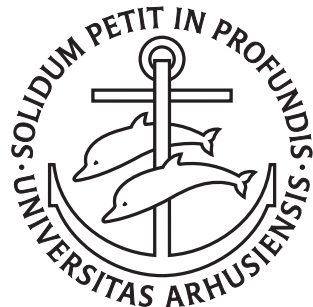


Essays on Advance Demand Information, Prioritization and Real Options in Inventory Management

Ph.D. Dissertation

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1

Introduction

Introduction

§1.1 Research topics

In general inventory control concerns balancing demand and supply where in order to do that in a smooth fashion, some stocks are needed. Most often, this is due to *economics of scale* and/or *uncertainties* in demand and/or supply. Economics of scale concerns that as there might be some fixed costs when issuing a procurement or a production order it is advisable to use some reasonable batch sizes, therefore it will take some time for demand to deplete the inventory and an inventory is occasionally build up. Uncertainty concerns both uncertainties in demand and in the timing and in the amounts delivered from the supply system. This give rise to safety stocks, an excess buffer to be kept to provide adequate customer service. The uncertainty in supply can concern limited capacity in the supply system (due to inflow of orders from other customers), quality problems, and uncertainties in production times or in transportation times. Uncertainty of demand is typically concerned with uncertainty about inter-arrival time of subsequent customer orders and uncertainty about the magnitudes of these.

In this thesis the economics of scale element is ignored so the focal point of the thesis is the uncertainty element. For the first three papers this concern a *more detailed description of the demand process*. Most often, as indicated above, there are only two random components namely inter-arrival times and customer order sizes, thus a demand model that is a compound stochastic process. However, other components could be added and in this thesis the focus is on two additional components:

heterogeneity and *advance demand information* (sometimes abbreviated ADI). Heterogeneity concerns, that there can be different customer groups with different random patterns, for instance concerning the two first components inter-arrival times and order size. Advance demand information concerns that, say a customer request occur at time point t , the customer also specifies time point $y + t$ where he wants to receive his order. Thus y can be interpreted as a *demand lead-time* of the customer. Though this is a common feature in many B2B (Business to Business) settings, it is ignored in most textbooks on inventory control models. In the thesis these two components will be studied with respect to *prioritization*. When studying heterogeneity in demands and inventory control, the literature assumes that the different customer groups should have different service, that is, sometimes a customer belonging to a lower prioritized customer group should be denied fulfillment, if the inventory level is critically low. Most often, this literature is not so explicit about putting specific *attributes* on what distinguishes the customer groups (except expressing this using different back-order or lost-sales costs adjoined to each customer group). In Paper 1 the attribute is the *size of customer order*. It seems reasonable to assume that large customer orders causes more disturbances upstream and the issuer of a larger order might be aware of this and thereby may have understanding for that his order gets lower priority than smaller orders. This is the premise for studying inventory control policies where degraded service is offered to customers of larger orders. The paper analysis two plausible policies for degrading the service for larger orders. In Paper 2 the attribute is the *demand lead-time* which is assumed to be a random variable but of course known when the customer request is received. Here the issue for the analysis is when should the customer order be reserved (or tagged) that is any later arriving request are later tagged customer order after the tagging takes place will be prioritized lower than this customer order. The paper examines three plausible reservation policies. In Paper 3 the issue is also prioritization, but it is analyzed in a static one-period model. Here the focal point is whether a supplier by offering different prices, before and after the end-customer demand is known to his (prioritized) buyers can induce these to make early orders, thereby providing the supplier

with advance demand information. So the analysis concern how to set the pre-order price optimally thereby also concerning whether it is optimal to gain knowledge about advance order information. As this analysis can be quite complex because it involves several independent decision-makers, the paper also addresses whether the supplier can derive his optimal pricing decision by using a more aggregated model. In Paper 4 the *uncertainty element is redirected to the supply side*, however, many of the ideas in Paper 3 are *mirrored*. The focal point is here a manufacturer who is uncertain about how much capacity in terms of raw material supply he can get. Therefore he might engage in contracting the future supply of raw material in terms of making option contracts. So some of his demand for raw material can be covered by his option contract while the remaining needs must be sourced at the open market where there might be some limitations (due to various capacity limitations, like competition from other similar manufacturers) on the amount he can get. One can interpret that option contracting enables the manufacturer with advance supply information. Also when comparing to the restricted model with no options one can assess what is the value of be able to use to option instrument in a similar fashion as the supplier in Paper 3 could assess the value of his early-ordering pricing.

Therefore the thesis (consisting of the four above mentioned papers) concerns *inventory control models under aspects advance demand information, heterogeneous demand, prioritization and option pricing*, which motivates the choice of title. Below, follows a short summary (in terms of abstract presentation) of the four papers.

§1.2 The structure of the thesis

For the inter-relationship between the papers of the thesis, see Figure 1.1. All 4 papers are within the topic of Advance demand information (ADI). Specifically, the first three papers (Paper 1,2,3) more focus on the prioritization, or customer segmentation, by taking customers into multiple demand classes, the supplier serve them by different prioritization.

Figure 1.1 is about here

The arrows in this figure shows the time-line (or sequence) of the research work. Paper 1 contributed on the attribute of order size. Based on the work of Paper 1, we extended the contribution to the attribute of uncertain demand lead-time in Paper 2. Simultaneously, we study the prioritization in a static case in a Newsvendor model, which is Paper 3. Later on, as a natural extension, we expanded some elements of this static model into a dynamic case, which is Paper 4.

§1.3 Summary of papers

Paper 1: Base Stock Policies with Degraded Service to Larger Orders

We study an inventory system controlled by a base stock policy assuming a compound renewal demand process. We extend the base stock policy by incorporating rules for degrading the service of larger orders. Two specific rules are considered, denoted $Postpone(q, t)$ and $Split(q)$, respectively. The parameter q distinguishes between regular orders (of size less than or equal to q) and larger orders. We develop mathematical expressions for the performance measures: order fill rate of the regular orders and average on-hand inventory level. We make numerical experiments where the postpone parameter t and the base stock levels of each rule are such that all customers (of both order types) are indifferent between the two rules. When comparing the difference in the average on-hand inventory levels, we can then make an assessment of the threshold value of the cost of splitting an order (which may otherwise be hard to quantify) in the rule $Split(q)$. Our numerical results indicate that this threshold value is increasing in the variance of the order sizes. Based on the numerical experiment our conclusion is therefore that when the variance of the order sizes is low, then $Postpone(q, t)$ seems to be a good option, while when the variance is high, then $Split(q)$ is more competitive.

Paper 2: Investigating Reservation Policies of Advance Orders in the presence of Heterogeneous Demand

We consider an inventory system where customers provide advance order information. Specifically each customer order has two attributes: a request date when the order is received and a due date specifying when the customer wants his order delivered. The inventory system is operated as a base stock system where replenishment orders are issued upon the receipt of an order. We assume the demand process is a Poisson process. As the demand lead-times, that is, the difference between the due date and the request date, can vary stochastically, the sequence of requests dates and due dates will differ. Therefore a potentially important issue is then how early should orders be reserved at the inventory. We explore three logical reservation policies. These policies also facilitates that the average on-hand inventory can be specified in a much simpler way than done in [Marklund \(2006\)](#). Assuming that the revenue of an order (assumed to be of unit size) depends on the demand lead-time we propose a profit optimization model, where the expected profit is the difference of the expected revenue and the expected inventory holding costs. We derive the profit function for each of the three proposed reservation policies. Our numerical results indicate that unless the revenue of an order with a large demand lead-time is very large compared to those with a smaller delivery lead-time, the items should only be reserved in a short time interval before the delivery and most often not at all.

Paper 3: Advance Demand Information, Capacity Restrictions and Customer Prioritization

We study a single supplier who must invest in capacity to manufacturer and sell products to buyers having different priorities. The buyers can place pre-orders before their demand is observed, and can also issue additional orders upon observing demand information. Since the supplier guarantees delivery of pre-ordered goods (these are not constrained by the supplier's capacity), buyers with lower priorities

may consider pre-ordering in order to secure inventory. We derive optimal policies for the supplier and buyers. We show surprisingly that it is optimal for the supplier to set the pre-order price so high that pre-ordering will not be used much even though pre-ordering should be of benefit to the supplier due to risk sharing. We also show that the supplier can make a pricing decision using an aggregate model.

Paper 4: Inventory Rationing with Real Options

We consider an inventory system consisting of two supply sources and one buyer. The first supply source is the real options contract provider while the other supply source is a capacitated external market. The buyer's unsatisfied demands are treated as lost sales. We formulate the problem as a dynamic programming model in order to perform some numerical investigations. In particular, we study the effect of using real options compared to a restricted situation where real options are not applied.

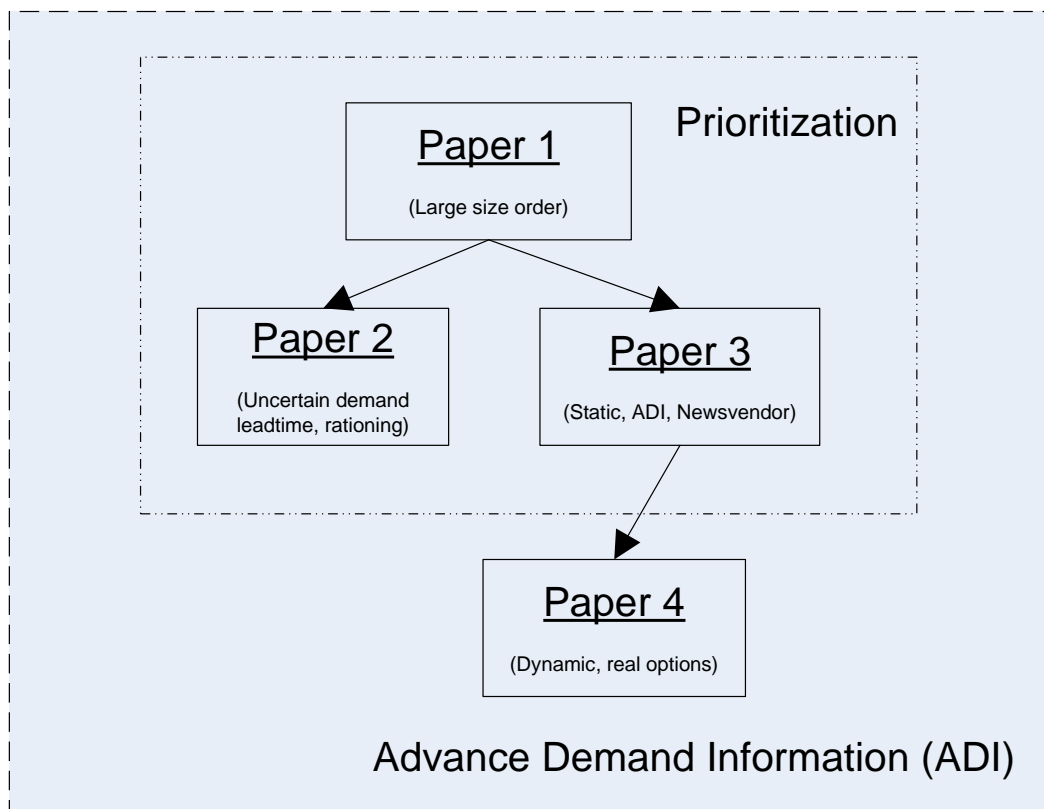


Figure 1.1: Inter-relationship between papers

2

Base Stock Policies with Degraded Service to Larger Orders

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Base Stock Policies with Degraded Service to Larger Orders

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Abstract

We study an inventory system controlled by a base stock policy assuming a compound renewal demand process. We extend the base stock policy by incorporating rules for degrading the service of larger orders. Two specific rules are considered, denoted $Postpone(q, t)$ and $Split(q)$, respectively. The parameter q distinguishes between regular orders (of size less than or equal to q) and larger orders. We develop mathematical expressions for the performance measures: order fill rate of the regular orders and average on-hand inventory level. We make numerical experiments where the postpone parameter t and the base stock levels of each rule are such that all customers (of both order types) are indifferent between the two rules. When comparing the difference in the average on-hand inventory levels, we can then make an assessment of the threshold value of the cost of splitting an order (which may otherwise be hard to quantify) in the rule $Split(q)$. Our numerical results indicate that this threshold value is increasing in the variance of the order sizes. Based on the numerical experiment our conclusion is therefore that when the variance of the order sizes is low, then $Postpone(q, t)$ seems to be a good option, while when the variance is high, then $Split(q)$ is more competitive.

Keyword

Base stock policy; Compound renewal process; Order fill rate; Differentiated service.

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§2.1 Introduction

It is well known that the larger demand variation, the higher inventory levels are needed in order to secure adequate service from an inventory system. One reason for large demand variation may be that the system occasionally receives large customer orders. Furthermore, these large orders may have negative effects further upstream in the supply chain. Therefore, as a manager of an inventory system, you would prefer to receive smaller orders at a more frequent pace rather than receiving orders with large variation at a less frequent pace. However, assuming that in the short run you cannot do anything to change the order behavior of your customers, it may be sensible to introduce a policy implying degraded service to larger orders. Customers submitting larger orders may also very well be aware of the inconvenience that they cause and be willing to accept degraded service. Furthermore, such an initiative may need the customers to change their ordering behavior such that in the long run one will in fact encounter less variation in the demand pattern.

The considerations raised here are inspired by a discussion that the authors had recently with the logistics personnel in a larger Danish company where the second author has been involved in an inventory control project. The aim of this project was to decide optimal base stock levels when the service measure is order fill rate, that is, the fraction of orders received where the whole order is delivered instantaneously from the inventory. This project was reported in [Larsen et al. \(2008\)](#) and some theoretical aspects of the project in [Larsen and Thorstenson \(2008\)](#). Obviously, larger orders can have very negative effects on the order fill rate service measure (thus, using this service measure, it is implicitly assumed that smaller orders are just as important as larger ones).

As the source of inspiration is this company project, it is natural that the mathematical model developed in this paper takes its point of departure in the model of this project. Therefore, we study a base stock system with control parameter S and where all replenishment orders, issued instantaneously upon receipt of an order, have a constant lead-time L . Our extension of the model is the introduction of

a given positive integer q , based on which orders are distinguished as being either regular orders (of a size less than or equal to q) or large orders (of a size larger than q). We introduce two rules, $Split(q)$ and $Postpone(q, t)$, which both seek to serve regular orders as well as possible while degrading the service of larger orders. $Postpone(q, t)$ operates as follows: the parameter t is in the interval between 0 and L . When receiving a large order, say at time point $\bar{\tau}$, the order is deliberately backlogged for t time units before it is attempted to be served. This implies that all regular orders arriving in the time interval $(\bar{\tau}, \bar{\tau} + t)$ are served ahead of this large order. One could interpret $Postpone(q, t)$ not as a rule for degrading service but as a special case of an advance demand information system where all receipts of larger orders are known t time units in advance. For studies of inventory systems in the presence of advance demand information, see among others [Hariharan and Zipkin \(1995\)](#), [Gallego and Özer \(2001\)](#) and [Marklund \(2006\)](#). As shown in [Hariharan and Zipkin \(1995\)](#) one could therefore make the interpretation that the effective replenishment lead-time of any larger order is $L - t$. The other rule $Split(q)$ operates as follows: each time a larger order is received, it is split into a suborder of size q and a suborder of the remaining size. The first suborder is then treated as a regular order while the second suborder is handled outside the inventory system, directly from the supply system, thus having a lead-time L . Therefore, the inventory system only faces orders that are less than or equal to q and all replenishment orders are the original order eventually truncated by q .

We believe that both rules could be reasonable choices for handling a situation where one wants to give degraded service to larger orders. As the two rules are quite different, determining which of the two to use is not immediately obvious. However, we here present an approach for accomplishing this. When choosing between the two rules, we propose that one should examine the following 4 elements; 1: the inconvenience caused to larger orders, 2: the service offered to regular orders, 3: the inventory investment and 4: the additional costs of splitting an order in the $Split(q)$ rule. Our approach involves all 4 elements. We believe that for a given q , a threshold value of t exists making the customers of the larger orders indifferent between the

two rules. Later in the paper, we provide a reasonable method for deciding this threshold value (though the method is somewhat approximate it is convenient for computations). When this threshold t value is determined, we compute base stock levels enabling the customers of the regular orders to reach the same order fill rate level, irrespective of which rule is used. So when going through these two steps, all customers should be indifferent between the two rules. The choice of which rule to apply is then based on which of the two performs best with respect to the last two elements. In our opinion the rule $Split(q)$ is a bit harder to administrate as it involves splitting and monitoring two suborders. Specifically, we imagine there is a cost incurred when doing a split. However, as this cost of splitting an order may be hard to quantify, we instead identify its threshold value, making the cost performance of the two rules equal (as can be seen later, the threshold value is actually measured relative to the inventory holding cost rate). Therefore, our numerical analysis is focused on identifying values of the input data where this threshold value is low (which suggests that $Postpone(q, t)$ is the best choice), and where it is high (which suggests that $Split(q)$ is the best choice). As a by-product, our numerical analysis also provides some insights into how the preference between the rules may alter upon observing a change in the behavior of the customers of the larger orders due to the downgrading of their order requests, for instance, if gradually one begins to see an increased arrival intensity of the order requests but in general a smaller average order size. Here our results seem to suggest that initially one might prefer the $Split(q)$ rule. However, due to a possible change in customer behavior, this could be altered to a preference for the $Postpone(q, t)$ rule.

There has been a large number of studies of inventory control in the presence of several (most often two) customer classes. For a good overview of the literature, see [Teunter and Haneveld \(2008\)](#). The aim of these studies is to design rationing rules concerning when to backlog (or reject) the demand of the least important customer class when the inventory level is critically low. In our work, we do not introduce any critical numbers on the inventory levels for when to deny service of (some) orders. Furthermore we do not explicitly model several customer classes. Therefore, our

work is not so closely related to the traditional studies of rationing policies in inventory control systems. When considering the rule $Postpone(q, t)$, our work is more in line with the paper of Wang et al. (2002) which is motivated by a company analysis reported in Cohen et al. (1999). They consider a base stock system with two customer classes where the service of one of the classes is first attempted after a given time period (in the paper denoted demand or delivery lead times) has elapsed since the receipt of the order. As the demand model therein is a Poisson process (thus without a compound element), the concern about degrading service of larger orders is obviously outside the scope of this paper. Furthermore, we also generalize the demand model by considering a renewal process instead of a Poisson process. As an aside, we note that recently a paper by Kocaga and Sen (2007) has been published which extends the model of Wang et al. (2002) by introducing critical number rules as seen in the rationing literature, featured in Teunter and Haneveld (2008). When considering the rule $Split(q)$, our work has some relation to studies of order splitting and multiple sourcing. For a review of these studies, see Thomas and Tyworth (2006). However, as we exclusively use $Split(q)$ in comparisons with another rule, $Postpone(q, t)$, which cannot be cast into the framework of the splitting/sourcing literature, our work cannot be compared to results obtained in this field.

In Section 2.2, we derive mathematical expressions for the order fill rate of the regular orders and the average on-hand inventory for the rules $Split(q)$ and $Postpone(q, t)$. For each rule, this is done in two stages: first for a general compound renewal process and then in the case of a compound Erlang process, making the mathematical expressions more computable. We finish this section by deriving a reasonable threshold value of t , which we find will make customers of larger orders indifferent between the two rules, and by deriving the threshold value of the split cost. Then in Section 2.3, we present the results of a numerical study. Finally, we state some concluding remarks in Section 2.4.

§2.2 Mathematical model

§2.2.0 Preliminaries

The demand process is a compound renewal process where the time between order arrivals is specified by a positive continuous random variable T . The size of a customer order is specified by a positive integer valued random variable X . As stated in the previous section, we assume a given positive integer q that distinguishes between regular and large orders. We imagine that q is reasonably large, for instance the 90% or the 95% quantile of X , as seen later in our numerical experiments.

Let the random variable X^{Reg} denote the size of a regular order. It has probability distribution

$$P(X^{Reg} = x) = \frac{P(X = x)}{P(X \leq q)} \quad x = 1, 2, \dots, q \quad (2.1)$$

For any non-negative integer m , define the random variable $Q(m)$ as

$$Q(m) = \sum_{i=1}^m X_i \quad (2.2)$$

where $X_i, i = 1, \dots, m$ are independent and identically distributed as X . Similarly, for any non-negative integer m , define the random variable $Q^{Reg}(m)$ as

$$Q^{Reg}(m) = \sum_{i=1}^m X_i^{Reg} \quad (2.3)$$

where $X_i^{Reg}, i = 1, \dots, m$ are independent and identically distributed as X^{Reg} . Per definition, $P(Q(0) = 0) = 1$ and $P(Q^{Reg}(0) = 0) = 1$.

For later use, we state

Lemma 2.1. *When $x \leq q$, it holds that $P(Q(m) = x) = [P(X \leq q)]^m P(Q^{Reg}(m) = x)$.*

Proof: See Appendix.

Let τ be an arbitrarily chosen time point which in our paper can either be a time point of an arrival of a customer with a regular order or a randomly chosen time point. For any non-negative real number s , the random variable $N(\tau)_s$ is the number of customer arrivals in the time interval $[\tau - s, \tau)$.

§2.2.1 Rule *Split*(q)

We first develop an expression for the order fill rate. Let $\bar{\tau}$ be the time point of an arrival of a customer with a regular order. Let the random variable \bar{D}_L denote the aggregate demand recorded in the inventory system in the time interval $[\bar{\tau} - L, \bar{\tau})$. As all larger orders are truncated by q (and the remaining order of any larger order is handled outside the inventory system), the probability distribution of \bar{D}_L can be specified as follows.

When $x = 0$ or $q = 1$

$$\mathrm{P}(\bar{D}_L = x) = \mathrm{P}(\bar{N}(\bar{\tau})_L = x) \quad (2.4a)$$

and when $x > 0$ and $q > 1$

$$\begin{aligned} \mathrm{P}(\bar{D}_L = x) &= \sum_{m=\lceil \frac{x}{q} \rceil}^x \mathrm{P}(N(\bar{\tau})_L = m) \sum_{y=\lceil \frac{qm-x}{q-1} \rceil}^m \binom{m}{y} \\ &\quad [\mathrm{P}(X \leq q)]^y [\mathrm{P}(X > q)]^{m-y} \mathrm{P}(Q^{\mathrm{Reg}}(y) = x - (m - y)q) \end{aligned} \quad (2.4b)$$

Roof(a), $\lceil a \rceil$ is the smallest integer greater than or equal to the real number a and $\binom{m}{y}$ is the binomial coefficient. The order fill rate service measure, abbreviated *OFR*, measuring the fraction of regular orders where the whole order is delivered instantaneously from the inventory, is then

$$\mathrm{OFR}_{\mathrm{split}(q)}(S) = \mathrm{P}(X^{\mathrm{Reg}} + \bar{D}_L \leq S) \quad (2.5)$$

We now develop expressions for the average on-hand inventory level. Let $\tilde{\tau}$ be

a randomly chosen time point. Let the random variable \tilde{D}_L denote the aggregate demand recorded in the inventory system in the time interval $[\bar{\tau} - L, \bar{\tau})$. It then follows that the probability distribution of \tilde{D}_L can be specified similarly as (2.4a) and (2.4b) when replacing $N(\bar{\tau})_L$ with $N(\tilde{\tau})_L$. The average on-hand inventory level is

$$I_{split(q)}(S) = \sum_{x=0}^{S-1} \mathbb{P}(\tilde{D}_L = x)(S - x) \quad (2.6)$$

For the case where T is a k -phased Erlang distribution with mean k/λ (that is, λ is the intensity of the underlying Poisson process), we have (see for instance Cox (1962), p. 39).

$$\mathbb{P}(N(\bar{\tau})_L = n) = e^{-\lambda L} \sum_{j=nk}^{(n+1)k-1} \frac{(\lambda L)^j}{j!} \quad n = 0, 1, 2, \dots \quad (2.7)$$

and

$$\mathbb{P}(N(\tilde{\tau})_L = n) = \begin{cases} e^{-\lambda L} \sum_{j=0}^{k-1} \frac{k-j}{k} \frac{(\lambda L)^j}{j!} & n = 0 \\ e^{-\lambda L} \sum_{j=1-k}^{k-1} \frac{k-|j|}{k} \frac{(\lambda L)^{j+nk}}{(j+nk)!} & n = 1, 2, \dots \end{cases} \quad (2.8)$$

One can thus get computable expressions in order to do numerical analysis of $S_{split}(q)$ for the case of a compound Erlang process.

§2.2.2 Rule *Postpone*(q, t)

First, we derive a measure for the order fill rate. Let $\bar{\tau}$ be the arrival point of a customer with a regular order. Let the random variable $\bar{C}_L(t)$ denote what we define to be the committed aggregate demand in the time interval $[\bar{\tau} - L, \bar{\tau})$. This is all

the recorded demand in the interval $[\bar{\tau} - L, \bar{\tau} - t)$ and all the recorded demand of the regular orders in the time interval $[\bar{\tau} - L, \bar{\tau})$. The reason why only demand of the regular orders is counted in the latter interval is that all larger orders received in this interval are still denied access to the inventory at time point $\bar{\tau}$. The probability distribution of $\bar{C}_L(t)$ can be specified as follows.

$$P(\bar{C}_L(t) = 0) = P(N(\bar{\tau})_L = 0) + \sum_{m=1}^{\infty} [P(X > q)]^m P(N(\bar{\tau})_L = m, N(\bar{\tau})_t = m) \quad (2.9a)$$

When $x > 0$

$$P(\bar{C}_L(t) = x) = \sum_{m=1}^{\infty} \sum_{r=\max\{m-x, 0\}}^m P(N(\bar{\tau})_L = m, N(\bar{\tau})_t = r) \cdot \sum_{y=0}^{\min\{x-m+r, r\}} \binom{r}{y} [P(X \leq q)]^y [P(X > q)]^{r-y} P(Q(m-r) + Q^{Reg}(y) = x) \quad (2.9b)$$

The two random variables $N(\bar{\tau})_L$ and $N(\bar{\tau})_t$ are positively correlated and $P(N(\bar{\tau})_L \geq N(\bar{\tau})_t) = 1$. The order fill rate service measure, measuring the fraction of regular orders where the whole order is delivered instantaneously from the inventory, is then

$$OFR_{postpone(q,t)}(S) = P(X^{Reg} + \bar{C}_L(t) \leq S) \quad (2.10)$$

We now derive a measure for the average on-hand inventory level. Let $\tilde{\tau}$ be a randomly chosen time point. Let the random variable $\tilde{C}_L(t)$ denote the committed aggregate demand in the time interval $[\tilde{\tau} - L, \tilde{\tau})$. As previously defined, this is all the recorded demand in the interval $[\tilde{\tau} - L, \tilde{\tau} - t)$ and all the recorded demand of the regular orders in the time interval $[\tilde{\tau} - t, \tilde{\tau})$. The probability distribution of $\tilde{C}_L(t)$ can be specified as in (2.9a) and (2.9b) when the random variables $N(\bar{\tau})_L$ and $N(\bar{\tau})_t$ are replaced by $N(\tilde{\tau})_L$ and $N(\tilde{\tau})_t$. Therefore the average on-hand inventory level is

specified as

$$I_{\text{postpone}(q,t)}(S) = \sum_{x=0}^{S-1} \mathbf{P}(\widetilde{C}_L(t) = x) (S - x) \quad (2.11)$$

A major problem in computing the expressions derived so far for $\text{Postpone}(q, t)$ is the presence of the joint probabilities $\mathbf{P}(N(\bar{\tau})_L = m, N(\bar{\tau})_t = r)$ and $\mathbf{P}(N(\bar{\tau})_L = m, N(\bar{\tau})_t = r)$. We now consider the case where T is a k -phased Erlang distribution with mean k/λ . Note that a k -phased Erlang distribution can be subdivided into k phases, each having a duration that is exponentially distributed with mean $1/\lambda$. Each time a phase completes, we can interpret it as an “arrival”, where it is only every k th arrival that is real while the others are fictitious. Thus, the process will at any time point be in one of the phases $1, 2, \dots, k$. Being in phase i means that $k - i$ fictitious arrivals have elapsed since the last real arrival, see Figure 2.1 (note that as we look backward in time when doing mathematical derivations, we also state the phase numbers accordingly).

Figure 2.1 is about here

When we use the word arrival in the following, it should be understood as a real arrival. Let τ be an arbitrarily chosen time point. Denote by $F(i, r, t|j)$ the conditional probability that given that we are in phase j at time point τ , we are in phase i ($i = 1, 2, \dots, k$) at time point $\tau - t$ and the total number of arrivals in the time interval $[\tau - t, \tau)$ is r . Then

$$F(i, r, t|j) = e^{-\lambda t} \frac{(\lambda t)^{rk+i-j}}{(rk+i-j)!} \quad (2.12)$$

$$i = 1, \dots, k; j = 1, \dots, k; r = I_{\{j>i\}}, \dots, \infty$$

where the function $I_{\{A\}}$ is 1 if condition A is true and 0 otherwise. For an explanation of the number $rk + i - j$ in (2.12), see Figure 2.2.

Figure 2.2 is about here

Let the random variable $\widehat{N}(i)_{L-t}$ denote the total number of arrivals in the time interval $[\tau - L, \tau - t)$ given that we are in phase i at time point $\tau - t$. This has probability distribution

$$\mathbf{P}(\widehat{N}(i)_{L-t} = u) = e^{-\lambda(L-t)} \sum_{v=\max\{uk+1-i,0\}}^{(u+1)k-i} \frac{[\lambda(L-t)]^v}{v!} \quad (2.13)$$

Let the random variable $C_L(t|r, u)$ denote the aggregate committed demand (see the previous definition) in the time interval $[\tau - L, \tau)$ given that one has observed r arrivals in the time interval $[\tau - t, \tau)$ and u arrivals in the time interval $[\tau - L, \tau - t)$. This has probability distribution

$$\mathbf{P}(C_L(t|r, u) = x) = \sum_{y=0}^{\min\{r, x-u\}} \binom{r}{y} [\mathbf{P}(X \leq q)]^y [\mathbf{P}(X > q)]^{r-y} \mathbf{P}(Q^{Reg}(y) + Q(u) = x) \quad (2.14)$$

Using τ as an arrival point of a customer with a regular order, we get the probability distribution $\overline{C}_L(t)$ can be specified as follows

$$\mathbf{P}(\overline{C}_L(t) = x) = \sum_{i=1}^k \sum_{u=0}^x \mathbf{P}(\widehat{N}(i)_{L-t} = u) \sum_{r=0}^{\infty} \mathbf{F}(i, r, t|1) \mathbf{P}(C_L(t|r, u) = x) \quad (2.15)$$

Using τ as a randomly chosen time point, we will at this time point be in phase is j with probability $1/k$. Therefore, the probability distribution of $\widetilde{C}_L(t)$ can be

specified as follows

$$P(\tilde{C}_L(t) = x) = \frac{1}{k} \sum_{j=1}^k \sum_{i=1}^k \sum_{u=0}^x P(\widehat{N}(i)_{L-t} = u) \sum_{r=0}^{\infty} F(i, r, t|j) P(C_L(t|r, u) = x) \quad (2.16)$$

This accomplishes that we can get computable expressions for doing numerical analysis of $Postpone(q, t)$ in the case of a compound Erlang process.

§2.2.3 Implementation

Some of the expressions presented so far can be further simplified when $k = 1$, that is the demand process is a compound Poisson process, by using the recursion scheme of [Adelson \(1966\)](#). Actually, we have developed computer codes, programmed in Visual Basic for Excel, both for the general Erlangean case (k any positive integer) and the case $k = 1$, in order to validate our computer codes as well as possible. This is done for both rules $Split(q)$ and $Postpone(q, t)$.

§2.2.4 The case $S \leq q$

For this case in particular, the rule $Split(q)$ is superfluous as not even the truncated suborder of a larger order has any chance of receiving full service. Note that $Postone(q, 0)$ represents the case where larger orders are not discriminated. We can prove

Proposition 2.2. *When $S \leq q$, the performance measures (OFR and average on-hand inventory) are identical for the rules $Postpone(q, 0)$ and $Split(q)$.*

Proof: *See Appendix.*

§2.2.5 Choice of t

We derive what we find is a reasonable threshold value of t , making customers of larger orders indifferent between $Postpone(q, t)$ and $Split(q)$. Consider a large

order whose size can be specified by the random variable X^{Lar} which has probability distribution

$$P(X^{Lar} = x) = \frac{P(X = x)}{P(X > q)} \quad x = q + 1, q + 2, \dots \quad (2.17)$$

In $Postpone(q, t)$ a measure for accumulated waiting time of all units of a larger order is $E[X^{Lar}] t$. This measure is slightly optimistic as we here assume that any larger order is instantaneously served after t time units of postponement. Similarly, in $Split(q)$ a measure for the accumulated waiting time can be specified as $E[\max\{X^{Lar} - q, 0\}] L$. Also this measure is slightly optimistic as we here assume that the truncated suborder (of size q) is served instantaneously. We assume that both measures are equally optimistic, the value of t that equalizes these two terms is

$$t = \frac{L \sum_{j=q+1}^{\infty} (j - q) P(X = j)}{\sum_{j=q+1}^{\infty} j P(X = j)} \quad (2.18)$$

The right hand side of (2.18) belongs to the interval between 0 and L . We find that the right hand side of (2.18) is a good estimate of a t value that makes customers of larger orders indifferent between $Postpone(q, t)$ and $Split(q)$. Developing exact expressions would be very difficult.

§2.2.6 Specification of a threshold split cost

Irrespective of which rule is applied, we assume that a holding cost rate h is charged on the average on-hand inventory. Furthermore, in the cost evaluation of $Split(q)$ we assume that a cost c_{split} is incurred each time a larger order is split. As the expected time between successive occurrences of larger orders is $E[T]/P(X >$

q), the cost evaluation of $Split(q)$ is

$$h I_{split}(S) + c_{split} \frac{P(X > q)}{E[T]} \quad (2.19)$$

while the cost evaluation of $Postpone(q, t)$ is $h I_{postpone(q,t)}(S)$. Assume now that for a given q, t has been settled according to (2.18), such that customers of larger orders are indifferent between the two rules. Assume then that for each rule, the base stock levels $S_{postpone(q,t)}$ and $S_{split(q)}$, respectively, have been established such that customers of regular orders enjoys the same order fill rate and are therefore also indifferent between the two rules. When equalizing the two cost expressions stated above and isolating the cost parameters, we then get a threshold value of the split cost (relative to the inventory holding cost rate)

$$c_{split}/h = \frac{[I_{postpone(q,t)} S_{postpone(q,t)} - I_{split(q)} S_{split(q)}] E[T]}{P(X > q)} \quad (2.20)$$

In principle, this threshold value could be negative. However, as the demand volume faced by the inventory is less under $Split(q)$ than under $Postpone(q, t)$ (due to some demand being filtered away from the inventory and sent directly to the supply system) the result is that generally when the parameters are set such that customers are indifferent between the two rules, the average on-hand inventory in $Postpone(q, t)$ is larger than the average on-hand inventory in $Split(q)$.

§2.3 Numerical results

As indicated in the previous section we have only done implementations of the model when T is k -phased Erlang distributed with mean k/λ . Throughout this section, we keep L fixed at level 4, we also assume that the random variable X is

(delayed) geometrically distributed. This means that it has probability distribution

$$P(X = j) = (1 - \rho)\rho^{j-1} \quad j = 1, 2, \dots \quad (2.21)$$

where the parameter ρ is in the interval between 0 and 1. The use of the word “delayed ” is due to (Zipkin (2000),p. 451). Then (2.18) simplifies to

$$t = \frac{1}{q + 1 - \rho q} L \quad (2.22)$$

and the demand rate d , defined as $d = E[X]/E[T]$, is

$$d = \frac{\lambda}{k(1 - \rho)} \quad (2.23)$$

When $k = 1$, this demand process is called a “stuttering ” Poisson process (See Axsäter (2006), p. 82). In Johnston et al. (2003), some empirical evidence is given for the relevance of such a demand model.

We now make a systematic evaluation of how the two rules compare to each other. We let the demand rate d attain the values 1.25, 2.5, 3.75, 5 (these values perhaps seem more obvious choices when multiplying by L as $dL = 5, 10, 15, 20$). For each choice of d , we then let $\rho = 0.5, 0.6, 0.7, 0.8, 0.9$. In order also to investigate the impact of deviating from the common Poisson process assumption, we let $k = 1$ and 2. The parameter λ is then given from (2.23). We let q be the α quantile (the least integer value x making $P(X \leq x) \geq \alpha$) of X where α is either 0.90 or 0.95. We denote the required order fill rate to the regular orders β and let it be either 0.9 or 0.95. Combining all these parameter values $(d, \rho, k, \alpha, \beta)$ gives a total of 160 data sets. Our numerical procedure can now be outlined as follows.

1. Given input $(d, \rho, k, \alpha, \beta)$.

2. Compute q as the least integer value x making $P(X \leq x) \geq \alpha$ and compute t by (2.22).
3. Compute $S_{postpone(q,t)}$ as the least integer value of S bringing $OFR_{postpone(q,t)}(S) \geq \beta$ and compute $S_{split(q)}$ as the least integer value of S bringing $OFR_{split(q)}(S) \geq \beta$. Evaluate the average on-hand inventories $I_{postpone(q,t)}(S)$ and $I_{split(q)}(S)$ by setting $S = S_{postpone(q,t)}$ and $S = S_{split(q)}$, respectively.
4. Evaluate the threshold split cost relative to the inventory cost by

$$c_{split}/h = \frac{[I_{postpone(q,t)}(S_{postpone(q,t)}) - I_{split(q)}(S_{split(q)})]k}{\lambda P(X > q)}$$

Here we follow closely the outline given in the introduction. We refrain from making a formal statistical analysis of variance (ANOVA) on the data material on the threshold split costs, as this will probably lead us nowhere as all interaction effects are statistically significant. Instead we inspect the data material more informally. It turns out that c_{split}/h very much depends on the parameter ρ and it is generally increasing in ρ . A typical illustration of this is given in Table 2.1. Though there are also a few exceptions, as illustrated in Table 2.2. The reason is that we are forced to choose the base stock values as integers. For instance, the reason why the threshold value c_{split}/h is about 50 when $\rho = 0.5$ in Table 2.2 is that the base stock value securing an order fill rate above 0.9 is actually well above, at level 0.93. As we will demonstrate later, this phenomenon is most likely to occur when the demand rate d is low, that is with low-frequent demand. When d is increased, the selected values of S will in general secure that the actual order fill rate is close to the required one. Because the pattern seen in Table 2.1 is the most typical for our data, which can also be verified from Table 2.3, we find it natural to group our observations of the c_{split}/h values on the values of the ρ parameter. In the following Tables 2.4(a)-2.7(b) and Figure 2.3 - 2.6, we explore how the other input factors influence the relationship between c_{split}/h and ρ . From Tables 2.4(a)-2.4(b) and Figure 2.3 we see that k has some significance. When having an Erlangean arrival

process ($k = 2$), the threshold value is in general larger, making $Split(q)$ more attractive for threshold values falling in between the threshold values of Table 2.4(a) and Table 2.4(b). Thus, the assumption about the demand process has significance for the conclusion. It emphasizes the importance of doing a careful analysis of the data of a demand process in order to specify the appropriate demand model, and not just automatically choosing a compound Poisson process because it is most convenient. From Tables 2.6(a)-2.6(b) and Figure 2.5, it also seems that the threshold values c_{split}/h are larger when having a high required order fill rate (0.95) on the regular orders than when the required order fill rate is only 0.90. One explanation may be that as more attention to regular orders is required (by increasing the order fill rate), it is of more advantage to direct some demand away from the inventory system by using the rule $Split(q)$. It does not seem as if the requirement on α is of great significance, see Tables 2.7(a)-2.7(b) and Figure 2.6. Finally, when examining Tables 2.5(a)-2.5(d) and Figure 2.4, we see, as stated above, that as d is increased, it appears more clearly that c_{split}/h is increasing in ρ . Our results also give some indication of what might happen if customers begin to react on our rules for degraded service to larger orders, for instance if customers of larger orders began subdividing their orders into several smaller ones. As an example, examine Tables 2.1 from bottom ($\rho = 0.9$) to top ($\rho = 0.5$). When following this trace, we initially have a high variation on the order sizes ($\rho = 0.9$) and as the c_{split}/h is high, it suggests that the $Split(q)$ rule might be a good option. As ρ is decreased, the variation decreases and thereby the definition of larger order is also redefined by decreasing q , and t is changed as well. In this process it turns out that c_{split}/h decreases considerably, now making $Postpone(q, t)$ more attractive.

§2.4 Concluding remarks

Using a very general demand model, we have developed mathematical expressions for the average on-hand inventory level and order fill rate service measure of the regular orders in a base stock inventory system when degraded service is en-

forced for customers of larger orders. The models (and corresponding computer codes) may be valuable tools for an organization to explore the impact of various strategies for degrading the service of larger orders; particularly in recognition of the fact that these orders may have a very disruptive effect upstream in the supply chain. We have refrained from doing a cost optimization with respect to service requirements. This is partly due to the fact that this is a quite complex operation as we have two service level constraints to consider, and partly due to the fact that we believe that a cost parameter such as the cost of splitting a larger order may be hard to quantify. Therefore, our approach is to specify the parameters t , $S_{postpone(q,t)}$ and $S_{split(q)}$ such that all customers (irrespective whether they issue a regular or a larger order) will consider the two rules to be equally good. Additionally, we explore the threshold value of the split cost, thereby establishing when each rule seems to perform best. Our numerical investigations reveal that the choice between $S_{split(q)}$ and $Postpone(q,t)$ seems to depend on the variance of the order sizes. The lower it is, the better option is the rule $Postpone(q,t)$. It also seems that deviating from the common Poisson process assumption as well as the choice of order fill rate offered to the regular orders does have some impact on this conclusion. Though we examine a fairly large number of data sets, we look exclusively at the geometric distribution to describe order sizes. So obviously, further numerical studies could be devoted to examining other distributions, for instance the negative binomial, the Poisson or the logarithmic distribution. As noted in the introduction, the rule $Postpone(q,t)$ has resemblance to an advance demand information system. Therefore, the mathematical model concerning this rule can easily be adapted to a more general study about aspects of advance demand information where it need not be solely the customers of larger orders that provide this information.

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§2.5 Appendix

Proof. **Proof of Lemma 2.1**

We prove by induction in m . When $m = 1$, the result is given from (2.1). Consider now an integer $m' \geq 1$ and assume that we have shown the result for $m = m' - 1$. For any $x = 0, 1, \dots, q$, it holds that

$$\begin{aligned} \mathbb{P}(Q(m') = x) &= \sum_{y=0}^x \mathbb{P}(Q(m' - 1) = y) \mathbb{P}(X = x - y) \\ &= [\mathbb{P}(X \leq q)]^{m'} \sum_{y=0}^x \mathbb{P}(Q^{Reg}(m' - 1) = y) \mathbb{P}(X^{Reg} = x - y) \\ &= [\mathbb{P}(X \leq q)]^{m'} \mathbb{P}(Q^{Reg}(m') = x) \end{aligned}$$

□

Proof. **Proof of Proposition 2.2**

We claim that for any $x = 0, 1, \dots, q-1$, it holds that $\mathbb{P}(\overline{D}_L = x) = \mathbb{P}(\overline{C}_L(0) = x)$. First, we consider the random variable $\overline{C}_L(0)$. When $t = 0$, the random variable $\overline{N}(\overline{\tau})_t$ is identical to zero. This means that the summation on the right hand side of (2.9a) vanishes to zero. Also, the right hand side of (2.9b) only gives non-zero terms if index $r = 0$. This again forces index m to be less than or equal to x and index y to be zero. Therefore, we get

$$\mathbb{P}(\overline{C}_L(0) = x) = \begin{cases} \mathbb{P}(N(\overline{\tau})_L = 0) & x = 0 \\ \sum_{m=1}^x \mathbb{P}(N(\overline{\tau})_L = m) \mathbb{P}(Q(m) = x) & x = 1, 2, \dots \end{cases} \quad (2.24)$$

Now we consider the random variable \overline{D}_L . If $q = 1$, our claim follows by

comparing (2.24) to (2.4a). Assume $q > 1$. When $x \leq q - 1$, the inequality $(qm - x)/(q - 1) > m - 1$ is equivalent to the inequality $q + m > x + 1$, which is true when $q > x$, implying $\lceil (qm - x)/(q - 1) \rceil = m$ (note $(qm - x)/(q - 1)$ is always less than or equal to m). When $q > x$, it also holds that $\lceil x/q \rceil = 1$. Therefore, we get from (2.4a) and (2.4b) that

$$\mathbb{P}(\bar{D}_L = x) = \begin{cases} \mathbb{P}(N(\bar{\tau})_L = 0) & x = 0 \\ \sum_{m=1}^x \mathbb{P}(N(\bar{\tau})_L = m) [\mathbb{P}(X \leq q)]^m \mathbb{P}(Q^{Reg}(m) = x) & x = 1, 2, \dots, q - 1 \end{cases} \quad (2.25)$$

Lemma 2.1 now verifies our claim. Similarly, we can prove that for any $x = 0, 1, \dots, q - 1$, it holds that $\mathbb{P}(\bar{D}_L = x) = \mathbb{P}(\bar{C}_L(0) = x)$. As it is the probabilities $\mathbb{P}(\bar{D}_L = x) = \mathbb{P}(\bar{C}_L(0) = x)$ for $x = 0, 1, \dots, S - 1$ that specify the *OFR* expressions in (2.5) and (2.10) and it is the probabilities $\mathbb{P}(\bar{D}_L = x) = \mathbb{P}(\bar{C}_L(0) = x)$ for $x = 0, 1, \dots, S - 1$ that specify the average on-hand inventory expressions in (2.6) and (2.11), the result follows from $S \leq q - 1$. \square

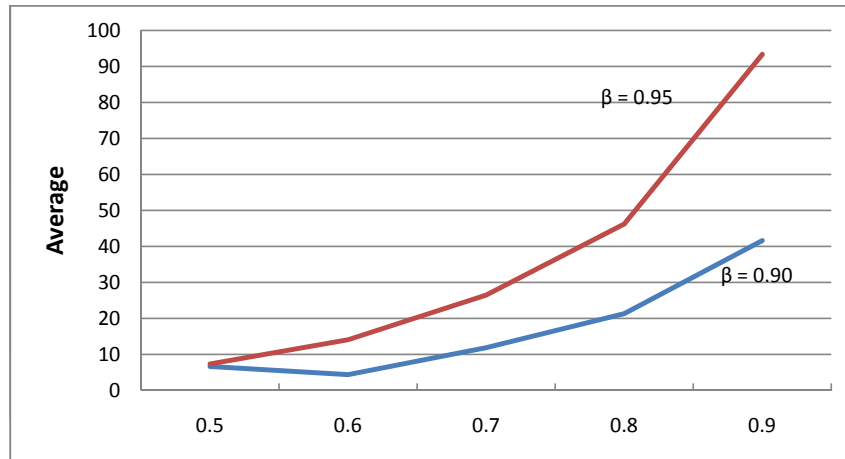


Figure 2.4: Average threshold split costs from Tables 2.5(a)-2.5(d) depicted as a function of ρ .

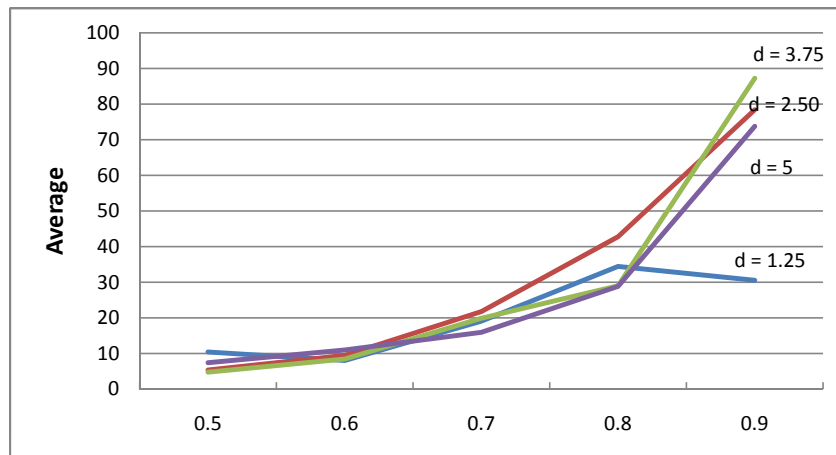


Figure 2.5: Average threshold split costs from Tables 2.6(a)-2.6(b) depicted as a function of ρ .

Table 2.1: Numerical results when $d = 1.25$, $k = 1$, $\alpha = 0.90$, $\beta = 0.95$. For simplicity, various subscripts are suppressed.

Input			<i>Postpone</i> (q, t)				<i>Split</i>			c_{split}/h
ρ	λ	q	t	S	I	<i>OFR</i>	S	I	<i>OFR</i>	
0.5	0.625	4	1.3333	13	8.3854	0.9512	13	8.3547	0.9605	0.7853
0.6	0.5	5	1.3333	15	10.4621	0.9586	14	9.4416	0.9555	26.2484
0.7	0.375	7	1.2903	17	12.5178	0.9522	16	11.4787	0.9509	33.6472
0.8	0.25	11	1.25	22	17.5582	0.9565	21	16.4942	0.9583	49.5463
0.9	0.125	12	1.25	32	27.7277	0.9516	31	26.5900	0.9508	92.4196

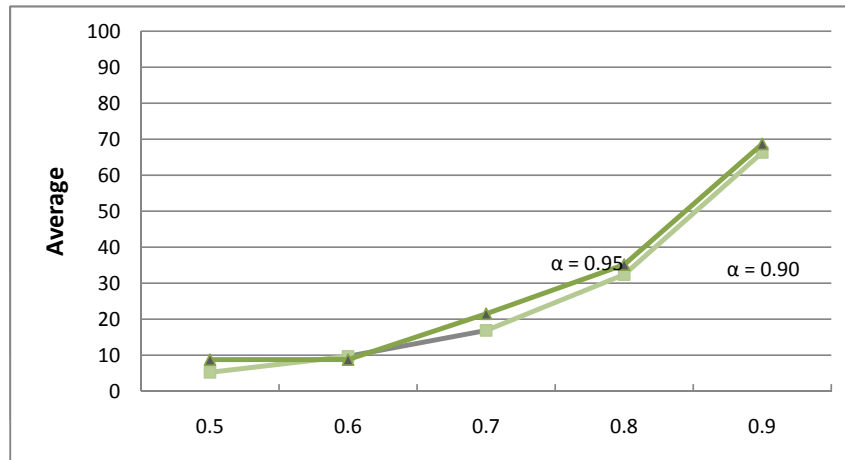


Figure 2.6: Average threshold split costs from Tables 2.7(a)-2.7(b) depicted as a function of ρ .

Table 2.2: Numerical results when $d = 1.25$, $k = 2$, $\alpha = 0.95$, $\beta = 0.90$. For simplicity, various subscripts are suppressed.

Input			<i>Postpone</i> (q, t)				<i>Split</i>			c_{split}/h
ρ	λ	q	t	S	I	<i>OF</i> R	S	I	<i>OF</i> R	
0.5	1.25	5	1.1429	11	6.2502	0.9328	10	5.2646	0.9078	50.4633
0.6	1	6	1.1765	11	6.4043	0.9021	11	6.3434	0.9134	2.6133
0.7	0.75	9	1.0811	13	8.4011	0.9063	13	8.3321	0.9142	4.5589
0.8	0.5	14	1.0526	16	11.4846	0.9070	16	11.3793	0.9101	9.5768
0.9	0.25	29	1.0256	24	19.6469	0.9024	24	19.5151	0.9012	22.3957

Table 2.3: All data on threshold split costs grouped on their ρ value. In all 32 observations in each group.

ρ	Average	St.deviation
0.5	6.9936	10.7894
0.6	9.2335	7.7857
0.7	19.1750	14.7692
0.8	33.7684	23.3968
0.9	67.4894	43.9498

Table 2.4: All data on threshold split costs with $k = 1, 2$ grouped on their ρ value. In all 16 observations in each group

(a) $k = 1$			(b) $k = 2$		
ρ	Average	St.deviation	ρ	Average	St.deviation
0.5	4.2966	7.0621	0.5	9.6906	13.2361
0.6	9.0079	7.6432	0.6	9.4591	8.1699
0.7	14.3175	11.4941	0.7	24.0326	16.3814
0.8	24.9927	17.4121	0.8	42.5441	25.7651
0.9	55.3669	35.1893	0.9	79.6119	49.3979

Table 2.5: All data on threshold split costs with $d = 1.25, 2.50, 3.75, 5$ grouped on their ρ value. In all 8 observations in each group

(a) $d = 1.25$			(b) $d = 2.50$		
ρ	Average	St.deviation	ρ	Average	St.deviation
0.5	10.4247	18.3465	0.5	5.3679	9.0228
0.6	8.03289	11.3072	0.6	9.466754	9.2272
0.7	19.0796	23.9360	0.7	21.7746	13.8800
0.8	34.4377	40.5262	0.8	42.7813	15.6782
0.9	30.5445	25.2403	0.9	78.4080	35.9589

(c) $d = 3.75$			(d) $d = 5$		
ρ	Average	St.deviation	ρ	Average	St.deviation
0.5	4.7663	6.1712	0.5	7.4155	6.0912
0.6	8.452221	5.1020	0.6	10.9823	4.8496
0.7	19.90063	12.4137	0.7	15.9453	5.1054
0.8	29.0074	16.9490	0.8	28.8473	10.0932
0.9	87.2418	54.5942	0.9	73.7633	38.1638

Table 2.6: All data on threshold split costs with $\beta = 0.90, 0.95$ grouped on their ρ value. In all 16 observations in each group

(a) $\beta = 0.90$			(b) $\beta = 0.95$		
ρ	Average	St.deviation	ρ	Average	St.deviation
0.5	6.6371	13.3852	0.5	7.3501	7.8199
0.6	4.385126	3.8374	0.6	14.0819	7.7718
0.7	11.8715	9.5540	0.7	26.4786	15.6757
0.8	21.3147	16.0571	0.8	46.2222	23.2941
0.9	41.6173	23.6862	0.9	93.3615	44.7540

Table 2.7: All data on threshold split costs with $\alpha = 0.90, 0.95$ grouped on their ρ value. In all 16 observations in each group

(a) $\alpha = 0.90$			(b) $\alpha = 0.95$		
ρ	Average	St.deviation	ρ	Average	St.deviation
0.5	5.2279	7.1466	0.5	8.7593	13.5226
0.6	9.6910	7.6634	0.6	8.7761	8.1302
0.7	16.8580	10.7187	0.7	21.4921	18.0127
0.8	32.3841	23.1956	0.8	35.1528	24.2733
0.9	66.2996	39.8561	0.9	68.6792	48.9940

3

Investigating Reservation Policies of Advance Orders in the presence of Heterogeneous Demand

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Investigating Reservation Policies of Advance Orders in the presence of Heterogeneous Demand

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Abstract

We consider an inventory system where customers provide advance order information. Specifically each customer order has two attributes: a request date when the order is received and a due date specifying when the customer wants his order delivered. The inventory system is operated as a base stock system where replenishment orders are issued upon the receipt of an order. We assume the demand process is a Poisson process. As the demand lead-times, that is, the difference between the due date and the request date, can vary stochastically, the sequence of requests dates and due dates will differ. Therefore a potentially important issue is then how early should orders be reserved at the inventory. We explore three logical reservation policies. These policies also facilitates that the average on-hand inventory can be specified in a much simpler way than done in [Marklund \(2006\)](#). Assuming that the revenue of an order (assumed to be of unit size) depends on the demand lead-time we propose a profit optimization model, where the expected profit is the difference of the expected revenue and the expected inventory holding costs. We derive the profit function for each of the three proposed reservation policies. Our numerical results indicate that unless the revenue of an order with a large demand lead-time is very large compared to those with a smaller delivery lead-time, the items should only be reserved in a short time interval before the delivery and most often not at all.

Keyword

Inventory control; Advance demand information; Heterogeneous demand

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§3.1 Introduction

We consider an inventory system where customers do not physically appear at the inventory system. Instead they submit their order requests; for instance by e-mail or via a web-page organized by the inventory system. They do not expect instantaneously delivery but instead when submitting their order requests they also specify a future time point when they want to receive their order. Such a situation could be realistic in a business to business (B2B) context. The inventory system could be placed somewhere upstream in a supply chain, say a central warehouse serving different independent retailers and retail chains as well. It could very well be that there are different levels of communication exchange between these retail chains and the central warehouse. In some cases there is a very close cooperation such that the retailer (or retail chain) well in advance, for instance by giving the central warehouse access to their information systems, specify future order requests, in combination with that orders are freezed when coming into a certain time window. With others there is less information exchange and therefore orders are received with a shorter notice. The situation could also occur in a business to consumer (B2C) setting, when sales are organized through the internet, as witnessed by well known business cases like Amazon and e-Bay.

Advance order information, or advance demand information as it is most often denoted in the literature, means that if a request for an order occurs, say at time point τ , the customer also specifies a time point, say time point $y + \tau$, where he wants to receive his order. Thus, y can be interpreted as the **demand lead-time** of the customer. In many textbooks on inventory control, like [Silver et al. \(1998\)](#) and [Zipkin \(2000\)](#) it is tacitly assumed that $y = 0$ (though the issue is raised in Exercise 7.8 of [Silver et al. \(1998\)](#) and Exercise 6.1 of [Zipkin \(2000\)](#)). The first paper to make a systematic analysis of advance demand information in inventory control systems is [Hariharan and Zipkin \(1995\)](#). Here it is assumed that all customers have the same demand lead-time and the improvement in performance is equivalent to an equal reduction in the replenishment lead-time. A similar study is done in [Gallego and Özer](#)

(2001) assuming a periodic review model where customers can have different demand lead-times. Here it is proven that a state dependent (s, S) policy is optimal. Lu et al. (2003) and Marklund (2006) extends the model of Hariharan and Zipkin (1995) by studying multi-level systems; the first studies an assembly system while the second studies a divergent distribution system. In particular, Marklund (2006) is our main source of inspiration for this study, because he introduces the idea of a reservation policy to be applied at the upper level echelon. However, he does not specify any specific reservation policy, and are most often studying the two extreme ones: the complete reservation policy and the no reservation policy. The papers above only consider the inventory control side. However, if customers use different demand lead-times probably also the revenues of the orders will depend on the demand lead-time. Except for a work by Chen (2001) studying advance demand information and market segmentation the revenue side has not been linked to the inventory side when studying inventory control in the presence of advance order information. Our aim is to make a more detailed analysis of the some of the ideas about reservation indicated in Marklund (2006) and link it to a revenue model acknowledging that the revenue varies with the demand lead-time, in order to develop a profit maximization model in order to find the optimal base stock level and the optimal reservation policy. For completeness of this literature review, for newer contributions on advance demand information see also Tan (2008), Wang and Toktay (2008), Gayon et al. (2009) and Papier and Thonemann (2010). In finalizing this literature review we should also mention that there is another line of research studying which sort of inventory control policies should be applied in presence of heterogeneous demand. When examining the literature (see Topkis (1968), Evans (1968), Kaplan (1969), Frank et al. (2003) and Tempelmeier (2006) for periodic review models and Nahmias and Demmy (1981), Dekker et al. (1998), Dekker et al. (2002), Melchioris et al. (2000), Melchioris (2003) and Deshpande et al. (2003) for continuous review models) all assume that it is possible to differentiate among the customer classes by introducing a rationing policy. A rationing policy means that low prioritized customers are denied access to the inventory when the on hand level

is critically low. The reservation policies we consider in this paper do have some resemblance to rationing. But the implementation is not based on critical numbers on the on-hand inventory but rather a dynamic updating on the priorities of the order by considering the time until the due-date.

We study the setting outlined in the start of this section. Furthermore we also assume that seen from the perspective of the inventory system, the demand lead-times of the customers are random. We will assume the inventory system is controlled by a base-stock system (thus assuming negligible fixed replenishment costs), where all replenishment are issued instantaneously upon the receipt of an order. The aim is to find a good base-stock policy in combination with setting the parameter of the chosen reservation policy such that the expected profit is maximized. The key element in our paper is to study reservation policies. One option is to let the order be immediately reserved (or **tagged**, as we sometimes call it in the following) from the inventory. It means that it cannot be accessed by other customers, whose requests arrive later. This is denoted **complete reservation** by [Marklund \(2006\)](#). Another option is first to reserve it after a certain time delay. We then denote it a partial reservation policy. This may be sensible if the demand lead time of the customer is long so keeping the amount free on inventory helps giving adequate service also to customers with a short demand lead-time. We study three ways of implementing a partial reservation policy, denoted Forward Delay (*FD*), Backward Delay (*BD*) and Proportional Delay (*PD*), respectively. Roughly speaking, the difference between *FD* and *BD*, is that under the first policy the order is first tagged at the inventory r time units (where r is a policy variable) **after** the request has been registered while in the second the order is first tagged at the inventory d time units (where d is a policy variable) **before** it is needed. One could also interpret it as in the first the request date is the anchor point while in the second the due date is the anchor point when deciding when to reserve the order. Finally, in the *PD* reservation policy the time until the tagging takes place depends on the actual demand lead time and it is proportional to it.

In Section [3.2](#) we will in more detail specify the mathematical assumption be-

hind the inventory system and specifying the profit function. As this profit function requires knowledge about the order fill rate offered to a customer having any demand lead-time as well as the average on-hand inventory, we show in Sections 3.3 and 3.4, respectively, how these expressions can be derived mathematically. In Section 3.5 we will present the results of our numerical investigations. Finally, Section 3.6 contains some concluding remarks.

§3.2 The mathematical model

We assume the arrival processes of order requests is a Poisson processes with intensity λ arrivals per time unit. The non-negative random variable Y , with density function $f(y)$, represents the demand lead-time of a customer. That is, if an order request occurs at time point τ and the customer has a demand lead-time y , then the customer wants to get his demand fulfilled at time point $\tau + y$. As Y is assumed stochastic it is understood that the customer's preferences for demand lead times is outside the control of the inventory system. In the special case where it is a constant it could represent a scenario that when for instance ordering from the internet only one option is given for choosing a delivery lead time. Most realistically the random variable Y is discrete, the customers can only choose between a selected number of delivery options, or there is only a limited number of different customer classes (within this class all customers have the same demand lead-time) . However, in order to be as general as possible we will assume Y is a continuous random variable.

The inventory is controlled by a base stock policy with parameter S (a non-negative integer). All unfulfilled demand is backlogged. It is assumed that replenishments, which all have a constant lead-time L , are done exactly at the time point when an order request occurs. Throughout the paper we assume that $P(Y < L) = 1$, as any order with a demand lead-time longer than the replenishment lead-time is a "known" order which can be handled outside the inventory system and is therefore not interesting when focusing on inventory control.

We now describe the reservation policies. Any order specified by the double

(τ, y) (where τ is the arrival time and y is the demand lead time) has a reservation delay $g(y)$ where $0 \leq g(y) \leq y$. It means that at time point $\tau + g(y)$ the order will be tagged and any orders having a reservation time after this time point are served after this order. The order gets complete fulfilment at its due date, time point $\tau + y$, if the inventory level at that time point minus the total volume of orders, not yet fulfilled and reserved prior to time point $\tau + g(y)$ is larger than or equal to 1. As the inventory control is a base stock policy with base stock level S and all replenishment orders are issued exactly when the requests occur, all requests received prior to time point $\tau + y - L$ are fulfilled and resupplied at the inventory at time point $\tau + y$ we can rephrase it to: Any order specified by the double (τ, y) (where τ is the arrival time and y is the demand lead time) and having a reservation delay $g(y)$ where $0 \leq g(y) \leq y$ receives complete fulfilment at time point $\tau + y$ if the total volume of orders received in the time $[\tau + y - L, \tau + g(y))$ and reserved prior to time point $\tau + g(y)$ is smaller than or equal to $S - 1$. In order to complete the description the inventory system, if an order cannot get complete fulfilment the whole order is put on a waiting list. When receiving incoming replenishments the waiting list is always attempted to be cleared and any waiting orders are prioritized according to when they were reserved.

We consider three reservation policies, abbreviated *FD*, *BD* and *PD*, respectively. The Forward Delay *FD* reservation policy is specified by a parameter r where $0 \leq r \leq L$. It means the aforementioned g is specified as $g(y) = \min(y, r)$. If y is small, $y \leq r$, it means the order is never reserved. A special case of this reservation policy, with a constant demand lead-time, is studied in a slightly different context in [Du and Larsen \(2011\)](#) where it is denoted “Postpone”. Here it was used to differentiate larger customer orders from smaller ones. The Backward Delay *BD* reservation policy is specified by a parameter d where $0 \leq d \leq L$. Here $g(y) = \max(y - d, 0)$. The logic behind the policy is to prioritize orders in accordance with their actual needs. If y is small, $y \leq d$, it means the order is reserved immediately when receiving the request. Finally the Proportional Delay *PD* reservation policy is characterized by a parameter α where $0 \leq \alpha \leq 1$ and $g(y) = \alpha y$.

If α is chosen in between 0 and 1, an order is always reserved prior to its due date and it is never reserved immediately when receiving the request. In that respect *PD* is different from *FD* and *BD*. This can also be seen as some of the mathematical derivations concerning this policy are done differently than for the two others. Note that when $r = 0$, $d = L$ and $\alpha = 0$ the reservation policies coincide and is a complete reservation policy according to the definition in [Marklund \(2006\)](#). If $r = L$, $d = 0$ and $\alpha = 1$ the reservation policies also coincide and can be characterized as a no reservation case which also have been studied in [Marklund \(2006\)](#).

One could ask is how to make a meaningful comparison between the three reservation policies. Here we compare the expected **tag time** of the policies. We define the tag time as the time interval between when the order is reserved and its need date. For *FD* the expected tag time is $\int_r^L (y - r)f(y)dy$ while for *BD* it is $\int_0^d yf(y)dy + dP(Y \geq d)$. So therefore when comparing the *FD* and the *BD* reservation policies, r and d should be chosen such that

$$\int_r^L (y - r)f(y)dy = \int_0^d yf(y)dy + dP(Y \geq d) \quad (3.1)$$

For *PD* the expected tag time is $(1 - \alpha)E[Y]$. Therefore, when comparing the *FD* and *PD* reservation policies, r and α should be chosen such that

$$\int_r^L (y - r)f(y)dy = (1 - \alpha)E[Y] \quad (3.2)$$

Later in [Section 3.5](#) we return to these expressions.

Now we address how to derive a meaningful performance evaluation of the inventory system. We assume that the inventory system gets different revenues of an order depending on its demand lead-time. Specifically we assume it has a functional form $P(y)$. Furthermore this revenue is only collected if the order is fulfilled in time, otherwise the revenue is $P_0 < \min \{P(y) : 0 \leq y \leq L\}$. Thus in the spirit of

Chen (2001) we assume revenue term P_0 to be independent of the original demand lead-time. In order to avoid too involved mathematical expressions and because it contains no loss of generality we will in the following set $P_0 = 0$. We also assume that items on inventory are incurred an inventory holding cost at rate h per unit. Denote by $I(S, r)$, $I(S, d)$ and $I(S, \alpha)$ the average on-hand inventory when base stock parameter is S (a non-negative integer) and using respectively reservation policy BD , FD or PD . Denote by $OFR(S, r, y)$, $OFR(S, d, y)$ and $OFR(S, \alpha, y)$ the order fill rate, which is the probability that an order will be fulfilled at its specified due date, when the base stock parameter is S and the customer has demand lead-time y and using respectively reservation policy BD , FD or PD . Then the relevant optimization problems under each of the reservation policies are:

Under reservation policy FD , find r and S that maximizes

$$\max \left\{ \lambda \int_0^L OFR(S, r, y) P(y) f(y) dy - hI(S, r) : 0 \leq r \leq L, S = 0, 1, 2, \dots \right\} \quad (3.3)$$

Under reservation policy BD , find d and S that maximizes

$$\max \left\{ \lambda \int_0^L OFR(S, d, y) P(y) f(y) dy - hI(S, d) : 0 \leq d \leq L, S = 0, 1, 2, \dots \right\} \quad (3.4)$$

Under reservation policy PD , find α and S that maximizes

$$\max \left\{ \lambda \int_0^L OFR(S, \alpha, y) P(y) f(y) dy - hI(S, \alpha) : 0 \leq \alpha \leq L, S = 0, 1, 2, \dots \right\} \quad (3.5)$$

In the following two sections we mathematically derive the order fill rates and the average on-hand inventories under each of the three reservation policies.

§3.3 Derivations of order fill rates

First we introduce some notation. Let $Po(m)$ be a Poisson distributed random variable with mean m . For $0 \leq b \leq c \leq L$ let the random variable $N(b, c)$ denote the number of order requests received in the time interval $[0, b)$ having a due date before time point c . Using the properties about a Poisson process we can

show, see Section 3.7.1 of Appendix A, that $N(b, c)$ is Poisson distributed with mean $\lambda \left[b P(Y \leq c - b) + \int_{c-b}^c (c - y) f(y) dy \right]$

§3.3.1 Partial reservation policy FD

Here we get, see Section 3.7.2 of Appendix B, , that the order fill rate (specified as a function of the base stock level S and the reservation policy variable r) for a customer with demand lead-time y is:

$$OFR(S, r, y) = P \left(Po \left(\lambda \left[L - y - (r - y)^+ + \int_0^r (r - x) f(x) dx \right] \right) \leq S - 1 \right) \quad (3.6)$$

Here $(r - y)^+$ is used as shorthand notation for $\max\{r - y, 0\}$ as we also sometimes do in the following. If we assume $r = L$, which means there is no reservation then we get

$$OFR(S, r, y) = P(Po(\lambda(L - E[Y])) \leq S - 1) \quad (3.7)$$

It tells that when there is no reservation, we (very naturally) do not care about the probability distribution of Y , only the mean value matters. This has also been shown in [Hariharan and Zipkin \(1995\)](#) and it is a realisation of Palms Theorem, see [Palm \(1938\)](#) and also for instance [Zipkin \(2000\)](#) p. 247. If we assume $r = 0$, which means complete reservation then we get,

$$OFR(S, r, y) = P(Po(\lambda(L - y)) \leq S - 1) \quad (3.8)$$

§3.3.2 Partial reservation policy BD

Here we get, see Section 3.7.3 of Appendix C, that the order fill rate (specified as a function of the base stock level S and the reservation policy variable d) for a customer with demand lead-time y is:

$$OFR(S, d, y) = P \left(Po \left(\lambda \left[L - \min(y, d) + \int_d^L (d - x) f(x) dx \right] \right) \leq S - 1 \right) \quad (3.9)$$

As a point of validation, note that when $d = 0$ (no reservation), then $OFR(S, d)$ is identical to the right hand side of (3.7) and when $d = L$ (complete reservation), then $OFR(S, d)$ is identical to the right hand side of (3.8).

§3.3.3 Partial reservation policy PD

Consider the time interval $[0, c)$ and let the random variable $M(c, \alpha)$ denote all customer requests received in this interval that under reservation policy *PD* will be reserved prior to time point c . In Section 3.7.4 of Appendix *D*, we show that when $\alpha > 0$, $M(c, \alpha)$ is Poisson distributed with mean $\lambda \left[cP(Y \leq \frac{c}{\alpha}) - \alpha \int_0^{\frac{c}{\alpha}} yf(y)dy \right]$. From Section 3.7.5 of Appendix *E*, we get that the order fill rate (specified as a function of the reservation parameter α and the base stock level S) for a customer with demand lead-time y is:

$$OFR(S, \alpha, y) = P(Po(\lambda[L - (1 - \alpha)y - \alpha E[Y]]) \leq S - 1) \quad (3.10)$$

As a point of validation, note that when $\alpha = 1$ (no reservation), then $OFR(S, \alpha)$ is identical to the right hand side of (3.7) and when $\alpha = 0$ (complete reservation), then $OFR(S, \alpha)$ is identical to the right hand side of (3.8).

§3.4 Derivations of average on-hand inventories

When doing this derivation we use the unit tracking methodology of [Axsäter \(1990\)](#), also used by [Marklund \(2006\)](#). Consider an order, Current-order, arriving at time point τ having a demand lead-time y and a reservation delay $g(y)$. To this order there is a Considered-item which replenishment is triggered by an order (the Triggering-order) received at time point t , where $t < \tau$. That is, the Considered-item is received at the inventory at time point $t + L$. If $t < \tau + y - L$ the Considered-item will sit in the inventory and incur a total inventory cost $h(\tau - t + y - L)$. At time point t there will be some orders that are received but not yet reserved at this

time point (inclusive the Triggering-order). Consider such an order with demand lead-time \hat{y} and reservation delay $g(\hat{y})$. This order is reserved prior to time point $\tau + g(y)$ if $t \leq \tau + g(y) - g(\hat{y})$. Therefore if $\tau + g(y) - g(\hat{y}) \geq \tau + y - L$ an inventory carrying Considered-item will have a triggering time point t so distant in time that all unreserved orders at time point t (received but not yet delivered) are reserved before Current-order will be reserved. This condition can be rewritten to $L \geq y - g(y) + g(\hat{y})$. Under reservation policy *FD*: $y - g(y) + g(\hat{y}) = y - \min\{y, r\} + \min\{\hat{y}, r\} = \max\{y - r, 0\} + \min\{\hat{y}, r\} \leq L - r + r = L$. Under reservation policy *BD*: $y - g(y) + g(\hat{y}) = y - \max\{y - d, 0\} + \max\{\hat{y} - d, 0\} = \min\{y, d\} + \max\{\hat{y} - d, 0\} \leq d + L - d = L$. Under reservation policy *PD*: $y - g(y) + g(\hat{y}) = y - \alpha y + \alpha \hat{y} = (1 - \alpha)y + \alpha \hat{y} \leq L$. thus all the three reservation policies fulfils this condition. Therefore for any of the three reservation policies, in order to have that Considered-item is designated for Current-order there must have occurred $S - 1$ order requests in the time interval $(t, \tau + g(y))$ that are reserved prior to time $\tau + g(y)$. As orders arrive with intensity λ , the probability that Triggering-order occur in an infinitesimal time interval of length ds around time point t is λds . Therefore, when using the parameterization $s = \tau - t$, we get for any of our three reservation policies that Considered-item incurs a total inventory cost of

$$I_{\text{Unit}}(S, y) = h\lambda \int_{L-y}^{\infty} \text{P}(XYZ)(s - L + y)ds \quad (3.11)$$

Where XYZ means “there must have occurred $S - 1$ order requests in the time interval $(t, \tau + g)$ that are reserved prior to time $\tau + g$ ”.

In order to make this specification more concrete one can reuse the derivations made for the order fill rates. Under reservation policy *FD* we get that the number of orders arriving in the time interval $(t, \tau + \min\{y, r\})$ that are reserved prior to time $\tau + \min\{y, r\}$ is Poisson distributed with mean $\lambda \left[s - (r - y)^+ + \int_0^r (r - x)f(x)dx \right]$. Under the reservation policy *BD*, the number of orders arriving in the time interval $(t, \tau + \min\{y, r\})$ that are reserved prior to time $\tau + (y - d)^+$ is Poisson distributed with mean $\lambda \left[s + (y - d)^+ + \int_d^{Lr} (d - x)f(x)dx \right]$. Finally under the reservation policy *PD*,

the number of orders arriving in the time interval $(t, \tau + \alpha y)$ that are reserved prior to time $\tau + \alpha y$ is Poisson distributed with mean $\lambda (s + \alpha(y - E[Y]))$. Denote by $E(S, \lambda)$ an S -phased Erlang distributed random variable with mean S/λ and by $g_{E(S, \lambda)}$ its density function. Then we get that Considered-item incurs a total inventory cost of

$$I_{\text{Unit}}(S, r, y) = h \int_{L-y}^{\infty} g_{E(S, \lambda)} \left(s - (r - y)^+ + \int_0^r (r - x)f(x)dx \right) (s - L + y) ds \quad (3.12)$$

$$I_{\text{Unit}}(S, d, y) = h \int_{L-y}^{\infty} g_{E(S, \lambda)} \left(s - (y - d)^+ + \int_d^L (d - x)f(x)dx \right) (s - L + y) ds \quad (3.13)$$

$$I_{\text{Unit}}(S, \alpha, y) = h \int_{L-y}^{\infty} g_{E(S, \lambda)} (s + \alpha(y - E(Y))) (s - L + y) ds \quad (3.14)$$

under the reservation policies FD , BD and PD respectively.

Furthermore one can show that the expected unit inventory costs under each of the policies are monotone w.r.t. the reservation parameters r , d and α , respectively.

First consider reservation policy FD . Here one can show

$$\frac{\delta I_{\text{Unit}}(S, r, y)}{\delta r} = \begin{cases} \text{P}(Y \geq r) \text{P}\left(E(S, \lambda) \geq L - r + \int_0^r (r - x)f(x)dx\right) & y < r \\ -\text{P}(Y \leq r) \text{P}\left(E(S, \lambda) \geq L - y + \int_0^r (r - x)f(x)dx\right) & y > r \end{cases} \quad (3.15)$$

Because

$$\begin{aligned} & \text{P}\left(E(S, \lambda) \geq L - r + \int_0^r (r - x)f(x)dx\right) \text{P}(Y \geq r) \\ & \leq \int_r^L \text{P}\left(E(S, \lambda) \geq L - y + \int_0^r (r - x)f(x)dx\right) f(y)dy \end{aligned} \quad (3.16)$$

we get that the expected unit inventory cost under policy FD

$$\int_0^L I_{\text{Unit}}(S, r, y)f(y)dy \quad (3.17)$$

is decreasing in r . A similar argument can be used to show that the expected unit inventory cost under policy BD

$$\int_0^L I_{\text{Unit}}(S, d, y)f(y)dy \quad (3.18)$$

is increasing in d . Because

$$\frac{\delta I_{\text{Unit}}(S, \alpha, y)}{\delta \alpha} = (E[Y] - y) P(E(S, \lambda) \geq L - (1 - \alpha)y - \alpha E[Y]) \quad (3.19)$$

we get that the expected unit inventory cost under policy PD

$$\int_0^L I_{\text{Unit}}(S, \alpha, y)f(y)dy \quad (3.20)$$

is convex and decreasing as a function of α .

Therefore seen exclusively from an inventory cost perspective, no reservation should be offered. The reason is that reservations cause a diminishing polling effect, because some inefficiency occurs in terms of at some times one can simultaneously have a positive on-hand inventory (but all are tagged) as well as a positive back-log.

In order to close the gap to the specification of the profit functions in section 3.2, note that for reservation policy FD

$$hI(S, r) = \lambda \int_0^L I_{\text{Unit}}(S, r, y)f(y)dy \quad (3.21)$$

and similar for the other reservation policies.

§3.5 Numerical results

Before one can do numerical investigations one has to consider what is an appropriate random variable to describe the demand lead-times? and what is an appropriate revenue function? Concerning the latter we assume an affine revenue function $P(y) = a - by$ where $\min\{a, a - bL\} > 0$. Thus we look at both the case of an increasing ($b < 0$) and a decreasing ($b > 0$) revenue function. Concerning the former, it seems reasonable to let Y have finite support on $[0, L]$ and a good choice could then be to let Y be beta distributed; for a specification of this distribution see pp 338-339 of [Law and Kelton \(1991\)](#). Note that the beta distribution as a special case contains the uniform distribution. In the following we decided that when sticking to the uniform distribution we developed a computational model, following the outline of Sections 3.2, 3.3 and 3.4, programmed in Visual Basic to carry out the numerical investigations, while, partly in order to save time, decided to do the numerical analysis by using simulations carried out in Arena, [Kelton et al. \(2007\)](#) for the general case. Without loss of generality we let $h = 1$, $L = 4$ and $\lambda = 2$ in all the numerical examples presented.

When Y is uniformly distributed between 0 and L , (3.1) is

$$d = L - \sqrt{2Lr - r^2} \quad (3.22)$$

and (3.2) is

$$\alpha = \frac{2Lr - r^2}{L^2} \quad (3.23)$$

In Tables 3.1(a-c) we present a numerical study of a situation where the revenue per order is decreasing in y . This is the assumption made in [Chen \(2001\)](#). The tables are organized such that for all policies the upper entry represents the situation with complete reservation and then by traversing downwards the element of reservation

is lessened until ending in the lowest entry with no reservation. Also the tables are organized such that when comparing across reservation policies in each entry, by (3.22) and (3.23), the expected reservation time is same.

Tables 3.1(a-c) are about here

We see that reservation is not of advantage. It is important to give order requests arriving with notice the best possible service as these also give the highest revenue. One could wonder if this is influenced by that equally many customers have short demand lead-times as long demand lead-times, that is, demand lead-times are uniformly distributed. If we assume that the revenue function is still decreasing but the majority of the customers have long demand lead-times, and presumably would be interested in achieving some advantages would that make reservation policies of advantage? We let Y be beta distributed with mean $0.8L$ and set up a simulation experiment to investigate this and the results can be seen in Tables 3.2(a-c) (organized differently compared to the other tables).

Tables 3.2(a-c) are about here

We still see that the no reservation option is optimal. Apparently though the individual customer might want advantages in terms of reservation, a “crowding out” effect makes such an initiative inefficient. Therefore we begin investigating the case where $b < 0$. An argument for having $b < 0$ could be that although the customers may pay a lower price for accepting a longer delivery time there might be some benefits in terms of reduced transshipment cost that overall makes the (net) revenue increasing in y . In Tables 3.3(a-c) we present our result for a case with $b < 0$.

Tables 3.3(a-c) about here

We notice that reservation policy FD is dominated by reservation policies BD and PD . This is also observed in Tables 3.1(a-c) and 3.2(a-c). Apparently, using information about the specific demand lead-time (in case PD) or information about the due date and then back schedule (the case BD) is better than to forward schedule (the case FD). When exploring how to set the reservation parameters we see a very moderate use of the reservation policy as the optimal r is close 1, the optimal d is close to 0 and the optimal α is close to 1. We also see that the highest profit is achieved when using the reservation policy PD . When decreasing b further, see Tables 3.4(a-c), we very obviously lower the optimal values of r and α and decrease the optimal value of d . We still see almost dominance of BD and PD over FD and the best overall solution is still found by using reservation policy PD . This finding that the PD reservation policy dominates is also seen in other experiments not reported in the paper.

Tables 3.4(a-c) about here

However, the general conclusion one can draw from these examples is that one should be cautious about applying reservation policies. Only in the case where the revenues associated with orders with large demand lead-times is significantly larger compared to those with a shorter demand lead-time, one should consider a (moderate) usage of a reservation policy, which then should be policy PD .

§3.6 Concluding remarks

We have studied an inventory model with advance order information model in most general way by letting the demand lead-times be specified by a random

variable. We have proposed three partial reservation policies for handling advance orders. We have for the case of a Poisson process of the order requests, for each of the reservation policies derived order fill rates for an order of any demand lead-time and specified the average on-hand inventory. Concerning the latter we have derived some simpler expressions than seen in [Marklund \(2006\)](#). Among the three reservation policies it is the proportional partial reservation (reservation policy *PD*) that dominates. Thus knowledge about the actual demand lead-time is more than knowledge about the due date (reservation policy *BD*) or knowledge about the request date (reservation policy *FD*). But we also see that irrespective of which reservation policy used, the orders should only be reserved in a short time (and often of zero time), thus receive limited protection from being overtaken by orders arriving with a shorter notice. It may seem as paradox. In order to give customers incentives to provide long demand lead times they should receive some advantages in terms of some prioritization, implemented as reservation policies like in this paper. However if there is also large revenues by serving customers with short demand lead-time this does not seem to be a viable approach.

Acknowledgement

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§3.7 Appendices

§3.7.1 Appendix A: Proof that $N(b, c)$ is Poisson distributed

First consider a simplified situation where Y can only attain three different values: y_1 , y_2 and y_3 with probabilities p_1 , p_2 and p_3 , respectively. Assume that $0 \leq y_1 \leq c - b \leq y_2 \leq c \leq y_3 \leq L$. Using the theory about splitting a Poisson process, the requests of customers all having a demand lead-time y_j arrive according to

a Poisson process with intensity λp_j , $j = 1, 2, 3$ and all three Poisson processes are independent. All the customer requests with a demand lead-time y_1 arriving in the time interval $[0, b)$ will have a need date before c . All the customer requests with a demand lead-time y_2 arriving in the time interval $[0, c - y_2)$ will have a need date before c (and the requests arriving in the interval $[c - y_2, b)$ will have a need date time point at or after c). Finally, no requests with a demand lead-time y_3 arriving in the time interval $[0, b)$ will have a need date time before c . From the theory about merging independent Poisson processes we therefore get that for this simplified situation $N(b, c)$ is Poisson distributed with mean $\lambda[bp_1 + (c - y_2)p_2]$. For any discrete distribution of Y , this argument can be generalized to the case that $N(b, c)$ is Poisson distributed with mean $\lambda[bP(Y \leq c - b) + \sum_{i: b < y_i \leq c} (c - y_i)P(Y = y_i)]$. When Y is a continuously distribution in order to make a rigorous proof, we need to do a argumentation through a differential equation:

Focusing on an infinitesimal increment in time by Δ , we get that

$$P(N(b, c) = x) = \begin{cases} \lambda\Delta P(Y \leq c - \Delta)P(N(b - \Delta, c - \Delta) = x - 1) & x \geq 1 \\ \quad + (1 - \lambda\Delta P(Y \leq c - \Delta))P(N(b - \Delta, c - \Delta) = x) & \\ (1 - \lambda\Delta P(Y \leq c - \Delta))P(N(b - \Delta, c - \Delta) = 0) & x = 0 \end{cases}$$

By collecting terms and letting Δ approach zero, we get

$$\begin{aligned} \frac{\delta P(N(b, c) = x)}{\delta b} + \frac{\delta P(N(b, c) = x)}{\delta c} \\ = \begin{cases} \lambda P(Y \leq c) [P(N(b, c) = x - 1) - P(N(b, c) = x)] & x \geq 1 \\ -\lambda P(Y \leq c) P(N(b, c) = 0) & x = 0 \end{cases} \end{aligned}$$

We can now verify that when $N(b, c)$ is Poisson distributed with mean $\lambda[bP(Y \leq c - b) + \int_{c-b}^c (c - y)f(y)dy]$, its probabilities solve the differential equation above. Strictly speaking, we only needed to provide proof for the most general case. However, for pedagogical reasons, we also provided a simpler proof for the case where Y is a discrete random variable.

§3.7.2 Appendix B: Derivation of *OFR* under *FD* reservation

Consider a customer arriving at time point τ with demand lead-time y and denote it Current-order. By distinguishing whether $y > r$ or $y \leq r$, we have identified which request will be served prior to Current-order, see the legend of Figure 3.1. These findings can be summarized as follows: all orders received in the time interval $[\tau + y - L, \tau - (r - y)^+)$ will be served ahead of Current-order while the orders received in the time interval $[\tau - (r - y)^+, \tau + \min(r, y))$ with a need date before time point $\tau + \min(r, y)$ will be served ahead of Current-order. The latter number ($0 = \tau - (r - y)^+, b = c = \min(r, y) + (r - y)^+ = r$) is specified by the random variable $N(r, r)$. Because we do not know the value of y and the size of Current-order in advance, the order fill rate is as specified in (3.6).

Figure 3.1 is about here

§3.7.3 Appendix C: Derivation of *OFR* under *BD* reservation

Consider a customer arriving at time point τ with demand lead-time y and denote it Current-order. Distinguishing between the cases $y > d$ and $y \leq d$ and summarizing the findings of Figure 3.2, we get that all orders received in the time interval $[\tau + y - L, \tau + \max(y, d) - L)$ will be served ahead of Current-order while the orders received in the time interval $[\tau + \max(y, d) - L, \tau + (y - d)^+)$ with a demand need date before time point $\tau + \max(y, d)$ will be served ahead of Current-order. The latter number ($0 = \tau + \max(y, d) - L, b = \tau + (y - d)^+ - \tau - \max(y, d) + L = L - d$ and $c = \tau + \max(y, d) - \tau - \max(y, d) + L = L$) is specified by the random variable $N(L - d, L)$. Because we do not know the value of y and the size of Current-order in advance, the order fill rate is as specified in (3.9).

Figures 3.2 (a) and (b) are about here

§3.7.4 Appendix D: Proof that $M(c, \alpha)$ is Poisson distributed

Assume first that Y is a discrete random variable. Using the same argumentation as in Section 3.7.1 of Appendix A about first splitting and merging independent Poisson processes, we get that $M(c, \alpha)$ is Poisson distributed with mean $\lambda \left[cP(Y \leq c/\alpha) - \alpha \sum_{i: y_i \leq c/\alpha} y_i P(Y = y_i) \right]$. For the case where Y is a continuous random variable, we have to provide rigorous arguments based on differential equations. By focusing on a small time increment Δ , we get

$$P(M(c, \alpha) = x) = \begin{cases} \lambda \Delta P(Y \leq (c - \Delta)/\alpha) P(M(c - \Delta, \alpha) = x - 1) + & x \geq 1 \\ (1 - \lambda \Delta P(Y \leq (c - \Delta)/\alpha)) P(M(c - \Delta, \alpha) = x) & \\ (1 - \lambda \Delta P(Y \leq (c - \Delta)/\alpha)) P(M(c - \Delta, \alpha) = 0) & x = 0 \end{cases}$$

By collecting terms and letting Δ approach zero, we get

$$\frac{\delta P(M(c, \alpha) = x)}{\delta c} = \begin{cases} \lambda P(Y \leq c/\alpha) P(M(c, \alpha) = x - 1) - P(M(c, \alpha) = x) & x \geq 1 \\ -\lambda P(Y \leq c/\alpha) P(M(c - \Delta, \alpha) = 0) & x = 0 \end{cases}$$

From this we derive that $M(c, \alpha)$ is Poisson distributed with mean $\lambda \int_0^c P(Y \leq y/\alpha) dy$. To ensure completeness of the specification, note that if $\alpha = 0$, then obviously $M(c, \alpha)$ is Poisson distributed with mean λc . In order to show the equivalence to the discrete case, note that

$$\int_0^c P\left(Y \leq \frac{c}{\alpha}\right) dy = cP\left(Y \leq \frac{c}{\alpha}\right) - \alpha \int_0^{\frac{c}{\alpha}} y f(y) dy$$

§3.7.5 Appendix E: Derivation of *OFR* under *PD* reservation

Consider a customer arriving at time point τ with demand lead-time y and denote it Current-order. As it will be tagged at time point $\tau + \alpha y$, we need to focus on the

number of requests received in the interval $[\tau + y - L, \tau + \alpha y)$ that are reserved before time point $\tau + \alpha y$. This number is specified by the random variable $M(L - (1 - \alpha)y, \alpha)$ ($0 = \tau + y - L$ and $c = \tau + \alpha y - \tau + y - L = L - (1 - \alpha)y$). Because we assume that $P(Y \leq L) = 1$, it holds that $c/\alpha \geq L$. Therefore, we get that the number of requests received in the interval $[\tau + y - L, \tau + \alpha y)$ that is reserved before Current-order is Poisson distributed with mean $\lambda[L - (1 - \alpha)y - \alpha E[Y]]$. Therefore, the order fill rate is as specified in (3.10).

§3.8 Figures and Tables

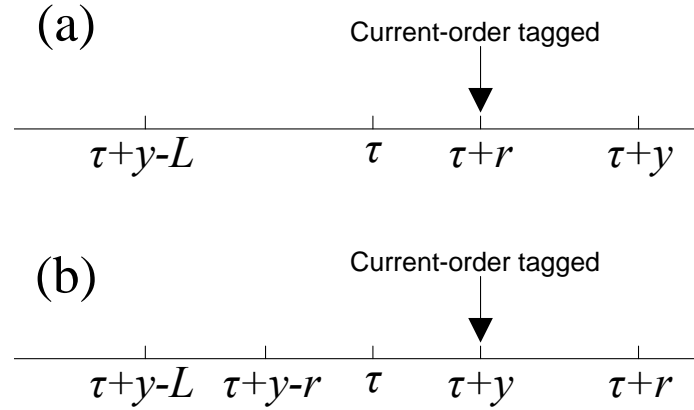


Figure 3.1: (a) All requests received in the time interval $[\tau + y - L, \tau)$ will be served ahead of Current-order. All requests received in the time interval $[\tau, \tau + r)$ with a demand realization before time point $\tau + r$ will be served ahead of Current-order. (b) All requests received in the time interval $[\tau + y - L, \tau + y - r)$ will be served ahead of Current-order. All requests received in the time interval $[\tau + y - r, \tau + y)$ with a demand realization before time point $\tau + y$ will be served ahead of Current-order.

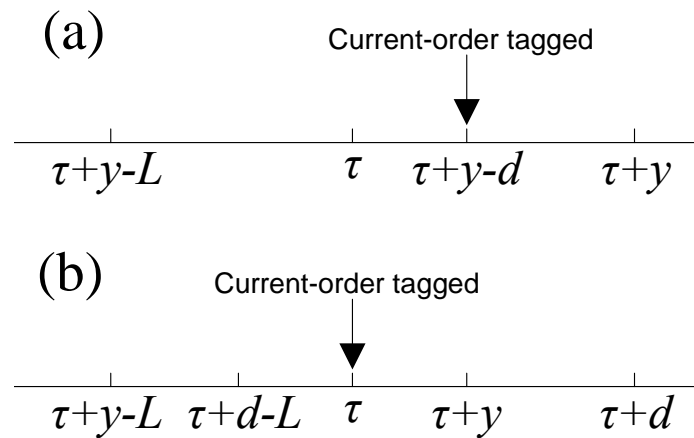


Figure 3.2: (a) All requests received in the time interval $[\tau + y - L, \tau + y - d)$ with a demand realization before time point $\tau + y$ will be served ahead of Current-order. (b) All orders received in the time interval $[\tau + y - L, \tau + d - L)$ will be served ahead of Current-order. All orders received in the time interval $[\tau + d - L, \tau)$ with a demand realization before time point $\tau + d$ will be served ahead of Current-order.

Table 3.1: **(a)** Reservation policy FD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$, $a = 5$, $b = 1$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 6$ which is the optimal base stock level.

r	Av. OH	Revenue	Profit
0	2.551279	3.613333	1.062054
0.5	2.523217	3.636836	1.113619
1	2.456787	3.737113	1.280327
1.5	2.377631	3.920839	1.543208
2	2.304997	4.152242	1.847244
2.5	2.249383	4.379493	2.13011
3	2.214189	4.560289	2.346101
3.5	2.198202	4.672722	2.47452
4	2.195435	4.710782	2.515348

Table 3.1: **(b)** Reservation policy BD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 6$ which is the optimal base stock level.

r	Av. OH	Revenue	Profit
4	2.551279	3.613333	1.062054
2.063508	2.337913	4.154912	1.816998
1.354249	2.246088	4.435627	2.189539
0.877501	2.210928	4.587004	2.376077
0.535898	2.199201	4.663519	2.464318
0.291901	2.196065	4.696843	2.500778
0.127017	2.195487	4.708194	2.512707
0.031373	2.195435	4.710627	2.515192
0	2.195435	4.710782	2.515348

Table 3.1: **(c)** Reservation policy *PD*, $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 6$ which is the optimal base stock level.

r	Av. OH	Revenue	Profit
0	2.551279	3.613333	1.062054
0.234375	2.416076	3.870386	1.45431
0.4375	2.319791	4.10916	1.789369
0.609375	2.257186	4.311343	2.054158
0.75	2.221165	4.468873	2.247709
0.859375	2.203644	4.582167	2.378523
0.9375	2.197061	4.656326	2.459264
0.984375	2.195536	4.697607	2.50207
1	2.195435	4.710782	2.515348

Table 3.2: **(a)** Reservation policy *FD*, $L = 4$, Y beta distributed on 0 to L with mean $0.8L$ and variance $2L^2/75$, $h = 1$, $\lambda = 2$. The expected profit per time is collect by simulation in Arena, using 10 replications, each with a run length of 100000 starting the inventory system having no reservations and no outstanding replenishment orders.

$S \setminus r$	0	1	2	3	4
2	0.8068	0.8066	0.8088	0.9215	1.1640
3	0.6781	0.6783	0.7105	0.9878	1.3036
4	0.2700	0.2708	0.3266	0.6353	0.8709

Table 3.2: **(b)** Reservation policy *BD*, $L = 4$, Y beta distributed on 0 to L with mean $0.8L$ and variance $2L^2/75$, $h = 1$, $\lambda = 2$. The expected profit per time is collect by simulation in Arena, using 10 replications, each with a run length of 100000 starting the inventory system having no reservations and no outstanding replenishment orders.

$S \setminus r$	4	3	2	1	0
2	0.8068	1.0132	1.8663	1.1625	1.1640
3	0.6781	1.0249	1.2527	1.3023	1.3036
4	0.2700	0.5876	0.8267	0.8700	0.8709

Table 3.2: (c) Reservation policy PD , $L = 4$, Y beta distributed on 0 to L with mean $0.8L$ and variance $2L^2/75$, $h = 1$, $\lambda = 2$. The expected profit per time is collect by simulation in Arena, using 10 replications, each with a run length of 100000 starting the inventory system having no reservations and no outstanding replenishment orders.

$S \setminus \alpha$	0	0.25	0.5	0.75	1
2	0.8084	0.8064	0.9592	1.0474	1.1669
3	0.6793	0.8519	1.0176	1.1753	1.3057
4	0.2710	0.4587	0.6340	0.9805	0.8720

Table 3.3: (a) Reservation policy FD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$, $a = 5$, $b = -1$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 7$ which is the optimal base stock level.

r	Av. OH	Revenue	Profit
0	3.343313	11.57702	8.23371
0.5	3.318559	11.59989	8.281331
1	3.264155	11.70108	8.436929
1.5	3.204561	11.8826	8.678035
2	3.154155	12.09962	8.945465
2.5	3.118144	12.29311	9.17497
3	3.096347	12.41701	9.320658
3.5	3.086514	12.4568	9.370286
4	3.084761	12.45056	9.365804

Table 3.3: **(b)** Reservation policy BD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$, $a = 5$, $b = -1$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 7$ which is the optimal base stock level.

d	Av. OH	Revenue	Profit
4	3.343313	11.57702	8.23371
2.063508	3.190394	12.26043	9.070033
1.354249	3.122085	12.43928	9.317195
0.877501	3.09595	12.47085	9.374901
0.535898	3.087416	12.46592	9.378506
0.291901	3.085195	12.45698	9.371783
0.127017	3.084796	12.45204	9.36724
0.031373	3.084761	12.45066	9.365903
0	3.084761	12.45056	9.365804

Table 3.3: **(c)** Reservation policy PD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$, $a = 5$, $b = -1$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 7$ which is the optimal base stock level.

α	Av. OH	Revenue	Profit
0	3.343313	11.57702	8.23371
0.234375	3.240126	11.92687	8.686747
0.4375	3.170256	12.203	9.032742
0.609375	3.126555	12.38227	9.255718
0.75	3.102021	12.46926	9.367236
0.859375	3.090245	12.49025	9.400002
0.9375	3.085846	12.47768	9.391838
0.984375	3.084828	12.45877	9.373944
1	3.084761	12.45056	9.365804

Table 3.4: **(a)** Reservation policy FD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$, $a = 3$, $b = -2$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 7$ which is the optimal base stock level.

r	Av. OH	Revenue	Profit
0	3.343313	12.06558	8.722263
0.5	3.318559	12.06499	8.746431
1	3.264155	12.09901	8.834853
1.5	3.204561	12.18817	8.983611
2	3.154155	12.31103	9.156872
2.5	3.118144	12.42357	9.305423
3	3.096347	12.48379	9.387438
3.5	3.086514	12.47724	9.390722
4	3.084761	12.45056	9.365804

Table 3.4: **(b)** Reservation policy BD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$, $a = 3$, $b = -2$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 7$ which is the optimal base stock level.

d	Av. OH	Revenue	Profit
4	3.343313	12.06558	8.722263
2.063508	3.190394	12.58612	9.395728
1.354249	3.122085	12.6146	9.492516
0.877501	3.09595	12.55138	9.455434
0.535898	3.087416	12.49683	9.409416
0.291901	3.085195	12.46613	9.380939
0.127017	3.084796	12.45375	9.368951
0.031373	3.084761	12.45077	9.366006
0	3.084761	12.45056	9.365804

Table 3.4: (c) Reservation policy PD , $L = 4$, Y uniform on 0 to L , $h = 1$, $\lambda = 2$, $a = 3$, $b = -2$. Av OH is the average on-hand inventory. Revenue is the expected revenue per time unit, and Profit is expected profit per time unit. The table is prepared for $S = 7$ which is the optimal base stock level.

α	Av. OH	Revenue	Profit
0	3.343313	12.06558	8.722263
0.234375	3.240126	12.31705	9.076924
0.4375	3.170256	12.49948	9.329226
0.609375	3.126555	12.59338	9.466825
0.75	3.102021	12.60652	9.504502
0.859375	3.090245	12.56809	9.477844
0.9375	3.085846	12.51239	9.426543
0.984375	3.084828	12.46745	9.382626
1	3.084761	12.45056	9.365804

4

Advance Demand Information, Capacity Restrictions and Customer Prioritization

History: This paper was initially started at the Tepper School of Business, Carnegie Mellon University during my visit in the Fall 2009. This work was presented at the European Conference on Operational Research (EURO), July 2010, Lisbon, Portugal; the International Society for Inventory Research (ISIR) biennial Conference - 16th Symposium, August 2010, Budapest, Hungary; and the INFORMS annual conference, November 2010, TX-Austin, USA.

Advance Demand Information, Capacity Restrictions and Customer Prioritization

Bisheng Du* Christian Larsen† Alan Scheller-Wolf‡

Abstract

We study a single supplier who must invest in capacity to manufacture and sell products to buyers having different priorities. The buyers can place pre-orders before their demand is observed, and can also issue additional orders upon observing demand information. Since the supplier guarantees delivery of pre-ordered goods (these are not constrained by the supplier's capacity), buyers with lower priorities may consider pre-ordering in order to secure inventory. We derive optimal policies for the supplier and buyers. We show surprisingly that it is optimal for the supplier to set the pre-order price so high that pre-ordering will not be used much even though pre-ordering should be of benefit to the supplier due to risk sharing. We also show that the supplier can make a pricing decision using an aggregate model.

Keywords

Advance demand information; Capacity restrictions; Customer prioritization; Pre-order

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§4.1 Introduction

The inspiration for this paper is based on observations from practice. A vital auto component and part supplier, located in China, serves several auto manufacturers in a specific area. With the dramatic growth of private cars in China, the demands for cars from some auto manufacturers therefore begin to exceed their supply, even in the global economic slump of the current financial crisis. Thus, in order to get more production, major car manufacturers are competing for their suppliers' capacity. One way they could try to ensure capacity would be to pre-order supplies well in advance of the actual sales period, and then later, when they know the actual demand, they make additional orders if necessary. In the case illustrated above, the motivation for pre-orders comes from the manufacturers (in the paper, "buyers"), which is because of the competition between auto makers. It may therefore be realistic that the supplier sets the prices. Hence, this analysis is seen from the supplier's perspective. It motivates the study in this paper, which considers a price setting component supplier with several customers, who makes both pre-orders (before observing demand) and after-orders (after observing demand). We limit ourselves to study only two buyers and one single selling season, thereby turning our model into a variant of a newsvendor. The key feature is that the buyers can make pre-orders before they observe their real demand, and later, when having observed their real demand they can issue an additional order to be fulfilled to the extent that the supplier has sufficient capacity. The pre-orders can be considered as advance demand information from the perspective of the supplier. Pre-ordering also enables risk-sharing between the supplier and the buyers. We focus on how the supplier can set the pre-order price optimally given a fixed (market determined) after-order price.

This paper is organized as follows. In Section 4.2, we review the literature on the inventory problem with advance booking and multiple demand classes. In Section 4.3, we analyze the two-buyer case with ADI (Advance Demand Information), find the optimal solution and give its interpretation. Later, in Section 4.4, we analyze

the aggregate buyer case with ADI and find the optimal solution. In Section 4.5, we conduct numerical investigation using different demand distributions. Finally, in Section 4.6, we present conclusions and suggest future research directions.

§4.2 Literature review

Our work follows two streams of research, which are inventory systems with advance booking and inventory systems with multiple demand classes, respectively. The closest work to ours is [Dong and Zhu \(2007\)](#) and [Cachon \(2004\)](#). Based on the work by [Cachon \(2004\)](#), [Dong and Zhu \(2007\)](#) present a Two-Wholesale-Price contract that divides the inventory system into Push, Pull and Partial Advance Booking (PAB) sections. Compared with [Dong and Zhu \(2007\)](#), our work extends their topic into a two demand classes case: there are two customers with different priorities. Therefore, [Dong and Zhu \(2007\)](#) is a special case of our work with only one customer. In this case, we recover the result of [Dong and Zhu \(2007\)](#). There are plenty of studies in the advance booking field. One of the earliest works is [Hariharan and Zipkin \(1995\)](#), who consider the effects of customer order information and leadtime on the whole inventory system. In addition, there are also some other research works, for instance, [Chen \(2001\)](#), [Gallego and Özer \(2001\)](#), [Frank et al. \(2003\)](#), [Hu \(2003\)](#), [Özer \(2003\)](#), [Gallego and Özer \(2003\)](#), [Özer and Wei \(2004\)](#), [Marklund \(2006\)](#), [Wang and Toktay \(2008\)](#), [Papier and Thonemann \(2010\)](#) have also done plenty of studies within this field.

The other stream important to our work is research on systems with multiple demand classes. Many publications have investigated this field, such as [Gayon et al. \(2009\)](#) who study a Make-to-Stock system, a supplier with limited capacity and ADI within multiple customer classes. [Arslan et al. \(2007\)](#) investigates a single product inventory model with multiple demand classes, which is a model with Poisson demand, deterministic replenishment leadtime, and a continuous-review (Q, R) policy with rationing. In addition, [Deshpande et al. \(2003\)](#), [Teunter and Haneveld \(2008\)](#), [Enders et al. \(2009\)](#) and [Zhou and Zhao \(2010\)](#) also study the multiple de-

mand classes problem from different perspectives. The work by [Şen and Zhang \(1999\)](#), which considers the newsboy problem with multiple demand classes, is closest to our work from a methodological point of view. They assume N demand classes, and each class is charged at different prices: increasing prices or decreasing prices. They derived both the fare allocation limit and the initial capacity and discussed managerial implication. For more details on the newsvendor problem, please refer to [Porteus \(1990\)](#) or [Porteus \(2002\)](#).

§4.3 The mathematical model

We assume that there are two buyers, one buyer “H” with a high priority and another buyer “L” with a low priority. They operate in two separated markets, each facing a random demand described by the random variables D_H and D_L , respectively. The buyers pre-order quantities Q_1^H and Q_1^L , respectively. Furthermore, in anticipation of further orders, the supplier builds capacity. With his capacity, the supplier should fulfill the orders from the higher priority customer at first, any excess capacity or inventory can be considered for the lower priority customer. We assume that both buyers have a given unit sales price r . We could have added more generality by assuming sales prices r_H and r_L , respectively. However, in order to rule out any arbitrage, we let the two buyers operate in separated markets using the same fixed unit sales price r . The supplier has unit production cost c . All entities can sell any left-overs at a price of s . As usual, we assume $s < c < r$. The order of events is now as follows.

1. The supplier sets pre-order and after-order unit prices w_1 and w_2 , where we mainly focus on how to optimize w_1 while w_2 is fixed.
2. Buyers H and L decide (simultaneously and independently of each other) pre-order amounts Q_1^H and Q_1^L , respectively.
3. The supplier decides how much capacity $Q_1^H + Q_1^L + Q_2$ to develop.

4. The two buyers observe actual demand d_H and d_L drawn from the random variables D_H and D_L .
5. Buyer H gets the amount $Q_1^H + R_H$ where the first Q_1^H units are charged the unit price w_1 and all remaining units $R_H = \min \{ \max \{ d_H - Q_1^H, 0 \}, Q_2 \}$ are charged the unit price w_2 . R_H is the additional sale (in excess of Q_1^H) to buyer H.
6. Buyer L gets the amount $Q_1^L + R_L$ where the first Q_1^L units are charged the unit price w_1 and all remaining units $R_L = \min \{ \max \{ d_L - Q_1^L, 0 \}, \max \{ Q_2 - R_H, 0 \} \}$ are charged the unit price w_2 . R_L is the additional sale (in excess of Q_1^L) to buyer L.
7. Buyer H sells the amount $\min \{ d_H, Q_1^H + R_H \}$ in his market at unit price r and the remaining amount $\max \{ Q_1^H + R_H - d_H, 0 \}$ to the scrap market at unit price s . Similarly, buyer L sells the amount $\min \{ d_L, Q_1^L + R_L \}$ in his market at unit price r and the remaining amount $\max \{ Q_1^L + R_L - d_L, 0 \}$ to the scrap market at unit price s . The supplier sells any left-overs $Q_2 - R_H - R_L$ to the scrap market at unit price s .

As the supplier knows that he can sell $Q_1^H + Q_1^L$ units, the decision variable Q_2 concerns how much additional capacity he will develop. We want to explore how the supplier can set his prices w_1 and w_2 optimally. To avoid a trivial solution, we will assume that there exist alternative sourcing opportunities concerning the after-orders. Thus, we assume that w_2 is fixed and the analysis is then concerned about how to set pre-order price w_1 optimally.

We here assume that when the buyers determine their pre-order quantities they have full information about the cost structure of the supplier and they are also aware that the supplier knows the probability distribution of their demands. Therefore, when deciding optimal pre-order quantities, they know the optimal reaction (in terms of the decision variable Q_2) of the supplier. Assume in the following that

$f(x)$ and $g(x)$ are the density functions of the random variables D_H and D_L , respectively.

There are two sources of revenue yielding the supplier's total profit: the first source is the revenue by selling products from supplier to buyers, the second source is the revenue by selling products to the secondary salvage market at a discounted price.

During the initial period, the supplier knows that she will sell $Q_1^H + Q_1^L$ items, which are the pre-orders from buyer H and buyer L, respectively. Recall that the quantity of products sold to buyer H are $Q_1^H + R_H$, $R_H = \min[(D_H - Q_1^H)^+, Q_2]$, the quantity of products sold to buyer L are $Q_1^L + R_L$, $R_L = \min[(D_L - Q_1^L)^+, Q_2 - R_H]$. Therefore, there are three situations of demand at H, D_H . (1) If $D_H \geq Q_1^H + Q_2$, H will buy all Q_2 additional units, no supply remain for L. (2) If $Q_1^H \leq D_H \leq Q_1^H + Q_2$, H will buy $D_H - Q_1^H$ additional units. All the remaining units, $Q_2 + Q_1^H - D_H$ will be bought by L if $D_L \geq Q_2 + Q_1^H - D_H + Q_1^L$. $Q_1^L \leq D_L \leq Q_2 + Q_1^H - D_H + Q_1^L$, and L will buy $D_L - Q_1^L$ additional units. If $D_L \leq Q_1^L$, then L will buy no additional units. (3) If $D_H \leq Q_1^H$, all the Q_2 additional units will be bought by L if $D_L \geq Q_1^L + Q_2$. If $Q_1^L \leq D_L \leq Q_1^L + Q_2$, then $D_L - Q_1^L$ additional units will be bought. If $D_L \leq Q_1^L$, then L will buy no additional units.

We now derive the expected value of R_H :

$$E(R_H) = \int_{Q_1^H}^{Q_1^H + Q_2} (x - Q_1^H) f(x) dx + q_2 P(D_H \geq Q_1^H + q_2) \quad (4.1)$$

And also, the expected value of R_L is:

$$E(R_L) = \int_{Q_1^H}^{Q_1^H + Q_2} \left[\int_{Q_1^L}^{Q_1^L + (q_2 - R_H)^+} (y - Q_1^L) g(y) dy + (q_2 - R_H)^+ P(D_L \geq Q_1^L + (q_2 - R_H)^+) \right] f(x) dx \\ + \int_0^{Q_1^H} \left[q_2 P(D_L \geq Q_1^L + q_2) + \int_{Q_1^L}^{Q_1^L + q_2} (y - Q_1^L) g(y) dy \right] f(x) dx$$

Note that the order of integrals is due to Buyer H having the highest priority.

In the first term of the above equation, $R_H = D_H - Q_1^H$ can be inserted inside the integral of R_L because R_H belongs to the interval $(Q_1^H, Q_1^H + q_2)$. Then R_L is equals to

$$E(R_L) = \int_{Q_1^H}^{Q_1^H+q_2} \left[(Q_1^H + q_2 - x)P(D_L \geq Q_1^H + Q_1^L + q_2 - x) + \int_{Q_1^L}^{Q_1^H+Q_1^L+q_2-x} (y - Q_1^L) g(y) dy \right] f(x) dx$$

$$+ P(D_H \leq Q_1^H) \left[q_2 P(D_L \geq Q_1^L + q_2) + \int_{Q_1^L}^{Q_1^L+q_2} (y - Q_1^L) g(y) dy \right] \quad (4.2)$$

The expected revenue for the supplier from selling to buyer H and buyer L is:

$$w_1(Q_1^H + Q_1^L) + w_2 [E(R_H) + E(R_L)] \quad (4.3)$$

The second source of the supplier's revenue comes from sales to the secondary market. This can be described by three components. (1) If $D_H \geq Q_1^H + q_2$, then 0 unit is scrapped. Therefore,

$$E(LO_1) = 0$$

Where LO_i ($i = 1, 2, 3$) donate the left over products sales to the secondary market under three situations.

(2) If $Q_1^H \leq D_H \leq Q_1^H + q_2$, then $Q_1^H + q_2 - D_H$ units are left over to L. If $D_L \leq Q_1^L$, he will buy none of these additional units, so the $Q_1^H + q_2 - x$ units are all scrapped. If $Q_1^L \leq D_L \leq Q_1^H + Q_1^L + q_2 - D_H$, then $Q_1^H + Q_1^L + q_2 - D_H - D_L$ units are scrapped. If $D_L \geq Q_1^H + Q_1^L + q_2 - D_H$, then no units are scrapped. Therefore,

$$E(LO_2) = \int_{Q_1^H}^{Q_1^H+q_2} \left[(Q_1^H + q_2 - x)P(D_L \leq Q_1^L) + \int_{Q_1^L}^{Q_1^H+Q_1^L+q_2-x} (Q_1^H + Q_1^L + q_2 - x - y) g(y) dy \right] f(x) dx$$

(3) If $D_H \leq Q_1^H$, all the q_2 additional units are left over for L. If $D_L \leq Q_1^L$, all these

q_2 units are scrapped. If $Q_1^L \leq D_L \leq Q_1^L + q_2$, then $Q_1^L + q_2 - D_L$ units are scrapped. If $D_L \geq Q_1^L + q_2$, then 0 unit is scrapped. Therefore,

$$E(LO_3) = P(D_H \leq Q_1^H) \left[q_2 P(D_L \leq Q_1^L) + \int_{Q_1^L}^{Q_1^L + q_2} (Q_1^L + q_2 - y) g(y) dy \right]$$

All in all, the revenue from sales to the secondary market is

$$s [E(LO_1) + E(LO_2) + E(LO_3)] \quad (4.4)$$

The optimal additional capacity to develop given the pre-order quantities Q_1^H and Q_1^L is found by the following optimization model:

$$\begin{aligned} \Pi_S(q_2) = & w_1(Q_1^H + Q_1^L) + w_2 [E(R_H) + E(R_L)] \\ & + s [E(LO_1) + E(LO_2) + E(LO_3)] - c (Q_1^H + Q_1^L + q_2) \end{aligned} \quad (4.5)$$

Denote the optimal solution $Q_2(Q_1^H, Q_1^L)$.

This is a concave function of q_2 with first-order derivative, which is $\frac{d\Pi_S(q_2)}{dq_2}$ (for more details, see Section 4.7.1 of Appendix):

$$w_2 - c - (w_2 - s) \left\{ P(D_H \leq Q_1^H) P(D_L \leq Q_1^L + q_2) + \int_{Q_1^L}^{Q_1^L + q_2} P(D_L \leq Q_1^H + Q_1^L + q_2 - x) f(x) dx \right\} \quad (4.6)$$

Therefore, we conclude that the optimal solution, Q_2 , is 0 if

$$P(D_H \leq Q_1^H) P(D_L \leq Q_1^L) > \frac{w_2 - c}{w_2 - s} \quad (4.7)$$

Otherwise, it is the unique value of q_2 that solves the equation

$$P(D_H \leq Q_1^H)P(D_L \leq Q_1^L + q_2) + \int_{Q_1^H}^{Q_1^H + q_2} P(D_L \leq Q_1^H + Q_1^L + q_2 - x) f(x) dx = \frac{w_2 - c}{w_2 - s} \quad (4.8)$$

Special case: If we assume that $Q_1^H = Q_1^L = 0$, the first order condition equals $\int_0^{q_2} P(D_L \leq q_2 - x) f(x) dx = \frac{w_2 - c}{w_2 - s}$, which means $P(D_H + D_L \leq q_2) = \frac{w_2 - c}{w_2 - s}$. So the supplier will base his capacity decision on the aggregate demand information, which is the same as the aggregate model we will present in the next section.

In this way we have derived a reaction function $Q_2(q_1^H, q_1^L)$ of any pair of pre-order amounts (q_1^H, q_1^L) .

Before we continue, let us examine the cases where $D_L \equiv 0$. In this case, (4.6) becomes

$$w_2 - c - (w_2 - s)P(D_H \leq Q_1^H + q_2) \quad (4.9)$$

This term is less than or equal to (4.6). Therefore, if $Q_2(q_1^H, D_L \equiv 0)$ is the reaction function of the supplier if buyer H is the only buyer, then we conclude that $Q_2(q_1^H, D_L \equiv 0) \leq Q_2(q_1^H, q_1^L)$. This means that the presence of a low prioritized buyer will always induce the supplier to provide more additional capacity than was made available if only one buyer was present. This argument does not depend on the pre-order price w_1 .

Given that both buyers know the optimal behavior of the supplier, they solve their own optimization problems.

Denoted by $Q_1^H(q_1^L)$, the optimal pre-order quantity of buyer H is derived so that he knows the optimal reaction of the supplier as well as the pre-order quantity q_1^L of buyer L. $Q_1^H(q_1^L)$ is the optimal solution to the following optimization problem.

Buyer H's optimization problem:

$$\begin{aligned} \Pi_H(q_1^H) = & r \left[\int_0^{q_1^H + Q_2(q_1^H, q_1^L)} x f(x) dx + (q_1^H + Q_2(q_1^H, q_1^L)) P(D_H \geq q_1^H + Q_2(q_1^H, q_1^L)) \right] \\ & - w_1 q_1^H - w_2 E(R_H) + s \int_0^{q_1^H} (q_1^H - x) f(x) dx \end{aligned} \quad (4.10)$$

At first glance, it appears that buyer H's profit function does not depend on the probability distribution of D_L . However, it does depend on D_L as $Q_2(q_1^H, q_1^L)$ depends on D_L , we denote the optimal solution $Q_1^H(q_1^L)$.

This optimization problem has first-order condition on Q_1^H (see Section 4.7.3 of Appendix).

Similarly, denoted by $Q_1^L(q_1^H)$ the optimal pre-order quantity of buyer L is derived so that he knows the optimal reaction of the supplier as well as the pre-order quantity q_1^H of buyer H. This is the optimal solution to the following optimization problem.

The revenue of buyer L from selling products in the market is dependent on three situations, based on the sales of buyer H.

(1) If $D_H \geq q_1^H + Q_2(q_1^H, q_1^L)$, then the products supplied to buyer L are q_1^L .

$$E(SL_1) = P(D_H \geq q_1^H + Q_2(q_1^H, q_1^L)) \left[\int_0^{q_1^L} y g(y) dy + P(D_L \geq q_1^L) q_1^L \right] \quad (4.11)$$

Where SL_i ($i = 1, 2, 3$) denote the products supplied to buyer L under three situations.

(2) If $q_1^H \leq D_H \leq q_1^H + Q_2(q_1^H, q_1^L)$, then $q_1^H + Q_2(q_1^H, q_1^L) - D_H$ units are left over

to L. Then the products supplied to buyer L are $E(SL_2)$.

$$E(SL_2) = \int_{q_1^H}^{q_1^H + Q_2(q_1^H, q_1^L)} \left[\int_0^{q_1^H + q_1^L + Q_2(q_1^H, q_1^L) - x} y g(y) dy \right. \\ \left. + P(D_L \geq q_1^H + q_1^L + Q_2(q_1^H, q_1^L) - x) (q_1^H + q_1^L + Q_2(q_1^H, q_1^L) - x) \right] f(x) dx \quad (4.12)$$

(3) If $D_H \leq q_1^H$, all the Q_2 additional units are left over to L. Then products from the supplier to buyer L are $E(SL_3)$.

$$E(SL_3) = P(D_H \leq q_1^H) \left[\int_0^{q_1^L + Q_2(q_1^H, q_1^L)} y g(y) dy \right. \\ \left. + P(D_L \geq q_1^L + Q_2(q_1^H, q_1^L)) (q_1^L + Q_2(q_1^H, q_1^L)) \right] \quad (4.13)$$

Buyer L's optimization problem becomes:

$$\Pi_L(q_1^L) = r [E(SL_1) + E(SL_2) + E(SL_3)] \\ - w_1 q_1^L - w_2 E(R_L) + s \int_0^{q_1^L} (q_1^L - y) g(y) dy \quad (4.14)$$

This optimization problem has first-order condition on Q_1^L (see Section 4.7.4 of Appendix).

In this way the two buyers can continue to iterate until they reach an equilibrium $(\overline{Q_1^H}, \overline{Q_1^L})$ characterized by $Q_1^H(\overline{Q_1^L}) = \overline{Q_1^H}$ and $Q_1^L(\overline{Q_1^H}) = \overline{Q_1^L}$. For each value of w_1 under this full information scenario, the supplier then needs to figure out what the equilibrium is, and then he can compute his expected profit. By varying w_1 , he can thus find his optimal pre-order price. It is quite a task to carry out such an analysis and furthermore there may be several equilibriums. In the following, we will therefore consider whether the analysis could be carried out by using a more aggregate model.

§4.4 An aggregate model

The aggregate model focuses on the supplier's point of view, especially when two buyers are collapsed into one, denoted by A (A stands for aggregate), who accordingly sells his product at unit price r in a market with random demand $D_A = D_H + D_L$. $h(x)$ and $H(x)$ denote the density and distribution function of D_A . The order of events is as follows:

1. The supplier sets unit pre-order and unit after-order prices w_1 and w_2 .
2. Buyer A decides a pre-order amount Q_1 .
3. The supplier decides how much capacity $Q_1 + Q_2$ to develop.
4. Buyer A observes the actual demand d drawn from the random variable D_A .
5. Buyer A gets the amount $y = \min\{\max\{d, Q_1\}, Q_1 + Q_2\}$ where all units in excess of Q_1 are charged the unit price w_2 and all remaining units are charged the unit price w_1 .
6. Buyer A sells the amount $\min\{d, y\}$ in his market at unit price r and the remaining amount $\max\{y - d, 0\}$ to the scrap market at unit price s . The supplier sells any left-overs $Q_1 + Q_2 - y$ at the scrap market.

So in this aggregate model, when the supplier conduct an analysis of how to set his preorder price optimally, he just assumes for simplicity that he only has one (super) buyer. The mathematics of this model is essentially developed in [Dong and Zhu \(2007\)](#). So here, we will just summarize the key findings. The reaction function can now be specified explicitly as (see also (4.9)):

$$Q_2(q_1) = \max \left\{ H^{-1} \left(\frac{w_2 - c}{w_2 - s} \right) - q_1, 0 \right\} \quad (4.15)$$

The profit function of the supplier, as a function of w_1 and incorporating the optimal behavior of the (super) buyer, can be specified explicitly as follows

$$\Pi_S(w_1) = \begin{cases} (w_1 - c)H^{-1}\left(\frac{r-w_1}{r-s}\right) & c \leq w_1 \leq \widetilde{w}_1(w_2) \\ (w_2 - s) \left[\int_0^{H^{-1}\left(\frac{w_2-c}{w_2-s}\right)} xh(x)dx - \int_0^{H^{-1}\left(\frac{w_2-w_1}{w_2-s}\right)} xh(x)dx \right] & \widetilde{w}_1(w_2) < w_1 \leq w_2 \\ (w_2 - s) \int_0^{H^{-1}\left(\frac{w_2-c}{w_2-s}\right)} xh(x)dx & w_2 \leq w_1 \leq r \end{cases} \quad (4.16)$$

As this function is specified in another way in [Dong and Zhu \(2007\)](#) (the similar expression is (3) in their paper), its derivation can be seen in Section 4.7.5 of Appendix. The function $\widetilde{w}_1(w_2)$ is the unique value of w_1 , which determines when the Buyer will only pre-order (in [Dong and Zhu \(2007\)](#) denoted PUSH) or when he will both pre-order and after-order (in [Dong and Zhu \(2007\)](#) denoted PAB). Note that the middle term in (4.16) vanishes if $\widetilde{w}_1(w_2) \geq w_2$. From the description given in (4.16), it is also easy to see that in the middle term the derivative of Π_S with respect to w_1 is $H^{-1}\left(\frac{w_2-c}{w_2-s}\right)$ which is non-negative. Therefore, the optimal pre-order price will never be found in the middle term (the PAB regime).

The optimal profit of the buyer:

$$\Pi_B(w_1) = \begin{cases} (r - s) \int_0^{\mathbf{F}^{-1}\left(\frac{r-w_1}{r-s}\right)} x f(x) dx & c \leq w_1 \leq \widetilde{w}_1(w_2) \\ (r - w_2) \int_0^{\mathbf{F}^{-1}\left(\frac{w_2-c}{w_2-s}\right)} x f(x) dx & \widetilde{w}_1(w_2) < w_1 \leq w_2 \\ + (w_2 - s) \int_0^{\mathbf{F}^{-1}\left(\frac{w_2-w_1}{w_2-s}\right)} x f(x) dx & \\ + (r - w_2) \frac{c-s}{w_2-s} \mathbf{F}^{-1}\left(\frac{w_2-c}{w_2-s}\right) & \\ (r - w_2) \int_0^{\mathbf{F}^{-1}\left(\frac{w_2-c}{w_2-s}\right)} x f(x) dx & w_2 \leq w_1 \leq r \\ + (r - w_2) \frac{c-s}{w_2-s} \mathbf{F}^{-1}\left(\frac{w_2-c}{w_2-s}\right) & \end{cases} \quad (4.17)$$

§4.5 A numerical example

We consider a case where $s = 2$, $c = 5$ and $r = 30$. The demand distributions are negative binomial distributions. We assume that D_H and D_L have the same distribution with shape parameter $n = 9$ and probability parameter $p = 0.5$, meaning that they both have mean value 9. Table 4.1 shows our results for the general model. For each value of w_2 , the optimal value of w_1 is stated (in this analysis restricted to be integer valued) and its resulting equilibrium values of (Q_1^H, Q_1^L, Q_2) and the resulting expected profits of the three entities, denoted by Π_S , Π_H and Π_L for the Supplier, Buyer H and Buyer L, respectively.

Tables 4.1 is about here

As expected, we see that buyer H always has a higher expected profit than buyer L. Furthermore, as buyer H has the first priority, he does not need to make pre-orders. When the after-order price is low, the supplier will not develop that much

capacity. Therefore, buyer L needs to pre-order. Then the optimal pre-order price is higher than the after-order price, and yet buyer L is still willing to pre-order so as to reserve capacity. We also see that buyer H benefits from an increase of w_2 from 8 to 10 because this will make the supplier invest in more unreserved capacity (variable Q_2), in which he can exploit better due to his first priority. When w_2 gets larger, we see that it is optimal to set the pre-order price so high that in reality the pre-order option is virtually, but not entirely ruled out. Note that Q_1^L appears to be one not due to rounding but because of the discrete nature of the numerical experiment that Q_1^L is slightly above 0. From Section 4.7.4 of Appendix, we can formally prove:

$$\begin{aligned} \frac{d\Pi_L(q_1^L)}{dq_1^L} \Big|_{q_1^L=0} &= r\mathbf{P}(D_H \geq q_1^H + Q_2) - w_1 \\ &\quad + w_2 \left[\mathbf{P}(D_H \leq q_1^H) + \mathbf{P}(q_1^H \leq D_H \leq Q_2 + q_1^H) \right] \\ &= r\mathbf{P}(D_H \geq q_1^H + Q_2) - w_1 + w_2\mathbf{P}(D_H \leq Q_2 + q_1^H) \end{aligned}$$

Thus even if $w_1 = w_2$, we get that the partial derivative of buyer L's profit function is positive around 0, and thus he will pre-order, though only in small amounts as seen in Table 4.2. This partial derivative has nice economic interpretations: If buyer L decides to pre-order one marginal unit (and currently has pre-ordered none), then he will of course incur a marginal loss w_1 . But if the demand of buyer H should exceed his maximum delivery $Q_2 + q_1^H$, then buyer L is secured a marginal sale giving him a marginal revenue r . On the other hand, if the demand of buyer H is less than his maximum delivery $Q_2 + q_1^H$, then buyer L can decrease his after-order by a marginal unit giving him a marginal decrease in his purchase costs of w_2 .

We now turn to the aggregate model. Here the aggregate demand is negatively binomially distributed with shape parameter $n = 18$ and probability parameter $p = 0.5$. Using the approach from above, we get the following results.

Tables 4.2 is about here

We see here almost the same results as in Table 4.2. Thus it indicates that the supplier, maybe except for the low values of w_2 , could have reached the same conclusion about how to set the pre-order price optimally by using the aggregate model. From this model it is even more apparent that the price setting supplier will set the pre-order price high enough to keep the option ruled out.

§4.6 Conclusions

We have derived a newsvendor model with two buyers having different priorities in case of shortages, and a price setting supplier offering a pre-order opportunity. Though the pre-order opportunity should favor the supplier, enabling him to allocate some risk to his buyers, it turns out that when he is a price setter, he will set the price for pre-ordering high enough that this option will not be used by the buyers. Of course, as the data set presented to support this claim is very limited (though some more data are available than presented in this paper), it is obvious that more comprehensive numerical studies should be undertaken. It also seems to be the case that the supplier can make his pre-order price decision using a more aggregate model. This is of course of great help as the detailed model is very complex and involves finding Nash-equilibria.

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§4.7 Appendices

§4.7.1 Supplier's function, first order condition on Q_2

The first order deviation on q_2 to the supplier's profit function is equivalent to:

$$w_2 \left[\frac{\partial E(R_H)}{\partial q_2} + \frac{\partial E(R_L)}{\partial q_2} \right] - c + s \left[\frac{\partial E(LO_1)}{\partial q_2} + \frac{\partial E(LO_2)}{\partial q_2} + \frac{\partial E(LO_3)}{\partial q_2} \right] = 0$$

Where,

$$\begin{aligned} \frac{\partial E(R_H)}{\partial q_2} &= q_2 f(Q_1^H + q_2) + P(D_H \geq Q_1^H + q_2) - q_2 f(Q_1^H + q_2) \\ &= P(D_H \geq Q_1^H + q_2) \end{aligned}$$

$$\begin{aligned} \frac{\partial E(R_L)}{\partial q_2} &= \int_{Q_1^H}^{Q_1^H + q_2} [P(D_L \geq Q_1^H + Q_1^L + q_2 - x) - (Q_1^H + q_2 - x)g(Q_1^H + Q_1^L + q_2 - x) \\ &\quad + (Q_1^H + q_2 - x)g(Q_1^H + Q_1^L + q_2 - x)] f(x) dx \\ &\quad + F(Q_1^H) [P(D_L \geq Q_1^L + Q_2) - q_2 g(Q_1^L + q_2) + q_2 g(Q_1^L + q_2)] \\ &= \int_{Q_1^H}^{Q_1^H + q_2} P(D_L \geq Q_1^H + Q_1^L + q_2 - x) f(x) dx + F(Q_1^H) P(D_L \geq Q_1^L + q_2) \end{aligned}$$

$$= F(Q_1^H + q_2) - F(Q_1^H) - \int_{Q_1^H}^{Q_1^H + q_2} G(Q_1^H + Q_1^L + q_2 - x) f(x) dx + F(Q_1^H) (1 - G(Q_1^L + q_2))$$

$$= F(Q_1^H + q_2) - F(Q_1^H) G(Q_1^L + q_2) - \int_{Q_1^H}^{Q_1^H + q_2} G(Q_1^H + Q_1^L + q_2 - x) f(x) dx$$

Therefore,

$$\frac{\partial E(R_H)}{\partial q_2} + \frac{\partial E(R_L)}{\partial q_2} = 1 - F(Q_1^H)G(Q_1^L + q_2) - \int_{Q_1^H}^{Q_1^H + q_2} G(Q_1^H + Q_1^L + q_2 - x) f(x) dx$$

Simultaneously, we get

$$\begin{aligned} \frac{\partial E(LO_1)}{\partial q_2} &= 0 \\ \frac{\partial E(LO_2)}{\partial q_2} &= \int_{Q_1^H}^{Q_1^H + q_2} [G(Q_1^L) + \int_{Q_1^L}^{Q_1^H + Q_1^L + q_2 - x} g(y) dy] f(x) dx = \int_{Q_1^H}^{Q_1^H + q_2} G(Q_1^H + Q_1^L + q_2 - x) f(x) dx \\ \frac{\partial E(LO_3)}{\partial q_2} &= F(Q_1^H) \left[G(Q_1^L) + \int_{Q_1^L}^{Q_1^L + q_2} g(y) dy \right] = F(Q_1^H)G(Q_1^L + q_2) \end{aligned}$$

Therefore, the first order to q_2 is:

$$\begin{aligned} \frac{d\Pi_S(q_2)}{dq_2} &= w_2 \left[1 - F(Q_1^H)G(Q_1^L + q_2) - \int_{Q_1^H}^{Q_1^H + q_2} G(Q_1^H + Q_1^L + q_2 - x) f(x) dx \right] - c \\ &\quad + s \left[F(Q_1^H)G(Q_1^L + q_2) + \int_{Q_1^H}^{Q_1^H + q_2} G(Q_1^H + Q_1^L + q_2 - x) f(x) dx \right] \end{aligned}$$

The second order derivative is:

$$\begin{aligned} \frac{d^2\Pi_S(q_2)}{dq_2^2} &= -(w_2 - s) \left[F(Q_1^H)g(Q_1^L + q_2) + G(Q_1^L)f(Q_1^H + q_2) \right. \\ &\quad \left. + \int_{Q_1^H}^{Q_1^H + q_2} g(Q_1^H + Q_1^L + q_2 - x) f(x) dx \right] \end{aligned}$$

Thus, $\frac{d^2\Pi_S(q_2)}{dq_2^2} < 0$, so the supplier's profit function is concave w.r.t. q_2 .

As the solution to the expression above is equal to $\frac{d\Pi_S(q_2)}{dq_2} = 0$, then we can get

$$w_2 - c = (w_2 - s) \left[F(Q_1^H)G(Q_1^L + q_2) + \int_{Q_1^H}^{Q_1^H + q_2} G(Q_1^H + Q_1^L + q_2 - x)f(x)dx \right] \quad (4.18)$$

If we assume that $D_L = 0$, which means that there is only one demand class, then the CDF of it, $G(\cdot) = 1$. The equation above is equal to

$$w_2 - c = (w_2 - s) \left[F(Q_1^H) + \int_{Q_1^H}^{Q_1^H + q_2} f(x)dx \right] \quad (4.19)$$

This is exactly the same as the one demand class for $Q_1^H = Q_1$.

$$F(Q_1^H + q_2) = \frac{w_2 - c}{w_2 - s} \quad (4.20)$$

§4.7.2 Sensitivity of the optimal additional capacity as function of the pre-orders

In this section, we develop $\frac{\partial Q_2}{\partial q_1^H}$ and $\frac{\partial Q_2}{\partial q_1^L}$ by use of an implicit function theorem.

For buyer H, from the implicit function theorem on q_1^H , we derive

$$\begin{aligned} \frac{\partial Q_2}{\partial q_1^H} &= - \frac{\frac{\partial}{\partial q_1^H}(4.18)}{\frac{\partial}{\partial Q_2}(4.18)} \\ &= - \frac{f(q_1^H + Q_2)P(D_L \leq q_1^L) + \int_{q_1^H}^{q_1^H + Q_2} g(q_1^H + q_1^L + Q_2 - x)f(x)dx}{P(D_H \leq q_1^H)g(q_1^L + Q_2) + \int_{q_1^H}^{q_1^H + Q_2} g(q_1^H + q_1^L + Q_2 - x)f(x)dx + P(D_L \leq q_1^L)f(q_1^H + Q_2)} \end{aligned} \quad (4.21)$$

Here we get, $-1 \leq \frac{\partial Q_2}{\partial q_1^H} \leq 0$ and $\frac{\partial Q_2}{\partial q_1^H} \Big|_{q_1^H=0} = -1$.

For buyer L, from the implicit function theorem on q_1^L , we derive

$$\begin{aligned} \frac{\partial Q_2}{\partial q_1^L} &= - \frac{\frac{\partial}{\partial q_1^L}(4.18)}{\frac{\partial}{\partial Q_2}(4.18)} \\ &= - \frac{\text{P}(D_H \leq q_1^H)g(q_1^L + Q_2) + \int_{q_1^H}^{q_1^H + Q_2} g(q_1^H + q_1^L + Q_2 - x)f(x)dx}{\text{P}(D_H \leq q_1^H)g(q_1^L + Q_2) + \int_{q_1^H}^{q_1^H + Q_2} g(q_1^H + q_1^L + Q_2 - x)f(x)dx + \text{P}(D_L \leq q_1^L)f(q_1^H + Q_2)} \end{aligned} \quad (4.22)$$

Also here we get, $-1 \leq \frac{\partial Q_2}{\partial q_1^L} \leq 0$ and $\frac{\partial Q_2}{\partial q_1^L}|_{q_1^L=0} = -1$.

§4.7.3 Buyer H's function, first order condition on Q_1^H

Buyer H's optimization problem, for simplicity, we use Q_2 instead of $Q_2(q_1^H, q_1^L)$, and use Q_2' denote the partial deviation of Q_2 w.r.t. q_1^H , $\frac{\partial Q_2(q_1^H, q_1^L)}{\partial q_1^H}$.

From (4.10), the first order deviation is

$$\begin{aligned} &= r \left[(q_1^H + Q_2) f(q_1^H + Q_2)(1 + Q_2') + (1 + Q_2') \text{P}(D_H \geq q_1^H + Q_2) \right. \\ &\quad \left. - (q_1^H + Q_2) f(q_1^H + Q_2)(1 + Q_2') \right] \\ &\quad - w_2 \left[- \int_{q_1^H}^{q_1^H + Q_2} f(x) dx + Q_2 f(q_1^H + Q_2)(1 + Q_2') + Q_2' \text{P}(D_H \geq q_1^H + Q_2) \right. \\ &\quad \left. - Q_2 f(q_1^H + Q_2)(1 + Q_2') \right] - w_1 + s F(q_1^H) \\ &= r(1 + Q_2') \text{P}(D_H \geq q_1^H + Q_2) - w_1 - w_2 \left[F(q_1^H) + Q_2' - (1 + Q_2')F(q_1^H + Q_2) \right] + s F(q_1^H) \\ &= r(1 + Q_2') - r(1 + Q_2')F(q_1^H + Q_2) - w_1 - w_2 F(q_1^H) - w_2 Q_2' \\ &\quad + w_2(1 + Q_2')F(q_1^H + Q_2) + s F(q_1^H) \\ &= (r - w_1) + (r - w_2) Q_2' - (r - w_2)(1 + Q_2')F(q_1^H + Q_2) - (w_2 - s)F(q_1^H) \\ &= (r - w_2)(1 + Q_2') \text{P}(D_H \geq q_1^H + Q_2) + (w_2 - w_1) - (w_2 - s)F(q_1^H) \end{aligned}$$

Which is

$$\frac{d\Pi_H(q_1^H)}{dq_1^H} = (r - w_2)(1 + Q_2')P(D_H \geq q_1^H + Q_2) + (w_2 - w_1) - (w_2 - s)F(q_1^H) \quad (4.23)$$

Where the optimal solution is $Q_1^H(q_1^L)$.

For the special case $q_1^H = 0$, then $Q_2'|_{q_1^H=0} = 0$ and $F(q_1^H) = 0$, we can get

$$\frac{d\Pi_H(q_1^H)}{dq_1^H}\bigg|_{q_1^H=0} = w_2 - w_1 \quad (4.24)$$

$$\begin{aligned} \frac{d^2\Pi_H(q_1^H)}{(dq_1^H)^2} &= \left[-(1 + Q_2')^2 f(q_1^H + Q_2) + Q_2'' P(D_H \geq q_1^H + Q_2) \right] (r - w_2) \\ &\quad - (w_2 - s) f(q_1^H) \end{aligned} \quad (4.25)$$

§4.7.4 Buyer L's function, first order condition on Q_1^L

The first order deviation on q_1^L to buyer L's profit function is equivalent to:

$$r \frac{\partial [E(SL_1) + E(SL_2) + E(SL_3)]}{\partial q_1^L} - w_1 - w_2 \frac{\partial E(R_L)}{\partial q_1^L} + s G(q_1^L) = 0 \quad (4.26)$$

$$\begin{aligned} \Pi_L(q_1^L) &= r [E(SL_1) + E(SL_2) + E(SL_3)] \\ &\quad - w_1 q_1^L - w_2 E(R_L) + s \int_0^{q_1^L} (q_1^L - y) g(y) dy \end{aligned} \quad (4.27)$$

Buyer L's optimization problem is below. For simplex, we use Q_2 instead of $Q_2(q_1^H, q_1^L)$ and use Q_2' donate the partial deviation of Q_2 w.r.t. q_1^L . Therefore, from (4.14), the first order deviation is

From equation (4.11) on page 80, we get

$$\begin{aligned} \frac{\partial E(SL_1)}{\partial q_1^L} &= P(D_H \geq q_1^H + Q_2) P(D_L \geq q_1^L) \\ &\quad - Q_2' f(q_1^H + Q_2) \left[\int_0^{q_1^L} y g(y) dy + P(D_L \geq q_1^L) q_1^L \right] \end{aligned}$$

From equation (4.12) on page 81, we get

$$\begin{aligned}
\frac{\partial E(SL_2)}{\partial q_1^L} &= \int_{q_1^H}^{q_1^H+Q_2} \mathbf{P}(D_L \geq q_1^H + q_1^L + Q_2 - x)(1 + Q_2') f(x) dx \\
&\quad + Q_2' \left[\int_0^{q_1^L} yg(y)dy + \mathbf{P}(D_L \geq q_1^L)q_1^L \right] f(q_1^H + Q_2) \\
&= (1 + Q_2') \int_{q_1^H}^{q_1^H+Q_2} \mathbf{P}(D_L \geq q_1^H + q_1^L + Q_2 - x) f(x) dx \\
&\quad + Q_2' \left[\int_0^{q_1^L} yg(y)dy + \mathbf{P}(D_L \geq q_1^L)q_1^L \right] f(q_1^H + Q_2)
\end{aligned}$$

From equation (4.13) on page 81, we get

$$\begin{aligned}
\frac{\partial E(SL_3)}{\partial q_1^L} &= \mathbf{P}(D_H \leq q_1^H)(1 + Q_2')\mathbf{P}(D_L \geq q_1^L + Q_2) \\
&= (1 + Q_2')\mathbf{P}(D_H \leq q_1^H)\mathbf{P}(D_L \geq q_1^L + Q_2)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial [E(SL_1) + E(SL_2) + E(SL_3)]}{\partial q_1^L} \\
&= \mathbf{P}(D_H \geq q_1^H + Q_2)\mathbf{P}(D_L \geq q_1^L) - Q_2' f(q_1^H + Q_2) \left[\int_0^{q_1^L} yg(y)dy + \mathbf{P}(D_L \geq q_1^L)q_1^L \right] \\
&\quad + (1 + Q_2') \int_{q_1^H}^{q_1^H+Q_2} \mathbf{P}(D_L \geq q_1^H + q_1^L + Q_2 - x) f(x) dx \\
&\quad + Q_2' \left[\int_0^{q_1^L} yg(y)dy + \mathbf{P}(D_L \geq q_1^L)q_1^L \right] f(q_1^H + Q_2) \\
&\quad + (1 + Q_2')\mathbf{P}(D_H \leq q_1^H)\mathbf{P}(D_L \geq q_1^L + Q_2)
\end{aligned}$$

Then,

$$\begin{aligned} & \frac{\partial[E(SL_1) + E(SL_2) + E(SL_3)]}{\partial q_1^L} \\ &= \mathbb{P}(D_H \geq q_1^H + Q_2) \mathbb{P}(D_L \geq q_1^L) + (1 + Q_2') \int_{q_1^H}^{q_1^H + Q_2} \mathbb{P}(D_L \geq q_1^H + q_1^L + Q_2 - x) f(x) dx \\ & \quad + (1 + Q_2') \mathbb{P}(D_H \leq q_1^H) \mathbb{P}(D_L \geq q_1^L + Q_2) \end{aligned}$$

From equation (4.2) on page 77, we get

$$\begin{aligned} \frac{\partial E(R_L)}{\partial q_1^L} &= \int_{q_1^H}^{q_1^H + Q_2} \left[Q_2' \mathbb{P}(D_L \geq q_1^H + q_1^L + Q_2 - x) \right. \\ & \quad \left. - (q_1^H + Q_2 - x) g(q_1^H + q_1^L + Q_2 - x) (1 + Q_2') \right. \\ & \quad \left. - \int_{q_1^L}^{q_1^H + q_1^L + Q_2 - x} g(y) dy + (q_1^H + Q_2 - x) g(q_1^H + q_1^L + Q_2 - x) (1 + Q_2') \right] f(x) dx \\ & \quad + \mathbb{P}(D_H \leq q_1^H) \left[Q_2' \mathbb{P}(D_L \geq q_1^L + Q_2) - (1 + Q_2') Q_2 g(q_1^L + Q_2) \right. \\ & \quad \left. - \int_{q_1^L}^{q_1^L + Q_2} g(y) dy + (1 + Q_2') Q_2 g(q_1^L + Q_2) \right] \\ &= \int_{q_1^H}^{q_1^H + Q_2} \left[Q_2' \mathbb{P}(D_L \geq q_1^H + q_1^L + Q_2 - x) - \int_{q_1^L}^{q_1^H + q_1^L + Q_2 - x} g(y) dy \right] f(x) dx \\ & \quad + \mathbb{P}(D_H \leq q_1^H) \left[Q_2' \mathbb{P}(D_L \geq q_1^L + Q_2) - \int_{q_1^L}^{q_1^L + Q_2} g(y) dy \right] \\ &= Q_2' \int_{q_1^H}^{q_1^H + Q_2} \mathbb{P}(D_L \geq q_1^H + q_1^L + Q_2 - x) f(x) dx - \int_{q_1^H}^{q_1^H + Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx \\ & \quad + G(q_1^L) [F(q_1^H + Q_2) - F(q_1^H)] \\ & \quad + Q_2' \mathbb{P}(D_H \leq q_1^H) \mathbb{P}(D_L \geq q_1^L + Q_2) - F(q_1^H) [G(q_1^L + Q_2) - G(q_1^L)] \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial E(R_L)}{\partial q_1^L} = & Q_2' \int_{q_1^H}^{q_1^H+Q_2} P(D_L \geq q_1^H + q_1^L + Q_2 - x) f(x) dx - \int_{q_1^H}^{q_1^H+Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx \\ & + Q_2' P(D_H \leq q_1^H) P(D_L \geq q_1^L + Q_2) + G(q_1^L) F(q_1^H + Q_2) - F(q_1^H) G(q_1^L + Q_2) \end{aligned}$$

The first order deviation on q_1^L to buyer L's profit function:

$$\begin{aligned} & r \left\{ (1 - F(q_1^H + Q_2))(1 - G(q_1^L)) + (1 + Q_2') [F(q_1^H + Q_2) - F(q_1^H)] \right. \\ & \left. - (1 + Q_2') \int_{q_1^H}^{q_1^H+Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx + (1 + Q_2') F(q_1^H) (1 - G(q_1^L + Q_2)) \right\} \\ & - w_1 + s G(q_1^L) - w_2 \left\{ Q_2' \int_{q_1^H}^{q_1^H+Q_2} (1 - G(q_1^H + q_1^L + Q_2 - x)) f(x) dx \right. \\ & \left. - \int_{q_1^H}^{q_1^H+Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx \right. \\ & \left. + Q_2' F(q_1^H) (1 - G(q_1^L + Q_2)) + G(q_1^L) F(q_1^H + Q_2) - F(q_1^H) G(q_1^L + Q_2) \right\} \end{aligned}$$

Which is equivalent to

$$\begin{aligned}
& r \left\{ 1 - F(q_1^H + Q_2) - G(q_1^L) + F(q_1^H + Q_2)G(q_1^L) + F(q_1^H + Q_2) - F(q_1^H) \right. \\
& + Q_2' \left[F(q_1^H + Q_2) - F(q_1^H) \right] - (1 + Q_2') \int_{q_1^H}^{q_1^H + Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx \\
& \left. + F(q_1^H) - F(q_1^H)G(q_1^L + Q_2) + Q_2' F(q_1^H) (1 - G(q_1^L + Q_2)) \right\} - w_1 + s G(q_1^L) \\
& - w_2 \left\{ Q_2' \left[F(q_1^H + Q_2) - F(q_1^H) \right] - Q_2' \int_{q_1^H}^{q_1^H + Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx \right. \\
& - \int_{q_1^H}^{q_1^H + Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx + Q_2' F(q_1^H) \\
& \left. - Q_2' F(q_1^H) G(q_1^L + Q_2) + F(q_1^H + Q_2) G(q_1^L) - F(q_1^H) G(q_1^L + Q_2) \right\}
\end{aligned}$$

Then,

$$\begin{aligned}
& r - rG(q_1^L) - w_1 + sG(q_1^L) + (r - w_2) \left\{ F(q_1^H + Q_2)G(q_1^L) - F(q_1^H)G(q_1^L + Q_2) \right. \\
& \left. - (1 + Q_2') \int_{q_1^H}^{q_1^H + Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx + Q_2' \left[F(q_1^H + Q_2) - F(q_1^H)G(q_1^L + Q_2) \right] \right\}
\end{aligned}$$

Then,

$$\begin{aligned}
& (r - w_1) - (r - s)G(q_1^L) + (r - w_2) \left[F(q_1^H + Q_2)G(q_1^L) - F(q_1^H + Q_2) \right] \\
& + (r - w_2)(1 + Q_2') \left\{ \left[F(q_1^H + Q_2) - F(q_1^H)G(q_1^L + Q_2) \right] - \int_{q_1^H}^{q_1^H + Q_2} G(q_1^H + q_1^L + Q_2 - x) f(x) dx \right\}
\end{aligned}$$

where the optimal solution is $Q_1^L(q_1^H)$.

Therefore, we can conclude,

$$\begin{aligned} \frac{d\Pi_L(q_1^L)}{dq_1^L} \Big|_{q_1^L=0} &= r\mathbf{P}(D_H \geq q_1^H + Q_2) - w_1 \\ &\quad + w_2 \left[\mathbf{P}(D_H \leq q_1^H) + \mathbf{P}(q_1^H \leq D_H \leq Q_2 + q_1^H) \right] \\ &= r\mathbf{P}(D_H \geq q_1^H + Q_2) - w_1 + w_2\mathbf{P}(D_H \leq Q_2 + q_1^H) \end{aligned} \quad (4.28)$$

Furthermore, if $q_1^L = q_1^H = 0$, then we can get

$$\frac{d\Pi_L(q_1^L)}{dq_1^L} \Big|_{q_1^L=q_1^H=0} = \mathbf{P}(D_H \geq Q_2) - w_1 + w_2\mathbf{P}(D_H \leq Q_2) \quad (4.29)$$

§4.7.5 One demand class

After the two parties develop their probabilistic forecast, but before the supplier makes his procurement decision, the buyer informs the supplier that he will buy at least Q_1 units for sure. The additional order size that the supplier must consider is Q_2 .

Then the supplier solves:

$$\begin{aligned} \Pi_{SA}(Q_2) &= w_1 Q_1 + w_2 \left[\int_{Q_1}^{Q_1+Q_2} (x - Q_1) f(x) dx + Q_2 \mathbf{P}(D \geq Q_1 + Q_2) \right] \\ &\quad - c(Q_1 + Q_2) + s \left[\int_{Q_1}^{Q_1+Q_2} (Q_1 + Q_2 - x) f(x) dx + Q_2 F(Q_1) \right] \end{aligned} \quad (4.30)$$

The subscript SA represents supplier (S) with aggregating (A) demand classes. Where $\int_{Q_1}^{Q_1+Q_2} (Q_1 + Q_2 - x) f(x) dx$ is the salvage amount when the buyer's total demand is more than Q_1 but less than $Q_1 + Q_2$, and therefore, the additional demand is $x - Q_1$, the salvage amount is equal to $Q_2 - (x - Q_1) = Q_1 + Q_2 - x$. And also, $Q_2 F(Q_1)$ is the salvage amount when the buyer's demand is less than Q_1 , so the salvage value is $s Q_2$.

Illustrated in another way: if total demand $\geq Q_1$, then, additional demand =

$x - Q_1$, and therefore, salvage = $Q_2 - (x - Q_1) = Q_1 + Q_2 - x$. If total demand $< Q_1$, then, additional demand = 0, and therefore, salvage = Q_2 .

Therefore, the optimal solution $Q_2^*(q_1)$ is the solution to the first order derivative on Q_2 .

$$P(D \leq q_1 + Q_2) = \frac{w_2 - c}{w_2 - s} \quad (4.31)$$

The equation (4.31) is similar to (2) in Cachon (2004) with different assumption about salvaging.

If $q_1 \leq F^{-1}\left(\frac{w_2 - c}{w_2 - s}\right)$, then $Q_2^*(q_1) = F^{-1}\left(\frac{w_2 - c}{w_2 - s}\right) - q_1$. If $q_1 > F^{-1}\left(\frac{w_2 - c}{w_2 - s}\right)$, then $\frac{\partial}{\partial q_2} < 0$. $\forall q_2 \geq 0$, thus $Q_2^*(q_1) = 0$. Therefore, in general, we get the optimal Q_2^* w.r.t. q_1 is:

$$Q_2^*(q_1) = \left[F^{-1}\left(\frac{w_2 - c}{w_2 - s}\right) - q_1 \right]^+ \quad (4.32)$$

The Buyer's profit function:

$$\begin{aligned} \Pi_{BA}(q_1) = & r \left[\int_0^{q_1 + Q_2} x f(x) dx + (q_1 + Q_2) P(D \geq q_1 + Q_2) \right] \\ & - w_1 q_1 - w_2 \left[\int_{q_1}^{q_1 + Q_2} (x - q_1) f(x) dx + Q_2 P(D \geq q_1 + Q_2) \right] \\ & + s \int_0^{q_1} (q_1 - x) f(x) dx \end{aligned} \quad (4.33)$$

The subscript BA represents buyer (B) with aggregating (A) demand classes.

Optimal pre-order decision by the buyer

There are three different optimal pre-order decisions by the buyer:

1. PUSH.

If $c \leq w_1 \leq \widetilde{w}_1(w_2)$, $q_1 = F^{-1}\left(\frac{r - w_1}{r - s}\right)$, and $Q_2^*(q_1) = 0$.

2. PAB.

If $\widetilde{w}_1(w_2) < w_1 \leq w_2$, $q_1 = F^{-1}\left(\frac{w_2-w_1}{w_2-s}\right)$, and $Q_2^*(q_1) = F^{-1}\left(\frac{w_2-c}{w_2-s}\right) - F^{-1}\left(\frac{w_2-w_1}{w_2-s}\right) \geq 0$.

3. PULL.

If $w_2 \leq w_1 \leq r$, $q_1 = 0$, $Q_2^* = F^{-1}\left(\frac{w_2-c}{w_2-s}\right)$.

The optimal profit of the supplier

Based on the optimal decisions made by the buyer, we find the optimal profit of the supplier with the corresponding interval of w_1 .

$$\Pi_{SA}^*(w_1) = \begin{cases} (w_1 - c)F^{-1}\left(\frac{r-w_1}{r-s}\right) & c \leq w_1 \leq \widetilde{w}_1(w_2) \\ (w_2 - s) \left[\int_0^{F^{-1}\left(\frac{w_2-c}{w_2-s}\right)} x f(x) dx - \int_0^{F^{-1}\left(\frac{w_2-w_1}{w_2-s}\right)} x f(x) dx \right] & \widetilde{w}_1(w_2) < w_1 \leq w_2 \\ (w_2 - s) \int_0^{F^{-1}\left(\frac{w_2-c}{w_2-s}\right)} x f(x) dx & w_2 \leq w_1 \leq r \end{cases}$$

The optimal profit of the buyer

$$\Pi_{BA}^* = \begin{cases} (r - s) \int_0^{F^{-1}\left(\frac{r-w_1}{r-s}\right)} x f(x) dx & c \leq w_1 \leq \widetilde{w}_1(w_2) \\ (r - w_2) \int_0^{F^{-1}\left(\frac{w_2-c}{w_2-s}\right)} x f(x) dx \\ \quad + (w_2 - s) \int_0^{F^{-1}\left(\frac{w_2-w_1}{w_2-s}\right)} x f(x) dx \\ \quad + (r - w_2) \frac{c-s}{w_2-s} F^{-1}\left(\frac{w_2-c}{w_2-s}\right) & \widetilde{w}_1(w_2) < w_1 \leq w_2 \\ (r - w_2) \int_0^{F^{-1}\left(\frac{w_2-c}{w_2-s}\right)} x f(x) dx + (r - w_2) \frac{c-s}{w_2-s} F^{-1}\left(\frac{w_2-c}{w_2-s}\right) & w_2 \leq w_1 \leq r \end{cases}$$

Table 4.1: For selected values of w_2 : the optimal pre-order price and resulting consequence when using the full model

w_2	w_1	Q_1^H	Q_1^L	Q_2	Π_S	Π_H	Π_L
8	10	0	10	10	69.28	170.24	147.14
10	11	0	5	15	78.32	175.01	146.17
12	13	0	3	18	107.86	160.57	137.31
14	14	0	2	20	138.18	143.43	128.46
16	16	0	1	22	172.07	125.79	115.23
18	18	0	1	22	206.51	107.82	98.76

Table 4.2: For selected values of w_2 : the optimal pre-order price and resulting consequence when using the aggregate model

w_2	w_1	Q_1	Q_2	Π_S	Π_A
8	8	23	0	69.00	344.10
10	10	0	19	71.46	321.14
12	12	0	21	104.42	301.35
14	14	0	22	138.04	272.05
16	16	0	23	172.05	241.05
18	18	0	23	206.49	206.61

5

Inventory Rationing with Real Options

History: This project started at Aarhus on 2011.

Inventory Rationing with Real Options

Bisheng Du*

Abstract

We consider an inventory system consisting of two supply sources and one buyer. The first supply source is the real options contract provider while the other supply source is a capacitated external market. The buyer's unsatisfied demands are treated as lost sales. We formulate the problem as a dynamic programming model in order to perform some numerical investigations. In particular, we study the effect of using real options compared to a restricted situation where real options are not applied.

Keyword

Inventory control; Real options; Dynamic programming

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§5.1 Introduction

In a supply chain there may be suppliers providing contracts for products and a “market” selling the actual products. In this paper, we present an inventory system including a real options contract in multiple periods which consist of two supply sources. A real options is the flexibility of a logistics manager for making decisions about the manufacturing in their factory, which have great effects on their downstream partners.

The real options we consider is a form of supply chain contract, many common characteristics exist between supply chain contracts and the derivatives literature: there are some options like contracts, for instance, buy-back policies, backup agreement, quantity flexibility, etc. But the options contracts above are embedded into a contract. The real options, as we investigate in this paper, which is isolated from the supply chain contract, can be traded in a market, e.g. [Shi et al. \(2004\)](#).

[Barnes-Schuster et al. \(2002\)](#) investigate a real options problem of a two-period model with correlated demand. [Wu and Kleindorfer \(2005\)](#) extend the capacity options into the electronic market with competitive buyers. For more detail on the real options, please refer to [Sick \(1995\)](#) and [Luenberger \(1998\)](#).

In this paper, we study an inventory system of two supply sources and one buyer, the buyer’s unsatisfied demands are lost sales. The two supply sources are specified as follows:

- Supplier: with whom we can make options contract. The options price is p per unit, and the exercise price is c_1 per unit. We assume the options contract has signed one time period in advance, which means the buyer should decide the exercise quantity, and then receive the delivery from the supplier in the next period.
- Market: the external market capacity C_{mar} is a random variable, the actual capacity is known at the beginning of each period. C_{mar} is a random variable. The purchase cost is c_2 per unit.

In Section 5.2 the assumptions are stated and the dynamic programming model is formulated. Some characterization and discussion of special cases are in Section 5.3. A numerical study can be found in Section 5.4 while the conclusion and final remarks are in Section 5.5.

§5.2 Model formulation

§5.2.0 Notations and parameters list

- p , the options contract price per unit.
- c_1 , the options exercise price per unit.
- c_2 , the purchase price per unit from external market.
- n , ($n = 1, 2, \dots, N$) denotes the number of remaining decisions.
- C_{mar} , a random variable, the external market capacity.
- D , a random variable, the current period demand.
- q_{1n} , the number of units with real options contract are exercised at the period n , where $n = 1, 2, \dots, N$.
- q_{2n} , the additional units purchase from the “Market” at the period n , where $n = 1, 2, \dots, N$.
- k , the real options’ capacity of period n , where $n = 1, 2, \dots, N$.
- C , the external market capacity of period n , where $n = 1, 2, \dots, N$.
- i , the ending inventory of period n , where $n = 1, 2, \dots, N$.
- c_{new} , the new external market capacity in the next period.
- k_{new} , the new options available in the next period.
- k_{max} , the maximum options available in the current period.

- i_{new} , the new inventory available in the next period.
- i_{max} , the maximum inventory available in the current period.
- r , the price selling to the buyers.
- h , the holding cost per period per unit.
- s , the price for the leftover in the last period sold as scrap in the secondary market.
- Δ , the unit overprice, $\Delta = p + c_1 - c_2$.

§5.2.1 Assumptions

Consider an inventory system that operates in a finite horizon of N periods, we assume the options contract with the supplier have one period time delay, which means the lead-time of products from the supplier is one period. In contrast, the products from the external market can be purchased and received simultaneously within the market's capacity.

Assumption 5.1. In principle we can build our model and have all situations: (a). $p + c_1 \geq c_2$, or (b). $c_1 \leq p + c_1 \leq c_2$.

Where p is the contract price per unit, c_1 is the exercise price per unit and c_2 is the purchase price per unit from external market. If $p + c_1 \geq c_2 > c_1$. There may also exist the same condition, which is an opposite situation of the basic assumption above, we may also discuss this case later. If it happens, the reason for the buyer to choose the supplier with a real options contract is to guarantee supply. In this case, if the buyer also sells it at the same price, which will reduce the buyer's profit margin, therefore, the buyer may also increase the retail price or adjust his distribution channel policy.

If $c_1 \leq p + c_1 \leq c_2$. The assumption $p + c_1 \leq c_2$ reflects that the product purchased from the market more than the products from the supplier. The condition $p + c_1 \leq c_2$ establishes that the buyer is price sensitive. If $p + c_1 = c_2$, then the

buyer can choose between the market without lead-time delay and the supplier with lot size guarantee. Given $c_1 \leq p + c_1$, if $p = 0$, the real options contract is changed, the options price equals the purchasing cost. On the other hand, the full price real options contract leverage the payment guaranty from both of them to the buyer.

Assumption 5.2. In each period, the buyer's on-hand inventory is only available to the current demands, the unsatisfied demands of previous periods are treated as lost sales. The inventory of the new period is $i_{new} = (i + q_1 + q_2 - d)^+$.

Any unsatisfied demands are treated as lost sales. Therefore, if there are demands which are not be fulfilled by the current period's leftover inventory, then the current period's unsatisfied demands will not be delivered in the next period.

§5.2.2 Dynamic programming formulation

The terminology used to define the dynamic programming model of this paper includes the following main elements:

Stage/Period (n): This inventory system is divided into N periods, there is an initial stage (Stage 1 or Period N) and a terminating stage (Stage N or Period 1). The index n ($n = 1, 2, \dots, N$) denotes the number of remaining decisions.

State: Each stage has a number of states associated with it. For instance, in period n , there are three states k, C, i , which represent the real options' capacity, external market capacity and the ending inventory of period n .

Decision variable: There are three decision variables in period n (q_{1n}, q_{2n} and k_{new}). q_{1n} is the number of units with real options contract are exercised at the period n , where $n = 1, 2, \dots, N$. q_{2n} is the additional units purchase from the "Market" at the period n , where $n = 1, 2, \dots, N$. k_{new} is the new options available in the next period.

Boundary conditions: There are the initial conditions at stage N (or Period 1) and the terminal conditions at stage 1 (or Period N), therefore, the expected profit expressions of stage N (or Period 1) and stage 1 (or Period N) are given from the general expression of period n .

Figure 5.1 is about here

We build the Dynamic Programming models below:

The expected profit of the general period n (where $n = 2, \dots, N - 1$) is

$$\begin{aligned}
 & V_n(k, C, i) \\
 = & \max_{q_{1n} \leq k, q_{2n} \leq C, k_{new} \leq k_{max}} -c_1 q_{1n} - c_2 q_{2n} + rE[\min(i + q_{1n} + q_{2n}, D)] \\
 & - hE[i + q_{1n} + q_{2n} - D]^+ - pk_{new} \\
 & + \sum_d \sum_{c_{new}} V_{n-1}(k_{new}, c_{new}, (i + q_{1n} + q_{2n} - d)^+) P(D = d)P(C_{mar} = c_{new})
 \end{aligned} \tag{5.1}$$

Where r is the price selling to the buyers, h is the holding cost per period per unit, c_{new} is the new capacity developed for the next period, D is the current period demand.

In period n , there are three states (k, C, i) and three decision variables $(q_{1n}, q_{2n}$ and $k_{new})$. k_{max} is an upper bound on the number of options that can be bought.

We also consider the expected wasted options, meaning the un-exercised amount of purchased options.

$$\begin{aligned}
 EWaste_n(k, C, i) &= k - q_{1n}^* \\
 &+ \sum_d \sum_{c_{new}} EWaste_{n-1}(k_{new}^*, c_{new}^*, (i + q_{1n}^* + q_{2n}^* - d)^+) P(D = d)P(C_{mar} = c_{new}^*)
 \end{aligned} \tag{5.2}$$

Where $k_{new}^*, q_{1n}^*, q_{2n}^*$ are the optimal decision in (5.1).

Boundary conditions of period N

In the first stage, which is the period N , the expected profit is:

$$V_N(C) = \max_{q_{2N} \leq C, k_{new} \leq k_{max}} -c_2 q_{2N} - p k_{new} + rE[\min(q_{2N}, D)] - hE[q_{2N} - D]^+ + \sum_d \sum_{c_{new}} V_{N-1}(k_{new}, c_{new}, (q_{2N} - d)^+) P(D = d) P(C_{mar} = c_{new}) \quad (5.3)$$

Thus we assume the initial inventory is 0.

In period N , there is only one state (C) and two decision variables (q_{2N} and k_{new}).

Similarly as above, the expected wasted options of period N is,

$$EWaste_N(C) = \sum_d \sum_{c_{new}} EWaste_{n-1}(k_{new}^*, c_{new}, (q_{2N}^* - d)^+) P(D = d) P(C_{mar} = c_{new}) \quad (5.4)$$

Where k_{new}^* , q_{2N}^* are the optimal decision in (5.3).

Boundary conditions of period 1

The expected profit when having a real options of k units and the current capacity from the “Market” is C_{mar} , initial inventory is i , the leftover in the last period sold as scrap s . Therefore, the expected profit of period 1 (the last stage) is

$$V_1(k, C, i) = \max_{q_{11} \leq k, q_{21} \leq C} -c_1 q_{11} - c_2 q_{21} + rE[\min(i + q_{11} + q_{21}, D)] + sE[i + q_{11} + q_{21} - D]^+ \quad (5.5)$$

In the period 1, there are three states (k , C , i) and two decision variables (q_{11} and q_{21}).

Similarly as above, the expected wasted options of period 1 is,

$$EWaste_1(k, C, i) = k - q_{11}^* \quad (5.6)$$

Where q_{11}^* is the optimal decision in (5.5).

§5.3 Optimal policy

§5.3.1 Special case: $k_{\max} = 0$, Capacitated Base Stock Policy

If $k_{\max} = 0$, then no real options can be made, which equals to $q_1 = 0$, and it means the buyer can only purchase from the market but have no probability to purchase products from the supplier. Therefore, we solve this multiple periods inventory problem by use backwards recursions method. Then, the expected profit of the last stage (period 1), which is just a newsvendor:

$$\begin{aligned} V_1(0, C, i) &= \max_{q_{21} \leq C} -c_2 q_{21} + rE[\min(i + q_{21}, D)] + sE[i + q_{21} - D]^+ \\ &= \max_{q_{21} \leq C} -c_2 q_{21} + r \left[\int_0^{i+q_{21}} xf(x)dx + \int_{i+q_{21}}^{\infty} (i + q_{21})f(x)dx \right] \\ &\quad + s \int_0^{i+q_{21}} (i + q_{21} - x)f(x)dx \end{aligned} \quad (5.7)$$

First order derivative on q_{21} , we can get:

$$\begin{aligned} \frac{\partial V_1(0, C, i)}{\partial q_{21}} &= -c_2 + r \left[(i + q_{21})f(i + q_{21}) + \int_{i+q_{21}}^{\infty} f(x)dx - (i + q_{21})f(i + q_{21}) \right] \\ &\quad + s \int_0^{i+q_{21}} f(x)dx \\ &= r - c_2 - (r - s) \int_0^{i+q_{21}} f(x)dx \end{aligned} \quad (5.8)$$

Therefore, the optimal quantity of q_{21} is:

$$F(i + q_{21}) = \frac{r - c_2}{r - s} \quad (5.9)$$

In the normal case, $q_{21} = F^{-1}\left(\frac{r-c_2}{r-s}\right) - i$; when there is no options, the optimal quantity of options is $q_{21} = 0$; when there is not enough capacity, the optimal quantity of options is $q_{21} = C$. Generally,

$$q_{21}^* = \min\left(C, \max\left(F^{-1}\left(\frac{r - c_2}{r - s}\right) - i, 0\right)\right) \quad (5.10)$$

Therefore, given i from period 2, we can get the optimal q_{21} of period 1, and also, we can use the given information from period 3, etc., therefore, we use backward recursions to solve it in periods until period $N - 1$.

The expected profit of the regular n_{th} (where $n = 2, \dots, N - 1$) period is

$$\begin{aligned} & V_n(k, C, i) \\ &= \max_{q_{2n} \leq c_{new}} -c_2 q_{2n} + rE[\min(i + q_{2n}, D)] - hE[i + q_{2n} - D]^+ - pk_{new} \quad (5.11) \\ & \quad + \sum_d \sum_{c_{new}} V_{n-1}(k_{new}, c_{new}, (i + q_{2n} - D)^+) P(D = d)P(C_{mar} = c_{new}) \end{aligned}$$

§5.3.2 Special case: $P(C = 0) = 1$

If $P(C = 0) = 1$, which means the buyer have no chance to purchase products from the market but only from the supplier with a real options contract. Then, the

expected profit of the last stage (period 1) is

$$\begin{aligned}
 V_1(k, 0, i) &= \max_{q_{11} \leq k} -c_1 q_{11} + rE[\min(i + q_{11}, D)] + sE[i + q_{11} - D]^+ \\
 &= \max_{q_{11} \leq k} -c_1 q_{11} + r \left[\int_0^{i+q_{11}} xf(x)dx + \int_{i+q_{11}}^{\infty} (i + q_{11})f(x)dx \right] \\
 &\quad + s \int_0^{i+q_{11}} (i + q_{11} - x)f(x)dx
 \end{aligned} \tag{5.12}$$

Therefore, the optimal quantity of q_{11} is:

$$F(i + q_{11}) = \frac{r - c_1}{r - s} \tag{5.13}$$

Similarly, the condition of $q_{11} = 0$, which means there is no options, and the condition of $q_{11} = k$, which means there is limited capacity of real options, are also discussed. Then the general expression of optimal quantity of q_{11} is,

$$q_{11}^* = \min\left(k, \max\left(F^{-1}\left(\frac{r - c_1}{r - s}\right) - i, 0\right)\right) \tag{5.14}$$

The expected profit of the regular n_{th} (where $n = 2, \dots, N - 1$) period is

$$\begin{aligned}
 &V_n(k, 0, i) \\
 &= \max_{q_{1n} \leq k} -c_1 q_{1n} + rE[\min(i + q_{1n}, D)] - hE[i + q_{1n} - D]^+ - pk_{new} \\
 &\quad + \sum_d \sum_{c_{new}} V_{n-1}(k_{new}, c_{new}, (i + q_{1n} - D)^+) P(D = d)P(C_{mar} = c_{new})
 \end{aligned} \tag{5.15}$$

§5.3.3 $k_{\max} > 0$ and $P(C = 0) < 1$

When $k_{\max} > 0$ and $P(C = 0) < 1$, which means there are both supply sources available: the real options provider and external market.

The expected profit of the last stage (period 1) is:

$$\begin{aligned}
V_1(k, C, i) &= \max_{q_{11} \leq k, q_{21} \leq C} -c_1 q_{11} - c_2 q_{21} + rE[\min(i + q_{11} + q_{21}, D)] \\
&\quad + sE[i + q_{11} + q_{21} - D]^+ \\
&= \max_{q_{11} \leq k, q_{21} \leq C} -c_1 q_{11} - c_2 q_{21} \\
&\quad + r \left[\int_0^{i+q_{11}+q_{21}} x f(x) dx + \int_{i+q_{11}+q_{21}}^{\infty} (i + q_{11} + q_{21}) f(x) dx \right] \\
&\quad + s \int_0^{i+q_{11}+q_{21}} (i + q_{11} + q_{21} - x) f(x) dx
\end{aligned} \tag{5.16}$$

Note that options are paid in a previous stage.

If $c_1 \leq c_2$, based on different priorities between them, the buyer will first have to choose the real options provider, after that, if there is some further need, he will purchase from the external market. Therefore, we will solve $\frac{\partial}{\partial q_{11}} = 0$ firstly, and then given the optimal q_{11}^* , we insert it to the profit expression to solve $\frac{\partial}{\partial q_{21}} = 0$. Obviously we will first use all options before buying in the market, therefore, the optimal q_{11} is given as:

$$q_{11}^* = \max \left\{ k, F^{-1} \left(\frac{r - c_1}{r - s} \right) - i \right\} \tag{5.17}$$

This optimal expression considered the limited real options quantity. Given q_{11}^* ,

the first order partial derivative on q_{21} is:

$$\begin{aligned}
 \frac{\partial V_1(k, C, i)}{\partial q_{21}} &= -c_2 + r \left[(i + q_{11}^* + q_{21})f(i + q_{11}^* + q_{21}) \right. \\
 &\quad \left. + \int_{i+q_{11}^*+q_{21}}^{\infty} f(x)dx - (i + q_{11}^* + q_{21})f(i + q_{11}^* + q_{21}) \right] \\
 &\quad + s \int_0^{i+q_{11}^*+q_{21}} f(x)dx \\
 &= r - c_2 - (r - s) \int_0^{i+q_{11}^*+q_{21}} f(x)dx
 \end{aligned} \tag{5.18}$$

Therefore, the optimal quantity of q_{21} , $q_{21}^*(q_{11}^*)$ is:

$$F(i + q_{11}^* + q_{21}^*(q_{11}^*)) = \frac{r - c_2}{r - s} \tag{5.19}$$

Therefore, the general solution is:

$$q_{21}^* = \min \left(C, \max \left(F^{-1} \left(\frac{r - c_2}{r - s} \right) - i - q_{11}^*, 0 \right) \right) \tag{5.20}$$

The case $c_1 \geq c_2$ could be analyzed similarly.

It is very hard to derive closed form expression for the optimal actions when rolling the time further back. Therefore we will in the following resort to numerical analysis.

§5.4 Numerical study

The focal question for OM analysis is: since options give a guarantee for future delivery, there might be over-price for the material delivered through options. We define the unit overprice Δ as $\Delta = p + c_1 - c_2$. Therefore the analysis is also done with regard to sensitivity analysis on this parameter. Because this overprice can stay constant for many combinations of p and c_1 , we also study the impact of the

composition of options price p and exercise price c_1 . Finally the main concern of the analysis is to study the effect of introducing an options arrangement: we always make comparisons to the case where no options are used. As this comparison can be quite involved as there are many states, we assume that the analysis is conducted prior to any knowledge about the capacity development. As a kind of performance measurement for an options contract, we use

$$PO = \sum_C V_N(C)P(C_{mar} = C) \quad (5.21)$$

$$EWaste = \sum_C EWaste_N(C)P(C_{mar} = C) \quad (5.22)$$

Similarly the performance of the no-options expected profit is computed as

$$PN = \sum_C V_N(C)|_{k_{max}=0} P(C_{mar} = C) \quad (5.23)$$

We assume the demand follows Poisson Distribution, $D \sim \text{Po}(\lambda)$, the Market capacity follows a Discrete Uniform Distribution, $C_{mar} \sim [c_{min}, c_{max}]$.

In this numerical investigation, we set the parameters as: $\lambda = 5, s = 1, r = 12, h = 2, i_{max} = 10, k_{max} = 10$. When $c_1 + p > c_2$, and $p < c_2$, we assume the difference between the options paid and the external price is Δ , then, $\Delta = c_1 + p - c_2$, which we may call it “overprice”. Therefore, e.g. $N = 3$. when $c_2 = 5$ and $\Delta = 1$, then $p = 1$, so $c_1 = 5$.

When there is $c_1 + p > c_2$, we compare them for $N = 4, 5, 6$. When $c_2 = 4$, see Table 5.1 on $N = 4$, Table 5.2 on $N = 5$, Table 5.3 on $N = 6$. Where $PO(4)$ stands for the profit with options contract when $c_1 = 4$. This explain how the tables, seen below, are constructed.

Tables 5.1- 5.6 are about here

When examining the tables we see that the increase in profit using options compared to not using options is modest in the range from 0% to 5%. Naturally the increase is highest when Δ is lowest and p is lowest. In these cases, most waste in terms of unused options can also be seen. In the opposite scenario where the overprice is negative, the options is very favorable. The results of this analysis are seen in the Tables 5.7 - 5.9.

Tables 5.7 - 5.9 are about here

Obviously, the gap between the options case and the no options case is very dramatic. But we also see that although the options are cheap still some waste in terms of unused options is encountered.

From Table 5.1 to Table 5.3, we can easily find, along with p increasing from 1 to 3, the profit/ratio/expected waste are all decreasing. But in general, those three tables are kept with the same trend. The result is obvious, because we set the maximum inventory and the market capacity, both of them are higher than the real needs. Therefore, if the buyer can freely choose between the two supply sources, then he has no constraints on supply, but just options prices. In this case, as the options price increases, they will loose partial profit by some unexpected waste options, certainly, when the options price increase, and they will also prefer to keep lower expected waste options by purchasing mostly from the options contract provider.

Next, we adjust $c_2 = 5$ to find the data by $N = 4, 5, 6$. When $c_2 = 5$, see Table 5.4 on $N = 4$. see Table 5.5 on $N = 5$. see Table 5.6 on $N = 6$.

If $c_1 + p < c_2$, we compare them for $N = 4, 5, 6$, see Table 5.7 on $N = 4$, Table 5.8 on $N = 5$, Table 5.9 on $N = 6$.

Figure 5.2 is about here

We compare the average profit among $N = 4, 5, 6$ when $c_2 = 4$, which is shown in Figure 5.2. If there is no options, the average profit keeps in the lowest level. Along with the decrease by Δ from 4 to 1, the average profit is increasing. We could say, with the alternative scenario of real options contract, the buyers have more sources to relax their risk, therefore, if there is no more difference on prices between the real options provider and the external market, they may have more flexibilities to choose between them, see the highest level when $\Delta = 1$. Otherwise, with the uncertainty of the external market and higher price gap between them, the average profit will keep in a lower level. Figure 5.3 shows similar trend for $c_2 = 5$.

Figure 5.3 is about here

There are some differences in Figure 5.4, the average profit are given by using $\Delta \leq 0$, which means $c_1 + p \leq c_2$, the buyer has more incentive to buy adjustable amount from the external market.

Figure 5.4 is about here

When we use the data collected by $p = 1, 2, 3, 4, 5, 6$ when $c_2 = 7$ and $N = 4$, we found that the profits increasing slopes are almost the same as the ones when the buyer adapt “No options” and $\Delta = 0, -1, -2, -3$. By contrast, when we use $c_2 = 4$, we can get same decreasing slopes when the buyer adapt $\Delta = 1, 2, 3, 4$ and “No options”.

Figure 5.5 is about here

§5.5 Conclusion and remarks

We have developed a dynamic programming model to study the use of options to secure future supply of raw material in a possibly constrained supply market. The numerical results show a small increase in profits from using this arrangement in the “normal” case where the total unit price is higher than the external market price. However, so far not have numerical experiments been conducted. Though the dynamic programming model is somewhat involved, some obvious opportunities exist for how it could be further considered. Firstly, the market capacity could evolve after a Markov chain instead of as in the present model that each realization of market capacity is independent of what was observed in the previous period. Secondly, the market price c_2 could be decreasing as a function of the available market capacity. Thirdly the options need not be exercised in the next period, but could have longer time interval where it is possible to exercise them. There are a lot of possible future avenues to continue this research work.

Acknowledgement

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§5.6 Figures and Tables

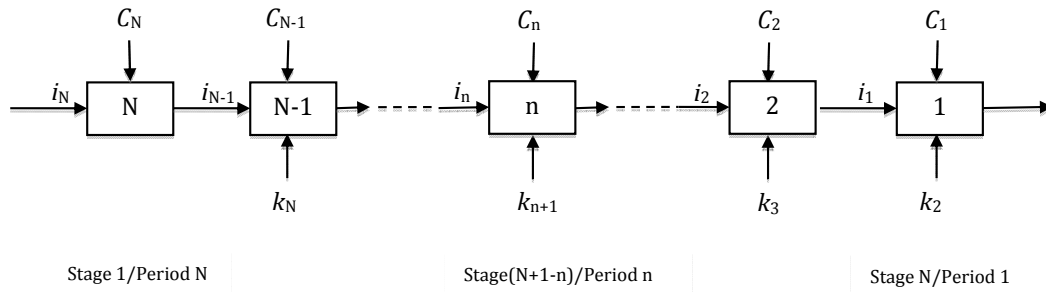


Figure 5.1: The DP structure of this inventory system with real options contract

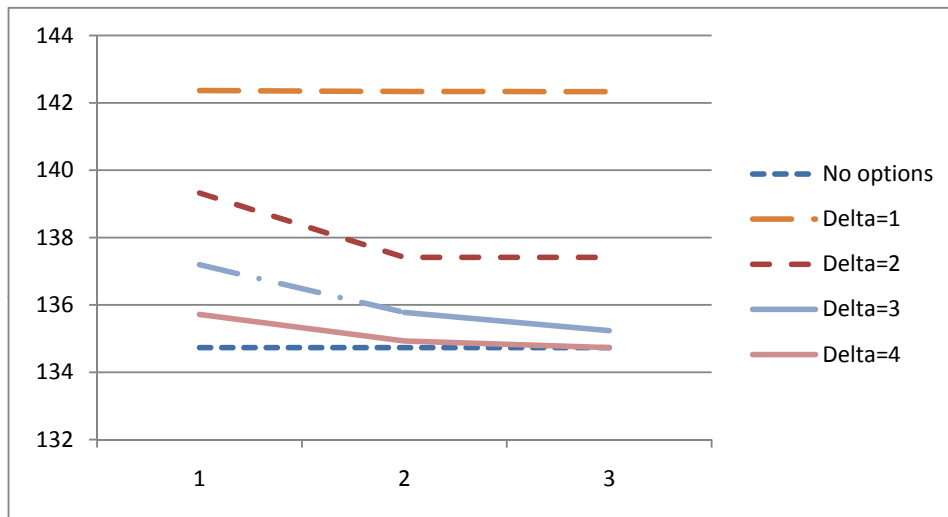


Figure 5.2: Average profit when c_2 is 4, we use the average value of $N = 4, 5, 6$. The horizontal axis is p value while the vertical axis is the profit.

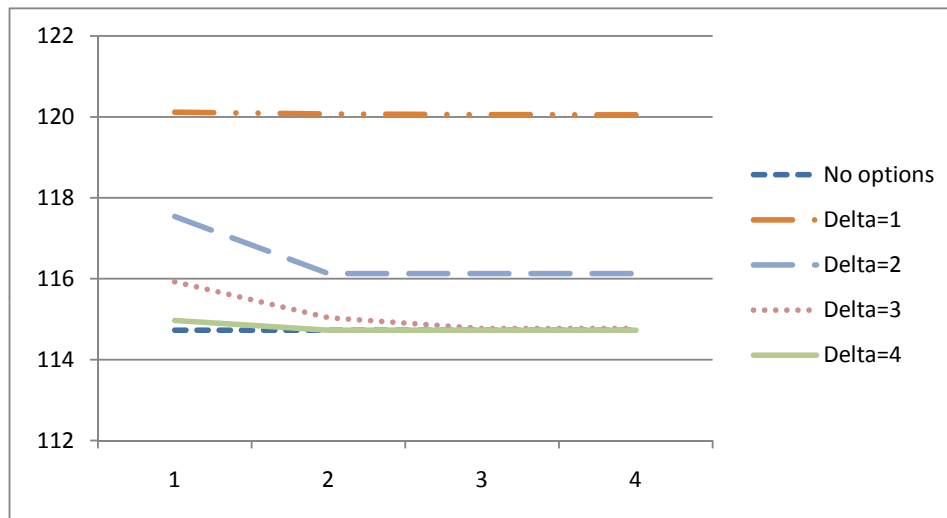


Figure 5.3: Average profit when c_2 is 5, we use the average value of $N = 4, 5, 6$. The horizontal axis is p value while the vertical axis is the profit.

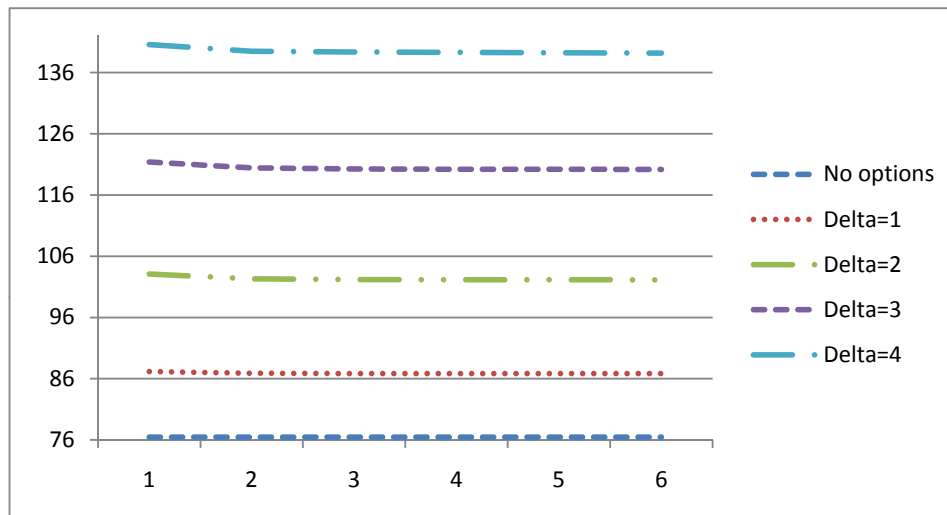


Figure 5.4: Average profit when c_2 is 7, we use the average value of $N = 4, 5, 6$. The horizontal axis is p value while the vertical axis is the profit.

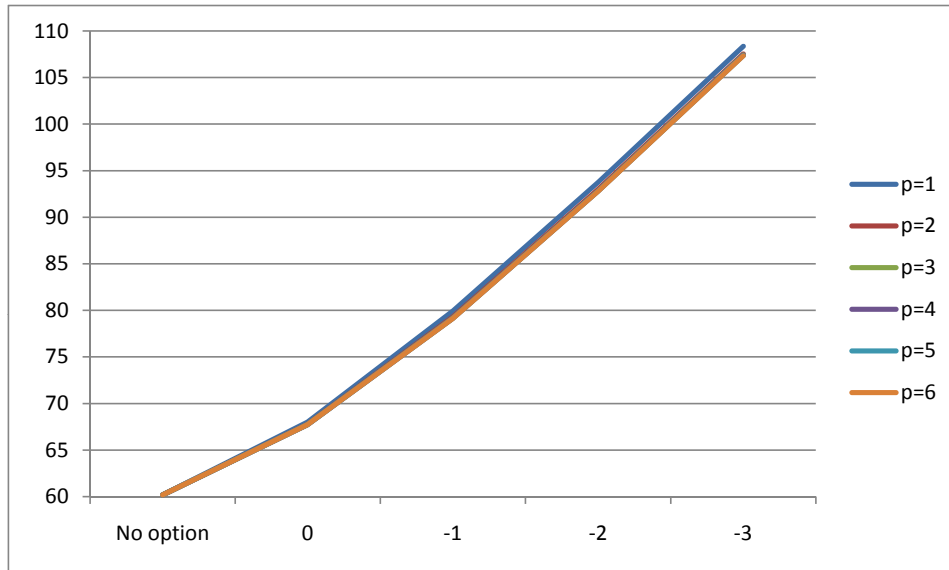


Figure 5.5: The profit trends when $c_2 = 7$ and $N = 4$ by use the options price $p = 1, 2, 3, 4, 5, 6$. The horizontal axis is p value while the vertical axis is the profit.

Table 5.1: $N = 4, c_2 = 4. (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$

Profit of No Options = 106.604			
$\Delta \backslash p$	1	2	$c_2 - 1$
1	PO(4)=112.419 %=5.455 EW=3.112	PO(3)=112.394 %=5.431 EW=0.006	PO(2)=112.391 %=5.429 EW=0.000
2	PO(5)=110.106 %=3.285 EW=2.461	PO(4)=108.621 %=1.892 EW=0.965	PO(3)=108.621 %=1.892 EW=0.000
3	PO(6)=108.480 %=1.760 EW=1.866	PO(5)=107.378 %=0.726 EW=0.700	PO(4)=106.954 %=0.328 EW=0.212
4	PO(7)=107.339 %=0.689 EW=1.249	PO(6)=106.739 %=0.127 EW=0.266	PO(5)=106.604 %=0.000 EW=0.000

Table 5.2: $N = 5, c_2 = 4. (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$

Profit of No Options = 134.721			
$\Delta \backslash p$	1	2	3
1	PO(4)=142.362 %=5.672 EW=3.947	PO(3)=142.335 %=5.651 EW=0.006	PO(2)=142.331 %=5.649 EW=0.000
2	PO(5)=139.324 %=3.417 EW=3.306	PO(4)=137.412 %=1.997 EW=1.218	PO(3)=137.412 %=1.997 EW=0.000
3	PO(6)=137.189 %=1.832 EW=2.416	PO(5)=135.775 %=0.782 EW=0.888	PO(4)=135.230 %=0.378 EW=0.262
4	PO(7)=135.712 %=0.736 EW=1.581	PO(6)=134.922 %=0.149 EW=0.360	PO(5)=134.721 %=0.000 EW=0.000

Table 5.3: $N = 6, c_2 = 4. (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$

Profit of No Options = 162.866			
$\Delta \backslash p$	1	2	3
1	PO(4)=172.305 %=5.796 EW=4.780	PO(3)=172.275 %=5.777 EW=0.006	PO(2)=172.272 %=5.775 EW=0.000
2	PO(5)=168.544 %=3.486 EW=4.153	PO(4)=166.203 %=2.049 EW=1.474	PO(3)=166.203 %=2.049 EW=0.000
3	PO(6)=165.903 %=1.865 EW=2.959	PO(5)=164.184 %=0.809 EW=1.070	PO(4)=163.522 %=0.403 EW=0.324
4	PO(7)=164.100 %=0.758 EW=1.911	PO(6)=163.127 %=0.160 EW=0.451	PO(5)=162.866 %=0.000 EW=0.000

Table 5.4: $N = 4, c_2 = 5. (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$

Profit of No Options = 90.615				
$\Delta \backslash p$	1	2	3	$c_2 - 1$
1	PO(5)=94.667 %=4.472 EW=3.379	PO(4)=94.625 %=4.425 EW=0.029	PO(3)=94.611 %=4.410 EW=0.005	PO(2)=94.609 %=4.408 EW=0.000
2	PO(6)=92.713 %=2.315 EW=1.861	PO(5)=91.620 %=1.109 EW=0.763	PO(4)=91.620 %=1.109 EW=0.000	PO(3)=91.620 %=1.109 EW=0.000
3	PO(7)=91.491 %=0.967 EW=1.155	PO(6)=90.825 %=0.232 EW=0.284	PO(5)=90.635 %=0.022 EW=0.075	PO(4)=90.635 %=0.022 EW=0.000
4	PO(8)=90.785 %=0.188 EW=0.500	PO(7)=90.615 %=0.000 EW=0.000	PO(6)=90.615 %=0.000 EW=0.000	PO(5)=90.615 %=0.000 EW=0.000

Table 5.5: $N = 5, c_2 = 5. (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$

Profit of No Options = 114.722				
$\Delta \backslash p$	1	2	3	4
1	PO(5)=120.117 %=4.703 EW=4.647	PO(4)=120.070 %=4.662 EW=0.029	PO(3)=120.056 %=4.650 EW=0.005	PO(2)=120.054 %=4.648 EW=0.000
2	PO(6)=117.536 %=2.453 EW=2.355	PO(5)=116.127 %=1.225 EW=0.974	PO(4)=116.127 %=1.225 EW=0.000	PO(3)=116.127 %=1.225 EW=0.000
3	PO(7)=115.921 %=1.045 EW=1.527	PO(6)=115.034 %=0.272 EW=0.387	PO(5)=114.766 %=0.038 EW=0.141	PO(4)=114.766 %=0.038 EW=0.000
4	PO(8)=114.963 %=0.210 EW=0.653	PO(7)=114.722 %=0.000 EW=0.000	PO(6)=114.722 %=0.000 EW=0.000	PO(5)=114.722 %=0.000 EW=0.000

Table 5.6: $N = 6, c_2 = 5. (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$

Profit of No Options = 138.844				
$\Delta \backslash p$	1	2	3	4
1	PO(5)=145.568 %=4.843 EW=5.915	PO(4)=145.515 %=4.805 EW=0.029	PO(3)=145.501 %=4.795 EW=0.005	PO(2)=145.499 %=4.793 EW=0.000
2	PO(6)=142.361 %=2.533 EW=2.855	PO(5)=140.636 %=1.291 EW=1.188	PO(4)=140.636 %=1.291 EW=0.000	PO(3)=140.636 %=1.291 EW=0.000
3	PO(7)=140.359 %=1.091 EW=1.894	PO(6)=139.253 %=0.295 EW=0.488	PO(5)=138.909 %=0.047 EW=0.198	PO(4)=138.909 %=0.047 EW=0.000
4	PO(8)=139.154 %=0.223 EW=0.804	PO(7)=138.844 %=0.000 EW=0.000	PO(6)=138.844 %=0.000 EW=0.000	PO(5)=138.844 %=0.000 EW=0.000

Table 5.7: $N = 4$, $c_2 = 7$. ($\lambda = 5$, $s = 1$, $r = 12$, $h = 2$, $i_{\max} = 10$, $k_{\max} = 10$)
 Profit of No Options = 60.193

$\Delta \setminus p$	1	2	3	4	5	$c_2 - 1$
0	PO(6)=68.019 %=13.002 EW=0.475	PO(5)=67.802 %=12.641 EW=0.042	PO(4)=67.760 %=12.571 EW=0.041	PO(3)=67.741 %=12.540 EW=0.006	PO(2)=67.738 %=12.535 EW=0.000	PO(1)=67.738 %=12.535 EW=0.000
-1	PO(5)=79.929 %=32.788 EW=1.128	PO(4)=79.294 %=31.733 EW=0.285	PO(3)=79.173 %=31.532 EW=0.046	PO(2)=79.143 %=31.482 EW=0.011	PO(1)=79.137 %=31.472 EW=0.005	PO(0)=79.132 %=31.464 EW=0.005
-2	PO(4)=93.755 %=55.757 EW=1.542	PO(3)=93.016 %=54.530 EW=0.332	PO(2)=92.840 %=54.237 EW=0.055	PO(1)=92.816 %=54.197 EW=0.010	PO(0)=92.805 %=54.179 EW=0.010	PO(-1)=92.795 %=54.162 EW=0.010
-3	PO(3)=108.344 %=79.994 EW=1.849	PO(2)=107.527 %=78.637 EW=0.319	PO(1)=107.416 %=78.453 EW=0.028	PO(0)=107.39 %=78.409 EW=0.028	PO(-1)=107.359 %=78.358 EW=0.028	PO(-2)=107.330 %=78.310 EW=0.028

Table 5.8: $N = 5, c_2 = 7, (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$
 Profit of No Options = 76.462

$\Delta \setminus p$	1	2	3	4	5	6
0	PO(6)=87.165 %=13.998 EW=0.634	PO(5)=86.895 %=13.645 EW=0.042	PO(4)=86.853 %=13.590 EW=0.041	PO(3)=86.834 %=13.565 EW=0.006	PO(2)=86.831 %=13.561 EW=0.000	PO(1)=86.831 %=13.561 EW=0.000
-1	PO(5)=103.076 %=34.807 EW=1.562	PO(4)=102.325 %=33.825 EW=0.289	PO(3)=102.197 %=33.657 EW=0.051	PO(2)=102.163 %=33.613 EW=0.015	PO(1)=102.153 %=33.600 EW=0.008	PO(0)=102.145 %=33.589 EW=0.008
-2	PO(4)=121.39 %=58.759 EW=2.266	PO(3)=120.404 %=57.469 EW=0.326	PO(2)=120.231 %=57.243 EW=0.059	PO(1)=120.202 %=57.205 EW=0.015	PO(0)=120.188 %=57.187 EW=0.015	PO(-1)=120.173 %=57.167 EW=0.015
-3	PO(3)=140.561 %=83.831 EW=2.658	PO(2)=139.487 %=82.427 EW=0.354	PO(1)=139.347 %=82.243 EW=0.055	PO(0)=139.292 %=82.172 EW=0.055	PO(-1)=139.237 %=82.100 EW=0.055	PO(-2)=139.182 %=82.028 EW=0.055

Table 5.9: $N = 6, c_2 = 7, (\lambda = 5, s = 1, r = 12, h = 2, i_{\max} = 10, k_{\max} = 10)$
 Profit of No Options = 92.735

$\Delta \setminus p$	1	2	3	4	5	6
0	PO(6)=106.310 %=14.638 EW=0.794	PO(5)=105.992 %=14.296 EW=0.042	PO(4)=105.950 %=14.250 EW=0.041	PO(3)=105.930 %=14.229 EW=0.006	PO(2)=105.927 %=14.225 EW=0.000	PO(1)=105.927 %=14.225 EW=0.000
-1	PO(5)=126.242 %=36.132 EW=1.997	PO(4)=125.345 %=35.165 EW=0.289	PO(3)=125.212 %=35.021 EW=0.060	PO(2)=125.174 %=34.980 EW=0.018	PO(1)=125.161 %=34.960 EW=0.012	PO(0)=125.148 %=34.952 EW=0.012
-2	PO(4)=149.031 %=60.706 EW=2.909	PO(3)=147.848 %=59.431 EW=0.342	PO(2)=147.664 %=59.232 EW=0.067	PO(1)=147.628 %=59.193 EW=0.022	PO(0)=147.606 %=59.170 EW=0.022	PO(-1)=147.584 %=59.146 EW=0.022
-3	PO(3)=172.777 %=86.313 EW=3.470	PO(2)=171.445 %=84.876 EW=0.383	PO(1)=171.276 %=84.694 EW=0.083	PO(0)=171.193 %=84.605 EW=0.083	PO(-1)=171.110 %=84.515 EW=0.083	PO(-2)=171.028 %=84.427 EW=0.083

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