Estimation of the Surface Area of a Three-Dimensional Set from Digitizations

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3rd Workshop on 3D Imaging, Analysis, Modeling and Simulation
Fontainebleau, April 20th 2010
Digitization of sets

How should one estimate the surface area $S(X)$ of an object $X$ from a binary digital image?

Digitization model $\tilde{X}_t$ of $X$:

**Gauss digitization**

\[ \tilde{X}_t = X \cap t\mathbb{Z}^d \]

**Volume-threshold digitization**

$\tilde{X}_t = \text{thresholding a gray-value image.}$
Fundamental problems with surface area estimation

1. Information loss due to discretization
   → work with sufficiently large resolution ($t \to 0$)

2. Jagging effect

\[ \hat{S} = 4.2, \quad \hat{S} = 5.04. \]

Pixel approximation ("Integral geometric approach"),
cf. [Michielsen et al. 2002], [K. Mecke, 2000]
Local and global approaches

Local approaches [Adding up local surface area contributions]
1. it does not give the correct result, even if the resolution becomes arbitrarily fine, “it is not multigrid convergent”
2. it depends (strongly) on the orientation of the grid,
3. fast calculation, using filters.

Global approaches [non-local]
1. multigrid convergent (when $X$ is convex),
2. relatively independent of grid orientation,
3. slow for complicated structures,
4. Examples (cf. [Klette & Rosenfeld, 2004]):
   DSS based (digital straight line segment),
   Tangent based (estimate tangent vector),
Marching cubes and Surface area estimation

[Lindblad & Nyström, 2002]: Determine the surface area contributions $\lambda$ of all $2^8 = 256$ small $2 \times 2 \times 2$-voxel cubes and add them up:

$$\hat{S} = t^2 \sum_{i=0}^{255} \lambda_i N_i$$

This estimator has an asymptotic worst case error of $12.8\%$. 

$\lambda_{LN} = 0.707 \quad \lambda_{LN} = 1.122 \quad \lambda_{LN} = 0.924$. 

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Random lattices

**General assumption:** the lattice is randomly translated.

- let $\xi$ be a random vector in the unit cube $[0, 1]^d$,
- the lattice $t(\xi + \mathbb{Z}^d)$ is a randomly translated lattice,
- $\tilde{X}_t = X \cap t(\xi + \mathbb{Z}^d)$ is a random set, and
- the estimator $\hat{S}(\tilde{X}_t)$ is a random variable.
Illustration in $\mathbb{R}^2$: Assume

$X = \text{half-plane}$ with outer unit normal $n$,

$\mathbf{C}_i = \text{configuration } i$,

e.g. $\mathbf{C}_i = \left( \bullet \bullet \bullet \right)$

Then, the “probability that $\mathbf{C}_i$ is observed” $\sim h_i(n)$.

(Configuration $\mathbf{C}_i$ not completely red or completely black: $i \neq 0, 255$.)
The key asymptotic result

Let $X \subset \mathbb{R}^3$ (with mild regularity conditions).

**Theorem [K. & Rataj, 2006].**
For $i = 1, \ldots, 254$, we have

$$\lim_{t \to 0^+} t^2 E N_i = \int_{\partial X} h_i \left( n(x) \right) \, dx$$

$N_i$ is the number of occurrences of $C_i$ in the digital image. $E$ denotes the usual expectation.
The key asymptotic result

Let \( X \subset \mathbb{R}^3 \) (with mild regularity conditions).

**Theorem [K. & Rataj, 2006].**
For \( i = 1, \ldots, 254 \), we have

\[
\lim_{t \to 0^+} t^2 E N_i = \int_{\partial X} h_i(\mathbf{n}(x)) \, dx
\]

where \( N_i \) is the number of occurrences of \( C_i \) in the digital image. \( E \) denotes the usual expectation.

**Usage:**

\[
\hat{S} := t^2 E \sum_{i=1}^{254} \lambda_i N_i \approx \int_{\partial X} \sum_{i=1}^{254} \lambda_i h_i(\mathbf{n}(x)) \, dx.
\]

\( H(\mathbf{n}(x)) \)

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Surface Area estimation from digitizations
For non-informative configurations \((h_i \equiv 0)\) set \(\lambda = 0\) for instance.

102 informative configurations \((h_i \not\equiv 0)\). Symmetry \(\Rightarrow\) only 5 different types of configurations.

It is therefore enough to find the 5 weights \(\lambda^1, \ldots, \lambda^5\).
The optimal weights

- **h**-functions are linear dependent:
  - **h**-functions of type 1 are linear combinations of the others,

- Weights minimizing $\| \sum_{i=1}^{254} \lambda_i h_i - 1 \|_\infty$ are

  $\lambda = \begin{align*}
  0 & & 1.3579 & & 2.3519 & & 0.9602 & & s
  \end{align*}$

  with $1.6631 \leq s \leq 1.7452$. 
The optimal weights

- $h$-functions are linear dependent:
  - $h$-functions of type 1 are linear combinations of the others,

- Weights minimizing $\| \sum_{i=1}^{254} \lambda_i h_i - 1 \|_\infty$ are

\[
\lambda = \begin{cases} 
    sr & 1.3579 \\
    2.3519 - sr & 0.9602 \\
    s(1 - r) & 0.9602
\end{cases}
\]

with $1.6631 \leq s \leq 1.7452$, $0 \leq r \leq 1$.

Then, the uniform deviation from the constant 1 is $< 4\%$. 
Comparison to other approaches

1. The presented estimator
   maximal asymptotic error 4.0%,

2. [Lindblad & Nyström, 2002]: “marching cube”-idea:
   maximal asymptotic error 12.8 %,

3. [Lindblad 2005]: Optimal weights for isotropic shapes:
   maximal asymptotic error 7.3 %,

4. [Schladitz & Ohser 2006]: Discretization of Crofton’s formula and the Euler number in linear sections;
   maximal asymptotic error 7.3 %.
### Some simulation examples

<table>
<thead>
<tr>
<th>Object</th>
<th>$t$</th>
<th>True value $S(X)$</th>
<th>Estimator $\hat{S}$</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>unit ball</td>
<td>0.1</td>
<td>12.57</td>
<td>12.73</td>
<td>1.3 %</td>
</tr>
<tr>
<td>half ball</td>
<td>0.1</td>
<td>9.42</td>
<td>9.37</td>
<td>0.5 %</td>
</tr>
<tr>
<td>cylinder</td>
<td>0.1</td>
<td>18.85</td>
<td>18.94</td>
<td>0.5 %</td>
</tr>
<tr>
<td>rotated cyl.</td>
<td>0.06</td>
<td>18.85</td>
<td>19.04</td>
<td>1.1 %</td>
</tr>
</tbody>
</table>
Surface orientation

\[
\lim_{t \to 0} EN_i = \int_{\partial X} h_i(n(x)) \, dx.
\]

**Idea:** Can be used to detect anisotropy by estimating the *rose of normal directions*:

**Simplest case:** Two-point configurations \( C = tv \)

\[
\lim_{t \to 0} EN = \frac{1}{2} \int_{\partial X} |v \cdot n(x)| \, dx.
\]
A model for the rose of directions

- Allowing only for configurations of pairs that are 26-neighbors, we obtain 26 different integrals.

- The software MAVI contains an algorithm for origin-symmetric roses of directions and 13 degrees of freedom.

- Use all $2 \times 2 \times 2$-configurations $\Rightarrow 50$ degrees of freedom.

- Imposing a model for the rose of direction (piecewise constant), parameter fitting is based on minimizing (essentially) the Kullback-Leibler divergence between configuration counts and asymptotic prediction.
An estimation example

A unit ball and a oblate spheroid, digitized with $t = 0.02$. 
Analysis of stacks from confocal microscopy

- Up to now surface area estimation assumes availability of 3D digitization (micro-CT, MRI).
- In principle also applicable to parallel stacks of 2D digitizations in confocal microscopy.

Blurring problem:
Planes almost touching the object yield blurred images.
Local stereology solves the Blurring Problem

Classical local stereology (without digitization):

- Choose a reference point \( O \) in the object (e.g. nucleus)
- Use stacks in a slice \( T_s \) of thickness \( 2s \) centered at \( O \) with isotropic random orientation.
- Employ Horvitz-Thompson estimation:

  Observations at a point \( x \in T_s \) must be weighted with the inverse of the sampling density \( f(\|x\|) = \min\{1, s/\|x\|\} \).
Combining Local and digital concepts

- Recall the surface area estimator

\[ \hat{S} = t^2 \sum_{i=1}^{254} \lambda_i N_i \]

from a full 3D digitization, \( N_i = \# \) configurations \( C_i \).

- Estimator from digitization in the slice \( T_s \):

\[ \hat{S} = t^2 \sum_{i=1}^{254} \lambda_i \tilde{N}_i \]

with \( \tilde{N}_i = \sum_{C_i} f(\| m(C_i) \|)^{-1} \).

Here each configuration \( C_i \) is weighted with the inverse sampling probability of its midpoint \( m(C_i) \).
If the weights $\lambda_i$ are chosen appropriately:

1. $\hat{S}$ is unbiased for all objects $X \subset sB^3$,
2. $\hat{S}$ is unbiased for all centered balls $X = rB^3$, $r > 0$,
3. For arbitrary objects, the asymptotic mean error is $< 4.7\%$,
4. The error bounds can be improved considerably, if the diameter of $X$ is known.

Here $B^3$ is the unit ball centered at $O$. 
A simulation example

Consider a cylinder $X$ with radius 1 and height 2, and random slices $T_s$ with thickness $2s = 1$.

As $X \subset \sqrt{2}B^3$, the theoretical asymptotic bound can be decreased from 4.7\% to 1.2\%.

<table>
<thead>
<tr>
<th></th>
<th>theoretical value</th>
<th>lattice dist. $t = 0.055$</th>
<th>lattice dist. $t = 0.02$</th>
</tr>
</thead>
<tbody>
<tr>
<td>surface area</td>
<td>18.850</td>
<td>18.115</td>
<td>18.553</td>
</tr>
<tr>
<td>mean error</td>
<td>&lt; 1.2%</td>
<td>3.8%</td>
<td>1.6%</td>
</tr>
</tbody>
</table>

Means from 1000 Monte Carlo simulations.