Asset Liquidity, Corporate Investment, and Endogenous Financing Costs

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Abstract

What is the effect of real asset liquidity on corporate investment? And how is the effect influenced by the liquid financial assets of the firm? Theoretical research usually does not analyze the effect of a firm’s existing real assets’ liquidity on new investment. Still, empirical evidence shows that the question is highly relevant. We give a new theoretical explanation for the puzzling empirical finding that investment-cash flow sensitivities can be negative. As we show, the magnitude of the sensitivity is determined by the firm’s asset liquidity. We distinguish two dimensions of asset liquidity: First, the degree of redeployability, and second, whether fire sale of assets can be used to avoid full liquidation. We derive testable implications for investment-cash flow sensitivities and debt usage, which are highly dependent on the liquidity of the existing assets. Our findings can be summarized as follows: First, firms with higher degree of redeployability invest less sensitive to their cash flow. Second, both the usage of debt financing and the sign of investment-cash flow sensitivities depend on asset liquidity and the level of internal funds in a non-monotonic way. Our findings suggest that asset liquidity is indeed an important factor towards resolving the empirical puzzle of negative investment-cash flow sensitivities.

JEL Classification: G31, G32, G33

Keywords: financing constraints, asset liquidity, redeployability, liquid funds, investment-cash flow sensitivity.
1 Introduction

What is the effect of real asset liquidity on corporate investment? And how is the effect influenced by the liquid financial assets of the firm? An influential paper by Almeida and Campello (2007) emphasizes the “credit multiplier effect”, meaning that investment in pledgeable assets increases a firm’s borrowing capacity and thus allows even further investment. This argument emphasizes the tangibility of newly invested assets. However, theoretical research usually does not analyze the effect of a firm’s existing real assets’ liquidity on new investment. Still, empirical evidence shows that the question is highly relevant. For example, Gan (2007) shows very clearly how an exogenous decline in firms’ collateral value leads to less new investment.

Our paper fills the gap and examines the interaction between the existing real assets’ liquidity and corporate investment. A key feature of our model is that financing costs are endogenous. More precisely, we show how financing costs result as a trade-off between underinvestment costs and asset liquidation costs. Therefore, asset liquidation costs affect a financially constrained firm’s investment policy in two aspects: First, firms with higher degree of redeployability invest less sensitive to their cash flow. Second, both the usage of debt financing and the sign of investment-cash flow sensitivities depend on asset liquidity and the level of internal funds in a non-monotonic way.

Only recently, there is empirical evidence showing that investment volume can be non-monotonic, and in particular increasing, in financing constraints. Thus, investment-cash flow sensitivities can be negative.\(^1\) Cleary et al. (2007) explain their evidence as an external investor’s trade-off between the cost of providing funds to a firm and the possible revenue received from the firm’s investment project. Hovakimian (2009) explains negative sensitivities with the life-cycle hypothesis: Firms with low cash are often young and have promising projects. Therefore it may be easier for them to get external funding than for more mature firms. Bhagat et al. (2005) find negative sensitivities as a result of firms being distressed, particularly having negative operating income. They explain this finding by the infusion of new equity to gamble for resurrection.

We give a new theoretical explanation for the puzzling empirical finding that investment-cash flow sensitivities can be negative. We show that even if a firm’s ex-post observed financing

\(^1\)Earlier research on the impact of financing constraints on corporate investment is dominated by the empirical question “Do investment-cash flow sensitivities provide useful measures of financing constraints?”, asked most prominently in the title of the article by Kaplan and Zingales (1997). They criticize the study by Fazzari et al. (1988) and show that investment-cash flow sensitivities do not necessarily increase in constraints. However, they still take for granted that investment volume itself is decreasing in constraints, i.e. sensitivities are always positive.
costs are strictly increasing in constraints, the ex-ante costs relevant for the decision can be non-monotonic.\footnote{The idea is similar to Strebulaev (2007)'s critique of empirical tests of capital structure theory: The ex-post observed capital structure is not reflecting the ex-ante faced trade-off, except for the rare case that a firm is observed exactly at a refinancing point.

To show that, we split up the endogenously implied financing costs and propose a trade-off between expected liquidation costs and underinvestment costs. For a given level of investment, the marginal benefits of more investment, i.e. reducing underinvestment, are independent of financing constraints. However, the marginal costs of debt financing due to the risk of liquidation are smaller if the firm already uses a higher level of debt financing. In relative terms, reducing underinvestment can therefore be more important for a more constrained firm. From this cost trade-off, we derive testable implications for investment-cash flow sensitivities and debt usage, which are highly dependent on the liquidity of the existing assets. We distinguish two dimensions of asset liquidity: First, the degree of redeployability, and second, whether fire sale of assets can be used to avoid full liquidation. While the nature of some assets may restrict a fractional fire sale, this may also be prohibited due to covenants, see e.g. Morelec (2001).

Our findings can be summarized as follows: First, investment is less sensitive to cash flow for firms with higher degree of redeployability. This is in contrast to Almeida and Campello (2007) which is due the fact that they consider quantity constraints, but they abstract from financing costs. Second, asset liquidity and the level of internal funds determine both the usage of debt financing and the sign of investment-cash flow sensitivities. More precisely, for firms that face full liquidation in financial distress, we find that firms with high internal funds avoid debt financing and have a one-to-one positive investment-cash flow sensitivity, whereas firms with low internal funds use debt financing and have a negative investment-cash flow sensitivity. For firms that can liquidate a fraction of their assets in a fire sale, we find that firms with high internal funds use risk-free debt financing and have a positive investment-cash flow sensitivity significantly below unity, whereas firms with low internal funds use risky debt financing and have a negative investment-cash flow sensitivity. Overall, our predictions are in line with the mentioned evidence by Cleary et al. (2007), Hovakimian (2009), and Bhagat et al. (2005), finding negative sensitivities for low-cash firms. As we show, the magnitude of the sensitivity is determined by the firm’s asset liquidity. Therefore, our findings suggest that asset liquidity is indeed an important factor towards resolving the empirical puzzle of negative investment-cash flow sensitivities.

A related strand of literature on asset liquidity and investment focuses on the liquidation
of assets and using the proceeds for subsequent investment, see Hovakimian and Titman (2006) and Gopalan et al. (2010). In contrast, we examine how the conditions for a possible future asset liquidation affect today’s investment. Other literature on asset liquidity focuses on the implications for capital structure, see e.g. Williamson (1988), Shleifer and Vishny (1992), Alderson and Betker (1995), Myers and Rajan (1998), Morellec (2001), and Sibilkov (2009), and on the implications for the cost of capital, see Ortiz-Molina and Phillips (2009). Note that in many of these papers, asset sales are seen as a means to divert value away from debtholders. In our setting, however, we assume symmetric information and perfectly enforceable contracts. Thus we abstract from the “dark side of liquidity”, as Myers and Rajan (1998) term the problem, and the costly liquidation of existing assets is only used if necessary to service the payments to debtholders. There is also empirical evidence by Brown et al. (1994) illustrating that the proceeds from asset sales might benefit the debtholders.

Our theoretical model is closely related to Cleary et al. (2007), who are to our knowledge the first to show that investment volume can be non-monotonic in financing constraints. They introduce the notion of a U-shaped investment curve in liquid funds, i.e. the lowest investment volume is reached for an intermediate level of liquid funds. While they use one specific liquidation rule reflecting neither fire sale of assets nor full liquidation, our main focus is on the effect of asset liquidity on investment policy. Another stream of theoretical corporate investment literature uses investment-timing models to show that investment can be increasing in constraints. This has been put forward by Boyle and Guthrie (2003) and Hirth and Uhrig-Homburg (2010b), who show that increasing constraints can lead to more investment today to avoid future constraints. The timing behavior derived in these papers can be interpreted as an inversely U-shaped investment curve in liquid funds, which is the opposite to the outcomes in Cleary et al. (2007) and our model. One reason for the significant discrepancy is that the mentioned timing models only consider one single project with a fixed investment amount. The decision to invest now or later may not always be equivalent to investing more or less at one point in time. By explicitly analyzing a decision about investment volume, our model implies empirical predictions in this dimension.

The paper proceeds as follows. In Section 2 we set up our model. Section 3 derives properties of an investment curve as a function of liquid funds. Section 4 complements the analytical properties with a numerical analysis of the cost trade-off faced by a financially constrained firm, focusing on one dimension of asset liquidity, namely the degree of fractional liquidation. In contrast, Section 5 focuses on the other dimension of asset liquidity, namely the degree of redeployability. We present empirically testable implications of our model in Section 6. Finally,
Section 7 concludes. In the Appendix, we relate our model to Cleary et al. (2007) in Section A and present the proofs for our propositions in Section B.

2 Model

We consider a firm with assets in place in a three-date model \((t = 0, 1, 2)\). The existing assets have a long-run value of \(G\), which corresponds to the value of future cash flows. This value is only fully realized if the firm is alive until \(t = 2\). Otherwise, there are liquidation costs as discussed below. In addition to the existing assets, the firm has liquid funds \(W\) at time \(t = 0\).

The liquid funds stem from previous operation of the firm. A positive \(W\) is due to a previously favorable outcome, whereas a negative \(W\) indicates for example a debt obligation. Moreover, the firm has a short-term investment opportunity. The opportunity must be exploited at time \(t = 0\) and will in return provide a cash flow at time \(t = 1\). The present value of the investment opportunity depends on the size of the investment, as well the state of the economy one period from now. Specifically, an investment of an amount \(I\) generates revenue \(F(I, \theta)\) after one period in state \(\theta\). We assume that \(\theta\) is a random variable distributed on \([\bar{\theta}, \bar{\theta}] \in \mathbb{R}_{0+}\) with density \(\omega(\theta)\) and cumulative density function \(\Omega(\theta)\). Moreover, we consider a production function with standard characteristics. A positive investment is needed for a positive probability of a positive value, i.e. \(F(0, \theta) = 0\). The production technology has decreasing returns to scale and \(F \in \mathcal{C}^{2,2}\) is increasing and concave in investment. Finally, we simplify the analysis by assuming that all agents are risk-neutral and the risk-free interest rate is 0.

2.1 Benchmark case: unconstrained firm

As a benchmark case we consider a financially unconstrained firm. This means that the entrepreneur can provide unlimited additional funds out of his own pockets. Then the assets in place are known to yield future cash flow worth \(G\) at time \(t = 2\).

Since the firm is financially unconstrained, the entrepreneur’s best strategy is to undertake the investment that maximizes the net present value of the investment opportunity, i.e.

\[
\max_I E[F(I)] - I,
\]

where \(E[\cdot]\) is the expectation operator wrt. \(\theta\). We denote the solution to the problem (1) as \(I^{**}\), which we call the first-best investment. The unconstrained entrepreneur’s problem is illustrated
in Figure 1. We will later return to the costs of not undertaking the first-best investment, but
first we set up the investment problem faced by the financially constrained firm.

entrepreneur invests $I^{**}$

entrepreneur gets return $F(I^{**}, \theta)$

entrepreneur realizes $G$

---

0

1

2

t

Figure 1: Time line and cash flows for the unconstrained firm.

2.2 Constrained firm

We now assume that the internal funds $W$ are insufficient to cover the first-best investment,
that is, we focus on the case $W < I^{**}$. Moreover, the entrepreneur is financially constrained,
which means that he cannot provide any additional funds out of his own pockets. If he has no
access to external capital, he therefore has to choose $I \leq W$. If even $W < 0$, liquidation of the
firm is inevitable.

If the entrepreneur wants to move closer to the first-best investment, i.e. to invest more
than $W$, he has to raise additional funds from a risk-neutral outside investor. We only consider
external financing using a standard one-period debt contract.\(^3\) Thus, the firm promises to pay
the principal, $D_1$, at time $t = 1$. In return it receives the present value at time $t = 0$ of the debt
contract, denoted as $D_0$. Correspondingly, we denote the value of the entrepreneur’s claim at
time $t$ as $E_t$.

We assume the firm has to use the proceeds from debt issuance either for investment or
as liquid funds carried over to the next period. Thus, we prohibit that the proceeds are paid
immediately to the entrepreneur as a dividend. Since we introduce liquidation costs below, it
is not optimal for the firm to issue debt in excess of the desired investment level and carry
over liquid funds. Hence, at time $t = 0$ the entrepreneur decides to undertake the investment
$I = W + D_0$, which is partly financed by borrowing the amount $D_0$.

The entrepreneur’s initial decision is depicted in Figure 2. The figure also illustrates the
time line and consequences of the possible realizations of the state of the economy. At $t = 1$, the
uncertainty has resolved, i.e. $\theta$ is known, and the investment generates a revenue $F(I, \theta)$. This

\(^3\)One could think of other financing contracts, but this is not in the focus of this paper. Therefore we restrict
ourselves to this simple, yet relevant case. We deliberately focus on a standard debt contract in order to illustrate
the key trade-off in a parsimonious model.
outcome is public knowledge. Whether the firm can fulfill its obligations and avoid liquidation is dependent on the realization of $\theta$. Formally, we distinguish between several different types of outcomes:

If the revenue $F(I, \theta)$ is sufficient to pay back the promised repayment of $D_1$ to the outside investor, then the entrepreneur keeps the remaining revenue $F(I, \theta) - D_1$ for himself. This is the case whenever $\theta \geq \theta_B$, where we define the bankruptcy threshold $\theta_B$ such that

$$F(I, \theta_B) = D_1.$$  \hspace{1cm} (2)

In this case, the firm continues operation, and at $t = 2$ the entrepreneur receives an additional payoff of $G$ from the existing assets. If the threshold is not underrun even for the worst state, i.e. $\theta_B \leq \theta$, then it is always possible to repay the outside investor using the investment revenue only, and the existing assets remain under the entrepreneur’s control in all states of the world.

In contrast, if the revenue $F(I, \theta)$ is not sufficient to pay the principal $D_1$, i.e. $\theta < \theta_B$, then the full revenue $F(I, \theta) < D_1$ is paid to the outside investor. It is common to assume that the bankruptcy costs are proportional to the asset value.\footnote{Such a friction can be explained for example as private benefits to the entrepreneur from running the firm, or as legal costs of bankruptcy. See Cleary et al. (2007) and Leland (1994).} Therefore we assume that the value of the existing assets to the outside investor is $(1 - \gamma)G$, with $\gamma \in (0, 1]$. The parameter $\gamma$ measures the fraction of asset value lost in bankruptcy, which is the central friction in our model. If the revenue is not sufficient to pay the principal, we distinguish between two sub-outcomes: First, if the achievable proceeds $(1 - \gamma)G$ from liquidating the existing assets are sufficient to cover
the remaining debt obligation to the outside investor, \( D_1 - F(I, \theta) \), then there is a positive value remaining to the entrepreneur. This is the case whenever \( \theta \in [\theta_R, \theta_B] \), where we define the creditor risk-free debt threshold \( \theta_R \) such that

\[
F(I, \theta_R) = D_1 - (1 - \gamma)G. \tag{3}
\]

If this threshold is not underrun even for the worst state, i.e. \( \theta_R \leq \theta \), then the outside investor receives full repayment in all states of the world, and debt is actually risk-free. Second, if the threshold (3) is underrun for the worst outcomes, i.e. \( \theta < \theta_R \), then the outside investor receives both the full revenue \( F(I, \theta) \) from investment and the proceeds \( (1 - \gamma)G \) from the assets initially in place. However, the creditor does not receive the full promised repayment \( D_1 \) and, hence, the entrepreneur receives nothing due to limited liability.

### 2.3 Entrepreneur’s problem

The entrepreneur’s optimization problem is to choose the investment amount \( I \), and thus also the amount of external financing \( I - W \), in order to maximize the value of his claim. However, the entrepreneur realizes that the debt issuance is subject to the outside investor’s participation constraint, and that debt financing exposes him to liquidation risk.

Note that in many papers on optimal capital structure, e.g. Leland (1994), the ex-ante optimal behavior is to maximize the total firm value, i.e. the sum of debt and equity value. This even holds for papers considering endogenous investment decisions, e.g. Hirth and Uhrig-Homburg (2010a). In contrast, our entrepreneur seems to be concerned about the equity value only. The reason for the difference is that most capital structure literature assumes that the proceeds from the debt issue are immediately paid out to equity holders. Therefore, the latter are maximizing ex-ante the sum of this payment and their value in the firm after debt issuance. In our case, the proceeds are directly invested into new assets rather than paid out and, thus, enter the equity value implicitly.

In the following we derive the debt value and the value of the entrepreneur’s claim. In states in which the payment promised to the debt holders exceeds the proceeds from the new investment project, the liquidity of the firm’s existing assets is relevant.

We distinguish two between dimensions of asset liquidity: The first dimension of asset liquidity is the degree of redeployability, i.e. the outside value of the firm’s assets, as discussed in Shleifer and Vishny (1992). Naturally, assets which are easily used by other firms in their production are relatively easy to sell. Conceivably, such assets are sold without a large discount,
i.e. marginal liquidation costs should be low for such assets. In our model, a higher degree of asset redeployability corresponds to a lower $\gamma$. The second dimension is the degree of fractional liquidation. Here, we distinguish between two extreme cases: Either full liquidation of all assets is required, or it is possible to sell any desired fraction of the assets in a fire sale. In the former case, a fractional sale of the assets to service the debt holders is not possible, for example due to covenants, see Morellec (2001) and DeAngelo et al. (2002), or the nature of the firm’s assets: While current assets like inventories are rather liquid and divisible, this may not be the case for long-term assets like property, plant, and equipment. Depending on industry and firm characteristics, it is possible that the assets consist to the major part of indivisible long-term assets. Then the firm has to be liquidated as a whole, and liquidation costs of $\gamma G$ apply to all of the existing assets $G$. The superscript $L$ denotes the full liquidation case. In the latter case, the firm may sell any fraction of the existing assets in a fire sale to service the debt holders. Then, liquidation costs apply only to the fraction $\varphi_S$ sold. The remaining fraction $(1 - \varphi_S)$ of the assets can be retained until $t = 2$ and is not subject to liquidation costs. The superscript $S$ denotes the fire sale of assets case. In practice, one would most likely encounter intermediate cases, and liquidation costs might be marginally increasing in the amount liquidated. For example, the liquidation of current assets will cause no or only little costs, while selling property, plant, and equipment may incur substantial costs. However, to keep tractability, we restrict our attention to the two cases explained above. In the following, we derive the values of debt and equity in these two cases.

### 2.3.1 Full liquidation of assets

Consider first the debt contract. The firm promises to pay the principal $D_1$ at time $t = 1$. However, if $\theta < \theta_R$, the debt holders only receive partial recovery. The recovery consists of the cash flow from the investment and the liquidation of assets. Since the risk-free rate of interest

\[ \text{Equation} \]
is zero, the value of debt at \( t = 0 \) given an investment of \( I \) is\(^5\)

\[
D^L_0(I) = \mathbb{E}[\min\{F(I) + (1 - \gamma)G, D_1\}] = \int_\theta^\bar{\theta} \min\{F(I, \theta) + (1 - \gamma)G, D_1\} \omega(\theta) d\theta \\
= \int_\theta^\bar{\theta} [F(I, \theta) + (1 - \gamma)G] \omega(\theta) d\theta + \int_\theta^\bar{\theta} D_1 \omega(\theta) d\theta \\
= \int_\theta^\bar{\theta} F(I, \theta) \cdot \omega(\theta) d\theta + (1 - \gamma)G \cdot (\Omega(\theta_R) + D_1 \cdot (1 - \Omega(\theta_R))).
\]

(4)

Consider now the entrepreneur’s problem. When the entrepreneur solves his investment problem, he decides upon an investment \( I \). If \( I > W \), additional funds equal to \( I - W \) are needed. Since external funds are covered by the outside investor, the entrepreneur issues a debt contract with a principal \( D_1 \) such that the firm receives the desired amount, i.e.

\[
D^L_0(I) = I - W.
\]

(5)

That is, for each level of liquid funds \( W \) and desired investment \( I \) and, hence, initial debt value \( D^L_0 \), we can apply (4) and (5) to implicitly define the principal \( D_1 \) required by the outside investor. The value of the entrepreneur’s claim consists of the expected payments at \( t = 0 \), \( 1 \), and \( 2 \). The entrepreneur receives nothing at \( t = 0 \), because all funds are invested. At \( t = 1 \), the entrepreneur receives the remaining value after paying off the debt holders. If liquidation is avoided at \( t = 1 \), he also receives the value \( G \) at \( t = 2 \). Thus, for a given investment of \( I \), the value of the entrepreneur’s claim is

\[
E^L_0(I) = \int_\theta^\bar{\theta} 0d\Omega(\theta) + \int_\theta^\bar{\theta} [F(I, \theta) + (1 - \gamma)G - D_1] \omega(\theta) d\theta + \int_\theta^\bar{\theta} [(F(I, \theta) - D_1) + G] \omega(\theta) d\theta \\
= \int_\theta^\bar{\theta} F(I, \theta) \cdot \omega(\theta) d\theta + (1 - \gamma)G \cdot (\Omega(\theta_R) - \Omega(\theta_L)) + (G - D_1) \cdot (1 - \Omega(\theta_R)).
\]

(6)

Observe we ensure that the entrepreneur’s limited liability constraint is satisfied.\(^6\) Given internal funds \( W \), we can write the entrepreneur’s problem as

\[
\max_I E^L_0(I)
\]

subject to the budget constraint

\[
I = W + D^L_0(I),
\]

(8)

\(^5\)There are two extreme cases, for which (4) reduces to much simpler expressions: If debt is creditor risk-free (i.e. \( \theta_R \leq \bar{\theta} \) and \( \Omega(\theta_R) = 0 \)) we have \( D^L_0(I) = D_1 \), or if in contrast debt never receives full recovery (i.e. \( \theta_R \geq \bar{\theta} \) and \( \Omega(\theta_R) = 1 \)) we have \( D^L_0(I) = \mathbb{E}[F(I)] + (1 - \gamma)G \). There are similar extreme cases for the equity values derived below.

\(^6\)In contrast, the related model by Cleary et al. (2007) implicitly assumes unlimited liability of equity holders. See Section A for a more detailed discussion.
where the debt value and, hence, also the value of the entrepreneur’s claim both depend on the principal \( D_1 \) induced by (4). We denote the solution to the problem (7) as \( I^*_L \), and we call it the second-best investment, i.e., the optimal investment for the constrained firm, given that there is full liquidation in financial distress.

### 2.3.2 Fire sale of assets

In the previous case we assumed that all of the firm’s assets had to be liquidated in financial distress. However, for some type of firms (or production) it is possible to sell parts of the firm’s assets unless it is prohibited by covenants.

To derive the values of debt and equity, we introduce the fraction of assets, \( \varphi_S \), needed to cover debt obligations. That is, \( \varphi_S \) must be such that exactly enough assets are sold to cover what the return from the investment cannot cover. Those assets that are sold are subject to liquidation costs. Due to these costs, it is never in the entrepreneur’s interest to liquidate more than necessary. Hence we have the condition

\[
D_1 - F(I, \theta) = \varphi_S(\theta)(1 - \gamma)G,
\]

if \( \theta_R \leq \theta < \theta_B \), i.e.

\[
\varphi_S(\theta) = \begin{cases} 
1 & \text{if } \theta < \theta_R \\
\frac{D_1 - F(I, \theta)}{(1 - \gamma)G} & \text{if } \theta_R \leq \theta < \theta_B \\
0 & \text{if } \theta \geq \theta_B 
\end{cases}.
\]

If the state of the economy turns out to be sufficiently good, then there is never a need for selling any asset, i.e. \( \varphi_S(\theta) = 0 \), \( \theta > \theta_B \). On the other hand, for the really bad states, \( \theta < \theta_R \), creditors are not fully paid albeit all assets are liquidated. The debt value is not affected by the changed liquidation policy, therefore

\[
D^S_0(I) = D^L_0(I).
\]

In contrast, compared to the case with full liquidation, the equity holders are better off, because only a fraction of the existing assets has to be liquidated when \( \theta \in (\theta_R, \theta_B) \). The value of their claim is

\[
E^S_0(I) = \frac{1}{1 - \gamma} \int_{\theta_R}^{\theta_B} F(I, \theta) \cdot \omega(\theta) d\theta + \int_{\theta_B}^{\theta} F(I, \theta) \cdot \omega(\theta) d\theta \\
+ G(1 - \Omega(\theta_R)) - \frac{D_1}{1 - \gamma} ((1 - \gamma) + \gamma(\Omega(\theta_B) - \Omega(\theta_R))).
\]
The objective for the entrepreneur is to maximize the equity value (10) subject to the cut-off levels as defined in (2) and (3) and, in addition, the incentive compatibility condition for debt holders, (5). We denote the solution to the maximization problem as $I^*_S$, and we call it the second-best investment, given that fire sales of assets are possible.

### 2.4 Separation of costs

Recall that we consider a firm consisting of an investment opportunity with expected revenue $E[F(I)]$ to be received at $t = 1$, liquid funds $W$, and existing assets with a long-run value of $G$.

First consider a financially unconstrained firm. This means that if $W < I$, then the amount missing to make the desired investment is balanced with other private funds or risk-free debt. Such a firm has a $t = 0$ equity value of

$$E^u_0(I) = E[F(I)] - I + W + G.$$  \hspace{1cm} (11)

In other words, the value of the unconstrained firm is equal to the net present value of the investment opportunity as in (1) added to the internal funds and the long-run value of the existing assets. If the firm is actually unconstrained, then it invests at the first-best level $I = I^{**}$, and it has the equity value $E^u_0(I^{**})$.

The value of the entrepreneur’s claim in the constrained firm is given by $E^C_0(I)$ in (6) and $E^S_0(I)$ in (10) for the cases of full liquidation and fire sale of assets, respectively. We state the following expressions with the superscript $c$ denoting the constrained firm, where $c$ represents either $L$ or $S$.

If the internal funds are non-negative, the entrepreneur may decide not to issue debt, i.e. choose $I = W$. In this case the values of the unconstrained firm and the constrained firm are identical, i.e.

$$E^C_0(W) = E^u_0(W) = E[F(W)] + G.$$  \hspace{1cm} (12)

In general, we define the total costs faced by a constrained firm as the total loss in equity value due to being constrained, i.e.

$$TC^c(I) = E^u_0(I^{**}) - E^C_0(I).$$  \hspace{1cm} (13)

In other words, the total costs, $TC^c(I)$, measure the distance between the first-best firm value and the constrained firm value given the investment $I$. The second-best investment level $I^*_c$ is
defined as the solution to the constrained firm’s problem

$$\max_I E_0^c(I)$$

which is equivalent to minimizing the costs, i.e.

$$\min_I TC^c(I).$$

We want to understand the underlying elements of the total costs. Therefore we separate (13) into two terms as follows:

$$TC^c(I) = UC^c(I) + LC^c(I).$$

Equation (14) shows that the origin of the total costs is twofold. On the one hand, there is a loss in value due to deviation from the first-best investment, i.e. choosing $I \neq I^{**}$. We define this first part as underinvestment costs

$$UC(I) = E_0^u(I^{**}) - E_0^u(I) = E[F(I^{**})] - I^{**} - (E[F(I)] - I).$$

These are costs from following an investment policy that is suboptimal from an unconstrained firm’s perspective. Note that the underinvestment costs do not bear a superscript, as they are identical for either type of constrained firm and not dependent on the liquidity of the existing assets. On the other hand, there is the second part,

$$LC^c(I) = E_0^c(I) - E_0^c(I),$$

which we define as liquidation costs. These are costs that arise directly from the present value of a possible future liquidation, while correcting for the loss in value due to suboptimal investment. In that sense, they correspond to an endogenized version of the financing cost function that is oftenly exogenously presumed in the literature, e.g. in Kaplan and Zingales (1997).

3 Properties of investment as a function of liquid funds

From now on, we impose some additional structure on the firm’s problem. In particular, we assume that the production technology can be described as $F(I, \theta) = \theta \sqrt{T}$. Thus, for a given state of the economy, the production function is increasing and concave in investment, and, for a given investment, we interpret a higher state of the economy as improved profitability of the
investment. Also, we let the state of the economy be uniformly distributed on \([\theta, \bar{\theta}]\). Thus, the density is \(\omega \equiv 1/(\bar{\theta} - \theta)\) with \(\theta \geq 0\).

For the given structure, we can derive several general properties of the optimal investment depending on the liquid funds in the firm. We are interested in seeing when the firm makes the first best investment and when the investment-cash flow sensitivity changes sign. Later, we return to assessing the consequences of the different debt contracts and liquidation options. Figure 3 shows in stylized form the investment behavior that we formally analyze later.

![Figure 3: Illustrative investment, full liquidation (solid curve) and asset sale (dashed curve).](image)

There is a level of liquid funds \(W_{\text{min}} < 0\) at which the sum of the project’s net present value given first-best investment and the existing assets’ liquidation value is just enough to outweigh the negative internal funds. As \(W_{\text{min}}\) is negative, the firm has to take debt to cover the liability. For lower levels of \(W\), the firm would not be able to attract new external funding. Even when investing first-best, the total firm value would be negative, and the debt holders would not get the promised repayment even in the best possible state. Therefore we consider only the region \(W \geq W_{\text{min}}\) as feasible.

The firm’s investment behavior depends on the liquidity of the existing assets. First, we look at the full liquidation case – the solid curve in Figure 3. In this case, first-best investment remains the optimal strategy as long as \(W < W^L_f\). For higher levels of liquid funds, the investment curve jumps up to a level above first-best, so there is overinvestment. The economic intuition for this behavior is the following: Overinvestment is indicating that the firm is funding new projects that have a negative net present value in expectation. However, in the favorable realizations, the entrepreneur does not only receive a positive value from the new projects, after
paying off the outside investor, but also avoids the costly liquidation of the existing assets. So in the overinvestment region, the benefits from avoiding costly liquidation in the favorable realizations outweigh the costs from taking projects with a negative net present value in expectation. The outside investor always receives zero profit according to the debt contract’s participation constraint (5). Therefore, the “gambling for resurrection” we observe here is not exploiting the outside investor, but it is purely shifting value from costly liquidation states to the entrepreneur. For \( W > W^L \), investment is decreasing in liquid funds, while the firm still issues credit-risky debt. At \( W = W^L \), the firm invests again first-best, and the behavior switches from overinvestment to underinvestment at that point. At \( W = W^L_R \), debt changes from credit-risky to being risk-free for the creditors when moving from lower to higher \( W \). Investment is further decreasing in liquid funds until a level of \( W = W \geq 0 \). Here, the optimal investment reaches the \( I = W \) line. This means that for \( W > W \), no debt financing is used and the optimal investment is equal to the firm’s liquid funds, and thus one-to-one increasing in the amount of liquid funds. Summarizing, investment reaches a minimum for \( W = W \).

Second, in the fire sale of assets case – the dashed curve in Figure 3 – the optimal strategy is to invest first-best as long as \( W < W^R_S \). For higher levels of liquid funds, investment is decreasing in \( W \). Assuming that the value of existing assets \( G \) is not too large, we show that investment is again increasing in liquid funds as soon as a critical level \( W^R_S \) is reached. Debt financing is still used at \( W^R_S \), but while the external investors face credit risk for \( W < W^R_S \), debt becomes creditor risk free for \( W > W^R_S \). This is an important difference to the full liquidation case, where debt financing is not used at all when exceeding the minimum point \( W = W \).

In the following, we state the hypothesized properties formally.

**Proposition 1.** Assume \( F(I, \theta) = \theta \sqrt{I} \) and \( \theta \sim U[\theta, \bar{\theta}] \) as above. Then the first best investment level is

\[
I^{**} = \left( \frac{E[\theta]}{2} \right)^2. \tag{17}
\]

Suppose default implies full liquidation of the firm’s assets yielding the liquidation value \((1-\gamma)G\). Then

1. Suppose internal funds are so low that liquidation is certain, but, on the other hand, liquidation value is high, i.e. \( \theta_B > \bar{\theta} \) and \((1-\gamma)G + F(\bar{\theta}, I^{**}) \geq I^{**} - W \), respectively. Then there is no credit risk and the firm invests first best.

2. Suppose in contrast that liquidation is not certain or liquidation value is not too high, i.e. \( \theta_B \leq \bar{\theta} \) or \((1-\gamma)G + F(\bar{\theta}, I^{**}) < I^{**} - W \). Then we have:
(a) The lowest level of internal funds allowing for investment is
\[ W_{\text{min}} = - \left( \frac{\mathbb{E}[\theta]}{2} \right)^2 + (1 - \gamma)G \],
(18)
that is, the total value of the project’s (first-best) net present value and the liquidation value is enough to outweigh the negative internal funds. At this level of internal funds the firm invests first best.

(b) Suppose debt is creditor risky, but liquidation is not certain, i.e. \( \theta < \theta_R < \theta_B \leq \bar{\theta} \). Then
\[ W < \hat{W}_L \]. At \( W = \hat{W}_L \) liquidation is almost certain (\( \theta_B = \bar{\theta} \)) and the firm invests \( I^L > I^{**} \), where \( \hat{I}^L \) solves
\[ 0 = -\omega(\hat{I}^L)^{3/2} + \frac{\omega}{2} \mathbb{E}[\theta] \hat{I}^L + \gamma \left[ \frac{\omega}{2} (1 - \gamma) G \right]^2 , \]
(19)
and
\[ \hat{W}_L = \frac{\omega}{2} (1 - \gamma)^2 G^2 (\hat{I}^L)^{-1/2} - \mathbb{E}[\theta] (\hat{I}^L)^{1/2} + \hat{I}^L - (1 - \gamma)G . \] (20)

i. the firm undertakes the first-best investment as long as \( W < \hat{W}_L \). At \( W = \hat{W}_L \) liquidation is almost certain (\( \theta_B = \bar{\theta} \)) and the firm invests \( I^L > I^{**} \), where \( \hat{I}^L \) solves
\[ 0 = -\omega(\hat{I}^L)^{3/2} + \frac{\omega}{2} \mathbb{E}[\theta] \hat{I}^L + \gamma \left[ \frac{\omega}{2} (1 - \gamma) G \right]^2 , \]
and
\[ \hat{W}_L = \frac{\omega}{2} (1 - \gamma)^2 G^2 (\hat{I}^L)^{-1/2} - \mathbb{E}[\theta] (\hat{I}^L)^{1/2} + \hat{I}^L - (1 - \gamma)G . \]

ii. the firm undertakes the first-best investment when \( W = \hat{W}_L > \hat{W}_L \), where
\[ W_L = - \left( \frac{\mathbb{E}[\theta]}{2} \right)^2 + (1 - \gamma)G \omega(\theta_R - \theta) = W_{\text{min}} + \frac{4(1 - \gamma)^2 G^2 \omega}{\mathbb{E}[\theta]} < 0 , \] (21)
and for this level of internal funds the cut-off for partial recovery of debt is
\[ \hat{\theta}_R = \bar{\theta} - (1 - \gamma)G \frac{4}{\mathbb{E}[\theta]} . \] (22)
and the cut-off for liquidation is
\[ \hat{\theta}_B = \bar{\theta} - (1 - \gamma)G \frac{2}{\mathbb{E}[\theta]} , \] (23)

iii. for \( W > \hat{W}_L \) we conjecture that investment is decreasing in internal funds. Let
\[ \hat{W}_L = I^{**}/4 - (1 - \gamma)G . \] (24)
Then investment is decreasing in internal funds for \( W \geq \hat{W}_L \).

A. If the liquidation value is high enough, i.e. \( I^{**} < \left[ \frac{\delta}{4} G^2(1 - \gamma)^2 \omega \right]^{2/3} \), then \( \hat{W}_L < W_L \). Hence, investment is decreasing for all \( W > W_L \).
B. If the liquidation value is not too high, i.e. \( I^{**} > \left[ \frac{8}{9} G^2 (1 - \gamma)^2 \omega \right]^{2/3} \), then \( \tilde{W}^L > W^L \). So
- for \( W \geq \tilde{W}^L \), investment is decreasing,
- if, in addition, \( I > I^{**}/4 \), then investment is decreasing also for \( W < \tilde{W}^L \), i.e. it is decreasing for all \( W > \hat{W}^L \).

(c) Let \( W^L_R \) be lowest level of funds at which the firm optimally invests such that the debt is creditor risk free, i.e. \( \theta_R = \theta \), when liquidation is not certain, \( \theta_B < \theta \). Then \( W^L_R \) is
\[
W^L_R = I^L_R - (\theta I^L_R^{1/2} + (1 - \gamma) G),
\]
with the associated second best investment
\[
I^L_R = \frac{1}{3} \left[ (a^2 + 2b) + (a^4 + 4a^2b + b^2 + 6ac)A_0^{-1/3} + A_0^{1/3} \right],
\]
where the constants are given as
\[
a = \frac{E[\theta]}{2} - \gamma G \omega, \quad b = \gamma G \omega \theta, \quad c = \gamma \omega (1 - \gamma) G^2,
\]
\[
A_0 = a^6 + 6a^4b + \frac{15a^2b^2}{2} - b^3 + 9a^3c + 18abc + \frac{27c^2}{2} = 3\sqrt{3} \left[ ab + c \right] \sqrt{(-a^2b^2 - 4b^3 + 4a^3c + 18abc + 27c^2)}.
\]

(d) Suppose debt is credit risk free, but liquidation is not certain, i.e. \( \theta_R < \bar{\theta} < \theta_B < \bar{\theta} \).

Let \( W \) be the highest level of internal funds at which the firm uses debt financing.

Then
\[
W = \begin{cases} \frac{E[\theta]}{2} - \gamma G \omega, & G < \frac{E[\theta]}{4\gamma \omega}, \\ \sqrt{\theta} \left( \sqrt{\frac{E[\theta]}{2}} - \sqrt{\gamma G \omega} \right)^4, & G \in \left( \frac{E[\theta]}{4\gamma \omega}, \frac{E[\theta]}{\gamma \omega} \right), \\ 0, & G \geq \frac{E[\theta]}{\gamma \omega}, \end{cases}
\]
where we assume \( \bar{\theta} = 0 \) for the derivation of the solution if \( G > \frac{E[\theta]}{4\gamma \omega} \). We note that \( W \geq 0 \) and that \( W \) decreases in \( G \). Moreover,
- i. investment is decreasing in internal funds for \( W^L_R < W \leq W \),
- ii. for \( W > W \) the firm does not issue debt and \( I = W \), i.e. investment is increasing in internal funds.

Proof: See Appendix B.1.

Similarly, we also state a number of properties for the case in which the firm can make a fire-sale to cover debt payments.
Proposition 2. We identify the following cases, dependent on the level of liquid funds $W$:

(i) Let $W_I^S < I^{**}$ be the highest level of funds at which the firm invests first best, where
\[
W_I^S = -\left\{ \left( \frac{\mathbb E[\theta]}{2} \right)^2 + (1 - \gamma)G \left[ 1 - (1 - \gamma)G \frac{\omega}{\mathbb E[\theta]} \right] \right\} = W_{\text{min}} + \frac{(1 - \gamma)^2 G^2 \omega}{\mathbb E[\theta]}.
\]
For all $W \in [W_{\text{min}}, W_I^S]$, we have that $I = I^{**}$, liquidation is inevitable, i.e. $\theta_B > \bar{\theta}$, and debt is credit risky ($\theta_R > \bar{\theta}$).

(ii) Let $W_R^S$ be lowest level of funds at which the firm optimally invests such that the debt is creditor risk free, i.e. $\theta_R = \bar{\theta}$. Then $W_R^S$ is
\[
W_R^S = I^S_R - \left( \frac{\bar{\theta} I^2_S}{2} + (1 - \gamma)G \right),
\]
with the associated second best investment
\[
I^S_R = \frac{1}{3} \left[ (a^2 + 2b) + (a^4 + 4a^2b + b^2 + 6ac)A_0^{-1/3} + A_b^{1/3} \right],
\]
where $\tilde{c} = c/2$, and the constants $a, b, c$ and $A_0$ (using $\tilde{c}$ instead of $c$) are given in Proposition 1.

(iii) For $W > W_R^S$ the firm issues creditor risk free debt and its investment depends on $G$. Assuming $\bar{\theta} = 0$ we have:

(a) If the liquidation value is not too small, i.e. $(1 - \gamma)G > -2W_R^S$, then investment is increasing for $W \in (W_R^S, 1)$,

(b) If the liquidation value is small, i.e. $(1 - \gamma)G < -2W_R^S$, then investment has a minimum in $W$ when
\[
W = \left( \frac{\bar{\theta}}{2\gamma} \right) 2 \left( \sqrt{1 - \gamma} - (1 - \gamma) \right)^2.
\]

Proof: See Appendix B.2.

4 The cost trade-off

The observed investment behavior is the result of a trade-off between the underinvestment costs and liquidation costs faced by the firm. For an analysis of the cost trade-off, we turn to numerical
Figure 4: Investment as a function of liquid funds.

As a function of liquid funds $W$, the figure shows the first-best investment $I^{**}$ (upper light green line), the minimum feasible investment (lower light green line), and the $I = W$ line (dashed purple line). Between these three lines, there are the investment of the constrained firms facing full liquidation ($I^*_L$, solid blue) or fire sale opportunities ($I^*_S$, dashed blue), respectively. Parameter values are $G = 2$ and $\gamma = 0.5$.

As in Section 3, we focus on a setting with $F(I, \theta) = \theta \sqrt{I}$, and $\theta$ is distributed uniformly on $[\theta = 0, \bar{\theta} = 4]$, i.e. $\omega(\theta) = \frac{1}{\bar{\theta} - \theta}$ and $\Omega(\theta) = \frac{1}{\bar{\theta} - \theta}(\theta - \theta)$. The first-best investment in this case is $I^{**} = 1$. We consider a case in which the new project is relatively small compared to the investments previously undertaken in the firm, i.e. $G > I^{**} = 1$. In particular, we focus on $G = 2$ and assume a 50% loss in case of default, i.e. $\gamma = 0.5$.

4.1 Optimal investment, ex-ante equity values, and costs

Figure 4 illustrates the optimal investment of the constrained firms facing full liquidation ($I^*_L$, solid blue) or fire sale opportunities ($I^*_S$, dashed blue), respectively, as a function of liquid funds $W$. Indeed, the numerical results are in line with the properties hypothesized in Section 3.

There is a cut-off point at $W = 0.56$. For $W \geq W$, the optimal investment given full

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7 We are currently improving the numerical method, which does not yet capture the fact that for very small liquid funds, i.e., $W \in (\hat{W}_L^f, W_L^f)$, there should be overinvestment in the case of full liquidation. However, we expect our empirical implications to be unchanged by this improvement.

8 This parametrization is the same as in Cleary et al. (2007), which ensures comparability with their results.
liquidation coincides with the firm’s internal liquid funds, i.e. the firm invests $I = W$ and does not use external financing. On the other hand, external financing is used for $W < W$. For a firm with access to fire sale, external financing is used on either side of $W$.

The observed investment behavior can explained by a trade-off between underinvestment costs and liquidation costs, as defined in (14). It is visualized in Figures 5(a) and 5(b) for a level of liquid funds $W = 0.6$. Figure 5(a) shows (from top to bottom) the equity values for an unconstrained firm investing at the first-best level $I^{**}$, an unconstrained firm investing at a given level $I$, a constrained firm with access to asset sale investing $I$, and a constrained firm investing $I$ and facing full liquidation. The differences between these equity values correspond to the costs in Figure 5(b): The blue line approaching zero for $I = 1$ illustrates the underinvestment costs, i.e., the difference between the two unconstrained equity values in Figure 5(a). The two light green lines reaching zero at $I = W = 0.6$ correspond to the liquidation costs, and the remaining two purple lines are the total costs. For liquidation costs and total costs, the solid and dashed lines represent the full liquidation case and the asset sale case, respectively.

Figure 5(b) illustrates that the marginal liquidation costs in the full liquidation case exceed the marginal underinvestment costs even for a tiny amount of debt financing. As soon as the firm invests more than its internal funds, the total costs thus increase, and the best choice for such a firm is to invest only at the level of the internal funds. For such a firm avoiding external financing, there is a positive investment-cash flow sensitivity of 1 (i.e., each additional dollar of cash leads to one additional dollar of investment). In contrast, a firm that can rely on asset sale in financial distress faces a much more modest increase in liquidation costs. For such a firm, it is therefore worthwhile to take a certain amount of debt and thus reduce underinvestment costs. It chooses an interior level of investment between $W$ and $I^{**}$. As can be seen in Figure 4, it also has a positive investment-cash flow sensitivity, which is however significantly less than 1 (i.e., each additional dollar of cash leads to less than one additional dollar of investment).

For $W = 0.1$, the corresponding financing cost graphs are shown in Figure 5(d). If only internal funds $W$ are invested, then we have significantly higher underinvestment costs. Also a firm facing full liquidation has now an incentive to take external financing: Its total costs can be reduced by investing more than the internal funds. Therefore, either type of constrained firm finds it optimal to choose an interior level of investment. The investment-cash flow sensitivity for a firm facing full liquidation is now negative: An additional dollar of cash induces the firm to invest less. Although the underinvestment problem then becomes more severe, the firm can reduce the risk of facing costly liquidation.
Figure 5: Equity values and financing costs as a function of investment amount $I$ for different levels of liquid funds $W$. Solid lines indicate the full liquidation case, while dashed lines correspond to the fire sale case. Note that the underinvestment costs coincide for the two cases and thus appear as only one solid blue line in each financing cost graph. Parameter values are $G = 2$ and $\gamma = 0.5$. 

(a) Equity values, $W = 0.6$  
(b) Financing costs, $W = 0.6$  
(c) Equity values, $W = 0.1$  
(d) Financing costs, $W = 0.1$  
(e) Equity values, $W = -1$  
(f) Financing costs, $W = -1$  
(g) Equity values, $W = -1.6$  
(h) Financing costs, $W = -1.6$
It is noteworthy that a firm might even invest on an absolutely higher level if it starts off, other things equal, with less liquid funds in the first place. Compare the firms facing full liquidation with $W = 0.1$ and $W = 0.6$: Starting off with a lower level of liquid funds, there is a huge underinvestment problem, which makes the firm take external funding and invest more. But then even at an investment level of $I = 0.6$, the total costs can be reduced further by investing more. The $W = 0.1$ firm is already dealing with a significant amount of debt financing. Therefore its marginal liquidation costs for another unit of debt are lower than for the $W = 0.6$ firm, which is still considering whether to take external financing at all, rather than avoiding the risk of costly liquidation. And while the latter actually finds it optimal not to take debt and invest at 0.6, the former ends up at an optimal investment level of about 0.78 – although it can be considered the more constrained firm.

Last we consider the region of negative liquid funds $W$. As an important difference to the positive liquid fund region, the liquidation costs can now be decreasing in the investment amount $I$, as illustrated in Figures 5(f) and 5(h): If the firm does not invest at all, then there is liquidation for sure, because the existing funding shortage has to be covered. For a high investment level, the liquidation costs might again increase in $I$, and then we have the usual trade-off between liquidation costs and underinvestment costs. This leads to the interior solution for the optimal investment. For lower $W$, we observe the firm facing full liquidation investing first-best, despite having very low liquid funds. As can be verified in Figure 5(g), this strategy provides indeed the highest expected equity value for such a firm. Consequently, its investment is not sensitive to cash flow. In contrast, a firm with access to asset sale exhibits negative investment-cash flow sensitivity.

### 4.2 Ex-post claim values and costs

Figure 6 shows the values of the different claims in the firm, given the optimal investment strategy according to Figure 4 is followed. Note that the first-best value is not constant, because it is defined as the value of an unconstrained firm consisting of existing assets $G$ and the NPV of the first-best investment according to (1), as well as liquid funds of $(W + D_0(I))$. The latter is corrected by the constrained firm’s debt value to reach comparability, as the constrained firm’s asset value is also increasing by debt issuance. It can be seen that indeed debt starts being used for the full liquidation case, when the liquid funds are low enough that the optimal investment leaves the $I = W$ line. Otherwise, the equity and debt values of the constrained firm are monotonic in $W$, and just looking at them would probably not give rise to assume
This figure shows the values of the different claims in the firm, given the optimal investment strategy according to Figure 4 is followed. From top to bottom, the purple line shows the value of a firm investing first-best. The red, blue, and green lines show the values of the sum of debt and equity, equity, and debt, respectively, in a constrained firm. The solid and dashed lines represent the full liquidation case and the fire sale case, respectively. Parameter values are $G = 2$ and $\gamma = 0.5$.

highly non-monotonic investment curves with both positive and negative investment-cash flow sensitivities. Moreover, when comparing the two different constrained cases, the values of debt and equity seem rather similar. We also examined the debt risk premium, which is a smooth and convex function in $W$.

Figures 7 and 8 show the realized ex-post financing costs for a given level of liquid funds $W$ and debt value $D_0$, respectively. Implicitly, the firm chooses the optimal investment to minimize the total costs, according to Figure 4. These figures could be interpreted as a confirmation of the conventional wisdom that financing costs are monotonically increasing in the financing amount. However, this is misleading! For a decision on the optimal investment, the firm considers instead the financing costs as a function of the investment amount, given in Figure 5. And these cost functions, which are determining the investment decision, are non-monotonic in the investment amount, and thus also non-monotonic in the financing amount.

To conclude this section, we give an explanation how the observed monotonic cost functions and a non-monotonic investment behavior can be seen as consistent: The key point is to consider
Figure 7: Financing costs as function of liquid funds.
The blue, light green, and purple lines illustrate the underinvestment costs, liquidation costs,
and total costs as a function of liquid funds $W$, given the optimal investment strategy according
to Figure 4 is followed. The solid and dashed lines represent the full liquidation case and the
fire sale case, respectively.

Figure 8: Financing costs as function of debt value.
The blue, light green, and purple lines illustrate the underinvestment costs, liquidation costs,
and total costs as a function of debt value $D_0$, given the optimal investment strategy according
to Figure 4 is followed. The solid and dashed lines represent the full liquidation case and the
fire sale case, respectively.
Figure 9: Optimal investment for varying redeployability.

As a function of liquid funds $W$, the figure shows the optimal investment for varying asset redeployability. A higher degree of redeployability corresponds to a lower $\gamma$. The solid and dashed lines represent the full liquidation case and the fire sale case, respectively. From top to bottom, the lines represent $\gamma = 0$ (green line, first best), 0.25, 0.5 (base case), 0.75, and 1 (the lowest line, where full liquidation case and the fire sale case are coinciding). Parameter values are $G = 2$ and $\gamma = 0.5$.

the ex-ante cost functions faced by the decision-maker, rather than the observed ex-post cost functions given the optimal investment. Our idea is similar to the critique of empirical tests of capital structure theory raised by Strebulaev (2007): The ex-post observed capital structure is not reflecting the ex-ante faced trade-off, except for the rare case that a firm is observed exactly at a refinancing point.

5 Optimal investment for varying redeployability

In the previous section, we distinguished the analysis along the second dimension of asset liquidity, namely the degree of fractional liquidation. Now we turn to analyze the first dimension of asset liquidity, namely the degree of redeployability. How is the optimal investment derived in the last section affected by varying the outside value of the firm’s assets? In our model, a higher degree of asset redeployability corresponds to a lower $\gamma$.

Figure 9 shows the optimal investment for varying asset redeployability as a function of liquid funds $W$. Both for firms that face full liquidation in financial distress and firms that can
liquidate a fraction of their assets in a fire sale, we find the same main result: The absolute sensitivity of investment with respect to liquid funds is decreasing (or constant) in the degree of asset redeployability.

Our finding can be explained as follows: Firms with high internal funds have a positive sensitivity of investment to cash. For the region in which firms facing full liquidation avoid debt financing, there is a one-to-one sensitivity independent of redeployability. However, the size of the region is decreasing in redeployability. Firms with highly redeployable assets (low $\gamma$) will obtain debt financing even for a rather mild cash shortfall. In contrast, firms with low redeployability (high $\gamma$) stay away from debt financing to avoid the risk of liquidation, even if they have to accept high underinvestment costs. On the other hand, once a firm with low redeployability decides to use debt financing, the pressure of higher underinvestment costs makes it also more beneficial to move closer to first-best investment. Therefore firms with low internal funds show a negative sensitivity of investment with respect to liquid funds that is, in absolute terms, decreasing in the degree of asset redeployability.

For firms that can liquidate a fraction of their assets in a fire sale, the situation is similar. While debt is always used, the U-shape of the investment curve is also becoming more and more pronounced if assets are less deployable. That means, firms with high liquidity compensate cash shortfalls less, if their assets are less redeployable. Again in consequence, these firms face a high level of underinvestment costs at the cut-off point where they change their debt policy and start overcompensating further cash shortfalls. For low levels of liquid funds, we therefore have a situation that is similar to in the full liquidation case: The negative sensitivity of investment with respect to liquid funds is more pronounced, because it pays off more to reduce underinvestment.

6 Empirical implications and discussion

6.1 Empirical implications

Our theoretical results lead to the following empirically testable implications for financially constrained firms with existing assets in place:

1. For firms that face full liquidation in financial distress

   a) Firms with high internal funds:

   Avoid debt financing and have a positive investment-cash flow sensitivity of 1 (i.e., each additional dollar of cash leads to one additional dollar of investment).
b) Firms with low internal funds:

Use debt financing and have a negative investment-cash flow sensitivity (i.e., more cash leads to less investment).

2. For firms that can liquidate a fraction of their assets in a fire sale

a) Firms with high internal funds:

Use risk-free debt financing and have a positive investment-cash flow sensitivity of significantly less than 1 (i.e., each additional dollar of cash leads to less than one additional dollar of investment).

b) Firms with low internal funds:

Use risky debt financing and have a negative investment-cash flow sensitivity (i.e., more cash leads to less investment).

3. Asset redeployability

Firms with higher degree of redeployability invest less sensitive to their cash flow.

6.2 Discussion

In a first simplified view of our implications, we highlight the general persistence of the U-shaped investment curve: Investment is decreasing in liquid funds for firms with low internal funds, and increasing in liquid funds for firms with high internal funds. This property holds for either extreme case of asset liquidity, and regardless of the nature of external financing. It is also very similar to the shape hypothesized by Cleary et al. (2007). Their empirical study supporting the U-shaped investment curve can therefore also be seen as support for our model.

The value added by our paper consists of the focus on asset liquidity.⁹ Here, our paper is contributing to the discussion initiated by Almeida and Campello (2007). They analyze the effect of asset tangibility on investment. In their model, they consider the same quantity as investment-cash flow sensitivity as we do. For an empirical test of our model, it would make sense to relate tangibility (or what we call redeployability) to the firm’s existing assets rather than to its new investment. That is actually done in the empirical part of Almeida and Campello (2007).¹⁰

⁹Some of the more subtle differences to Cleary et al. (2007) are that our debt contract does not imply unlimited liability of equity holders, and that we do not require negative liquid funds as a general prerequisite to observe a U-shaped investment curve. See Appendix A for a more detailed discussion.

¹⁰See the discussion in Almeida and Campello (2007, 2.2.3.1).
They theoretically hypothesize and empirically find that a constrained firm’s investment is more sensitive to cash flow if its assets are more tangible. This finding is strikingly different from our implication that firms with higher degree of redeployability invest less sensitive to their cash flow.

A main difference on the modeling side is the focus on quantity vs. cost constraints.\(^\text{11}\) Almeida and Campello (2007) model only quantity constraints and no financing costs. The optimal investment is therefore a boundary solution – the firm always invests such that the quantity constraint is binding. Therefore, investment is adjusted the more pronounced for increasing liquid funds, the more tangible the assets are, i.e., investment-cash flow sensitivity is increasing in asset tangibility. In contrast, we consider financing costs to be a very relevant factor in the investment decision. In our model, the amount of liquid funds needed to internally fund the first-best investment is independent of the existing assets’ redeployability. But for a given shortfall of liquid funds below that level, the cut on investment is the more pronounced, the less liquid the existing assets are. This corresponds to an investment-cash flow sensitivity that is decreasing in asset redeployability.\(^\text{12}\)

Another important difference between Almeida and Campello (2007) and our model is that we capture the puzzling empirical finding that investment-cash flow sensitivities can be negative (see for example Bhagat et al. (2005), Cleary et al. (2007), and Hovakimian (2009)). For the region of low liquid funds, investment-cash flow sensitivities are negative both in our model and in the empirical evidence. We point out that our Implication 3 holds both for positive and negative sensitivities, meaning that for both cases, the absolute sensitivity is decreasing in asset redeployability.

We emphasize that the “credit multiplier effect” identified by Almeida and Campello (2007) is more applicable to young firms, for which the new investment is exceeding the value of the existing assets, and thus the dominant effect of new investment is to increase the collateral value and thus also today’s funding capacity. In contrast, our implications are more likely to hold for more mature, established firms. If the value of the existing assets relative to the new investment is substantial, then the argument of avoiding debt financing in order not to set the existing assets at risk is relevant.

Our argument is related to the classical article by Myers and Majluf (1984), although the

\(^{11}\)See also the discussion in Almeida and Campello (2007, 1.2.2).

\(^{12}\)Here we argue in the case in which firms that can liquidate a fraction of their assets in a fire sale; if firms that face full liquidation in financial distress, the sensitivity is the same (=1) as long as no debt is used.
economic context is different: There, the existing shareholders pass on positive NPV projects, if the outsiders’ undervaluation of the existing assets is too severe and thus not compensated by the gain from the new investment. Also in our model, the underinvestment problem is more pronounced for a high value of the existing assets relative to the new investment. Very similarly, the firm trades off (possible) losses on the existing assets against the underinvestment costs. From the Myers and Majluf (1984) problem, we can also transfer one possible remedy: By using project financing, the connection between the existing assets and the new investment, and thus the underinvestment problem, can possibly be mitigated.

7 Conclusion

In this paper we analyze the relation between the liquidity of a firm’s existing real, financial assets, and corporate investment. We split up the financing costs faced by a constrained firm into two components, namely expected liquidation costs and underinvestment costs. We identify the trade-off between these two types of costs as the central reason for the U-shaped investment curve in liquid funds. From this cost trade-off, we derive testable implications for investment-cash flow sensitivities and debt usage, which are highly dependent on the liquidity of the existing assets.

We distinguish between two dimensions of asset liquidity. First, the degree of redeployability, and second, whether fire sale of assets can be used to avoid full liquidation. We find that investment is less sensitive to cash flow for firms with higher degree of redeployability. Further, we show that firms that are forced to liquidate all their assets in financial distress decide not to take external financing at all, if they have a high level of internal funds, since the risk of costly liquidation dominates the cost of underinvestment. In fact they have a one-to-one positive investment cash-flow sensitivity. However, there is a cut-off point of liquid funds at which the trade-off switches sides. Firms with lower liquid funds benefit more by getting closer to the first-best investment. Therefore, investment volume is increasing in constraints, i.e. we have a negative investment cash-flow sensitivity. However, a very different situation arises for firms that can use a fire sale of their assets and, thus, avoid full liquidation in financial distress. Such firms use external financing even for small cash shortages in which case they have a positive investment cash-flow sensitivity, but much lower than firms facing the risk of full liquidation. Still they also have a cut-off point of liquid funds at which the trade-off switches sides, which is for these firms when debt becomes risky. Hence, these firms can also face a negative investment
Overall, we suggest that asset liquidity is indeed an important factor towards resolving the empirical puzzle of negative investment-cash flow sensitivities. In particular, we demonstrate that even with a standard debt contract, investment is actually U-shaped in the firm’s liquid funds. However, there is no general rule as to whether the policy switch point should be in the region of positive or negative liquid funds. In addition, we show that despite the non-monotonic ex-ante financing costs driving investment, the observed ex-post cost functions may be perfectly monotonic and allow no meaningful conclusion on firms’ investment policies. Hence, this result must be carried as a caveat for empirical investigations analyzing the firm’s investment-cash flow sensitivity.
References


A Optimal debt contract and limited liability

In this section, we compare our model to a setting in which the financing contract is not a standard debt contract, but an ex-ante optimal contract. Such a setting is analyzed in Cleary et al. (2007), given that there is asymmetric information regarding the outcome of the investment, which is observed by the firm, but unobservable to the external investor. Depending on the repayment offered by the firm, their contract specifies a probability

$$\varphi_O = \begin{cases} \frac{D_1 - F(I, \theta)}{G} & \text{if } \theta < \theta_B \\ 0 & \text{if } \theta \geq \theta_B \end{cases}$$

that the firm is liquidated and taken over by the debt holders. The authors provide an alternative interpretation, where $\varphi_O$ denotes the fraction of assets that the debt holders receive in a partial liquidation. In that sense, their setting is quite similar to our fire sale of assets case. The superscript $O$ denotes the original Cleary et al. (2007) model, in contrast to a modified version we present below. The values of debt and equity in such a setting can be written as

$$D_0^O(I) = \int_{\theta_B}^{\theta} (F(I, \theta) + \varphi_O(1 - \gamma)G) \cdot \omega(\theta)d\theta + D_1 \cdot (1 - \Omega(\theta_B))$$

(33)

and

$$E_0^O(I) = \mathbb{E}[F(I, \theta)] + G - D_1.$$  

(34)

Figure 10 shows the resulting optimal investment in Cleary et al. (2007) in comparison with our solution shown in Figure 4. One feature that is emphasized by Cleary et al. (2007) is that the minimum investment always occurs in the region of negative liquid funds. They can even prove that this property holds in general for their framework. In contrast, the minimum investment for our full liquidation case is in the region of positive liquid funds, and this could also happen for the fire sale of assets case, depending on parameter values.

Our main criticism towards the Cleary et al. (2007) model is that it implicitly assumes unlimited liability of equity holders. In states in which the fraction of assets $\varphi_O$ to be liquidated exceeds 100\%, the equity holders have to inject additional funds, to ensure that the debt holders still receive the value stated in (33). Note that the equity value (34) is independent of the liquidation costs $\gamma$ for a given $D_1$, although in some states there is only fractional liquidation of assets, so one would usually expect that the equity holders care about the remaining asset value. On the other hand, the debt value can be rewritten as

$$D_0^O(I) = \int_{\theta_B}^{\theta} (F(I, \theta) + (D_1 - F(I, \theta))(1 - \gamma)) \cdot \omega(\theta)d\theta + D_1 \cdot (1 - \Omega(\theta_B)),$$
Figure 10: Comparison with investment in Cleary et al. (2007).

As a function of liquid funds $W$, the figure shows the first-best investment $I^{**}$ (upper light green line), the minimum feasible investment (lower light green line), and the $I = W$ line (dashed purple line). Between these three lines, there are the investment of the constrained firms facing full liquidation ($I^*_L$, solid blue) or fire sale opportunities ($I^*_S$, dashed blue), respectively. The red curve indicates the optimal investment in Cleary et al. (2007). Parameter values are $G = 2$ and $\gamma = 0.5$. 
which shows that it is independent of the value of the existing assets $G$ and thus the absolute size of the collateral value. These are quite unusual properties of debt and equity values, and they are due to the unlimited liability of the equity holders. To show that the latter has an important effect on the model’s results, we modify the Cleary et al. (2007) model such that the payoff to equity holders is non-negative in all states. The modified fraction of assets that the debt holders receive in a partial liquidation is then

$$
\varphi_M = \begin{cases} 
1 & \text{if } \theta < \theta_C \\
\frac{D_1 - F(I, \theta)}{G} & \text{if } \theta_C \leq \theta < \theta_B \\
0 & \text{if } \theta \geq \theta_B
\end{cases}
$$

We introduce a new cut-off value $F(I, \theta_C) = D_1 - G$, denoting the $\theta$ state in which the liquidation rule gives 100% of the existing assets to the debt holders. The values of debt and equity in this modified setting can be written as

$$
D_0^M(I) = \int_{\theta_C}^{\theta_B} (F(I, \theta) + \varphi_M(1 - \gamma)G) \cdot \omega(\theta)d\theta + D_1 \cdot (1 - \Omega(\theta_B)) \quad (35)
$$

and

$$
E_0^M(I) = \int_{\theta_C}^{\theta_B} F(I, \theta) \cdot \omega(\theta)d\theta + (G - D_1) \cdot (1 - \Omega(\theta_C)) \quad (36)
$$

This means, the debt value remains as defined in (33), except for the new definition of the liquidation fraction, which is now bounded by 100%. The modified equity value reflects that it is zero (rather than negative) for all $\theta < \theta_C$.

We show the comparison between the original and the modified Cleary et al. (2007) model in Figure 11. The base parametrization in Figure 11(a) shows mainly that with limited equity liability, investment is not feasible for low $W$ in the modified model. So given limited liability, one could argue that using a standard debt contract – as in our setting – improves welfare, because even for firms with low $W$, access to external financing and thus investment is still feasible in our model. The standard debt contract allows the firm to offer all the available value of the existing assets, i.e., $(1 - \gamma)G$, as collateral, while the debt holders receive less value in the modified model for $\theta_C < \theta < \theta_B$.

Comparing the two figures in Figure 11 shows that for the original Cleary et al. (2007) model, the optimal investment is independent of the value of the existing assets $G$. As explained before, the debt value (33) is independent of $G$. Moreover, the equity value (34) is linear in $G$, which makes the value of the existing assets irrelevant for maximization of the equity value.
As a function of liquid funds $W$, the two figures show the first-best investment $I^{**}$ (upper light green line), the minimum feasible investment (lower light green curve), and the $I = W$ line (dashed purple line). Between these three boundary curves, there are the solid and dashed red curves indicating the optimal investment in Cleary et al. and the modified model, respectively.

with respect to $I$. In contrast, Figure 11(b) shows that for a low value of the existing assets $G$, the modified Cleary et al. model differs even more significantly from the original model. The minimum investment is situated now in the positive $W$ region, contrary to the generally proved property that the minimum investment in the original Cleary et al. model always occurs in the region of negative liquid funds. The $W$ level where the minimum investment occurs for the original model is now out of the region of feasible financing contracts. Even for mildly negative $W$, external financing is not available anymore and the firm has to be liquidated without making use of the investment opportunity.

Cleary et al. (2007) propose that the U-shaped investment curve is due to considerations of the external investor regarding the trade-off between the cost of providing funds and the possible revenue from the investment project. However, recall from the full liquidation case that the level of liquid funds $\mathbb{W}$, where we observe the minimum investment, is characterized by using no debt for $W \geq \mathbb{W}$ and risk-free debt for $W < \mathbb{W}$. This means that the external investor is either not involved at all, or he can be sure to receive back the notional amount of the debt. Therefore, we point out that there can be situations in which the U-shaped investment curve is exclusively driven by trade-off considerations of the entrepreneur rather than the external investor. In our model, it is the trade-off between foregoing NPV of the investment (underinvestment costs) and the risk of losing part of the existing assets’ value (liquidation costs). As illuminated above, this trade-off can be relevant even if the external investor is not participating in the risk of the firm.
B Proofs

B.1 Proof of Proposition 1

Proof. From Section 2.1 we have the unconstrained problem (1). With our parametrization we get

$$\max_I \int_0^\theta F(I, \theta)\omega(\theta)d\theta - I$$

that is

$$\max_I \mathbb{E}[\theta]I^{1/2} - I.$$ 

The first and second order conditions are, respectively,

$$\frac{\mathbb{E}[\theta]}{2} I^{-1/2} - 1 = 0,$$

$$\frac{\mathbb{E}[\theta]}{-4} I^{-3/2} < 0,$$

so the FOC is sufficient. Hence

$$I^{**} = \left(\frac{\mathbb{E}[\theta]}{2}\right)^2,$$  \hspace{1cm} (37)

as asserted.

From Section 2.3.1 we have that the general optimization program is to maximize (6) subject to (2), (3), and (4), i.e.

$$\max_I \int_\theta^\theta F(I, \theta)\omega(\theta)d\theta + \left[1 - \gamma \Omega(\theta_B) + 1 - \Omega(\theta_R)\right] G - \left[1 - \Omega(\theta_R)\right] D_1,$$  \hspace{1cm} (38)

s.t.

$$D_1 = F(I, \theta_B)$$  \hspace{1cm} (39)

$$D_1 = F(I, \theta_R) + (1 - \gamma)G$$  \hspace{1cm} (40)

$$D_0 \triangleq \int_\theta^\theta F(I, \theta)\omega(\theta)d\theta + (1 - \gamma)G \cdot \Omega(\theta_R) + D_1 \cdot (1 - \Omega(\theta_R)) = I - W.$$  \hspace{1cm} (41)

We now turn to the various cases.

1. Assume $\theta_B \geq \bar{\theta}$ and $(1 - \gamma)G + F(\bar{\theta}, I^{**}) \geq I^{**} - W$. The external funds needed to fund the first best investment is $D_0 = I^{**} - W$. However, as this is less than the liquidation value, the debt is creditor risk free. Hence, $\theta_R \leq \bar{\theta}$, so $D_1 = D_0$. Using this in (38) we get

$$\max_I \int_\theta^\theta F(I, \theta)\omega(\theta)d\theta + (1 - \gamma)G - (I - W),$$  \hspace{1cm} (42)

which yields the first best investment.
2. Assume either liquidation is not certain, i.e. $\theta_B \leq \bar{\theta}$, or liquidation value is not very high, i.e. $(1 - \gamma)G + F(\bar{\theta}, I^{**}) < I^{**} - W$.

(a) We begin with the lowest admissible level of internal funds, so assume $\theta_B > \bar{\theta}$ and $\theta_R \in [\bar{\theta}, \bar{\theta}]$. Using (38)–(41) we have the objective function and the creditors’ participation constraint so the program becomes

$$\max_I \int_{\theta_R}^{\bar{\theta}} F(I, \theta) \omega(\theta) d\theta + \left[1 - \Omega(\theta_R)\right] \left\{ (1 - \gamma)G - \left(\theta_R^{1/2} + (1 - \gamma)G\right) \right\},$$

s.t.

$$\int_{\theta_R}^{\bar{\theta}} F(I, \theta) \omega(\theta) d\theta + (1 - \gamma)G \cdot \Omega(\theta_R) + \left(\theta_R^{1/2} + (1 - \gamma)G\right) \cdot (1 - \Omega(\theta_R)) = I - W,$$

that is

$$\max_I \int_{\theta_R}^{\bar{\theta}} F(I, \theta) \omega(\theta) d\theta - \left[1 - \Omega(\theta_R)\right] F(I, \theta_R), \tag{43}$$

s.t.

$$\int_{\theta_R}^{\bar{\theta}} F(I, \theta) \omega(\theta) d\theta + (1 - \gamma)G - I + W = -F(I, \theta_R)\left(1 - \Omega(\theta_R)\right). \tag{44}$$

Plugging in the creditors’ participation constraint (44) in the objective we get

$$\max_I \int_{\theta_R}^{\bar{\theta}} F(I, \theta) \omega(\theta) d\theta + \int_{\bar{\theta}}^{\theta_R} F(I, \theta) \omega(\theta) d\theta + (1 - \gamma)G - I + W, \tag{45}$$

hence

$$\max_I E[F(I, \theta)] + (1 - \gamma)G - I + W, \tag{46}$$

which is equal to (42) so we get the first best investment $I^{**}$, cf. (37). We now seek the lowest admissible level of internal funds. To do this, we derive $\theta_R$. Using the first best investment in (44) yields

$$\int_{\theta_R}^{\bar{\theta}} \frac{\theta E[\theta]}{2} \omega d\theta + \theta_R \frac{E[\theta]}{2} \omega(\bar{\theta} - \theta_R) + (1 - \gamma)G - \left(\frac{E[\theta]}{2}\right)^2 + W = 0, \tag{47}$$

thus

$$\omega \frac{E[\theta]}{2} \left[\frac{1}{2}(\theta_R^2 - \bar{\theta}^2) + \theta_R(\bar{\theta} - \theta_R)\right] - \left[\left(\frac{E[\theta]}{2}\right)^2 - W - (1 - \gamma)G\right] = 0, \tag{48}$$
hence

\[ \theta_R^2 - 2\theta \theta_R + \theta^2 + \frac{4}{\omega E[\theta]} \left( \frac{E[\theta]}{2} \right)^2 - W - (1 - \gamma)G = 0, \]  

(49)

which is a parabola in \( \theta_R \). Employing the smallest positive root as the solution—in order to undertake as little debt as possible—the solution is

\[ \theta_R = \bar{\theta} - \sqrt{\left( \bar{\theta}^2 - \bar{\theta}^2 \right) - \frac{4}{\omega E[\theta]} \left( \frac{E[\theta]}{2} \right)^2 - W - (1 - \gamma)G}. \]  

(50)

The lowest level of internal funds sufficient to undertake debt occurs when \( \theta_R = \bar{\theta} \), i.e. not only will the firm default for sure, it is also the case that debt holders only receive full recovery in the best possible state. Thus we consider \( \theta_R(W) \triangleq \bar{\theta} \), and using (50) we obtain

\[ \bar{\theta} - \sqrt{\left( \bar{\theta}^2 - \bar{\theta}^2 \right) - \frac{4}{\omega E[\theta]} \left( \frac{E[\theta]}{2} \right)^2 - W - (1 - \gamma)G} = \bar{\theta}, \]

and rewriting yields

\[ W = - \left( \frac{E[\theta]}{2} \right)^2 + (1 - \gamma)G \triangleq W_{\text{min}}, \]  

(51)

as asserted.

(b) Consider the case with \( \underline{\theta} \leq \theta_R < \theta_B \leq \bar{\theta} \). Ultimately, we want to consider the sensitivity of investment w.r.t. internal funds. To do this, we consider the Lagrange function

\[ \mathcal{L}(I, \theta_R, \theta_B, W) \]

\[ = \frac{\omega}{2} \left( \bar{\theta}^2 - \bar{\theta}_R^2 \right) I^{1/2} + (1 - \gamma)G - (1 - \gamma) \Omega(\theta_R)G - (1 - \gamma) \Omega(\theta_B)G + \theta_B I^{1/2} \]

\[ + \lambda_1 \left\{ \left( \theta_B - \theta_R \right) I^{1/2} - (1 - \gamma)G \right\} \]

\[ + \lambda_2 \left\{ \frac{\omega}{2} \left( \bar{\theta}_R^2 - \bar{\theta}^2 \right) I^{1/2} + (1 - \gamma)G \Omega(\theta_R) + \theta_B I^{1/2} \right\} , \]

(52)

where we have eliminated \( D_1 \) by combining (39) and (40) to

\[ (\theta_B - \theta_R) I^{1/2} = (1 - \gamma) G \]

(53)

and we use \( \lambda_1 \) and \( \lambda_2 \) as the Lagrange multipliers associated with the two remaining conditions, i.e. (41) and (53). The first order condition w.r.t. investment is

\[ \frac{\omega}{2} \left( \bar{\theta}^2 - \bar{\theta}_R^2 \right) I^{1/2} - \theta_B (1 - \Omega(\theta_R)) I^{1/2} + \lambda_1 (\theta_B - \theta_R) I^{-1/2} \]

\[ + \lambda_2 \left\{ \frac{\omega}{2} \left( \bar{\theta}_R^2 - \bar{\theta}^2 \right) I^{1/2} + \theta_B (1 - \Omega(\theta_R)) I^{-1/2} - 1 \right\} = 0, \]  

(54)
and the one w.r.t. $\theta_R$ is

$$-2\theta_R \frac{\omega}{2} I^{1/2} - (1 - \gamma) \omega G + \theta_B \omega I^{1/2} + \lambda_1 \{-I^{1/2}\} + \lambda_2 \left\{ \frac{\omega}{2} 2\theta_R I^{1/2} + (1 - \gamma) G \omega - \theta_B I^{1/2} \omega \right\} = 0,$$  

(55)

and w.r.t. $\theta_B$ we have

$$-\gamma \omega G - (1 - \Omega(\theta_R)) I^{1/2} + \lambda_1 I^{1/2} + \lambda_2 I^{1/2} (1 - \Omega(\theta_R)) = 0,$$  

(56)

and regarding the first multiplier we obtain

$$(\theta_B - \theta_R) I^{1/2} - (1 - \gamma) G = 0,$$  

(57)

whereas the second multiplier condition yields

$$\omega \frac{(\theta_B^2 - \theta^2)}{2} I^{1/2} + (1 - \gamma) G \Omega(\theta_R) + \theta_B I^{1/2} (1 - \Omega(\theta_R)) - I + W = 0.$$  

(58)

Using (53) in (55) we get

$$\omega ((\theta_B - \theta_R) I^{1/2} - (1 - \gamma) G) + \lambda_1 \{-I^{1/2}\} + \lambda_2 \omega \left\{ (\theta_R - \theta_B) I^{1/2} + (1 - \gamma) G \right\} = 0,$$

so $\lambda_1 = 0$. From (56) we then get the other multiplier

$$\lambda_2 = 1 + \frac{\gamma \omega GI^{-1/2}}{(1 - \Omega(\theta_R))} \geq 1,$$  

(59)

and using (57) we define

$$\frac{(\theta_B - \theta_R) - (1 - \gamma) G I^{-1/2}}{\xi_k(I, \theta_R, \theta_B, W)} = 0,$$  

(60)

and similarly we define the function $h$ from (58) as

$$\frac{\left[ \omega \frac{(\theta_B^2 - \theta^2)}{2} + \theta_B (1 - \Omega(\theta_R)) \right] I^{1/2} + (1 - \gamma) G \Omega(\theta_R) - I + W}{\xi_k(I, \theta_R, \theta_B, W)} = 0.$$  

(61)

Applying $\lambda_2$ in (54) and rewriting gives us

$$\frac{\left[ \omega \frac{(\theta_B^2 - \theta^2)}{2} - \theta_B (1 - \Omega(\theta_R)) \right]}{(1 - \Omega(\theta_R))} \frac{1}{2} I^{-1/2} + \left( 1 + \frac{\gamma \omega GI^{-1/2}}{(1 - \Omega(\theta_R))} \right) \left\{ \frac{\omega}{2} (\theta_B^2 - \theta^2) + \theta_B (1 - \Omega(\theta_R)) \right\} \frac{1}{2} I^{-1/2} - 1 \right\} = 0.$$  

39
rearranging yields
\[\frac{1}{2} (\theta_R^2 - \theta_B^2) - \theta_B (1 - \Omega(\theta_R)) + \frac{1}{2} (\theta_R^2 - \theta_B^2) + \theta_B (1 - \Omega(\theta_R))] \frac{1}{2} I^{-1/2} - 1
\]
\[+ \gamma \omega GI^{-1/2} \left\{ \left[ \frac{1}{2} (\theta_R^2 - \theta_B^2) + \theta_B (1 - \Omega(\theta_R)) \right] \frac{1}{2} I^{-1/2} - 1 \right\} = 0,
\]
and we get
\[\left( \frac{E[\theta]}{2} - I^{1/2} \right) (1 - \Omega(\theta_R)) + \gamma \omega G \left\{ \left[ \frac{1}{2} (\theta_R^2 - \theta_B^2) + \theta_B (1 - \Omega(\theta_R)) \right] \frac{1}{2} I^{-1/2} - 1 \right\} = 0.
\]
(62)
i. To derive \( \hat{W}^L \) we solve (60)–(62) assuming \( \theta_B = \bar{\theta} \). From (60) we immediately get that
\[\theta_R = \bar{\theta} - (1 - \gamma)GI^{-1/2},\]
and inserting this in (61) gives \( W \) as a function of investment. To derive the investment level \( \hat{I}^L \) we employ (62) together with (63). This yields the equation for \( \hat{I}^L \) as stated in the proposition. Moreover, as we argue below, the FOCs imply that investment is decreasing in \( W \). Thus, investment should increase if we consider \( W \) just below \( \hat{W}^L \). But this would imply \( \theta_B > \bar{\theta} \) which leads to the same conclusion as the first part of Proposition 2.

ii. Suppose now that \( I = I^{**} = \left( \frac{E[\theta]}{2} \right)^2 \). We then derive \( W, \theta_R, \) and \( \theta_B \) such that the first order conditions are satisfied. We first turn to condition (62). This gives us
\[g(I^{**}, \theta_R, \theta_B, W) = 0 \]
(64)
and inserting the first best investment we get
\[+ \gamma \omega G \left\{ \left[ \frac{1}{2} (\theta_R^2 - \theta_B^2) + \theta_B (1 - \Omega(\theta_R)) \right] \frac{1}{2} E[\theta] - 1 \right\} = 0,
\]
(65)
thus
\[\frac{1}{2} (\theta_R^2 - \theta_B^2) + \theta_B (1 - \Omega(\theta_R)) = E[\theta].\]
(66)
Using this in (61) we see that
\[E[\theta] \frac{E[\theta]}{2} + (1 - \gamma) G \Omega(\theta_R) - \left( \frac{E[\theta]}{2} \right)^2 + W = 0,
\]
(67)
40
whence it follows that
\[
W = -\left[\left(\frac{\mathbb{E}[\theta]}{2}\right)^2 + (1 - \gamma)G\Omega(\theta_R)\right] \triangleq W^L, \tag{68}
\]
as asserted. From (60) we obtain
\[
\theta_B = \theta_R + (1 - \gamma)\frac{2}{\mathbb{E}[\theta]}.	ag{69}
\]
Applying this in (66) yields
\[
\frac{\omega}{2} (\theta_R^2 - \theta^2) + \left(\theta_R + (1 - \gamma)\frac{2}{\mathbb{E}[\theta]}\right) \omega(\theta - \theta_R) - \mathbb{E}[\theta] = 0, \tag{70}
\]
which is a parabola in \(\theta_R\). The solution is
\[
\theta_R = \bar{\theta} - (1 - \gamma)\frac{G^2}{\mathbb{E}[\theta]} - \sqrt{\left(\bar{\theta} - (1 - \gamma)\frac{G^2}{\mathbb{E}[\theta]}\right)^2 - \left(\bar{\theta}^2 - (1 - \gamma)\frac{G^4}{\mathbb{E}[\theta]} + \frac{2\mathbb{E}[\theta]}{\omega}\right)},
\]
and using the fact that \(\mathbb{E}[\theta] = \omega(\bar{\theta}^2 - \bar{\theta}^2)/2\) we obtain
\[
\theta_R = \bar{\theta} - (1 - \gamma)\frac{4}{\mathbb{E}[\theta]} \triangleq \hat{\theta}_R, \tag{71}
\]
which completes the proof of this case since (60) with \(I = I^{**}\) gives us \(\hat{\theta}_B\).

iii. Recall the first order conditions from the full program yield the conditions
\[
k(I, \theta_R, \theta_B, W) \triangleq \theta_B - \theta_R - (1 - \gamma)GI^{-1/2} = 0, \tag{72}
\]
\[
h(I, \theta_R, \theta_B, W) \triangleq \left[\frac{\omega}{2}(\theta_R^2 - \theta^2) + \theta_B(1 - \Omega(\theta_R))\right] I^{1/2} + (1 - \gamma)G\Omega(\theta_R) - I + W = 0, \tag{73}
\]
\[
g(I, \theta_R, \theta_B, W) \triangleq \left[\frac{\mathbb{E}[\theta]}{2} I^{-1/2} - 1\right] I^{1/2}(1 - \Omega(\theta_R)) + \gamma\omega G \left\{\left[\frac{\omega}{2}(\theta_R^2 - \theta^2) + \theta_B(1 - \Omega(\theta_R))\right] \frac{1}{2} I^{-1/2} - 1\right\} = 0, \tag{74}
\]
Using subscripts for partial derivatives and suppressing the arguments we observe that
\[
k_I = \frac{1}{2}(1 - \gamma)GI^{-3/2} > 0, \quad k_{\theta_R} = -1, \quad k_{\theta_B} = 1, \quad k_W = 0, \tag{75}
\]
\[
h_I = \left[\frac{\omega}{2}(\theta_R^2 - \theta^2) + \theta_B(1 - \Omega(\theta_R))\right] \frac{1}{2} I^{-1/2} - 1 < 0, \tag{76}
\]
where the last inequality follows from the cogent observation that
\[
g(I, \theta_R, \theta_B, W) = \left(\frac{\mathbb{E}[\theta]}{2} - I^{1/2}\right)(1 - \Omega(\theta_R)) + \gamma\omega G \cdot h_I = 0,
\]
and since the second best investment is less than the first best investment, the
first term is positive, implying that the second term is negative as desired. Fur-
thermore,

\[
h_\theta_B = (1 - \Omega(\theta_R))I^{1/2} > 0, \quad h_W = 1,
\]

\[
h_\theta_R = \omega((\theta_R - \theta_B)I^{1/2} + (1 - \gamma)G) \geq 0,
\]

(77)

\[
g_I = -\frac{1}{2}I^{-1/2}(1 - \Omega(\theta_R)) - \gamma \omega G \left[\frac{\omega}{2}(\theta_R^2 - \theta^2) + \theta_B(1 - \Omega(\theta_R))\right] \frac{1}{4}I^{-3/2} < 0,
\]

(78)

\[
g_\theta_R = -\omega \left[\frac{E[\theta]}{2} - I^{1/2}\right] + \omega \gamma G(\theta_B - \theta_R) \frac{1}{2}I^{-1/2} < 0,
\]

(79)

\[
g_\theta_B = \gamma \omega G(1 - \Omega(\theta_R)) \frac{1}{2}I^{-1/2} > 0, \quad g_W = 0.
\]

(80)

Our goal is to consider \(dI/dW\). To do so we apply total differentiation on (72)–(74) a number of times. First, using (72) we obtain

\[
d\theta_R = -\frac{1}{k_{\theta_R}}(k_I dI + k_{\theta_B} d\theta_B + k_W dW),
\]

(82)

and from (74) we get

\[
g_I dI + g_{\theta_R} d\theta_R + g_{\theta_B} d\theta_B + g_W dW = 0,
\]

(83)

which from (82) yields

\[
g_I dI - \frac{g_{\theta_R}}{k_{\theta_R}}(k_I dI + k_{\theta_B} d\theta_B + k_W dW) + g_{\theta_B} d\theta_B + g_W dW = 0,
\]

(84)

hence

\[
d\theta_B = -\frac{-1}{k_{\theta_R} g_{\theta_B} - g_{\theta_B} k_{\theta_R}} \left[(k_{\theta_R} g_I - g_{\theta_B} k_I) dI + (k_{\theta_R} g_W - g_{\theta_R} k_W) dW\right].
\]

(85)

Using (73) and inserting the expressions for \(\theta_B\) and \(\theta_R\) from above, tedious, but simple, calculations provide us with the result

\[
\frac{dI}{dW} = \frac{(k_{\theta_R} h_{\theta_B} - k_{\theta_B} h_{\theta_R})(k_{\theta_R} g_W - g_{\theta_R} k_W) - (k_{\theta_B} h_W - h_{\theta_R} k_W)(k_{\theta_R} g_{\theta_B} - g_{\theta_R} k_{\theta_B})}{(k_{\theta_R} h_I - h_{\theta_R} k_I)(k_{\theta_R} g_{\theta_B} - g_{\theta_B} k_{\theta_R}) - (k_{\theta_B} h_{\theta_R} - h_{\theta_B} k_{\theta_R})(k_{\theta_R} g_I - g_{\theta_R} k_I)},
\]

and using that \(k_W = h_{\theta_R} = g_W = 0\) and \(k_{\theta_B} = -1, k_{\theta_R} = h_W = 1\) we get

\[
\frac{dI}{dW} = \frac{-(g_{\theta_B} + g_{\theta_R})}{h_I (g_{\theta_B} + g_{\theta_R}) - h_{\theta_R} (g_I + g_{\theta_R} k_I)} = \frac{-(g_{\theta_B} + g_{\theta_R})}{g_{\theta_B} (h_I - h_{\theta_R} k_I) + (g_{\theta_B} h_I - h_{\theta_B} g_I)}.
\]

(86)
As \( g_{\theta R} < 0, h_I < 0, k_I > 0, h_{\theta B} > 0 \) we see that \( g_{\theta R}(h_I - h_{\theta B}k_I) > 0 \). We now first show that the numerator is negative and then obtain conditions for the last term in the denominator to be positive. Using the expressions for \( g_{\theta B} \) and \( g_{\theta R} \) we get

\[
g_{\theta B} + g_{\theta R} = \gamma \omega G(1 - \Omega(\theta_R)) \frac{1}{2} I^{-1/2} - \omega \left[ \frac{E[\theta]}{2} - I^{1/2} \right] + \omega \gamma G(\theta_B - \theta_R) \frac{1}{2} I^{-1/2},
\]

\[
= -\omega \left[ \frac{E[\theta]}{2} - I^{1/2} \right] - \omega \gamma G(\theta_B - \theta_R) \frac{1}{2} I^{-1/2},
\]

\[
= -\omega \gamma G \left( \frac{1}{1 - \Omega(\theta_R)} \right) \left\{ \left[ \frac{\omega}{2} (\theta_R^2 - \theta^2) + \theta_B (1 - \Omega(\theta_R)) \right] \frac{1}{2} I^{-1/2} - 1 \right\}
\]

\[
- \omega G(\theta_B - \theta_R) \frac{1}{2} I^{-1/2},
\]

\[
= \frac{\gamma \omega^2 G}{(1 - \Omega(\theta_R))} \left[ \left[ \frac{\omega}{2} (\theta_R^2 - \theta^2) + \theta_B (1 - \Omega(\theta_R)) \right] \frac{1}{2} I^{-1/2} - 1 \right],
\]

and using that

\[
\frac{\omega}{2} (\theta_R^2 - \theta^2) + (1 - \Omega(\theta_R)) \theta_B = \frac{\omega}{2} (\theta^2 - \theta^2) + \frac{\omega}{2} (\theta_B - \theta_R)^2 > \frac{\omega}{2} (\theta^2 - \theta^2) = E[\theta]
\]

we conclude that

\[
g_{\theta B} + g_{\theta R} > \frac{\gamma \omega^2 G}{(1 - \Omega(\theta_R))} \left[ E[\theta] \frac{1}{2} I^{-1/2} - 1 \right] > 0,
\]

(87)

since the second best investment is less than the first best investment. Finally, we need to consider \( (g_{\theta B}h_I - h_{\theta B}g_I) \). Assume that \( I > 4/3 \). Then

\[
g_{\theta B}h_I - h_{\theta B}g_I
\]

\[
= (\gamma \omega G(1 - \Omega(\theta_R)) \frac{1}{2} I^{-1/2}) \left[ \left( \frac{\omega}{2} (\theta_R^2 - \theta^2) + \theta_B (1 - \Omega(\theta_R)) \right) \frac{1}{2} I^{-1/2} - 1 \right]
\]

\[
+ (1 - \Omega(\theta_R)) I^{1/2} \left( \frac{1}{2} I^{-1/2} + \gamma \omega G \left[ \frac{\omega}{2} (\theta_R^2 - \theta^2) + \theta_B (1 - \Omega(\theta_R)) \right] \frac{1}{4} I^{-3/2} \right)
\]

\[
= - (1 - \Omega(\theta_R))^2 \left[ \frac{E[\theta]}{2} - I^{1/2} \right] \frac{1}{2} I^{-1/2}
\]

\[
+ (1 - \Omega(\theta_R))^2 \frac{1}{2} + (1 - \Omega(\theta_R)) \gamma \omega G \left[ \frac{\omega}{2} (\theta_R^2 - \theta^2) + \theta_B (1 - \Omega(\theta_R)) \right] \frac{1}{4} I^{-1}.
\]

Hence a sufficient condition for \( g_{\theta B}h_I - h_{\theta B}g_I > 0 \) is

\[
- (1 - \Omega(\theta_R))^2 \left[ \frac{E[\theta]}{2} - I^{1/2} \right] \frac{1}{2} I^{-1/2} + (1 - \Omega(\theta_R))^2 \frac{1}{2} > 0
\]
1 > \frac{E[\theta]}{2}I^{-1/2} - 1

that is

\[ 4I > \left(\frac{E[\theta]}{2}\right)^2 = I^{**}, \quad (88) \]

which holds by assumption. Condition (88) depends on the endogenous investment level so we want to rewrite this to obtain a condition depending on exogenous parameters only. As \( I > (1 - \gamma)G + W \) (since \( \theta_R > \bar{\theta} \)) we get the condition

\[ (1 - \gamma)G + W > I^{**}/4 \]

that is

\[ W > I^{**}/4 - (1 - \gamma)G = \tilde{W}^L. \quad (89) \]

Thus, for \( W > \tilde{W}^L \) we know that the denominator in (86) is positive so investment decreases in internal funds. Finally, we want to see when liquidation value is high enough so that \( \tilde{W}^L < W^L_I \). Plugging in the expressions for \( \tilde{W}^L \) and \( W^L_I \) we get \( \tilde{W}^L < W^L_I \) if and only if

\[ I^{**} \frac{1}{4} - (1 - \gamma)G < -(I^{**} + (1 - \gamma)G) + 4 \frac{(1 - \gamma)^2 G^2 \omega}{E[\theta]}. \]

Rewriting this we obtain

\[ I^{**} < \left[ \frac{8}{5} (1 - \gamma)^2 G^2 \omega \right]^{2/3}. \quad (90) \]

Thus,

- if (90) holds, \( \tilde{W}^L < W^L_I \), so investment decreases for all \( W \geq W^L_I \).
- otherwise, we have an interval \((W^L_I, \tilde{W}^L)\) in which we have not proved the sensitivity. However, if \( I > I^{**}/4 \), then we know that investment is decreasing by (88).

(c) We consider the case with \( \underline{\theta} \leq \theta_R < \theta_B < \bar{\theta} \) and we are specifically interested in the case with \( \theta_R = \underline{\theta} \). We use the first order conditions (74)–(72) in this specific case. First, (72) yields

\[ \theta_B = \underline{\theta} + (1 - \gamma)GI^{-1/2}, \quad (91) \]
and plugging this into (74) gives us

\[
\left[ \frac{\mathbb{E}[\theta]}{2} I^{-1/2} - 1 \right] I^{1/2} + \gamma \omega \mathcal{G} \left\{ (\theta + (1 - \gamma)GI^{-1/2}) \frac{1}{2} I^{-1/2} - 1 \right\} = 0, \quad (92)
\]

\[
\frac{\mathbb{E}[\theta]}{2} - \gamma \omega \mathcal{G} - I^{1/2} + \gamma \omega \frac{2}{G} (\theta I^{-1/2} + (1 - \gamma)GI^{-1}) = 0, \quad (93)
\]

multiplying with \( I \) we get

\[
\left( \frac{\mathbb{E}[\theta]}{2} - \gamma \omega \mathcal{G} \right) I - I^{3/2} + \gamma \omega \frac{2}{G} \theta I^{1/2} + \gamma \omega \frac{2}{G} (1 - \gamma)G^2 = 0, \quad (94)
\]

giving us a third degree polynomial in \( I^{1/2} \) with solution \( I^K \) as stated in (26). Finally, condition (73) together with (91) give us

\[
\left( \theta + (1 - \gamma)GI_H^{-1/2} \right) I^{1/2}_I - I_R + W = 0, \quad (95)
\]

i.e.

\[
W = I^K_R - \left( \theta I^{1/2}_R \right) (1 - \gamma)G \triangleq W^L_R, \quad (96)
\]

as asserted.

We now claim that \( W^L_I = W^L_R \) for \((1 - \gamma)G = (\bar{\theta} - \theta) \frac{\mathbb{E}[\theta]}{4}\). So suppose \((1 - \gamma)G = (\bar{\theta} - \theta) \frac{\mathbb{E}[\theta]}{4}\). Using this in \( \bar{\theta}_R \) yields \( \bar{\theta}_R = \theta \). Hence, by (68) we get

\[
W^L_I = - \left( \frac{\mathbb{E}[\theta]}{2} \right)^2 = -I^{**}. \]

Let us conjecture that \( I = I^{**} \) solves (92). In this case, the first term cancels out, so we must show that

\[
\left( \theta + (1 - \gamma)GI^{-1/2} \right) \frac{1}{2} I^{-1/2} - 1 = 0, \quad (97)
\]

thus

\[
\left( \theta I^{1/2} + (1 - \gamma)G \right) = 2I, \quad (98)
\]

and using the expression for \((1 - \gamma)G \) and \( I^{**} \) we get

\[
(\bar{\theta} + \theta) \frac{\mathbb{E}[\theta]}{4} = 2 \left( \frac{\mathbb{E}[\theta]}{2} \right)^2, \quad (99)
\]
which holds as $E[\theta] = (\bar{\theta} + \theta)/2$. To calculate $W_R^L$ we use (96) with $I_R = I^{**} = \left(\frac{E[\theta]}{2}\right)^2$.

Then

$$W_R^L = \left(\frac{E[\theta]}{2}\right)^2 - \left[\theta \left(\frac{E[\theta]}{2}\right) + (\bar{\theta} - \theta) \left(\frac{E[\theta]}{4}\right)\right]$$

$$= \left(\frac{E[\theta]}{2}\right)^2 - \left(\frac{2}{E[\theta]}\right)^2 \left[\theta - \left(\frac{1}{E[\theta]}\right)\right] = -I^{**},$$

i.e. $W_R^L = W_I^L$ as asserted.

(d) Suppose $\theta_R < \theta$, i.e. case 2d. The program becomes

$$\max \int_\theta^\bar{\theta} F(I, \theta)\omega(\theta)d\theta + \left[1 - \gamma \Omega(\theta_B)\right] G - D_1,$$

s.t.

$$D_1 = F(I, \theta_B)$$

$$D_0 = D_1 = I - W.$$ (101)

Using the fact that debt is creditor risk free as well as the form of the production function and the outcome distribution we obtain

$$E_0^L(I) \triangleq \frac{\omega}{2} \left(\frac{\theta^2 - \theta^2}{\theta}ight) I^{1/2} + \left[1 - \gamma \omega \{(I - W)I^{-1/2} - \bar{\theta}\}\right] G - (I - W),$$ (103)

which we must maximize wrt. $I$. The first order condition can be written as

$$\frac{E[\theta]}{2} - \gamma \omega \left\{1 + \frac{W}{I}\right\} - I^{1/2} = 0,$$ (104)

so letting $I = W$ (for $W > 0$) implies

$$\frac{E[\theta]}{2} - \gamma \omega G - W^{1/2} = 0,$$ (105)

that is

$$W = \left(\frac{E[\theta]}{2} - \gamma \omega G\right)^2.$$ (107)

The second order condition (SOC) reveals that the value function $E_0^L(I)$ given by (103) is concave in $I$ for $G < \frac{E[\theta]}{\gamma \omega^4}$ and convex for $G > \frac{E[\theta]}{\gamma \omega^4}$. Therefore, for $G < \frac{E[\theta]}{\gamma \omega^4}$ we have that

$$W \triangleq \left(\frac{E[\theta]}{2} - \gamma \omega G\right)^2.$$ (108)
To continue with the analysis we restrict attention to the case \( \theta = 0 \). Suppose \( G \in (\frac{\mathbb{E}[\theta]}{\gamma \omega}, \frac{\mathbb{E}[\theta]}{\gamma \omega}) \). In this case the SOC can change sign so the first order condition is not sufficient for identifying the maximum of the value function. Moreover, even if we have a local maximum, it could be dominated by \( I = W \) as a corner solution. To determine \( W \) in this case, we consider for which level of \( I \) the equity holders’ value is the same whether they issue debt or not. The \( W \) that allows exactly one solution for \( I \) is the critical \( W \): For \( W > W \), the value is maximized by issuing no debt and investing \( I = W \), whereas for \( W < W \) the firm is better off by issuing debt and investing on a level \( I > W \).

If only internal funds are invested, then the value is

\[
E^L_0(W) = G + \mathbb{E}[\theta]W^{1/2},
\]

but if debt is issued, then \( I > W \), and the value is

\[
E^L_0(I) = \mathbb{E}[\theta]I^{1/2} + [1 - \gamma \omega(I - W)I^{-1/2}] G - (I - W),
\]

by (103). We now derive the \( I \) which equates the equity value in two cases:

\[
E^L_0(W) - E^L_0(I) = 0
\]

i.e.

\[
G + \mathbb{E}[\theta]W^{1/2} - \left( \mathbb{E}[\theta]I^{1/2} + [1 - \gamma \omega(I - W)I^{-1/2}] G - (I - W) \right) = 0.
\]

Using the fact that \( \frac{I^{1/2} - W^{1/2}}{I - W} = \frac{I^{1/2}}{I^{1/2} + W^{1/2}} \), we obtain

\[
-\mathbb{E}[\theta]I^{1/2} + (I^{1/2} + \gamma \omega G)(I^{1/2} + W^{1/2}) = 0,
\]

and since the denominator is positive we have the requirement (after rearranging terms)

\[
I - \left( \mathbb{E}[\theta] - \gamma \omega G - W^{1/2} \right) I^{1/2} + \gamma \omega GW^{1/2} = 0.
\]

Thus, the \( I \)s solving the above equation yield the same equity value as if debt had not been used. If the discriminant is zero, then there is exactly one level of investment, which we define as \( I_W \). This occurs when

\[
\left( \mathbb{E}[\theta] - \gamma \omega G - W^{1/2} \right)^2 - 4\gamma \omega GW^{1/2} = 0.
\]
The \( W \) satisfying this equation is what we are looking for. Rearranging we get

\[
W - 2(\mathbb{E}[\theta] + \gamma \omega G)W^{1/2} + (\mathbb{E}[\theta] - \gamma \omega G)^2 = 0
\]

hence

\[
W = \left( \sqrt{\mathbb{E}[\theta]} - \sqrt{\gamma \omega G} \right)^4, \tag{111}
\]

as the smallest root in the polynomial should be taking in order to get a \( W < I^{**} \).

For \( G > \mathbb{E}[\theta]/(\gamma \omega) \), equity value is convex in \( I \). From the LHS in (106) it follows that the derivative, \( dE_L^I(I)/dI \), is negative. Thus the optimal solution is to invest “as little as possible”, i.e. \( I = W \).

i. Suppose \( \theta_R < \theta < \theta_B < \bar{\theta} \). From the first order condition (104) we know that

\[
\mathbb{E}[\theta] - \gamma \omega G \{1 + \frac{W}{I}\} - 2I^{1/2} = 0. \tag{112}
\]

Define \( \mathcal{F} \) as the left-hand side

\[
\mathcal{F}(W, I) \triangleq \mathbb{E}[\theta] - \gamma \omega G \{1 + \frac{W}{I}\} - 2I^{1/2} \tag{113}
\]

First, fix \( I = I_W \) as defined above, i.e. the FOC is solved with \( I_W = \mathbb{W} \). We observe that (i) \( \mathcal{F}(\mathbb{W}, I_W) = 0 \) and that (ii) \( \mathcal{F}(W, I_W) \) is an affine function of \( W \) with slope \( -\gamma \omega G/I_W \), i.e. it is decreasing in \( W \). Consider a decrease in \( W \) below \( \mathbb{W} \). It follows that the first order condition is violated with a positive LHS and, thus, \( I \) must be higher than \( I_W \) to have the first order condition satisfied. This argument applies for any \( I \) and starting with a \( W \) such that \( \mathcal{F}(W, I) = 0 \). Whence it follows that a decrease in \( W \) implies an increase in \( I \) and, hence, investment is decreasing in internal funds, as asserted.

ii. On the other hand, consider a \( W > \mathbb{W} \). For \( G < \frac{\mathbb{E}[\theta]}{4\gamma \omega} \), we start at \( I = W \). Then

\[
\mathcal{F}(W, I)|_{I=W} = \mathbb{E}[\theta] - \gamma \omega G \{1 + \frac{W}{W}\} - 2W^{1/2} = 2 \left( W^{1/2} - W^{1/2} \right) < 0, \tag{114}
\]

as \( W > \mathbb{W} \). Thus, the firm would want to decrease its investment, i.e. to issue less debt. But since we consider \( I = W \) this implies that the firm does not issue any debt. Since \( W < I^{**} \), the firm minimizes underinvestment costs by setting \( I = W \), as asserted. For \( G > \frac{\mathbb{E}[\theta]}{4\gamma \omega} \), we recall that \( \mathbb{W} \) is the largest \( W \) that allows \( E_L^I(I) = E_L^I(W) \) with an \( I > W \). Hence, for \( W > \mathbb{W} \) we have \( E_L^I(I) < E_L^I(W) \) for all \( I > W \). Therefore, value maximization leads to \( I = W \), as asserted.

\[\square\]

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B.2 Proof of Proposition 2

Proof. (i) We must find the level of internal funds for which liquidation is inevitable except for the case when the best possible state occurs, i.e. $W^S_I$ solves $\theta_B(W^S_I) = \bar{\theta}$. Using this together with the combining condition for $\theta_B$ and $\theta_R$, cf. (60), we get

$$\theta_R = \theta_B - (1 - \gamma)GI^{-1/2} = \bar{\theta} - (1 - \gamma)G\frac{2}{E[\theta]},$$

which combined with $\theta_R$ from (50) yields

$$\bar{\theta} - \left\{ \frac{(\bar{\theta}^2 - \bar{\theta}^2)}{\omega E[\theta]} \right\} - \frac{4}{\omega E[\theta]} \left[ \left( \frac{E[\theta]}{2} \right)^2 - W - (1 - \gamma)G \right] = \bar{\theta} - (1 - \gamma)G\frac{2}{E[\theta]},$$

iff

$$\frac{2E[\theta]}{\omega} - \left\{ \frac{(\bar{\theta}^2 - \bar{\theta}^2)}{\omega E[\theta]} \right\} - \frac{4}{\omega E[\theta]} \left[ \left( \frac{E[\theta]}{2} \right)^2 - W - (1 - \gamma)G \right] = (1 - \gamma)^2G^2\frac{4}{E[\theta]^2},$$

and rearranging gives us

$$W = - \left\{ \left( \frac{E[\theta]}{2} \right)^2 + (1 - \gamma)G \left[ 1 - (1 - \gamma)G\frac{\omega E[\theta]}{E[\theta]} \right] \right\} \triangleq W^S_I,$$

as asserted.

(ii) Referring back to (4) and (10), the general optimization program is

$$\max \frac{1}{1 - \gamma} \int_{\theta_B}^\theta F(I, \theta) \cdot \omega(\theta)d\theta + \int_{\theta_B}^{\bar{\theta}} F(I, \theta) \cdot \omega(\theta)d\theta$$

$$+ G(1 - \Omega(\theta_R)) - \frac{D_1}{1 - \gamma} \left[ (1 - \gamma) + \gamma\Omega(\theta_B) - \Omega(\theta_R) \right],$$

s.t.

$$D_1 = F(I, \theta_B)$$

$$D_1 = F(I, \theta_R) + (1 - \gamma)G$$

$$D_0 = I - W.$$  

Consider the case of creditor risk free debt, i.e. $\theta_R < \bar{\theta}$. In this case the debt holders’ participation constraint yields $D_1 = I - W$. The program is then rewritten to

$$\max \frac{1}{1 - \gamma} \int_{\theta_B}^\theta F(I, \theta) \omega(\theta)d\theta + \int_{\theta_B}^{\bar{\theta}} F(I, \theta) \omega(\theta)d\theta$$

$$+ G - \frac{I - W}{1 - \gamma} \left[ (1 - \gamma) + \gamma\Omega(\theta_B) \right],$$
Multiplying with \( I \) and if we also use the probability function \( \Omega(\theta) = (\theta - \bar{\theta}) \omega \) the program becomes

\[
\max_I \frac{1}{1 - \gamma} \int_{(I-W)} I^{-1/2} \mathcal{I}^{1/2} \omega I d\theta + \int_{(I-W)} I^{-1/2} \omega I^1 \omega I d\theta \\
+ G - \frac{I - W}{1 - \gamma} [(1 - \gamma) + \gamma \omega ((I - W)I^{-1/2} - \bar{\theta})]
\]

i.e.

\[
\max_I [\theta^2 \gamma (I - W)I^{-1/2} - \frac{1}{1 - \gamma} \theta^2] \frac{\omega}{2} I^{1/2} + G \\
-(I - W)[1 + \frac{\gamma}{1 - \gamma} \omega ((I - W)I^{-1/2} - \bar{\theta})].
\]

From this we derive the first order condition

\[
(\theta^2 - \frac{1}{1 - \gamma} \theta^2) \frac{\omega}{2} I^{1/2} + \frac{\gamma}{1 - \gamma} \omega \left\{ 2(I - W)I^{-1/2} - \frac{1}{2} (I - W)^2 I^{-3/2} \right\} \\
- \left[ 1 + \frac{\gamma}{1 - \gamma} \omega ((I - W)I^{-1/2} - \bar{\theta}) \right] - (I - W) \frac{\gamma}{1 - \gamma} \omega \frac{1}{2} (I^{-1/2} + WI^{-3/2}) = 0.
\]

Multiplying with \( I^{3/2} \) and using the fact (when \( I = I_R^S \)) that

\[
I_R^S - W_R^S = (1 - \gamma) G + \theta (I_R^S)^{1/2},
\]

we get (writing \( I \) instead of \( I_R^S \) to shorten notation)

\[
(\theta^2 - \frac{1}{1 - \gamma} \theta^2) \frac{\omega}{2} I^{1/2} + \frac{\gamma}{1 - \gamma} \omega \left\{ 2(I - W)G + \theta I^{1/2} \right\} \\
- \left[ I^{3/2} + \frac{\gamma}{1 - \gamma} \omega \left\{ (1 - \gamma) G + \theta I^{1/2} \right\} - \theta I^{1/2} I \right],
\]

and through several algebraic manipulations we obtain

\[
-I^{3/2} + \frac{\omega}{4} \left[ \theta^2 - \bar{\theta}^2 - 4 \gamma G \right] I + \gamma \omega \frac{\theta I^{1/2}}{2} + \frac{\omega}{4} (1 - \gamma) \gamma G^2 = 0.
\]

Finally, using the fact that \( \frac{\omega}{4} (\theta^2 - \bar{\theta}^2) = \frac{\mathbb{E} [\theta]}{2} \) yields

\[
-I^{3/2} + \left[ \frac{\mathbb{E} [\theta]}{2} - \omega \gamma G \right] I + \gamma \omega \frac{\theta I^{1/2}}{2} + \frac{1}{2} \omega (1 - \gamma) \gamma G^2 = 0,
\]
whence it follows that $\tilde{a} = a, \tilde{b} = b,$ and $\tilde{c} = c/2$ referring to the constants in (94). Thus, $I_S^R$ has the solution written in the proposition. Employing this solution in (128) provides us with $W_S^R$:

$$W_S^R = I_S^R - (1 - \gamma)G + \bar{\theta}(I_S^R)^{1/2}. \quad (130)$$

(iii) We now consider how internal funds affect investment when debt is creditor risk free, i.e. $I(W)$ for $W \geq W_S^R$. Rewriting the first order condition (127) we get

$$0 = \left(\bar{\theta}^2 - \frac{\bar{\theta}^2}{1 - \gamma}\right)\frac{\omega}{4} + \left(\frac{\gamma}{1 - \gamma}\omega\bar{\theta} - 1\right)I^{1/2} - \frac{\gamma}{1 - \gamma}\omega\left\{3I - 2W - W^2I^{-1}\right\}, \quad (131)$$

and the implicit function theorem yields

$$\frac{dI(W)}{dW} = \frac{-\frac{\gamma}{1 - \gamma}\omega\left\{1 + WI(W)^{-1}\right\}}{\left(\frac{\gamma}{1 - \gamma}\omega\bar{\theta} - 1\right)\frac{1}{2}I(W)^{-1/2} - \frac{\gamma}{1 - \gamma}\omega\left\{3 + W^2I(W)^{-2}\right\}}. \quad (132)$$

Suppose $\bar{\theta} \leq (1 - \gamma)\bar{\theta}$. Then $\left(\frac{\gamma}{1 - \gamma}\omega\bar{\theta} - 1\right) \leq 0$ implying that the denominator in (132) is negative. Whence it follows that

$$\frac{dI(W)}{dW} > 0 \quad (133)$$

iff

$$-\frac{\gamma}{1 - \gamma}\omega\left\{1 + WI(W)^{-1}\right\} < 0 \quad (134)$$

iff

$$W > -I(W). \quad (135)$$

It follows that if $W_S^R < -I(W_S^R)$, then investment sensitivity changes sign for a $W > W_S^R$.

To get a more intuitive condition, we rewrite:

$$W_S^R + I(W_S^R) > 0, \quad (136)$$

using (128) we get

$$(1 - \gamma)G + \bar{\theta}(I_S^R)^{1/2} + 2W_S^S > 0, \quad (137)$$

and assuming that $\bar{\theta} = 0$ yields

$$(1 - \gamma)G > -2W_S^S, \quad (138)$$

(139)
which is the desired condition. As \( \theta = 0 \) implies \( \theta \leq (1 - \gamma) \overline{\theta} \) is satisfied we are done. Finally, consider the case \((1 - \gamma)G < -2W_S R\). Then we know that \(dI(W)/dW\) changes sign when \(W = -I(W)\). Using this (and remembering \( \theta = 0 \)) in the first order condition (132) we find

\[
\bar{\theta}^2 \omega \frac{I_1}{4} - \frac{1}{2} - \frac{\gamma}{1 - \gamma} I = 0,
\]

(140)

and since \(1/\omega = \overline{\theta}\) we get

\[
I + \frac{1 - \gamma}{\gamma} \overline{\theta} I^{1/2} - \frac{1 - \gamma}{\gamma} \overline{\theta}^2 = 0
\]

(141)

which gives us the solution

\[
I = \left( \frac{\overline{\theta}}{2\gamma} \right)^2 \left( \sqrt{1 - \gamma} - (1 - \gamma) \right)^2.
\]

(142)

Thus, investment sensitivity changes sign when

\[
W = - \left( \frac{\overline{\theta}}{2\gamma} \right)^2 \left( \sqrt{1 - \gamma} - (1 - \gamma) \right)^2 < 0.
\]

(143)

\( \square \)