the ODE

\[
\dot{\Phi}(t) = \bar{A}(t) \Phi(t), \quad \Phi(0) = 1,
\]

(B8)

and,

\[
\bar{A}(t) \equiv -\frac{\alpha^2 \sigma_{it}^2}{\beta^2} (T - t)^{k-2q},
\]

\[
G(t) \equiv \frac{\alpha \sigma_{it}^2}{\beta} (T - t)^{\frac{k}{2} - q},
\]

\[
\Theta(t) \equiv -\frac{\alpha^2 \sigma_{it}^2 dT}{\beta^2} (T - t)^{k-2q}.
\]

See subsection B.2.7 for the derivation of solving for ODE (B8).

Thus,

\[
\int_0^t \Phi^{-1}(s) \Theta(s) ds
\]

\[
= -\int_0^t \frac{T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k - 2q + 1) - (T - s)^{k-2q+1}}{\alpha^2 \sigma_{i0}^2 (k - 2q + 1)} \frac{\alpha^2 \sigma_{it}^2 dT}{\beta^2} (T - s)^{k-2q} ds
\]

\[
= -\int_0^t \frac{\sigma_{i0}^2 \alpha^2 dT}{\beta^2} (T - s)^{k-2q} ds = \int_0^t \frac{\sigma_{i0}^2 \alpha^2 dT}{\beta^2} (T - s)^{k-2q} d(T - s)
\]

\[
= \frac{\sigma_{i0}^2 \alpha^2 dT}{\beta^2} \frac{1}{k - 2q + 1} \left[ (T - t)^{k-2q+1} - T^{k-2q+1} \right],
\]

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and

$$\int_0^t \Phi^{-1}(s) G(s) \, dW_s$$

$$= \int_0^t \frac{\alpha^2 \sigma_0^2 \beta^{2q+1}}{(k-2q+1)^{\beta^2}} \frac{\alpha^2 \sigma_0^2 (T-s)^{k-2q+1} + \alpha \sigma_0^2 (T-s)^{\frac{1}{2}} - q} {\beta^2} \, dW_s$$

$$= \frac{\alpha^2 \sigma_0^2}{\beta^2} \int_0^t (T-s)^{\frac{1}{2} - q} \, dW_s.$$ 

Note $h_{it} \Phi(t) = h_{i0}$, thus I have

$$h_{it} m_{it} = h_{i0} \Phi(t) m_{i0} + h_{it} \Phi(t) \int_0^t \Phi^{-1}(s) \Theta(s) \, ds + h_{it} \Phi(t) \int_0^t \Phi^{-1}(s) G(s) \, dW_s$$

$$= h_{i0} m_i + h_{i0} \int_0^t \Phi^{-1}(s) \Theta(s) \, ds + h_{i0} \int_0^t \Phi^{-1}(s) G(s) \, dW_s$$

$$= h_{i0} m_i + \frac{\alpha^2 d_T}{\beta^2} \frac{1}{k-2q+1} \left( (T-t)^{k-2q+1} - T^{k-2q+1} \right) + \frac{\alpha}{\beta} \int_0^t (T-s)^{\frac{1}{2} - q} \, dW_s$$

$$= h_{i0} m_i + J_1(t, W),$$

(B9)

where

$$J_1(t, W) \equiv \frac{\alpha^2 d_T}{\beta^2} \frac{1}{k-2q+1} \left( (T-t)^{k-2q+1} - T^{k-2q+1} \right) + \frac{\alpha}{\beta} \int_0^t (T-s)^{\frac{1}{2} - q} \, dW_s,$$

is also common for all the investors.

Hence, substitute Eq. (B6) and Eq. (B9) into Eq. (B5), I obtain

$$p_t = \overline{m} + \overline{J_1(t, W)} - rZ/I \overline{h} + J_0(t),$$

(B10)

from which, it is easy to see the stock price is common for all the investors.

**B.2.6 Proof the Stock Price $p_t = \overline{m}^{\text{h}} - r\overline{\sigma}^2 Z/I$**

Note

$$\overline{m}^{\text{h}} - r\overline{\sigma}^2 Z/I = \frac{1}{T} \sum_{i=1}^{T} \frac{h_{it} m_{it} - rZ/I}{h_t} = \frac{1}{T} \sum_{i=1}^{T} \frac{h_{it} m_{it} - rZ/I}{h_t}$$

$$= \frac{1}{T} \sum_{i=1}^{T} \left( h_{i0} m_i + J_1(t, W) - rZ/I \right) - rZ/I \overline{h}_t + J_0(t) = \overline{h}_t + J_0(t)$$

This completes the proof of Proposition 1.
B.2.7 Solve for the ODE (B8)

Solve for the ODE (B8), I obtain

$$\dot{\Phi}(t) = A(t) \Phi(t) \iff d\ln \Phi(t) = \dot{\Phi}(t) dt \iff d\ln \Phi(t) = -\frac{\alpha^2 \sigma_{it}^2}{\beta^2} (T - t)^{k-2q} dt,$$

thus, substitute for the posterior variance, I obtain

$$d\ln \Phi(t) = \frac{(k - 2q + 1) \sigma_{i0}^2}{\sigma_{i0}^2 T^{k-2q+1} - \sigma_{i0}^2 (T - t)^{k-2q+1} + \frac{\beta^2}{\alpha^2} (k - 2q + 1)} (T - t)^{k-2q} dt$$

$$= \frac{(k - 2q + 1) \sigma_{i0}^2}{\sigma_{i0}^2 T^{k-2q+1} - \sigma_{i0}^2 (T - t)^{k-2q+1} + \frac{\beta^2}{\alpha^2} (k - 2q + 1)} \frac{1}{k - 2q + 1} d(T - t)^{k-2q+1}$$

$$= -\frac{1}{T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k - 2q + 1) - (T - t)^{k-2q+1}} d \left[ \left( T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k - 2q + 1) \right) - (T - t)^{k-2q+1} \right]$$

Therefore, I have a general solution for the ODE

$$\ln \Phi(t) = \ln \left[ \left( T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k - 2q + 1) \right) - (T - t)^{k-2q+1} \right]^{-1} + C_1$$

$$\iff \Phi(t) = \left[ \left( T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k - 2q + 1) \right) - (T - t)^{k-2q+1} \right]^{-1} \exp C_1,$$

where $C_1$ is a constant determined by the boundary condition.

Substitute the boundary condition at $t = 0$, I get

$$\Phi(0) = \left[ \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k - 2q + 1) \right]^{-1} \exp C_1 = 1$$

$$\iff C_1 = \ln \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k - 2q + 1),$$

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thus, I obtain the expression of \( \Phi(t) \) as

\[
\Phi(t) = \frac{\beta^2 (k - 2q + 1)}{\alpha^2 \sigma_0^2 T^{k-2q+1} + \beta^2 (k - 2q + 1) - \alpha^2 \sigma_0^2 (T - t)^{k-2q+1}}.
\]

### B.3 Proof of Proposition 3

The Pareto optimal portfolios should be self-financing, and satisfy both optimal wealth processes and the market clearing conditions. It is proofed as follows.

By Eq. (B10), the differential form of the stock price follows

\[
dp_t = \frac{1}{h + J_0(t)} dJ_1(t, W)
= \frac{1}{h + J_0(t)} \left( - \int_0^t \frac{\alpha^2 d_T}{\beta^2} (T - s)^{p-2q} ds + \frac{\alpha}{\beta} \int_0^t (T - s)^{\frac{p}{2}-q} dW_s \right)
= \frac{1}{h + J_0(t)} \left( - \frac{\alpha^2 d_T}{\beta^2} (T - t)^{p-2q} dt + \frac{\alpha}{\beta} (T - t)^{\frac{p}{2}-q} dW_t \right),
\]

thus, the SDE of the stock price under \( P \)-measure gives rise to the implied market price of risk

\[
\phi_t = -\frac{\alpha d_T}{\beta} (T - t)^{\frac{p}{2}-q}.
\]

Note the Brownian motion under \( Q \)-measure is

\[
dW_t^* = dW_t + \phi_t dt,
\]

substitute the Eq. (B13) into (B11), yields dynamics of the stock price under \( Q \)-measure,

\[
dp_t^* = \sigma_t^* dW_t^*, \quad \sigma_t^* \equiv \frac{\alpha (T - t)^{\frac{p}{2}-q}}{\beta(h + J_0(t))}.
\]

In the spirit of Eq. (B1), I have

\[
\dot{X}_i(t) = E^Q(e_i | F_t) - E^Q(e_i), \ t \in [0, T],
= E^Q(c_{iT}^* - \bar{\omega}_{iT} | F_t) - E^Q(c_{iT}^* - \bar{\omega}_{iT})
= E^Q(c_{iT}^* | F_t) - E^Q(\bar{\omega}_{iT} | F_t) - E^Q(c_{iT}^*) + E^Q(\bar{\omega}_{iT})
= E^Q(c_{iT}^* | F_t) - E^Q(c_{iT}^*),
\]

thus, according to the Martingale Presentation Theorem, there exists some process \( x_i(t) \), allows the optimal wealth process for the investor \( i \) to have a stochastic integral representation
\[ E^Q(c^*_{iT}) - E^Q(c^*_{iT}) = \int_0^t x_i(s)dp_i^* = \int_0^t x_i(s)\sigma_i^*dW_i^*, \forall t \in [0, T] \text{ a.s.} \]

Moreover, since \( E^Q(c^*_{iT}) \) is the initial wealth of the investor \( i \), thus,

\[ E^Q(c^*_{iT}) = E(c^*_{iT}c^*_{iT}) = E(c^*_{iT}c^*_{iT}), \quad \zeta^i_0 = 1, \]

thus yields

\[ E^Q[c^*_{iT}|\mathcal{F}_t] = E(c^*_{iT}c^*_{iT}) + \int_0^t x_i(s)\sigma_i^*dW_i^*, \quad (B15) \]

which indicates the martingale under \( Q \)-measure has a stochastic integral representation.

Note Ocone and Karatzast (1991) generalize the Clark-Ocone formula and provide explicit expression of the integrand in the martingale representation formulae under \( Q \)-measure.

Before showing their results, I first demonstrate the definition of the Malliavin derivative\(^9\) as follows. Let \( \tilde{W} = \int_0^T \tilde{h}(s)dW_s^* \) be defined for \( \tilde{h}(s) \in L^2([0, T]) \). For a smooth Brownian functional \( \tilde{F} \), i.e., a random variable of the form

\[ \tilde{F} = \tilde{f}\left( \tilde{W}\left( \tilde{h}_1 \right), ..., \tilde{W}\left( \tilde{h}_n \right) \right), \]

where \( \tilde{f} \) is a smooth bounded function with bounded derivatives of all orders, the Malliavin derivatives is defined by

\[ \partial_t \tilde{F} = \sum_{i=1}^n \partial_i \tilde{f}\left( \tilde{W}\left( \tilde{h}_1 \right), ..., \tilde{W}\left( \tilde{h}_n \right) \right) \tilde{h}_i(t), \]

where \( \partial_i \) stands for the \( i \)-th partial derivative. Note that \( \partial_t \left( \int_0^T \tilde{h}(s)dW_s^* \right) = \tilde{h}(t) \) and in particular \( \partial_s (W_t^*) = 1 \) for \( s \leq t \).

Employing the tool of Malliavin derivatives, I summarize the results of Proposition 2.5, Corollary 2.6, and Eq. (3.11) in Ocone and Karatzast (1991) in the following Lemma.

**Lemma 4** Consider the setting in which the market price of risk \( \{\phi(t); 0 \leq t \leq T\} \) is a deterministic bounded process, and the riskless interest rate is constantly 0.

(a) The process of market price of risk\(^10\) \( \phi(t) \in L^p_{1,1} \) and there exist some \( \tau > 1, \mu > 1 \)

---


\(^10\)Note for notations, norm \( \|\cdot\| \) denotes the \( L^2([0, T]) \) norm, and \( D_{1,1} \) is the Banach space and the closure
and some $\epsilon$ such that $(1/\epsilon) + (1/\tau) < 1$, s.t., for a random variable $\Xi$,

$$E \left[ \int_0^T \| \partial \phi (s) \|^2 ds \right]^{1/2} < \infty, \quad E \left[ \| \partial \Xi \|^m \right] < \infty, \quad \Xi \in L^c (P), \text{ and } \Xi \in D_{1,1}.$$

(b) Product $\Xi \cdot \Lambda (T) \in D_{1,1}$, where

$$\Lambda (T) = \exp \left( - \int_0^T \phi (t) dW (t) - \frac{1}{2} \int_0^T [\phi (t)]^2 dt \right).$$

(c) Conditions (a) and (b) are sufficient for the establishment of the stochastic integral representation formula

$$\Xi = E (\Xi \Lambda (T)) + \int_0^T E^Q (\partial_t \Xi | \mathcal{F}_t) dW_t^*,$$

and it follows also that

$$E^Q (\Xi | \mathcal{F}_t) = E (\Xi \Lambda (T)) + \int_0^t E^Q (\partial_s \Xi | \mathcal{F}_s) dW_s^*, \quad 0 \leq t \leq T. \quad (B16)$$

Note in this paper, both the market price of risk $\phi (t)$, and the riskless interest rate are deterministic, thus, I can apply the stochastic integral representation formula in Lemma 4. Compare Eq. (B16) to Eq. (B15), note $\Lambda (T) = \zeta^*_T$, hence, I obtain the optimal portfolio in the stock market as

$$x^*_{it} = (\sigma^*_t)^{-1} E^Q (\partial_t (c^*_T | \mathcal{F}_t), \quad (B17)$$

where $\partial_t (c^*_T)$ is the Malliavin derivative of the optimal terminal consumption $c^*_T$. This result enable me to achieve very explicit expressions for the optimal portfolios.

Note at the terminal date when the dividend is perfectly known, $p^*_T = d_T$, and by Eq.
(B14), the stock price under $Q$-measure is given as

$$p_T^* = p_0^* + \int_0^T \sigma_t^* dW_t^*.$$  

To apply the Malliavin derivative, let

$$\tilde{h}(t) = \sigma_t^*, \quad \tilde{W}(\tilde{h}) = \int_0^T \sigma_t^* dW_t^* = \int_0^T \tilde{h}(t) dW_t^*,$$  

hence, the stock price is

$$p_T^* = p_0^* + \tilde{W}(\tilde{h}).$$  

Moreover, to express $c_{it}^*$ as a function of $\tilde{W}(\tilde{h})$, let

$$\tilde{F} = \tilde{f} \left( \tilde{W}(\tilde{h}) \right) = \theta_{i0}^t \left( p_0^* + \tilde{W}(\tilde{h}) \right)^2 + x_{i0}^t \left( p_0^* + \tilde{W}(\tilde{h}) \right) + \gamma_{i0},$$  

thus, according to the definition of the Malliavin derivative, I have

$$\mathcal{C}_t(c_{it}^*) = \partial_t \left( \theta_{i0}^t (d_T)^2 + x_{i0}^t d_T + \gamma_{i0}^t \right) = \partial_t \left( \theta_{i0}^t (p_T^*)^2 + x_{i0}^t p_T^* + \gamma_{i0}^t \right)$$

$$= \partial_t \left( \theta_{i0}^t \left( p_0^* + \tilde{W}(\tilde{h}) \right)^2 + x_{i0}^t \left( p_0^* + \tilde{W}(\tilde{h}) \right) + \gamma_{i0}^t \right) = \partial_t \tilde{F}$$

$$= \partial \tilde{f} \left( \tilde{W}(\tilde{h}) \right) \cdot \tilde{h} = \left( 2\theta_{i0}^t \left( p_0^* + \tilde{W}(\tilde{h}) \right) + x_{i0}^t \right) \cdot \tilde{h} = \left( 2\theta_{i0}^t p_T^* + x_{i0}^t \right) \sigma_t^*,$$

note the volatility of the stock price under $Q$-measure, $\sigma_t^*$, is a deterministic function of date $t$, hence, the optimal portfolio in stock for investor $i$ is

$$x_{it}^* = (\sigma_t^*)^{-1} E^Q \left[ \left( 2\theta_{i0}^t p_T^* + x_{i0}^t \right) \sigma_t^* | \mathcal{F}_t \right]$$

$$= E^Q \left[ 2\theta_{i0}^t p_T^* + x_{i0}^t | \mathcal{F}_t \right] = E^Q \left[ 2\theta_{i0}^t d_T + x_{i0}^t | \mathcal{F}_t \right] = 2\theta_{i0}^t p_t + x_{i0}^t.$$  

Now turn to the portfolio in the riskless bond. Note the proof of Proposition 3 shows that the price of payment of $d_T^2$, i.e.,

$$\frac{E^i [\zeta_{it}^* d_T^2 | \mathcal{F}_t]}{\zeta_{it}^*} = \sigma_t^2 + p_t^2,$$
hence, the optimal wealth at date $t$ follows

$$E^Q[c^*_t | \mathcal{F}_t] = \frac{1}{\zeta_{it}^*} E^t [\zeta_{it}^* c^*_t | \mathcal{F}_t] = \frac{1}{\zeta_{it}^*} E^t \left[ \zeta_{it}^* \left( \theta_{i0}^t d_{i0}^T + x_{i0}^t d_T + \gamma_{i0}^t \right) | \mathcal{F}_t \right]$$

$$= \theta_{i0}^t (\sigma_i^2 + p_i^2) + x_{i0}^t p_t + \gamma_{i0}^t,$$

given the price of the bond is always 1. Thus, the optimal portfolio in the riskless bond is

$$\gamma_{it}^* = E^Q[c^*_t | \mathcal{F}_t] - x_{it} p_t = \theta_{i0}^t \sigma_i^2 + \gamma_{i0}^t - \theta_{i0}^t p_t^2.$$

Therefore, similar to the general proof for Proposition 1 (compare with Eq. (B1) and Eq. (B3)), the optimal portfolios here in a special case with exponential-utility investors and normally distributed dividend are self-financing and are able to construct the optimal wealth process. I now verify that the Pareto optimal portfolios satisfy the market clearing condition, i.e., (1). Note that

$$\sum_{i=1}^I x_{it}^* = 2 p_t \sum_{i=1}^I \theta_{i0}^t + \sum_{i=1}^I x_{i0}^t = Z,$$

$$\sum_{i=1}^I \gamma_{it}^* = (\sigma_i^2 - p_i^2) \sum_{i=1}^I \theta_{i0}^t + \sum_{i=1}^I \gamma_{i0}^t = 0,$$

thus, the market clearing conditions establish. This completes the proof.

**B.4 Proof of Proposition 4**

The price of the dividend derivative is given as

$$\pi_t = \frac{E^i [\zeta_{iT}^* d_{i0}^2 | \mathcal{F}_t]}{\zeta_{it}^*},$$

I obtain state-price deflator at date $t$ in the previous subsection, thus I only need to calculate the expected risk-adjusted payment of the dividend derivative.
Use the notation in the previous subsection, I have

\[ E_i \left[ \zeta_T^* d_T^2 | \mathcal{F}_t \right] \]
\[ = \frac{1}{\psi_{i0}} \int_{-\infty}^{+\infty} d_T^2 \exp - [\alpha_0 (d_T + \alpha_1)^2 + \alpha_2] \, d_T \]
\[ = \exp \left[ -\alpha_2 \right] \int_{-\infty}^{+\infty} d_T \exp - \left[ (a_1 d_T + b_1)^2 \right] \, d_T^2 \]
\[ = \frac{\exp \left[ -\alpha_2 \right]}{a_1^2} \int_{-\infty}^{+\infty} d_T \exp - \left[ (a_1 d_T + b_1)^2 \right] \left[ (a_1 d_T + b_1)^2 - 2a_1 b_1 d_T - b_1^2 \right] \]
\[ = \frac{\exp \left[ -\alpha_2 \right]}{a_1^2} \int_{-\infty}^{+\infty} d_T \exp - \left[ (a_1 d_T + b_1)^2 \right] \left[ (a_1 d_T + b_1)^2 - 2b_1 (a_1 d_T + b_1) \right], \]

let \( u = a_1 d_T + b_1 \), and \( d_T = \frac{u - b_1}{a_1} \), note \( u \to +\infty \), as \( d_T \to +\infty \), \( u \to -\infty \), as \( d_T \to -\infty \), thus,

\[ E_i \left[ \zeta_T^* d_T^2 | \mathcal{F}_t \right] = \exp \left[ -\alpha_2 \right] \int_{-\infty}^{+\infty} \frac{u - b_1}{a_1} \exp \left[ -u^2 \right] \, d \left[ u^2 - 2b_1 u \right] \]
\[ = \frac{\exp \left[ -\alpha_2 \right]}{a_1^3} \left( \int_{-\infty}^{+\infty} u \exp \left[ -u^2 \right] \, d \left[ u^2 - 2b_1 u \right] - b_1 \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] \, d \left[ u^2 - 2b_1 u \right] \right) \tag{B18} \]

Note the first integral in (B18)

\[ \int_{-\infty}^{+\infty} u \exp \left[ -u^2 \right] \, d \left[ u^2 - 2b_1 u \right] \]
\[ = \int_{-\infty}^{+\infty} u \exp \left[ -u^2 \right] \, d u - 2b_1 \int_{-\infty}^{+\infty} u \exp \left[ -u^2 \right] \, d u \]
\[ = \int_{0}^{+\infty} u \exp \left[ -u^2 \right] \, d u + \int_{-\infty}^{0} u \exp \left[ -u^2 \right] \, d u - b_1 \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] \, d u \]
\[ = \int_{0}^{+\infty} u \exp \left[ -u^2 \right] \, d u - \int_{-\infty}^{0} u \exp \left[ -(u)^2 \right] \, d u - b_1 \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] \, d u \]
\[ = 2 \int_{0}^{+\infty} u \exp \left[ -u^2 \right] \, d u - b_1 \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] \, d u, \]

let \( v = u^2 \), note \( v \to +\infty \), as \( u \to +\infty \), \( v \to +\infty \), as \( u \to -\infty \), thus,
\[ \int_{-\infty}^{+\infty} u \exp \left[ -u^2 \right] d \left[ u^2 - 2b_1 u \right] \]
\[ = 2 \int_{0}^{+\infty} \sqrt{v} \exp \left[ -v \right] dv - b_1 \int_{+\infty}^{+\infty} \exp \left[ -v \right] dv \]
\[ = 2 \int_{0}^{+\infty} \sqrt{v} \exp \left[ -v \right] dv - 0 \quad \forall = \frac{1}{2} \]
\[ = 2 \int_{0}^{+\infty} \sqrt{\frac{1}{2} x} \exp \left[ -\frac{1}{2} x \right] d \frac{1}{2} x \]
\[ = \sqrt{\frac{1}{2}} \int_{0}^{+\infty} \sqrt{x} \exp \left[ -\frac{1}{2} x \right] dx. \]

Note
\[ 1 \quad \sqrt{2\pi} \int_{0}^{+\infty} x^{\frac{1}{2}} \exp \left( -\frac{1}{2} x \right) dx \]
\[ = \frac{1}{\sqrt{2\pi}} \left[ -x^{\frac{1}{2}} \exp \left( -\frac{1}{2} x \right) dx \right]_{0}^{+\infty} + \int_{0}^{+\infty} \frac{1}{2} x^{-\frac{1}{2}} \exp \left( -\frac{1}{2} x \right) dx \]
\[ = \frac{1}{\sqrt{2\pi}} \left[ (0 - 0) + \int_{0}^{+\infty} x^{-\frac{1}{2}} \exp \left( -\frac{1}{2} x \right) dx \right] \]
\[ = \int_{0}^{+\infty} \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} \exp \left( -\frac{1}{2} x \right) dx = \int_{0}^{+\infty} f_X (x) dx = 1, \]

where \( f_X (x) \) is the probability density function of \( X \) which is a Chi-square random variable with 1 degrees of freedom.

Therefore,
\[ \int_{-\infty}^{+\infty} u \exp \left[ -u^2 \right] d \left[ u^2 - 2b_1 u \right] = \sqrt{\frac{1}{2}} \times \sqrt{2\pi} = \sqrt{\pi}. \]

Note the second integral in (B18) is derived in previous subsection, i.e.,
\[ \int_{-\infty}^{+\infty} \exp \left[ -u^2 \right] d \left[ u^2 - 2b_1 u \right] = -2b_1 \sqrt{\pi}, \]
hence,
\[ \mathbb{E}^i \left[ \zeta_{it} d^2 \mathcal{F}_t \right] = \frac{\exp \left[ -\alpha_2 \right]}{a_1^{\psi_i \psi_0}} \left( \sqrt{\pi} + 2b_1^2 \sqrt{\pi} \right). \]

Therefore, the price for the dividend derivative
\[
\pi_t = \frac{E^i [\zeta^*_t d^2_t | \mathcal{F}_t]}{\zeta^*_it} = \frac{\exp[-\alpha_2]{\sqrt{\pi} + 2b^2_t \sqrt{\pi}}}{{\alpha}\sqrt{\alpha_0} \exp[-\alpha_2]} = \frac{1}{2a_1^2} (1 + 2b^2_t)
\]

Moreover, by investing in \(2\theta_{i0}^t p_t\) share of the stock, and \(\theta_{i0}^t (\sigma_t^2 - p_t^2)\) share of the riskless bond continuously, the value of the portfolios is \(\theta_{i0}^t (p_t^2 + \sigma_t^2) = \theta_{i0}^t \pi_t\), which is the value of \(\theta_{i0}^t\) share of the dividend derivative at date \(t\). This completes the proof.
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