DEGREE OF A ISOLATED REAL POINT OR A SINGULAR COMPLEX POINT ON A PLANE CURVE DEFINED OVER $\mathbb{Q}$

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ABSTRACT. Let X be a curve in the affine plane defined by a reduced polynomial of degree d with rational coefficients. Assume that P is an isolated real point or a singular complex point on the curve X. The coordinates of P are algebraic numbers over the rationals of degree at most d^2.

The result in the complex case is obtained by elimination theory and the use of resultants. The result in the real case follows from the complex case using analytic uniformization.

This result in the real case has applications in the theory of games, see [2].

FiXme Note: Kan anvendelser i game-theory uddybes, evt. med referencer? Peter Bro Miltersen

FiXme Note: Uddyb begreberne inden præsentation af sætningerne?

Theorem 1. *Let X be an affine algebraic set in the affine plane defined by a reduced polynomial $f(X,Y) \in \mathbb{Q}[X,Y]$ of degree d. Assume that $P \in X(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$ is an isolated real point. The coordinates of P are algebraic over $\mathbb{Q}$ of degree at most d^2.*

In [2], Theorem 21 showns that $(2d+1)^2$ is an upper bound for the algebraic degree of the coordinates. This result is obtained by different methods.

The proof is presented at the end of this paper.

Theorem 2. *Let X be an affine algebraic set in the affine plane defined by a reduced polynomial $f(X,Y) \in \mathbb{Q}[X,Y]$ of degree d. Assume that $P \in X(\mathbb{C}) \subset \mathbb{A}(\mathbb{C})$ is a singular complex point. The coordinates of P are algebraic over $\mathbb{Q}$ of degree at most d^2.*

Recall that a polynomial f is said to be reduced (or square-free) if it can be factored in a product of distinct irreducible polynomials:

$$f = g_1 \cdot \dots \cdot g_r, \quad g_i \neq g_j \,\text{for}\, i \neq j$$

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Lemma 3. *Let k be a field of characteristic 0 and let $f \in k[X,Y]$ be a polynomial such that*

$$f = g \cdot h, \quad g, h \in k[X,Y],$$

where g is irreducible and h is not divisible by g. Assume that X appears in g. Then $\frac{\partial f}{\partial X} \in k[X,Y]$ is not divisible by g.

Proof. Assume that $\frac{\partial f}{\partial X} \in k[X,Y]$ is divisible by g. By the product rule, we have

$$\frac{\partial f}{\partial X} = \frac{\partial g}{\partial X} \cdot h + g \cdot \frac{\partial h}{\partial X}.$$

The $\frac{\partial g}{\partial X} \cdot h$ is divisible by g and since g is irreducible and does not divide h, g divides $\frac{\partial g}{\partial X}$. As $\frac{\partial g}{\partial X}$ has smaller degree than g, this is impossible and we have a contradiction. □

Lemma 4. *Let k be a field of characteristic 0 and let $f \in k[X,Y]$ be a reduced polynomial, such that Y is not a factor in f. Then f and $\frac{\partial f}{\partial X} \in k[X,Y]$ have no non-trivial common factor.*

Proof. Consider the factorization of f into distinct irreducible factors:

$$f = g_1 \cdot \dots \cdot g_r, \quad g_i \neq g_j \text{ for } i \neq j.$$

As Y is not a factor, X appears in all the irreducible factors $g_1, \dots g_r$. By Lemma 3 none of the irreducible factors divide $\frac{\partial f}{\partial X}$, so f and $\frac{\partial f}{\partial X} \in k[X,Y]$ have no non-trivial common factor. □

Remark 5. The theory of resultants are central to the proof, and we recollect part of the theory, see [3], p. 135, [5], Fünftes Kapitel, §34 and [1], Chap. 3, Chap. 6 and Chap. 8.

Let $f, g \in k[X,Y]$ be two polynomials with coefficients in a field k both of positive degree in X. Write f and g as polynomials in Y:

$$\begin{aligned} f(X,Y) &= a_0(X)Y^l + \dots + a_l(X), \quad a_i(X) \in k[X], \quad a_0(X) \neq 0 \\ g(X,Y) &= b_0(X)Y^m + \dots + b_m(X), \quad b_j(X) \in k[X], \quad b_0(X) \neq 0 \end{aligned}$$

Then resultant $\operatorname{Res}(f,g,Y)(X) \in k[X]$ of f and g with respect to Y is the polynomial obtained as the determinant:

$$\det \begin{pmatrix} a_0(X) & & & & b_0(X) & & & \\ a_1(X) & a_0(X) & & & b_1(X) & b_0(X) & & \\ & a_1(X) & \ddots & & & b_1(X) & \ddots & \\ \vdots & & \ddots & a_0(X) & \vdots & & \ddots & b_0(X) \\ & \vdots & & a_1(X) & & \vdots & & b_1(X) \\ a_l(X) & & & & b_m(X) & & & \\ & a_l(X) & & \vdots & & b_m(X) & & \vdots \\ & & \ddots & & & & \ddots & \\ & & & a_l(X) & & & & b_m(X) \end{pmatrix}$$

where the empty spaces are filled by zeros. The resultant has the following properties:

i) $\mathrm{Res}(f, g, Y)(X) \in k[X]$ is of degree at most $l \cdot m$, see [1], Chap. 8, §7, Lemma 5.
ii) $\mathrm{Res}(f, g, Y)(X) = 0$ if and only if f and g have a common factor in $k[X, Y]$ which has positive degree in Y, see [1], Chap. 6, §6, Proposition 1.
iii) There are polynomials $a, b \in k[X, Y]$ such that

$$\mathrm{Res}(f, g, Y)(X) = a(X, Y)f(X, Y) + b(X, Y)g(X, Y),$$

see [1], Chap. 3, §5, Proposition 9.

Proof. (Theorem 2) We prove the claim for the coordinate x, the proof for y is entirely similar. Assume Y is an irreducible factor in f:

$$f = g_1 \cdot \cdots \cdot g_{r-1} \cdot Y.$$

As for the singular loci we have

$$\mathrm{Sing}(V(f)) = \mathrm{Sing}(V(g_1 \cdot \cdots \cdot g_{r-1})) \cup (V(g_1 \cdot \cdots \cdot g_{r-1}) \cap V(Y)).$$

A point $P \in V(g_1 \cdot \cdots \cdot g_{r-1}) \cap V(Y)$ has coordinates $(x, 0)$, where x is a root in the univariate polynomial $g_1(X) \cdot \cdots \cdot g_{r-1}(X) \in \mathbb{Q}[X]$ of degree at most $d - 1$, so the claim is proved for such points.

We are therefore reduced to prove the theorem in the case where Y is not a factor in f. In that case X appears in all the irreducible factors $g_1, \ldots g_r$.

By Lemma 3 the polynomials f and $\frac{\partial f}{\partial X} \in k[X, Y]$ have no non-trivial common factor. Consider the resultant

$$R(X) = \mathrm{Res}(f(X, Y), \frac{\partial f}{\partial X}(X, Y), Y)(X) \in \mathbb{Q}[X]$$

By Remark 5 we have that $R(X) \neq 0$, $\deg R(X) \leq d^2$ and there exist polynomials $a, b \in \mathbb{Q}[X, Y]$ such that

$$R(X) = a(X, Y)f(X, Y) + b(X, Y)\frac{\partial f}{\partial X}(X, Y) \tag{1}$$

Substituting the coordinates (x, y) in (1) above, we obtain

$$R(x) = a(x, y)f(x, y) + b(x, y)\frac{\partial f}{\partial X}(x, y) = 0,$$

as (x,y) is a singular point on $V(f)$ and we conclude that x is algebraic over $\mathbb{Q}$ of degree at most d^2.

□

Proposition 6. *Let $X \subset \mathbb{A}^n$ be an affine algebraic set in the affine n-space defined by an ideal $I \subset \mathbb{Q}[X_1, \ldots, X_n]$ generated by polynomials with coefficients i $\mathbb{Q}$. Let $P \in X(\mathbb{R})$ be a real point in $X(\mathbb{R})$, that is isolated in the classical topology. Then P is a singular point on $X(\mathbb{C})$.*

Proof. Assume on the contrary that $P = (a_1, \dots, a_n)$ is a smooth point on $X(\mathbb{C})$ of dimension r. We apply *analytic uniformization*, see [4], (1.26) to obtain a contradiction.

Let $x_{i_1}, \dots, x_{i_r}$ be any sets of coordinates such that $dx_{i_1}, \dots, dx_{i_r}$ are independent linear functions on the tangent space T_P. Then there is an $\varepsilon > 0$ and uniguely determined convergent power series $g_j(x_{i_1}, \dots, x_{i_r})$, for all $j \in \{1, \dots, n\} - \{i_1, \dots, i_r\}$, such that if $|t_1|, \dots, |t_n| < \varepsilon$.

$$(a_1 + t_1, \dots, a_n + t_n) \in X \Leftrightarrow t_j = g_j(t_{i_1}, \dots, t_{i_r})$$

As X is defined over $\mathbb{Q}$ and in particular over $\mathbb{R}$, the powerseries g_j have (by uniqueness) real coefficients. In conclusion, $X(\mathbb{R})$ can locally around P be parametrized by real convergent power series. This contradicts that P is isolated in $X(\mathbb{R})$. $\square$

Proof. (Theorem 1). The claim follows directly from Proposition 6 and Theorem 2. $\square$

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