Søren Glud Johansen & Anders Thorstenson

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Abstract

We show that well-known textbook formulae for determining the optimal base stock of the inventory system with continuous review and constant lead time can easily be extended to the case with periodic review and stochastic, sequential lead times. The provided performance measures and conditions for optimality are exact.

Keywords: Inventory control, Optimisation, Stochastic

1. Introduction

We consider a single-item inventory system with periodic review. We assume that the system is controlled by a base-stock policy and that unsatisfied demands are backordered and satisfied as soon as possible. The inventory position (stock on hand + stock on order - amount backordered) is monitored at each review instant, when the policy places a replenishment order to restore the base-stock \( s \).

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The demand is Poisson with rate \( \lambda \) and the lead times \( L \) of replenishment orders are generated by an exogenous, sequential supply system (Zipkin, 2000, Section 7.4). The latter assumption implies that different replenishment orders do not cross in time.

The base-stock policy with review interval \( R \) is commonly referred to as an \((R, S)\) policy, where \( S = s \). This policy is frequently used in practice, and it is often applied as a benchmark in theoretical studies. Furthermore, for the inventory system described and with the standard cost structure assumed, the \((R, S)\) policy is the optimal control policy (Zipkin, 2000, Section 9.6.3.1).

2. The extended lead time

Like Zipkin (Zipkin, 2000, Section 6.7.3) and Rao (Rao, 2003), we model the times from the demand instants until the next review instant as realizations of a random variable \( U \) which is uniformly distributed over the interval from 0 to \( R \). Therefore, the generic variable \( L' \) for the time from a demand instant until the delivery instant of the replenishment order triggered by this demand can be specified as \( U + L \). Zipkin and Rao both assume that lead time \( L \) is constant. We call \( L' \) the extended lead time and observe that like the ordinary lead times, the extended lead times for customer demands can be seen as generated by an exogenous, sequential supply system. This observation provides that the exact results derived for the corresponding inventory system with continuous review can be used to derive similar results for the system with periodic review. The results for the latter system are obtained by focusing on customer demands and letting the lead times in the former system be distributed as \( L' \) rather than as \( L \).

The generic variable \( D_U \) for the demand during the extended lead time equals \( D_U + D_L \), where \( D_U \) is the generic variable for the demand during an interval of random length \( U \) and \( D_L \) is the generic variable for the ordinary lead time demand. The probability mass function (pmf) of \( D_U \) is

\[
f_U(x) = \frac{1}{R} \left[ \frac{1}{x!} \right] e^{-\lambda u} du = \sum_{y=0}^{\infty} \frac{(\lambda R)^y e^{-\lambda R}}{y!}, \quad x = 0, 1, \ldots
\]

(1)
The pmf \( f_U(x) \) is strongly unimodal. A proof is provided in the Appendix. Assuming that lead time \( L \) is gamma distributed, the variable \( D_L \) has the negative binomial distribution with parameters \( a = (E[L])^2 / \text{Var}[L] \) and \( p = \lambda / (\lambda + a/E[L]) \) (Zipkin, 2000, Section 7.5.1.1). Hence, if \( \text{Var}[L] > 0 \), we have the following useful recursion for the pmf of \( D_L \)

\[
f_L(y) = \frac{\alpha - 1 + y}{y} p f_L(y-1), \quad y = 1, 2, \ldots,
\]

where \( f_L(0) = (1-p)^a \). If \( \text{Var}[L] = 0 \), \( D_L \) has the Poisson distribution with mean \( \lambda E[L] \). In either case, the pmf \( f_L(y) \) is also strongly unimodal (Keilson and Gerber, 1971). Finally, because \( D_L' \) is the convolution of \( D_U \) and \( D_L \), the pmf of \( D_L' \) is

\[
f_L'(z) = \sum_{x=0}^{z} f_U(x) f_L(z-x), \quad z = 0, 1, \ldots
\]

which is strongly unimodal by Theorem 2 in Keilson and Gerber (ibid.).

3. Performance measures

Important performance measures of the considered inventory system can now be obtained from Zipkin (Zipkin, 2000, Corollary 7.4.3). The stock-out frequency is

\[
\bar{A}(s) = \Pr \{ D_L \geq s \},
\]

the average backorders are

\[
\bar{B}(s) = E\left[ \max \{ D_L - s, 0 \} \right],
\]

and the average inventory is

\[
\bar{T}(s) = s - E[D_L] + \bar{B}(s) = E\left[ \max \{ 0, s - D_L \} \right].
\]
Let $h$ and $b$ denote the unit holding and backorder costs per unit time, respectively, and let $\pi$ denote the cost incurred for each unit backordered. If no further costs are incurred, then the long-run average cost incurred per unit time is

$$C(s) = hT(s) + bB(s) + \pi\lambda\lambda(s).$$

(7)

The base-stock model with this cost structure corresponds to Rosling’s Model 2, Case (ii) (Rosling, 2002). Hence, based on his Proposition 2-2, we may conclude that $C(s)$ is quasi-convex in $s$, because the pmf of $D_L'$ is strongly unimodal, as noted above. (Quasi-convexity of $C(s)$ is equivalent to $-C(s)$ being unimodal.)

Then, since

$$\Delta C(s) = C(s+1) - C(s) = (h + b) \Pr\{D_L \leq s\} - (b + \pi\lambda \Pr\{D_L = s\}),$$

(8)

the optimal base-stock level is obtained as

$$s^* = \min\left\{s \mid \Delta C(s) > 0\right\}.$$

(9)

An example is depicted in Figure 1 which shows that $C(s)$ is not convex in $s$. Figure 1 also shows the misrepresentation that occurs if lead times are assumed to be independent, implying by Palm’s theorem (Palm, 1938) that the approximate cost rate $C_a(s)$ depends on the distribution of $L'$ only through its mean $E[L'] = R/2 + E[L]$. 

\[ \text{Figure 1}\]
For $\pi = 0$, we obtain the extended standard solution

$$s^* = \min \left\{ s : \Pr \{ D_L \leq s \} > \frac{b}{h+b} \right\}. \quad (10)$$

As another special case, consider the case when $b = 0$ (and $\pi > 0$). Eq. (9) may then be written as

$$s^* = \min \left\{ s : \frac{\Pr \{ D_L = s \}}{\Pr \{ D_L \leq s \}} < \frac{h}{\pi \lambda} \right\}. \quad (11)$$

The ratio $\Pr \{ D_L = s \}/\Pr \{ D_L \leq s \}$ is non-increasing in $s$, because the pmf of $D_L$ is strongly unimodal (Rosling, 2002). Note that Eq. (11) is an extension to the case of periodic review and stochastic lead times of the expression found in the well-known textbook by
Silver et al (Silver et al, 1998, Section 8.3.1) in the case of continuous review and a constant lead time. The extensions in Eqs. (9)-(11) are intuitively appealing, since they are obtained from the standard continuous-review formulae by simply replacing $D_L$, the demand during the ordinary lead time, by $D_L'$, the demand during the extended lead time.

4. Concluding remarks

The extended lead time has been defined as the time between a customer demand instant and the delivery instant of the replenishment order triggered by that demand. By applying this concept, exact performance measures for the base-stock inventory system with continuous review have been extended to a system with periodic review and stochastic, sequential lead times. Simple conditions for optimization have also been derived. The crucial assumption for these results is that lead times are sequential so that orders do not cross in time. The results obtained can be generalized further to accommodate compound Poisson demand (Johansen and Thorstenson, 2005).

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References


**Appendix**

**Proposition:** The pmf $f_U(x)$ as defined in Section 2 is strongly unimodal.

**Proof:** By Theorem 3 in Keilson and Gerber (Keilson and Gerber, 1971), a necessary and sufficient condition for the pmf of the discrete variable $D_U$ to be strongly unimodal is that it is log-concave, i.e., that

$$
\left[ f_U(x) \right]^2 \geq f_U(x+1) \cdot f_U(x-1), \ x = 1, 2, \ldots
$$

(A1)

The proof is by contradiction: Assume that Condition (A1) is not true. From Eq. (1) we then have for some $x$ that

$$
\left[ \sum_{y=x+1}^{\infty} (\lambda R)^{y-1} y! e^{-\lambda R} \right]^2 < \left[ \sum_{y=x+2}^{\infty} (\lambda R)^{y-1} y! e^{-\lambda R} \right] \left[ \sum_{y=x}^{\infty} (\lambda R)^{y-1} y! e^{-\lambda R} \right]
$$

\[\downarrow\]

$$
\frac{(\lambda R)^x}{(x+1)!} \left[ \sum_{y=x+1}^{\infty} (\lambda R)^{y-1} y! e^{-\lambda R} \right] < \left[ \sum_{y=x+2}^{\infty} (\lambda R)^{y-1} y! e^{-\lambda R} \right] \left( \sum_{y=x}^{\infty} (\lambda R)^{y-1} y! e^{-\lambda R} \right) x!
$$

(A2)

\[\downarrow\]

$$
\frac{1}{x+1} \left[ \sum_{y=x+1}^{\infty} (\lambda R)^{y} y! e^{-\lambda R} \right] < \left[ \sum_{y=x+1}^{\infty} (\lambda R)^{y} (y+1)! e^{-\lambda R} \right].
$$

As can be inferred from its last member, Condition (A2) is not satisfied. Hence, the assertion is false and Condition (A1) must be true. \[\square\]
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