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The realized empirical distribution function of stochastic variance with application to goodness-of-fit testing *

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Abstract

We propose a nonparametric estimator of the empirical distribution function (EDF) of the latent spot variance of the log-price of a financial asset. We show that over a fixed time span our realized EDF (or REDF)—inferred from noisy high-frequency data—is consistent as the mesh of the observation grid goes to zero. In a double-asymptotic framework, with time also increasing to infinity, the REDF converges to the cumulative distribution function of volatility, if it exists. We exploit these results to construct some new goodness-of-fit tests for stochastic volatility models. In a Monte Carlo study, the REDF is found to be accurate over the entire support of volatility. This leads to goodness-of-fit tests that are both correctly sized and relatively powerful against common alternatives. In an empirical application, we recover the REDF from stock market high-frequency data. We inspect the goodness-of-fit of several two-parameter marginal distributions that are inherent in standard stochastic volatility models. The inverse Gaussian offers the best overall description of random equity variation, but the fit is less than perfect. This suggests an extra parameter (as available in, e.g., the generalized inverse Gaussian) is required to model stochastic variance.

JEL Classification: C10, C50.

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1 Introduction

Stochastic volatility is a central concept in financial economics with numerous implications for capital allocation, asset- and derivatives pricing, or risk management. It is therefore essential to select a model for stochastic volatility that is able to reproduce the main features observed in data from financial markets, such as the implied volatility surface, as closely as possible. There is, however, no consensus about which model is "best," as the extensive list of papers on stochastic volatility suggests (the literature is too big to count, but a partial and necessarily incomplete set of specifications proposed in the literature can be found in Andersen, Benzoni, and Lund, 2002; Barndorff-Nielsen and Shephard, 2002; Chernov, Gallant, Ghysels, and Tauchen, 2003; Christoffersen, Jacobs, and Mimouni, 2010; Comte and Renault, 1998; Gatheral, Jaisson, and Rosenbaum, 2014; Heston, 1993; Hull and White, 1987, and the references therein).

The goodness-of-fit of a stochastic volatility model can be assessed in several ways. If the parameters are estimated with a method of moment-based estimator, a standard diagnostic check is available via a test of overidentifying restrictions (see also, e.g., Gallant, Hsieh, and Tauchen, 1997). This approach can be highly inefficient, however, as it depends on the selection of moment conditions and the associated estimation of the weighting matrix, which may produce significant size distortions in finite samples (Andersen and Sørensen, 1996).

Another idea is to compare the model-implied distribution of volatility with a nonparametric empirical measure of it (e.g., Aït-Sahalia, 1996). In a continuous-time setting, for instance, a model is often formulated via a stochastic differential equation for the variance process (or a transformation thereof, such as the square root or natural logarithm). This, in turn, implies that the marginal distribution of spot volatility (at least in the stationary case) belongs to a particular class of distributions. So if the underlying volatility was observed, a specification test based on the distance—with respect to a suitable norm—between the empirical distribution function (EDF) and the marginal distribution imposed by the model would be feasible. However, as spot volatility is not directly observed, this approach is not immediately applicable.

The recent access to financial high-frequency data has alleviated these concerns, as it enables the computation of an essentially error-free measure of realized volatility, allowing for a more direct evaluation of the goodness-of-fit of stochastic volatility models. Inspired by the above, a common approach is to compare (conditional) moments of the integrated variance of a parametric model with a nonparametric estimator hereof (see, e.g., Bollerslev and Zhou, 2002; Corradi and Distaso, 2006; Dette and Podolskij, 2008; Dette, Podolskij, and Vetter, 2006; Todorov, 2009; Todorov, Tauchen, and Grynkiv, 2011; Vetter and Dette, 2012). Zu (2015) proposes a test based on a de-convolution kernel density estimator of the distribution of the integrated variance, Lin, Lee, and Guo (2013, 2016) resort to the empirical characteristic function, while Bull (2017) proposes a wavelet-based test. As pointed out by Todorov and Tauchen (2012), however, the mapping between the probability distribution of the spot and integrated variance is in general not one-to-one. Thus, goodness-of-fit tests based on the latter suffer from lack of power against some alternatives, due to the smoothing entailed by integrating spot volatility over a discrete time interval.

In this article, we therefore propose to construct goodness-of-fit tests for stochastic volatility models that are based on a realized EDF (REDF hereafter) of spot volatility. We build on the work of Li, Todorov, and Tauchen (2013, 2016), who show that estimation of the EDF amounts to pinning down the occupation–or local—time of volatility. We first retrieve an estimate of the latent spot variance at any point in time by employing a Foster and Nelson (1996) rolling window-type estimator, which is computed on small blocks of high-frequency data. The main distinction is that we allow the asset price to be recorded at the highest resolution—i.e. the tick-by-tick level—so that instantaneous variance can be recovered as accurately as possible. This is crucial if the whole distribution, including the tails, is of interest and not only measures of central tendency, such as the mean or median.

At this sampling frequency, however, the data are invariably contaminated by frictions in the trading process, collectively known as microstructure noise (see, e.g. Hansen and Lunde, 2006). This complicates the issue a lot. The derived problem of estimating the integrated—or cumulative—variance process in the presence of noise has received substantial attention in the literature (see, e.g. Barndorff-Nielsen, Hansen, Lunde, and Shephard, 2008; Jacod, Li, Mykland, Podolskij, and Vetter, 2009; Zhang, Mykland, and Aït-Sahalia, 2005). We here adapt the pre-averaging approach of Jacod, Li, Mykland, Podolskij, and Vetter (2009) and Podolskij and Vetter (2009a,b) and show how it can be applied to design a consistent spot variance estimator from polluted high-frequency data (an alternative procedure is presented in Zu and Boswijk, 2014). The REDF is then built from the recovered sample path.

We show that under mild conditions the REDF is a consistent noise- and jump-robust estimator of the EDF over any fixed time interval, thus expanding Li, Todorov, and Tauchen (2013, 2016) to the noisy framework. Next, we let the time horizon tend to infinity and are able to prove a functional CLT for the REDF. This amounts to establish convergence of what van der Vaart and Wellner (2007) call an empirical process indexed by an estimated function. Here, however, time is continuous and the observations are not i.i.d.

The theoretical results are refined into two new goodness-of-fit tests for measuring the discrepancy between the marginal density implied by a candidate stochastic volatility model and the non-parametric gauge at the EDF represented by the REDF. The first is reminiscent to a Kolmogorov-Smirnov statistic for an observed process, while the second is based on a weighted L^2 norm. Both are trivial to compute once the REDF has been constructed, but there are a number of subtle technical problems that render the asymptotic distribution hard to evaluate, making it difficult to set critical regions and determine p-values. This is to some extent related to the complexities encountered in a classical setup, when the parameters of the model under the null are estimated. We resolve the issue via a parametric bootstrap (based on, e.g., Bull, 2017, and described in the Supplementary Appendix), which leads to fast evaluation of the t-statistic.

The paper is organized as follows. In Section 2, we cover the setting, assumptions, and introduce the EDF of volatility. In Section 3, the properties of the REDF are analysed. The asymptotic theory for our goodness-of-fit tests is also developed here. Section 4 is devoted to a Monte Carlo study of the REDF and an assessment of the size and power properties of the associated goodness-of-fit tests. The REDF is found to be accurate over the entire support of volatility. This leads to goodness-of-fit tests that are both correctly sized and relatively powerful against common alternatives. An empirical application is conducted in Section 5, where we recover the REDF from stock market high-frequency data and test the goodness-of-fit of several marginal distributions that are induced by mainstream stochastic volatility models. The inverse Gaussian offers the best overall description of random equity variation, but the fit is less than perfect. This suggests an extra parameter (as available in, e.g., the generalized inverse Gaussian) is required to model stochastic variance. We conclude in Section 6. The proofs appear in a supplementary web appendix.

2 The setting

Let $X = (X_t)_{t \geq 0}$ denote the efficient log-price process of a financial asset, which is defined on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\big)$. As consistent with no-arbitrage (e.g., Delbaen and Schachermayer, 1994), we assume X is an Itô semimartingale:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t, \quad \text{for} \quad t \in [0, T], \tag{2.1}$$

where X_0 is $\mathcal{F}_0$-measurable, $(b_t)_{t \geq 0}$ is a locally bounded, predictable drift, $(\sigma_t)_{t \geq 0}$ is an adapted, càdlàg volatility, $(W_t)_{t \geq 0}$ a Brownian motion, and $(J_t)_{t \geq 0}$ is a jump process:

$$J_t = \int_0^t \int_{\mathbb{R}} \delta(s, z) 1_{\{|\delta(s,z)| \leq 1\}} (\mu - \nu)(d(s, z)) + \int_0^t \int_{\mathbb{R}} \delta(s, z) 1_{\{|\delta(s,z)| > 1\}} \mu(d(s, z)), \tag{2.2}$$

where μ is a Poisson random measure on $(\mathbb{R}_+, \mathbb{R})$, $\nu(d(s,z)) = ds \otimes \lambda(dz)$ is a compensator, and λ is a σ-finite measure on $\mathbb{R}$. Also, $\delta : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a predictable function. We assume there exists a sequence $(\tau_k)_{k \geq 1}$ of $\mathcal{F}_t$-stopping times $\nearrow \infty$ such that $|\delta(\omega, s, z)| \wedge 1 \leq \Gamma_k(z)$ for all (ω, s, z) with $s \leq \tau_k(\omega)$ and $\int_{\mathbb{R}} \Gamma_k(z)^2 \lambda(\mathrm{d}z) < \infty$ for all $k \geq 1$.

We collect assumptions about the drift and volatility of X below.

Assumption 1 *We assume that for each $p \geq 1$ and $t \geq 0$, $\mathbb{E}\big[|b_t|^p\big] + \mathbb{E}\big[|\sigma_t|^p\big] \leq C$ for some constant C. In addition, σ has to fulfill at least one of the following conditions:*

(i) *There exists $H \in (0,1)$ such that for each $p \geq 1, s \geq 0, r > 0$, and a constant C (dependent on H and p):*

$$\mathbb{E}\left[\Big(\sup_{s \leq t \leq s+r} |\sigma_t - \sigma_s|\Big)^p\right] \leq C r^{pH}. \tag{2.3}$$

(ii) *σ has the representation:*

$$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \tilde{\sigma}'_s dB_s + \int_0^t \int_{\mathbb{R}} \tilde{\delta}(s,z)(\mu - \nu)(d(s,z)), \tag{2.4}$$

where σ_0 is $\mathcal{F}_0$-measurable, $\mathbb{E}\big[|\tilde{b}_t|^p\big] + \mathbb{E}\big[|\tilde{\sigma}_t|^p\big] + \mathbb{E}\big[|\tilde{\sigma}'_t|^p\big] \leq C$ for some constant C and for each $p \geq 1$ and $t \geq 0$, while B is a Brownian motion independent of W, $\tilde{\delta}$ is a predictable function, and $|\tilde{\delta}(\omega, t, z)| \wedge 1 \leq \tilde{\Gamma}(z)$ for all (ω, t, z) and some $\tilde{\Gamma} : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} \tilde{\Gamma}(z)^2 \lambda(dz) < \infty$.

Assumption 1 places some moment conditions on b_t and σ_t. While it is general and encompasses many volatility models applied in practice, it may feel a bit restrictive. The assumption can be relaxed, but this entails that a constant $\iota > 0$ (restricting the rate at which the sample size $T \to \infty$ and time gap $\Delta_n \to 0$, cf. Section 3) has to be bounded from below, while it can be arbitrarily small with the above setup.

Assumption 1(i) is a smoothness condition. It implies σ is continuous, so that it can be approximated by a locally constant process over short time intervals. The condition covers both "rough" volatility models and long-memory processes (e.g., Comte and Renault, 1998; Gatheral, Jaisson, and Rosenbaum, 2014) with Hurst exponent $H < 1/2$ and $H > 1/2$ (standard Brownian motion is the intermediate value $H = 1/2$), respectively.

Assumption 1(ii), which says σ is an Itô semimartingale, is common in the literature. It allows volatility to exhibit fairly unrestricted jump dynamics (at the expense of excluding fractional Brownian motion as in 1(i)) and also enables a leverage effect, as it does not restrict the correlation structure between the increments of X and σ.

Next, we define the EDF of volatility, which in the continuous-time setting is given by the function:

$$F_T(x) = \frac{1}{T} \int_0^T 1_{\{V_t \leq x\}} dt, \tag{2.5}$$

where $V_t \equiv \sigma_t^2$.[1]

Assumption 2 *F_T is a.s. continuous.*

This condition is fulfilled by most stochastic volatility models. It does not require σ to be continuous nor does it impose the existence of a density.

The right-hand side of (2.5) can be written $T^{-1} \int_0^T g(V_t)dt$, where $g(v) = 1_{\{v \leq x\}}$.[2] Then, if F_T is absolutely continuous with respect to the Lebesgue measure—which is stronger than Assumption 2—and g is a bounded or non-negative Borel function:

$$\frac{1}{T} \int_0^T g(V_t)dt = \frac{1}{T} \int_{\mathbb{R}_+} g(x)dF_T(x) = \frac{1}{T} \int_{\mathbb{R}_+} g(x)f_T(x)dx, \tag{2.6}$$

where $f_T = F_T'$. The EDF thus encapsulates all the information about $(V_t)_{t \geq 0}$ available in $[0, T]$ and can be viewed as a pathwise version of the distribution function of volatility.

2.1 A discrete and noisy high-frequency record of X

The main difficulty is that F_T is latent, because σ is not observable. The aim of this paper is therefore to construct a consistent estimator of F_T, while making the above minimal assumptions about X.

Of course, in an ideal world with no markets frictions and continuous trading—i.e. if the entire trajectory of X is available—we can recover the volatility process perfectly and also the exact time and size of jumps in X (in constrast to the drift, which cannot be consistently estimated in finite time, not even if it is constant, see, e.g., Merton, 1980; Foster and Nelson, 1996), but this setup is not realistic. In practice, we operate with discrete high-frequency data, which we assume are available at times $i\Delta_n$, for $i = 0, 1, \dots, n$, where Δ_n is the time gap between consecutive observations and

[1] In probability theory, $F_T(x) = \int_0^T 1_{\{Y_t \leq x\}} dt$ is called the occupation—or local—time of the stochastic process Y (e.g., Geman and Horowitz, 1980). This convention was adopted by Li, Todorov, and Tauchen (2013, 2016), who applied it to high-frequency volatility estimation. We normalize F_T by T here, as we are heading toward a setting with stationary volatility and an asymptotic theory with $T \to \infty$.

[2] Much work in the high-frequency literature studies integrals of the form $\int_0^T g(V_t)dt$, where g is smooth (e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard, 2006; Jacod and Rosenbaum, 2013). Here, in contrast, g is discontinuous, which makes the theory a lot more inaccessible.

$n = \lfloor T/\Delta_n \rfloor$ is the sample size (equidistant sampling is convenient here, but it can weakened). In a near-ideal world, the database then constitutes a high-frequency record of X, i.e. $(X_{i\Delta_n})_{i=0}^n$, and the estimator proposed by Li, Todorov, and Tauchen (2013, 2016) can be applied without further ado.

The microstructure of financial markets adds measurement error, however, which implies that we do not observe X directly (e.g., due to bid-ask bounce or price discreteness). Instead, we record a contaminated log-price Z, which we assume is related to X as follows:

$$Z_{i\Delta_n} = X_{i\Delta_n} + U_{i\Delta_n}, \quad \text{for} \quad i = 0, 1, \dots, n, \tag{2.7}$$

where U is a centered (i.e., $\mathbb{E}(U_{i\Delta_n}) = 0$) and i.i.d. noise process with $U \perp\!\!\!\perp X$.

Assumption 3 $\mathbb{E}(|U|^r) < \infty$ *for every* $r > 0$.

The moment condition is not standard in the high-frequency literature, but it is merely made for technical convenience. In the proofs, we only assume moments of U exist up to some order (for instance, in Lemma 2.1 we require the 4th moment to be finite).

We define the noisy log-return:

$$\Delta_i^n Z = Z_{i\Delta_n} - Z_{(i-1)\Delta_n}, \quad \text{for} \quad i = 1, \dots, n. \tag{2.8}$$

To deal with the noise, we pre-average the return series (see, e.g., Jacod, Li, Mykland, Podolskij, and Vetter, 2009; Podolskij and Vetter, 2009a,b):

$$\bar{Z}_i = \sum_{j=1}^{k_n - 1} g\left(\frac{j}{k_n}\right) \Delta_{i+j}^n Z, \quad \text{for} \quad i = 0, \dots, n - k_n + 1, \tag{2.9}$$

where k_n is a positive integer, while g is a kernel function.

Any $g : [0,1] \mapsto \mathbb{R}$ with g continuous and piecewise C^1 with Lipschitz derivative g', such that $g(0) = g(1) = 0$ and $\int_0^1 g(x)^2 dx > 0$ is permitted. We follow the default choice in the literature by setting $g(x) = \min(x, 1 - x)$.

The selection of k_n entails a trade-off. The intuition is that while pre-averaging lessens the noise, it also smooths out the underlying volatility of X. This can render it hard to construct estimates of spot volatility that fit the tails of the distribution if a too wide pre-averaging window is applied. As shown by Jacod, Li, Mykland, Podolskij, and Vetter (2009), an optimal k_n is achieved via:

$$k_n = \frac{\theta}{\sqrt{\Delta_n}} + o(\Delta_n^{-1/4}), \tag{2.10}$$

where $\theta > 0$ is a tuning parameter, which controls the balance struck between the above forces in small samples. As consistent with prior work, we base our analysis on $\theta = 1/3$ and set $k_n = \lfloor \theta/\sqrt{\Delta_n} \rfloor$.

2.2 The realized EDF

We now exploit the pre-averaged high-frequency data to form local estimates of V_t. The estimator we propose is both inherently jump- and noise-robust and can therefore be plugged into a "realized" version of the EDF (or REDF, as defined in (2.16)). It is computed on small blocks of pre-averaged returns. We denote the number of increments in a block by a sequence of positive integers h_n with $h_n \Delta_n \to 0$ and $h_n / k_n \to \infty$, as $\Delta_n \to 0$, so the time span of the block is decreasing, but there is an increasing amount of data within it.

Then, we set:

$$\tilde{V}_{i\Delta_n} = \frac{1}{h_n \sqrt{\Delta_n}} \sum_{m=0}^{h_n - 1} \left(\bar{Z}_{i+m} \right)^2, \quad \text{for} \quad i = 0, 1, \ldots, n - h_n - k_n + 1. \tag{2.11}$$

As the notation suggests, $\tilde{V}_{i\Delta_n}$ is an estimator of the spot variance at time $i\Delta_n$, i.e. $V_{i\Delta_n}$. It is convenient to extend this definition to $t \in [0, T]$. We do this by setting $\tilde{V}_t = \tilde{V}_{i\Delta_n}$, for $t \in ((i-1)\Delta_n, i\Delta_n]$, while for $t > (n - h_n - k_n + 1)\Delta_n$: $\tilde{V}_t = \tilde{V}_{(n-h_n-k_n+1)\Delta_n}$, thus holding the final spot variance estimate fixed up to time T.

Finally, we introduce the following constants, which depend on the weight function:

$$\psi_1 = \int_0^1 g'(s)^2 ds \quad \text{and} \quad \psi_2 = \int_0^1 g(s)^2 ds.^3 \tag{2.12}$$

With this notation in hand, we have the following result for our preliminary estimator of the spot volatility.

Lemma 2.1 *Suppose that Assumption 3 holds. For each* $t \in [0, T]$, *as* $\Delta_n \to 0$,

$$\tilde{V}_t \xrightarrow{\mathbb{P}} \theta \psi_2 V_t + \frac{1}{\theta} \psi_1 \omega^2, \tag{2.13}$$

where $\omega^2 = \text{var}(U_{i\Delta_n})$.

To strip out the residual noise variation, we need an estimator of ω^2. There are several options around (e.g., Bandi and Russell, 2006; Hansen and Lunde, 2006; Oomen, 2006) and each one has

[3] In the proofs, after freezing volatility locally, $\psi_1^n = k_n \sum_{j=1}^{k_n} \left(g(\frac{j}{k_n}) - g(\frac{j-1}{k_n}) \right)^2$ and $\psi_2^n = \frac{1}{k_n} \sum_{j=1}^{k_n - 1} g^2(\frac{j}{k_n})$ appear in the conditional expectation of $\hat{V}_{i\Delta_n}$. As $\Delta_n \to 0$, $\psi_i^n \to \psi_i$ and the error has order $\psi_i^n - \psi_i = O(\Delta_n^{1/2})$, for $i = 1, 2$. This means we can work with ψ_i in the expressions below, and this substitution has no impact on the asymptotic analysis. In contrast, ψ_i^n can differ materially from ψ_i if k_n is small. As a practical recommendation, it is therefore better to work with ψ_i^n.

it own merits and disadvantages (see, e.g., Gatheral and Oomen, 2010, for a comparison). In this paper, we adopt the following:

$$\hat{\omega}^2 = -\frac{1}{n-1}\sum_{i=2}^{n} \Delta_i^n Z \Delta_{i-1}^n Z \xrightarrow{\mathbb{P}} \omega^2, \tag{2.14}$$

so that

$$\hat{V}_t = \frac{1}{\theta\psi_2}\tilde{V}_t - \frac{\psi_1}{\theta^2\psi_2}\hat{\omega}^2 \xrightarrow{\mathbb{P}} V_t. \tag{2.15}$$

The subtraction of the bias implies $\hat{V}_t$ can be negative in finite samples. Nevertheless, not a single point estimate fell below zero neither in our simulations nor empirical work.

We then construct the REDF:

$$F_{n,T}(x) = \frac{1}{T}\int_0^T 1_{\{\hat{V}_t \leq x\}} dt, \tag{2.16}$$

3 Asymptotic properties of the REDF

In this section, we cover the asymptotic theory of the REDF. In Section 3.1, we deal first with consistency for the EDF in the infill setting, where we assume that the noisy log-price is recorded over ever shorter intervals but the total time elapsed is constant, as formalized by $\Delta_n \to 0$ with T fixed. In the following Section 3.2, we then further assume high-frequency data are collected on an expanding time window by also letting $T \to \infty$ and deduce convergence toward the marginal distribution function. We exploit these theoretical insights to develop some new goodness-of-fit tests for stochastic volatility models in Section 3.3.

3.1 Infill setting

We start this section with showing that $F_{n,T}$ is a consistent estimator of the EDF.

Theorem 3.1 *Suppose that Assumption 2 – 3 hold true. Then, for each $T > 0$, as $\Delta_n \to 0$,*

$$\sup_{x \in \mathbb{R}_+} \left| F_{n,T}(x) - F_T(x) \right| \xrightarrow{\mathbb{P}} 0. \tag{3.1}$$

It is worth noting that $F_{n,T}$ is consistent for F_T, even though it is based on pre-averaged returns that are not filtered for jumps. This intrinsic robustness is a relatively unique feature of this type of estimator, although it is shared by the realized Laplace transform of Todorov and Tauchen (2012). In their work it is due to the properties of the cosine function, while here it stems from the construction of the spot volatility estimator in (2.11).

This is opposite to the estimation of integrated variance, where realized variance—either noise-robust or not—converges to the quadratic variation, which includes the sum of the squared jumps in X, and we need to enforce some robustification (e.g., truncation or bipower variation) to recover the continuous variation of X (e.g., Barndorff-Nielsen and Shephard, 2004; Jing, Liu, and Kong, 2014; Mancini, 2009; Podolskij and Vetter, 2009a). As an aside, we note Li, Todorov, and Tauchen (2013) only prove consistency of the jump-truncated statistic in the presence of discontinuous X and no microstructure noise, but this can also be extended to the non-truncated estimator. Nonetheless, it may be preferable to truncate in practice, as it can act as a safeguard against finite sample distortions induced by jumps. Even if X is continuous, there is no cost in doing so asymptotically. We follow that idea in the empirical section.[4]

We define $Q_{n,T}(\alpha) = \inf\{x : F_{n,T}(x) \geq \alpha\}$ and $Q_T(\alpha) = \inf\{x : F_T(x) \geq \alpha\}$ as the uniquely determined α-quantile of $F_{n,T}$ and F_T, for $\alpha \in (0,1)$. Then, using the uniform convergence in probability of $F_{n,T}$, we deduce $Q_{n,T}(\alpha)$ is a consistent estimator of $Q_T(\alpha)$. As it is an implication of Lemma 21.2 in van der Vaart (1998), we omit a proof.

Corollary 3.2 *Suppose that Assumption 2 – 3 hold. Then, for each $T > 0$, if $Q_T(\alpha)$ is continuous at α and as $\Delta_n \to 0$:*

$$Q_{n,T}(\alpha) \overset{\mathbb{P}}{\to} Q_T(\alpha). \tag{3.2}$$

3.2 Joint infill and long-span setting

The goal here is to extend the above analysis (with T fixed) to estimation of the stationary distribution function, F. To do so, we study an asymptotic framework, where the time span $T \to \infty$ jointly with $\Delta_n \to 0$. The key result is a feasible CLT for the REDF, which enables us to compute the accuracy with which the marginal distribution can be recovered from the data. To facilitate the derivation of this theory we require some additional assumptions on the volatility process.

Assumption 4 *$(V_t)_{t\geq 0}$ is stationary and strongly mixing with mixing coefficient function $\alpha(t) = O(t^{-\gamma})$, as $t \to \infty$ and for some $\gamma > 1$.*

The stationarity condition in Assumption 4 is standard. We also restrict the memory of the volatility process, so that $(V_t)_{t\geq 0}$ is not "too" strongly dependent. This implies that F_T (as an estimator of the marginal distribution, F) is consistent, as $T \to \infty$. We can exploit this to deduce that $F_{n,T}$ also

[4] Moreover, unless X is continuous, truncation is sort of necessary if the aim is to derive rates of convergence for measures of the continuous variation of X.

converges in probability to F, once we show that the estimation error embedded in the recovery of the volatility path is asymptotically negligible (as $\Delta_n \to 0$ fast enough, because the discretization error accumulates with T). Moreover, it is an essential part in showing the weak convergence of the empirical processes that we construct in the derivation of our goodness-of-fit test statistics in Section 3.3.

The mixing condition is slightly weaker than the comparable assumptions made in the literature on joint infill and long-span asymptotics (see e.g. Todorov and Tauchen (2012) and Andersen, Thyrsgaard, and Todorov (2018)). It encompasses several stochastic volatility models, for instance a large class of processes driven by Brownian motion (e.g., Heston, 1993) or the Lévy-driven Ornstein-Uhlenbeck model of Barndorff-Nielsen and Shephard (2001), where volatility is governed by general (positive) processes.[5,6] As a result, the setup is not restrictive in practice as it can capture a wide variety of marginal distributions.

Next, we replace the pathwise smoothness imposed on the EDF of volatility from Assumption 2 with the following condition.

Assumption 5 *F is differentiable with bounded derivative f.*

Finally, we will for technical convenience also assume that the price process X is continuous. [7]

Theorem 3.3 *Suppose that Assumption 3 – 5 hold true. If either of the following conditions is fulfilled:*

(i) *Assumption 1*(i) *with $H \in (0,1)$, $h_n \asymp \Delta_n^{-\frac{4H+1}{4H+2}}$ and $T^{1/2+\iota}\Delta_n^{\frac{H}{4H+2}-\iota} \to 0$ as $\Delta_n \to 0$ and $T \to \infty$ for some $\iota > 0$, or*

(ii) *Assumption 1*(ii)*, $h_n \asymp \Delta_n^{-3/4}$ and $T^{1/2+\iota}\Delta_n^{1/8-\iota} \to 0$ as $\Delta_n \to 0$ and $T \to \infty$ for some $\iota > 0$.*

Then, for fixed $x \in \mathbb{R}_+$, it holds that

$$\sqrt{T}\big(F_{n,T}(x) - F(x)\big) \xrightarrow{d} N(0, \Sigma(x)), \tag{3.3}$$

[5]Assume that V is a solution of the stochastic differential equation: $dV_t = \tilde{b}(t, V_t)dt + \tilde{\sigma}(t, V_t)dW_t$, where $\tilde{\sigma}(t, V_t) \leq K\big(1 + |V_t|^{1/2}\big)$, for some $K > 0$. If there exists an S such that for all $|s| \geq S$ and $t \geq 0$, $b(t, s) \leq -\gamma$, for some $\gamma > 0$, Theorem 2 in Veretennikov (1988) implies that $\alpha(t) = O(t^{-\gamma})$.

[6]If V is of the form $dV_t = -\kappa V_t dt + dZ_{\kappa t}$, where $\kappa > 0$ and Z is a positive Lévy process (e.g., a subordinator) with Lévy measure ν, then it follows from Jongbloed, van der Meulen, and van der Vaart (2005) that V is stationary if $\int_2^\infty \ln(x)\nu(dx) < \infty$. If further $\mathbb{E}\big[|V_1|^p\big] < \infty$, for some $p > 0$, there exists $a > 0$ such that $\alpha(t) = O(e^{-at})$. The decay of the mixing coefficient can even be deduced if the driving process in volatility is a fractional Brownian motion (see, e.g., Magdziarz and Weron, 2011).

[7]This assumption can be relaxed at the cost of making the rates in 3.3 more tedious to derive.

where $\Sigma(x) = 2\int_0^\infty \big(F_t(x,x) - F(x)^2\big)dt$ *with* $F_t(x,y) = \mathbb{P}\left(V_0 \leq x, V_t \leq y\right)$.

Notice that the asymptotic variance, $\Sigma(x)$, is non-stochastic, so that the usual machinery of stable convergence from high-frequency analysis is not required to conclude that $\sqrt{T}\big(F_{n,T}(x) - F(x)\big)/\sqrt{\Sigma(x)} \xrightarrow{d} N(0,1)$.

In Theorem 3.3, the block length h_n is always selected so that the discrepancy between the empirical process associated with the REDF and the EDF vanishes at the fastest possible rate. In order to control this error, we have to restrict the relative speed at which $T \to \infty$ and $\Delta_n \to 0$. The least binding is to take h_n as indicated. Note that h_n is an increasing function of the smoothness of volatility. In particular, if σ is a continuous semimartingale (i.e., $H = 1/2$), the rate at which $\sqrt{T}\big(F_{n,T}(x) - F_T(x)\big)$ converges to zero is $T^{1/2+\iota}\Delta_n^{1/8-\iota}$. If volatility is smoother ($H \to 1$), the rate of convergence improves to $T^{1/2+\iota}\Delta_n^{1/6-\iota}$, whereas it gets arbitrarily slow as $H \to 0$. The intuition is that it is virtually impossible to retrieve spot variance from a "rough" path. On the other hand, if $H = 1/2$ the optimal choice of h_n and the associated speed condition is unchanged, irrespective of whether volatility is allowed to jump or not.

The noise-robustness of $F_{n,T}$ plays a critical role in determining the above condition. To compare, if there was no error in our measurement of X, a standard non-noise-robust realized measure suffices for estimation of spot volatility (see, e.g., Jacod and Protter, 2012). Here, the rate condition improves to $T^{1/2+\iota}\Delta_n^{1/4-\iota} \to 0$ with $H = 1/2$, as is consistent with the fixed T asymptotic theory in Li, Todorov, and Tauchen (2013). The deterioration of the speed condition is due to the added complexity from retrieving volatility robustly, which cuts the rate of convergence in half. This problem is well-known in the literature on estimation of integrated variance (see, e.g., Zhang, Mykland, and Aït-Sahalia, 2005; Barndorff-Nielsen, Hansen, Lunde, and Shephard, 2008; Jacod, Li, Mykland, Podolskij, and Vetter, 2009). Moreover, while a convergence rate of (almost) $\Delta_n^{1/8}$ is slow, it is optimal for extraction of spot variance in noisy diffusion models (e.g., Aït-Sahalia and Jacod, 2014, p. 296).

The asymptotic variance in Theorem 3.3 depends crucially on the memory of the process. If a series is highly dependent, which is true empirically for volatility, it becomes harder to recover the distribution of the process, which results in larger values of $\Sigma(x)$ and wider confidence intervals for $F(x)$. The only way to compensate for this effect is to collect a larger sample.

$\Sigma(x)$ can be computed either analytically or, for example, by numerical integration. As shown by Dehay (2005), however, for finite T:

$$\text{var}\left(\sqrt{T}\big(F_T(x) - F(x)\big)\right) = 2\int_0^T \left(1 - \frac{t}{T}\right)\left(F_t(x,x) - F(x)^2\right)dt. \tag{3.4}$$

The reduction by $(1 - t/T)$ in the true variance of F_T is an edge effect. We compute (3.4) in the simulation section, as it provides an infeasible assessment for the accuracy of the approximation in (3.3) for finite T. On the other hand, if Δ_n is not small enough to ignore the pathwise discretization error, there is an extra source of randomness in $F_{n,T}$ induced by the estimation of spot volatility. As such, (3.4) may actually understate the variation of $F_{n,T}$.

To estimate $\Sigma(x)$, we fix $\xi \in (0, 1/3)$ and set:

$$\Sigma_{n,T}(x) = 2 \int_0^{T^\xi} \left(1 - \frac{t}{T^\xi}\right) \big(F_{t,n,T}(x) - F_{n,T}(x)^2\big)\, dt, \tag{3.5}$$

where

$$F_{t,n,T}(x) = \frac{1}{T - T^\xi} \int_0^{T - T^\xi} 1_{\{\hat{V}_{s+t} \leq x, \hat{V}_s \leq x\}} ds. \tag{3.6}$$

Then, as shown in the proof of Theorem 3.3, $\Sigma_{n,T}(x) \xrightarrow{\mathbb{P}} \Sigma(x)$.

In practice, we may also be interested in determining how accurate the associated quantile(s) of F are estimated (e.g., for risk management reporting). These questions are, of course, intertwined. Indeed, from the assumed existence of the density f and Corollary 21.5 in van der Vaart (1998), we deduce the following result.

Corollary 3.4 *Suppose that the conditions of Theorem 3.3 are fulfilled. Then, for any $\alpha \in (0, 1)$ with $f\big(Q(\alpha)\big) > 0$, it holds that*

$$\sqrt{T}\big(Q_{n,T}(\alpha) - Q(\alpha)\big) \xrightarrow{d} N\left(0, \frac{\Sigma\big(Q(\alpha)\big)}{f\big(Q(\alpha)\big)^2}\right). \tag{3.7}$$

Thus, inference about any given quantile is feasible, so long as an estimator of the density function of the marginal distribution evaluated at that quantile is available.

3.3 Goodness-of-fit of the volatility distribution

The preceding theory shows that $F_{n,T}$ converges to the stationary distribution function of volatility, as $\Delta_n \to 0$ and $T \to \infty$. In this section, we build upon this analysis to construct realized extensions of some classical goodness-of-fit tests for the volatility process. To this end, we first derive a functional central limit theorem for the process $G_{n,T} = \Big\{\sqrt{T}\big(F_{n,T}(x) - F(x)\big), x \in \mathbb{R}_+\Big\}$.

Theorem 3.5 *Suppose that the conditions of Theorem 3.3 are fulfilled. Then, $G_{n,T}$ converges weakly in $C_b(\mathbb{R}_+)$ equipped with the uniform metric to a Gaussian process, G_F, with mean zero and covariance function $\Sigma(x, y) = \text{cov}\big(G_F(x), G_F(y)\big)$.*

Note that Theorem 3.5 can also be extended to the empirical process associated with the quantile estimator: $G_{n,T}^Q = \Big\{\sqrt{T}\big(Q_{n,T}(\alpha) - Q(\alpha)\big), \alpha \in (0,1)\Big\}$.

As shown in Lemma A.3 in the appendix, $\Sigma(x,y)$ can be consistently estimated by

$$\Sigma_{n,T}(x,y) = \int_0^{T^\xi} \big(F_{t,n,T}(x,y) + F_{t,n,T}(y,x) - 2F_{n,T}(x)F_{n,T}(y)\big)dt, \tag{3.8}$$

for $\xi \in (0, 1/3)$, where

$$F_{t,n,T}(x,y) = \frac{1}{T - T^\xi} \int_0^{T-T^\xi} 1_{\{\hat{V}_{s+t} \leq x, \hat{V}_s \leq y\}} ds. \tag{3.9}$$

Appealing to Theorem 3.5, we can now construct the test statistics for our goodness-of-fit tests, which evaluate the fit of an assumed stochastic volatility model:

$$rKS(F) = \sup_{x \in \mathbb{R}_+} \big|G_{n,T}(x)\big| \qquad \text{and} \qquad rL^2(F) = \int_{\mathbb{R}_+} G_{n,T}(x)^2 w(x) dx, \tag{3.10}$$

where $w : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous weight function.

rKS resembles a classical Kolmogorov-Smirnov test for an observed process, whereas rL^2 is a realized version of the goodness-of-fit test based on a weighted L^2 norm. The latter nests several existing tests, such as the Cramer-von-Mises (i.e., $w(x) = f(x)$) or Anderson-Darling (i.e., $w(x) = f(x)/[F(x)(1 - F(x))]$). rKS has the advantage that it is rather trivial to implement, once the REDF is constructed. On the other hand, the literature suggests that an L^2 statistic, at least in the i.i.d. setting, may be more powerful in finite samples, but it can also be tedious to compute, if a closed-form expression for w is unavailable. This is, for example, the case for a Cramer-von-Mises test if the marginal density is unknown or hard to evaluate.

Corollary 3.6 *Suppose that the conditions of Theorem 3.3 are fulfilled. Then, it holds that*

$$rKS(F) \xrightarrow{d} \sup_{x \in \mathbb{R}_+} |G_F(x)| \qquad \text{and} \qquad rL^2(F) \xrightarrow{d} \int_{\mathbb{R}_+} |G_F(x)|^2 w(x) dx. \tag{3.11}$$

Now, suppose we are interested in testing the fit of a stochastic volatility model, indexed by a known parameter vector υ. Later in this section, we return to the question of how parameter estimation affects the proposed tests. We let $F_\upsilon(x) = \mathbb{P}(V_t \leq x; \upsilon)$ describe the marginal distribution of the model.[8] The null—and associated alternative—hypothesis is then written as:

$$\mathcal{H}_0 : F = F_\upsilon \qquad \text{and} \qquad \mathcal{H}_a : F \neq F_\upsilon. \tag{3.12}$$

[8] As $\upsilon \mapsto F_\upsilon$ is not one-to-one, in general, the model itself is not fully identified by F_υ, unless further constraints are imposed on the drift and volatility (e.g., Aït-Sahalia, 1996; Bibby, Skovgaard, and Sørensen, 2005).

It follows that—under $\mathcal{H}_0$—$rKS(F_\upsilon) \xrightarrow{d} \sup_{x\in\mathbb{R}_+} |G_{F_\upsilon}(x)|$, whereas $rKS(F_\upsilon) \xrightarrow{\mathbb{P}} \infty$ under $\mathcal{H}_a$, because the alternative means there is at least one x, such that $F_{n,T}(x) \overset{\mathbb{P}}{\nrightarrow} F_\upsilon(x)$. An equivalent result holds for $rL^2(F)$. As large values of the t-statistics discredit that $F = F_\upsilon$, the tests are therefore one-sided.

A drawback of this approach is that the limiting distribution of $rKS(F)$ and $rL^2(F)$ depends on the model under $\mathcal{H}_0$. This differs from the classical theory, for example the Kolmogorov-Smirnov statistic in the i.i.d. case with F continuous and known (see, e.g., van der Vaart, 1998). Even there, however, the asymptotic distribution is not readily available if the parameters describing F_υ are estimated, as we do below, although in that setting it often suffices to prepare a single table of family-wise critical values in advance—e.g., as in Lilliefors (1967, 1969)—if F_υ belongs to a location-scale family. This is due to the fact that the limiting distribution, in that case, does not depend on nuisance parameters of F, only the functional form of the probability distribution (e.g., David and Johnson, 1948). This is not true here, and we are therefore forced to retrieve critical values through simulation on a case-by-case basis. In Appendix B, we offer a detailed recipe of how this procedure works both for Corollary 3.6 and 3.8.

In principle, we can construct a t-statistic that can be evaluated independently of F. To explain how, define $||f||_w = \sqrt{\int_{\mathbb{R}_+} w(x)f(x)^2dx}$, such that $\mathbb{E}\big[||G_F||_w^2\big] < \infty$.

The other ingredient is a normalizing constant:

$$\tilde{\Sigma} = \int_{\mathbb{R}_+} \Sigma(x)w(x)dx, \tag{3.13}$$

with $\tilde{\Sigma} < \infty$. Then, we set:

$$T(F) = \frac{||G_{n,T}||_w^2}{\tilde{\Sigma}}. \tag{3.14}$$

We can then define a test statistic that rejects $\mathcal{H}_0$, if we observe that $T(F) > \chi^2_{1-\alpha}$, where $\chi^2_{1-\alpha}$ is the $(1-\alpha)$-quantile of a chi-squared distribution with one degree of freedom. Let $\alpha_{T(F)}$ be the significance level achieved by such a test. The following proposition summarizes the details.

Proposition 3.7 *Suppose that the conditions of Theorem 3.3 are fulfilled. Then, for any* $\alpha \in (0, 0.215)$*, it holds that under* $\mathcal{H}_0$*:*

$$\lim_{\Delta_n\to 0, T\to\infty} \alpha_{T(F)} \leq \alpha, \tag{3.15}$$

while under $\mathcal{H}_a$*:*

$$\mathbb{P}\big(T(F) > c\big) \to 1, \tag{3.16}$$

for any $c > 0$.

We can thus construct a goodness-of-fit test for which critical values are readily available, however it achieves a significance level of at most α and may therefore be overly conservative for some F. Moreover, while the constant $\tilde{\Sigma}$ can be calculated under $\mathcal{H}_0$, so that no estimation is required, it is complicated to derive for many common stochastic volatility models. We therefore do not explore this test further.

In practice, the parameter vector υ is typically unknown and not fixed a priori. It therefore has to be estimated, which invalidates the previous analysis (assuming F_υ known). The intuition is that the estimation moves the model-implied marginal distribution closer to the EDF, making inference based on the preceding theory too conservative, as is well-known from the traditional literature on goodness-of-fit testing.

The complication brought about by parameter estimation can be viewed as a different testing problem that applies if we are interested in testing an entire class of stochastic volatility processes, as opposed to a particular member of that class.

The null is thus a composite hypothesis:

$$\mathcal{H}_0 : \exists \upsilon \in \Upsilon : F = F_\upsilon \qquad \text{and} \qquad \mathcal{H}_a : \forall \upsilon \in \Upsilon : F \neq F_\upsilon, \tag{3.17}$$

where Υ is the admissible parameter space (i.e., $\Upsilon = \{(\kappa, \upsilon_0, \xi) \in \mathbb{R}_+^3 : 2\kappa\upsilon_0 \geq \xi^2\}$ for the Heston (1993) model due to the Feller condition).

We assume a consistent estimator of υ, say $\hat{\upsilon}$, has been found and introduce a pseudo empirical process $\tilde{G}_{n,T} = \left\{\sqrt{T}\big(F_{n,T}(x) - F_{\hat{\upsilon}}(x)\big), x \in \mathbb{R}_+\right\}$. At a theoretical level, the problem then arises because the parameters of the underlying distribution typically cannot be recovered at a convergence rate faster than $T^{-1/2}$, i.e $\hat{\upsilon} - \upsilon = O_p(T^{-1/2})$. This cancels with the $\sqrt{T}$ scaling in the construction of $\tilde{G}_{n,T}$, so that the estimation error inevitably affects the asymptotic distribution.[9]

Corollary 3.8 *Suppose that the conditions of Theorem 3.3 are fulfilled. In addition, we assume $\hat{\upsilon}$ is asymptotically linear with influence function ψ and that $\dfrac{\partial F_\upsilon}{\partial \upsilon}(x)$ is bounded and continuous in both υ and x over the set $\mathbb{R}_+ \times \bar{\Upsilon}$ for a neighbourhood $\bar{\Upsilon}$ around υ. Then, $\tilde{G}_{n,T}$ converges weakly in $C_b(\mathbb{R}_+)$ equipped with the uniform metric to a Gaussian process of the form $G_{F_\upsilon} - \dfrac{\partial F_\upsilon}{\partial \upsilon}^\top \displaystyle\int \psi dG_{F_\upsilon}$.*

[9] A standard rate of convergence is required in the following. Although we are not aware about the presence of so-called "super" consistent estimators in the stochastic volatility literature—at least not in the stationary setting—we point out that if $\hat{\upsilon} - \upsilon = o_p(T^{-1/2})$ the sampling error is asymptotically negligible and the problem reverts back to Corollary 3.6. On the other hand, a slower rate of convergence implies the error is blown up by the $\sqrt{T}$ scaling and then our approach just goes out the window.

The requirement that $\hat{v}$ should be asymptotically linear is fulfilled by most $\sqrt{T}$-consistent estimators proposed in the literature (e.g., Bickel, Klaassen, Ritov, and Wellner, 1998).

4 Simulation study

We now appraise our estimator of the EDF of spot variance introduced at the end of Section 2 via Monte Carlo simulation. We also evaluate the size and power properties of the two goodness-of-fit tests for the marginal distribution of volatility proposed in Section 3.3.

Starting at $X_0 = 0$, the efficient log-price is a Heston (1993)-type process:

$$\begin{aligned} dX_t &= \sqrt{V_t} dW_t, \\ dV_t &= \kappa\big(v_0 - V_t\big)dt + \xi\sqrt{V_t}dB_t, \end{aligned} \tag{4.1}$$

where W and B are correlated standard Brownian motions with $\mathbb{E}[dW_t dB_t] = \rho dt$ and $\rho = -\sqrt{0.5}$ (recall that $\sigma_t = \sqrt{V_t}$). The other parameters are $\kappa = 0.05$, $v_0 = 1$, and $\xi = 0.2$, which is calibrated to our empirical data and broadly aligns with previous studies (see, e.g, Aït-Sahalia and Kimmel, 2007).

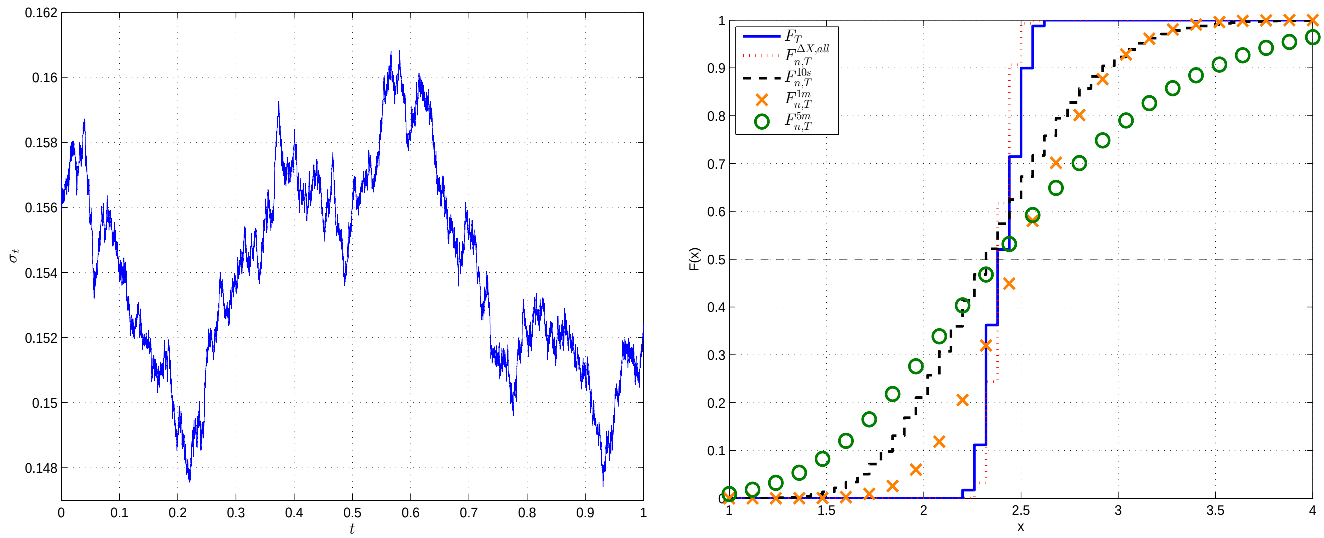
In this model, the law of V_t (conditional on V_s, for $s < t$) is—up to a scale factor—non-central chi-square with $d = 4v_0\kappa\xi^{-2}$ degrees of freedom and non-centrality parameter $\lambda = \dfrac{4\kappa e^{-\kappa(t-s)}}{\xi^2\big(1 - e^{-\kappa(t-s)}\big)}V_s$ (e.g., Cox, Ingersoll, and Ross, 1985). As $t \to \infty$, the conditional distribution converges to a $\text{Gamma}(2\kappa v_0\xi^{-2}, 2\kappa\xi^{-2})$ (interpreted as shape and scale parameters), which is also the marginal (or unconditional) law of V_t for any finite t, if V_0 is drawn from this distribution. We do that here. The variance of $F_T(x)$ in (3.4) is then computed from these expressions by numerical integration of the joint density function.

We simulate 10,000 "continuous-time" paths with $23,400$ updates per unit of time T. We discretize the whole system by an Euler approach. To gauge how close the REDF is to the population counterpart, we extract a coarser grid of $n = 2,340T$ log-price increments. This emulates a financial market, where new transactions arrive regularly every tenth second over a 6.5 hour trading day, which is aligned to the sample sizes in our empirical work. We add i.i.d. noise $Z_{i\Delta_n} = X_{i\Delta_n} + U_{i\Delta_n}$ with $U_{i\Delta_n} \sim N(0,\omega^2)$, where ω^2 is set by fixing the noise-to-signal ratio (e.g., Oomen, 2006): $\gamma = \sqrt{\dfrac{\omega^2}{\Delta_n \int_0^1 V_t dt}}$. As consistent with Christensen, Oomen, and Podolskij (2014) and our findings below, we assume that $\gamma = 0.5$.[10] Thus, $(Z_{i\Delta_n})_{i=0}^n$ comprises our discretely observed sample used to estimate the spot variances.

[10] We made a robustness check with $\gamma = 2.0$ as in Aït-Sahalia, Jacod, and Li (2012). The pre-averaging estimator

Figure 1: Estimation of the EDF.

Panel A: Sample path of σ_t. Panel B: EDF and REDF.



Note. This figure plots a simulated path of σ_t in Panel A. The associated EDF of V_t, F_T, appears in Panel B (solid line). We estimate F_T with $F_{n,T}$ over 1,000 simulations and plot the average (dashed line). The version proposed in Li, Todorov, and Tauchen (2013, 2016) based on 1- and 5-minute sparse sampling is shown as a comparison ('×' and '∘'). The dotted line is an infeasible estimate that has access to the whole record of X, $(X_{i\Delta_n})_{i=0}^n$. It illustrates how minimal smoothing renders it hard to track the tails of the volatility distribution, even in absence of noise.

In Figure 1, we examine how $F_{n,T}$ recovers F_T. We set $T = 1$ and freeze volatility across replica to keep the estimation target fixed. The sample path of σ_t is shown in Panel A, while the associated EDF appears in Panel B. We report our estimator and compare it to Li, Todorov, and Tauchen (2013). We construct the latter statistic from 1- and 5-minute low-frequency returns. As recommended in their work and to facilitate comparison, h_n is set throughout so that it spans about 1.5 hours worth of data.

At the 1-minute resolution, the Li, Todorov, and Tauchen (2013) estimator is severely distorted by market frictions, which leads to a systematic upward bias. This was to be expected, as it is not resistant to noise. On the other hand, while the 5-minute statistic does not accumulate any discernable bias, it delivers imprecise estimates, resulting in a highly overdispersed REDF. In contrast, as our noise-robust estimator is able to capitalize on tick-by-tick information, it is more accurate. This suggests the inferior rate of convergence embedded in pre-averaging is more than offset by being able to employ all the data.

Next, we vary σ_t randomly in each iteration and estimate F_T as above. In Figure 2, we plot the

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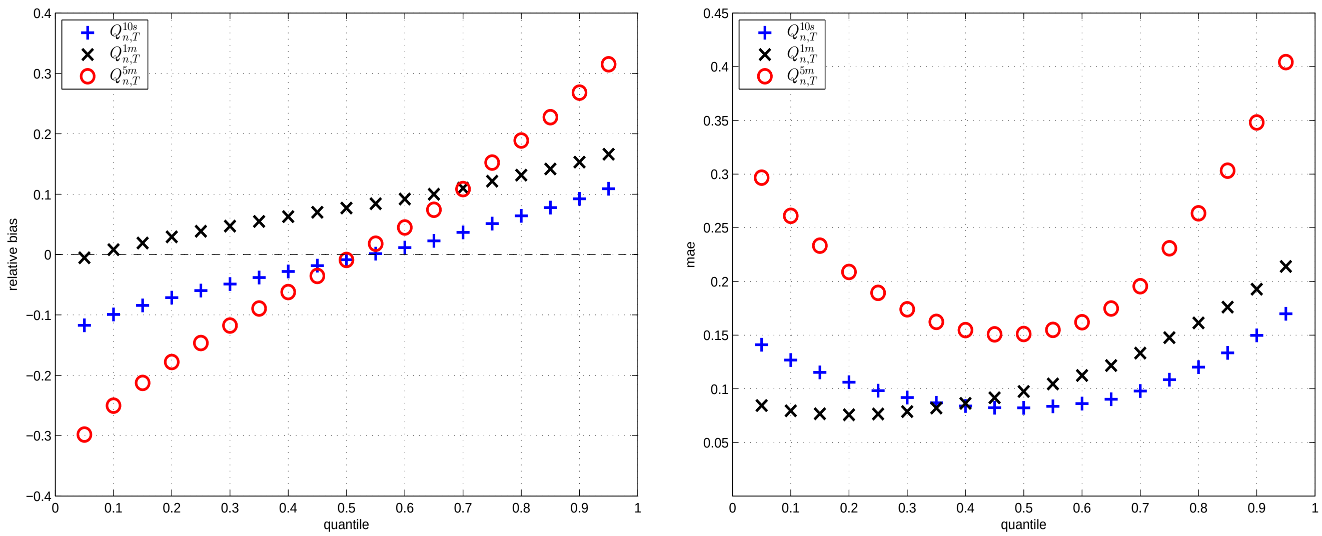
was hardly affected by this change, while the sparse estimator was substantially more biased, due to the amplification of the noise.

Figure 2: Mean relative bias and absolute error.

Panel A: Relative bias. Panel B: Absolute error.



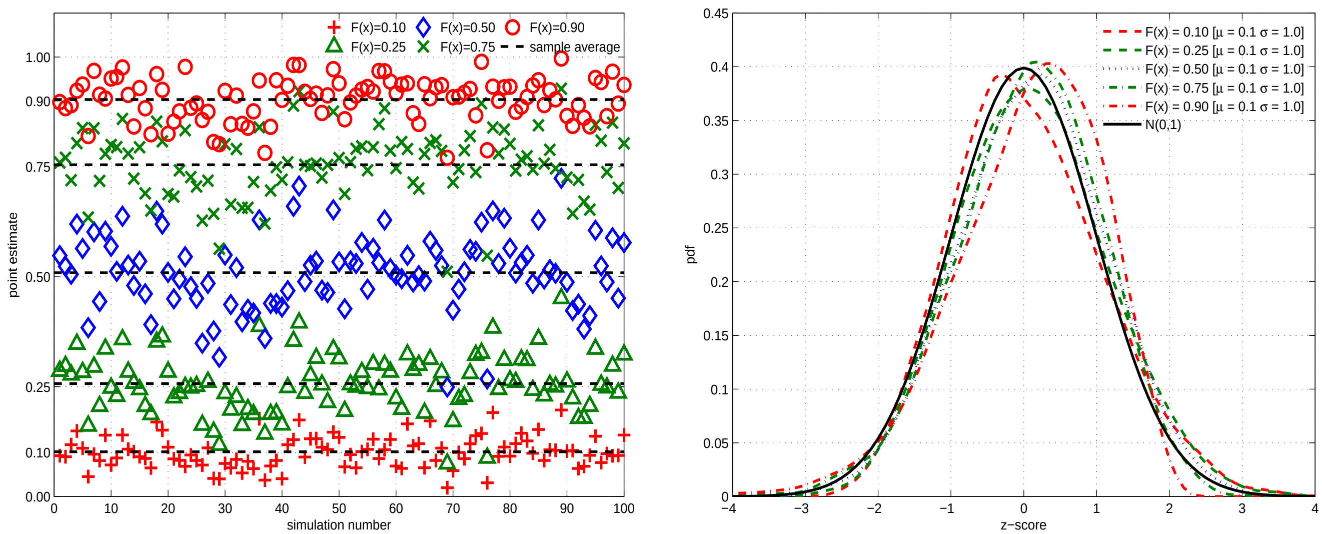
Note. This figure plots the average relative bias in Panel A and absolute error in Panel B of our realized estimator of the EDF of spot variance ('+'). As a comparison, we also plot the Li, Todorov, and Tauchen (2013) no-noise estimator based on 1- and 5-minute sparse sampling ('×' and '∘').

relative error $Q_{n,T}(\alpha)/Q_T(\alpha)-1$ in Panel A and absolute error $\left|Q_{n,T}(\alpha)-Q_T(\alpha)\right|$ in Panel B, which are averaged across Monte Carlo trials, for the vector of quantiles $\alpha=(0.05,0.10,\ldots,0.95)'$. As is evident—and consistent with Figure 1—our estimator is less biased, especially in the tails. Overall, the REDF proposed in this paper is roughly unbiased over a large spectrum of the distribution with a relative error within a few percent. This is further corroborated by Panel B. In general, while all estimators do a relatively poorer job at fitting the tails of the distribution of spot variance, there are again substantial improvements by applying our estimator.

We now turn to estimation of the marginal distribution of volatility based on Theorem 3.3. We work with $T=250$, corresponding to the number of trading days in about a year. This relatively low value is meant to be conservative and illustrate the potential of our estimator. As a comparison, $T=2,077$ in our empirical work in Section 5. Note that n is kept fixed relative to T, so that Δ_n is constant. In Figure 3, we report point estimates—for the first 100 realizations—and sample averages across all 10,000 simulations of $F_{n,T}(x)$ and kernel densities of the asymptotic pivot $\sqrt{T}\big(F_{n,T}(x)-F(x)\big)/\sqrt{\Sigma(x)}\xrightarrow{d}N(0,1)$ evaluated at five distinct points, which cover a variety of low-to-high volatility states, i.e. $F(x)=0.10,0.25,0.50,0.75$ and 0.90. We see that in general the REDF is unbiased and correctly scaled. If we look far in the tails, in particular the right-hand one, the density estimate is slightly skewed towards the center, which is intuitive due to the natural

Figure 3: Inference about marginal distribution, $T = 250$.

Panel A: Point estimate. Panel B: Kernel density.



Note. In Panel A, we report point estimates of $F_{n,T}(x)$ for the first one hundred Monte Carlo replica and sample averages across all simulations, for $F(x) = 0.10, 0.25, 0.50, 0.75$ and 0.90. In Panel B, we plot kernel density estimates (with bandwidth selection via Silverman's rule of thumb) of the standardized statistic $\sqrt{T}\big(F_{n,T}(x) - F(x)\big)/\sqrt{\Sigma(x)}$. The Gaussian curve is superimposed as a visual reference point. Throughout, $\Delta_n = 1/2,340$ and $T = 250$.

bounds on $F_{n,T}$. Still, the limiting distribution provides a good description of the finite sample variation of $F_{n,T}$.

Last but not least, we explore the properties of the realized Kolmogorov-Smirnov and Cramer-von-Mises (i.e., a weighted L^2 norm with $w(x) = f(x)$) goodness-of-fit test. This piece of analysis is based on a shorter run of 1,000 Monte Carlo replica due to the increased computational cost of the resampling scheme. We draw 500 bootstrap samples in each simulation trial and inspect both the setting where v is known and estimated. We follow Corradi and Distaso (2006) and recover the parameters via $\hat{v}$ using GMM to match the sample mean, variance, plus the first and second autocovariance of the bias-corrected pre-averaged bipower variation of Podolskij and Vetter (2009a) to the corresponding model-based moments of integrated variance (the former is consistent for the latter).

To be able to define a measure of the testing power we simulate from another model, i.e. the log-normal—or exponential Ornstein-Uhlenbeck—process, which is the solution of the stochastic differential equation:

$$d \ln V_t = \kappa(v_0 - \ln V_t)dt + \xi dB_t, \tag{4.2}$$

where $v = (\kappa, v_0, \xi) = (0.08, -0.3, 0.45)$ is comparable to what we observe in the real data. We leave the leverage correlation unchanged at $\rho = -\sqrt{0.5}$.

In Table 1, we report the size and power of the test for a nominal significance level of $\alpha = 5\%$ and $T = 250$, 500, 1,000, and 2,000. As seen, the test is roughly unbiased and has a rejection rate close to α. We next comment on the power, as reported in the second half of Table 1. It is calculated as the rejection rate achieved by simulating log-normal volatility and testing the square-root process. $\mathcal{H}_0$ in the setting without estimation is here based on the configuration of the Heston (1993) model with the true parameters used to simulate volatility for the size analysis. As shown, the test has moderate power if the sample size is small, but the rejection rate increases steadily as T grows larger. At $T = 2,000$, the power is in the 60–90% range. The largest choice of T corresponds to about eight years worth of high-frequency data, which is widely available for many assets in practice. The impact of parameter estimation is to "flatten" the power curve by adding an upward bias for small T, while dampening it for large T. We conjecture that this effect is caused by the inherent sampling error in the pre-averaged bipower variation, which leads to systematic upward biases in the estimate of both the mean reversion and volatility-of-volatility coefficient κ and ξ.[11] The misspecification of the dynamic properties of the system invariably yields simulated critical values that are slightly off and this effect appears more pronounced under the alternative. Finally, we notice the realized Kolmogorov-Smirnov test has slightly higher power than our weighted L^2 statistic, which is a somewhat surprising finding compared to the classical results for i.i.d. data.

5 Empirical application

We here illustrate the application of our new nonparametric estimator of the EDF of spot variance and the associated goodness-of-fit tests to real high-frequency data. We construct the $F_{n,T}$ measure from tick data based on selected exchange-traded funds (ETFs) that cover different sectors of the U.S. stock market. In addition to a proxy for the market index (ticker symbol SPY), we add the nine industry portfolios of the S&P 500, yielding a total of ten securities.[12] High-frequency data was

[11] The Heston (1993) model has an ARMA(1,1) representation for the integrated variance. We can exploit this structure by casting the whole system into state-space representation and then use a Kalman filter to extract a filtered (or smoothed) volatility series before estimating the model, as discussed in Barndorff-Nielsen and Shephard (2002) (see also, e.g., Todorov, Tauchen, and Grynkiv, 2011, for an alternative approach). This tends to deliver less biased parameter estimates. However, we do not pursue this idea here.

[12] There is also a 10th sector ETF (XLRE), which is a diversified portfolio of companies exposed to real estate. However, as the inception date of this fund is rather recent, we exclude it from consideration due to the limited availability of high-frequency data. Further information about the ETFs is available at http://www.sectorspdr.com/.

Table 1: Properties of the realized goodness-of-fit tests.

		Size				Power			
		$rKS(F)$		$rL^2(F)$		$rKS(F)$		$rL^2(F)$	
		v	$\hat{v}$	v	$\hat{v}$	v	$\hat{v}$	v	$\hat{v}$
$T =$	250	0.063	0.083	0.056	0.076	0.114	0.398	0.091	0.302
	500	0.050	0.066	0.052	0.064	0.198	0.432	0.127	0.334
	1000	0.026	0.031	0.030	0.040	0.453	0.541	0.299	0.407
	2000	0.047	0.040	0.050	0.041	0.923	0.717	0.849	0.573

Note. The table reports rejection rates of the goodness-of-fit tests for stochastic volatility models proposed in Section 3.3. $rKS(F)$ ($rL^2(F)$) is the realized Kolmogorov-Smirnov (Cramer-von-Mises) test. The number of Monte Carlo simulations is 1,000 with 500 bootstrap samples used to compute critical values for the t-statistic in each trial. v ($\hat{v}$) is based on the true (estimated) parameter vector. The power is calculated as the rejection rate achieved by simulating log-normal volatility and testing whether the Heston (1993) process is the true model. Further details are available in the main text.

acquired from the TAQ database and comprise a complete series of trades and quotes for each stock for the sample period July 2008 – September 2016. The analysis here is based on transaction data, which were filtered for outliers as in Christensen, Oomen, and Podolskij (2014) and subsequently pre-ticked to an equidistant 10-second grid from 9:30am to 4:00pm EST, resulting in $T = 2,077$ days of $n = 2,340$ intraday returns.[13] A list of ticker symbols and descriptive statistics of the sample are presented in Table 2.

The calculation of $\hat{V}_t$ proceeds as in Section 2 – 4 in terms of tuning parameters, but we add truncation as an extra level of protection. A pre-averaged increment is removed if $|\bar{Z}_i| \geq q_{1-\alpha}\sqrt{\widehat{IV}}\Delta_n^{\varpi}$, where $q_{1-\alpha}$ is the $1-\alpha$ quantile of the standard normal distribution, $\widehat{IV}$ is the daily pre-averaged bipower variation, and $\varpi \in (0, 0.25)$ controls how fast the cutoff goes to zero with Δ_n ($\bar{Z}_i = O_p\big(\Delta_n^{1/4}\big)$, if X is continuous). We set $\alpha = 0.001$ and $\varpi = 0.20$, which is a standard configuration. We also account for the diurnal pattern in intraday spot volatility. We follow the standard approach in the literature, which is to compute the average value of $\hat{V}_t$ at each time point of the day over the sample. The diurnal factor is then estimated by normalizing the sum of these averages to one. We correct $\hat{V}_t$ by this quantity and use the adjusted time series to construct $F_{n,T}$. In Panel A of Figure 4, we show a kernel-based estimate of the probability density function implied by the variance of

[13] Trading at NYSE terminates early at 1:00pm on a few regularly scheduled business days each year in observance of upcoming holidays. We do not collect data after the exchange has officially closed, as there is typically very little liquidity, and therefore use a shorter sample of $n = 1,260$ log-price increments on these days.

Table 2: Descriptive statistics of high-frequency data.

code	effective sample	σ	γ	goodness-of-fit test (p-value)					
				gamma		log-normal		inverse gaussian	
				$rKS(F)$	$rL^2(F)$	$rKS(F)$	$rL^2(F)$	$rKS(F)$	$rL^2(F)$
XLB	0.749	0.219	0.232	0.001	0.000	0.009	0.002	0.166	0.094
XLE	0.929	0.249	0.117	0.000	0.000	0.035	0.018	0.389	0.344
XLF	0.920	0.264	0.910	0.000	0.000	0.000	0.000	0.017	0.008
XLI	0.789	0.188	0.335	0.000	0.000	0.022	0.016	0.277	0.261
XLK	0.728	0.174	0.524	0.000	0.000	0.146	0.101	0.021	0.043
XLP	0.690	0.131	0.568	0.011	0.003	0.063	0.039	0.014	0.023
XLU	0.754	0.176	0.389	0.000	0.000	0.211	0.149	0.040	0.066
XLV	0.737	0.151	0.358	0.000	0.000	0.017	0.012	0.111	0.075
XLY	0.746	0.186	0.245	0.006	0.006	0.001	0.000	0.138	0.089
SPY	1.000	0.175	0.135	0.009	0.002	0.110	0.061	0.493	0.541

Note. The table reports descriptive statistics of our equity high-frequency data. "code" is the ticker symbol, "effective sample" is the fraction of previous-tick interpolated 10-second prices originating from a new transaction rather than a repetition (a measure of liquidity), σ is annualized volatility based on a pre-averaged bipower variation of Podolskij and Vetter (2009a), while γ is the noise-to-signal ratio defined in the main text. These numbers are computed daily and averaged over the sample. The p-value is for the realized Kolmogorov-Smirnov and Cramer-von-Mises goodness-of-fit test with marginal distribution under $\mathcal{H}_0$ indicated by the designated column label.

SPY, as representative of our data.

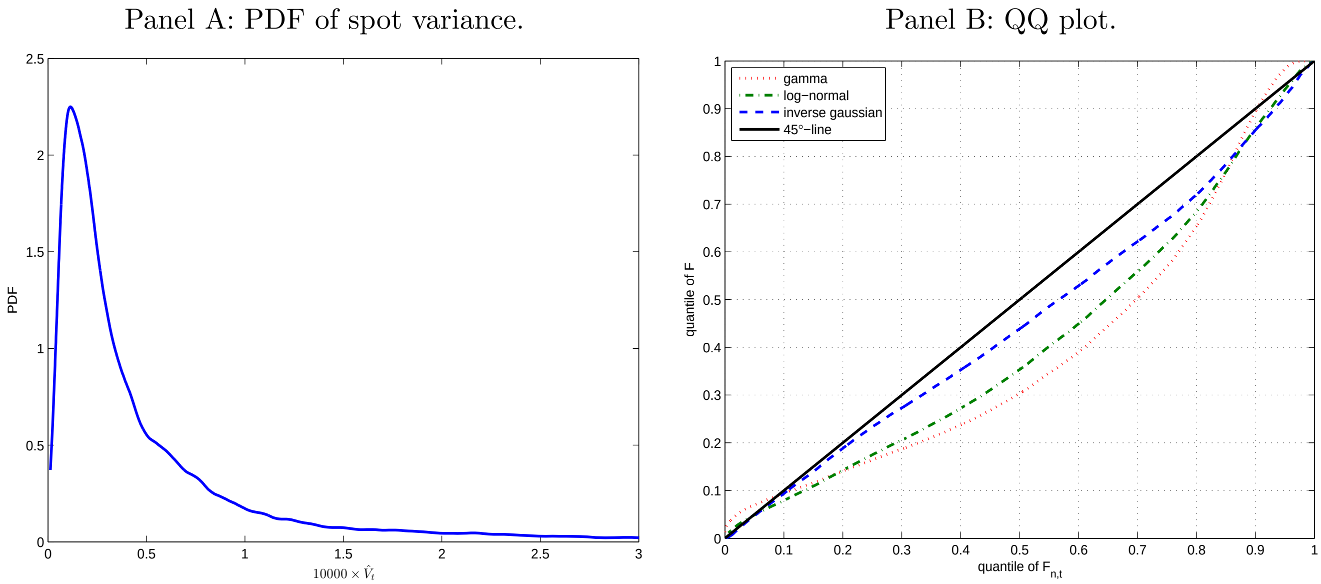
We compute $rKS(F)$ and $rL^2(F)$ with three choices of F under $\mathcal{H}_0$. The first are the gamma and log-normal distribution inherent in the square-root and log-normal diffusion models that were introduced in the simulation section. Furthermore, we also inspect the goodness-of-fit of a non-Gaussian Ornstein-Uhlenbeck process (Barndorff-Nielsen and Shephard, 2001). The latter is a pure-jump specification (i.e., in contrast to the other candidates, it has no Brownian part) with SDE:

$$dV_t = -\kappa V_t dt + dL_t, \tag{5.1}$$

where L_t is a subordinator with Lévy measure $\nu_L(dx)$.

To maintain a tractable setting, we proceed as in Todorov, Tauchen, and Grynkiv (2011). We assume the distribution of the increments of the volatility process corresponds to that of a tempered

Figure 4: Properties of stochastic variance in SPY.



Note. In Panel A, we plot a kernel-based estimate of the probability density function of spot variance for SPY. In Panel B, we create a QQ plot, where the quantiles of $F_{n,T}$ (on the x-axis) are mapped against the quantiles of a fitted distribution function F (on the y-axis). The gamma, log-normal and inverse Gaussian distribution are included for the latter comparison.

stable process (e.g., Rosiński, 2007), which—in particular—means it has a Lévy density:

$$\nu_V(dx) = c\frac{e^{-\lambda x}}{x^{1+\alpha}}dx, \quad \text{for } x > 0, \tag{5.2}$$

where $c > 0$, $\lambda > 0$ and $\alpha < 1$. The Lévy measure of the background driving process $\nu_L(dx)$ can then be backed out from $\nu_V(dx)$, see e.g., Barndorff-Nielsen and Shephard (2001). We fix $\alpha = 0.5$, so that the distribution of volatility is inverse Gaussian, i.e. $V_t \sim IG(\mu, \nu)$ with:

$$\mu = c\sqrt{\frac{\pi}{\lambda}}, \qquad \nu = 2\pi c^2, \tag{5.3}$$

and

$$f(x; \mu, \nu) = \sqrt{\frac{\nu}{2\pi x^3}} \exp\left\{ -\frac{\nu}{2\mu^2 x}(x-\mu)^2 \right\}, \quad \text{for } x > 0, \tag{5.4}$$

is the density function of V_t.

The procedure in Section 4 is applied to report a p-value for the t-statistic. The exception is that we boost the number of boostrap samples to 1,000 for higher precision. The non-Gaussian Ornstein-Uhlenbeck process is discretized and simulated as explained in the appendix of Todorov, Tauchen, and Grynkiv (2011).[14]

[14]As in Meddahi (2003), we exploit the ARMA(1,1) structure for the integrated variance of this model to com-

In Table 2, we present the outcome of these efforts, while Panel B of Figure 4 shows a QQ-style plot, where the quantiles of $F_{n,T}$ (on the x-axis) are compared to the quantiles of the best fitting distribution function F in each class (on the y-axis). The graph should align with the 45°-line if the model is correct. As readily seen, the gamma distribution does not capture stochastic equity variance very well. The main problem with the square-root process is that it cannot generate a sufficient level of volatility-of-volatility (via ξ) without violating the Feller restriction, which is binding in practice. The p-values of the log-normal model are higher on average, but it also does not deliver an acceptable fit of the data across sectors. Although there are a few close calls, for example XLK, XLU and SPY, the model is generally rejected at standard levels of significance. In contrast, the p-values of the inverse Gaussian are typically much higher. This is in line with Todorov and Tauchen (2012), and it appears relatively consistent across the ETF space. Nevertheless, the latter sometimes also struggles a bit (depending a bit on which t-statistic we ask), for instance it does not capture return variation exhibited by the highly volatile financial sector ETF (XLF).

6 Conclusion

We construct a nonparametric jump- and noise-robust realized measure of the EDF of the latent volatility of a general Itô semimartingale sampled at high-frequency on a fixed time interval. We extend previous work of Li, Todorov, and Tauchen (2013, 2016) in the noise-free setting and prove that in the presence of microstructure noise the pre-averaged version of their estimator is consistent in the infill asymptotic limit. The resistance to market frictions enables our statistic to fully exploit information about volatility available in tick-by-tick high-frequency data, which improves its accuracy in a simulation study.

In a subsequent analysis, we let the time span tend to infinity. We show that our estimator then converges to the marginal distribution function of volatility. This result is subject to some rate conditions, which are more restrictive in the presence of jumps or roughness in volatility. We establish a functional CLT, from which we prove the validity of a family of goodness-of-fit tests for stochastic volatility models. The limiting distribution of our t-statistic depends on the true model under the null hypothesis, but critical values can be found via simulation. We show how the

pute the relevant moments: $\mathbb{E}[IV_t] = c\sqrt{\frac{\pi}{\lambda}}$, $\text{var}(IV_t) = \frac{c\sqrt{\pi}}{\lambda^{3/2}\kappa^2}(e^{-\kappa} + \kappa - 1)$, $\text{cov}(IV_t, IV_{t-1}) = \sqrt{\frac{\pi c^2}{4\lambda^3}}\frac{(1-e^{-\kappa})^2}{\kappa^2}$, and $\text{cov}(IV_t, IV_{t+j}) = e^{-\kappa}\text{cov}(IV_t, IV_{t+j-1})$ for $j \geq 2$. We also calculated these expressions based on the ARMA coefficients reported in the appendix of Todorov, Tauchen, and Grynkiv (2011), but the results differ and we suspect there are some typos in that paper.

procedure can be adapted to the case with parameter estimation.

We apply the theory to high-frequency data from several ETFs that track different sectors of the U.S. stock market. The marginal distribution embedded in several classes of popular stochastic volatility models is tested against our nonparametric measure. There is strong evidence against the gamma or log-normal distribution, while the inverse Gaussian—implied by the class of non-Gaussian Ornstein-Uhlenbeck processes of Barndorff-Nielsen and Shephard (2001) with tempered stable increments—has more support. These findings are consistent with Todorov, Tauchen, and Grynkiv (2011), but in disagreement with Corradi and Distaso (2006).

Our paper can serve as a starting point for building more refined models of stochastic volatility. It appears natural, for example, to inspect the properties of the generalized inverse Gaussian (GIG) as the marginal distribution of volatility. It has the PDF:

$$f(x;a,b,p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{(p-1)} e^{-(ax+b/x)/2}, \quad \text{for } x > 0, \tag{6.1}$$

where K_p is the modified Bessel function of the second kind, while $a \geq 0, b \geq 0$ and p a real number are the parameters, such that $ab \neq 0$. The GIG is self-decomposable (necessary and sufficient for stationary solutions of (5.1) to exist) and nests the gamma distribution as $b \to 0$ (with $a > 0$) and the inverse Gaussian if $p = -1/2$. The extra degree of freedom may just be enough for a complete description of equity variance. We leave this assessment for future research.

A Appendix

We here prove the theoretical results presented in the main text. As usual, the localization procedure in Jacod and Protter (2012, Section 4.4.1) implies we can assume that b_t, σ_t, $\tilde{b}_t$, $\tilde{\sigma}_t$, $\tilde{\sigma}_t'$ and the jump component of X are bounded for the infill setting with T fixed, i.e. Lemma 2.1, Theorem 3.1 and Corollary 3.2. Also, to preserve notation C denotes a generic constant that changes value from one line to the next.

Proof of Lemma 2.1

Fix a $t > 0$. If $t \in ((i-1)\Delta_n, i\Delta_n]$, we set $t_n = i\Delta_n$ to highlight the dependence on n and note that $V_{t_n} \to V_t$, because σ is càdlàg. Thus, our job is to establish:

$$\tilde{V}_{i\Delta_n} - \theta\psi_2 V_{t_n} - \frac{1}{\theta}\psi_1\omega^2 \xrightarrow{\mathbb{P}} 0.$$

We start by introducing an approximation of the pre-averaged return series $\bar{Z}_i$, in which the stochastic volatility process is fixed locally in each computation:

$$\bar{Y}_{i,i+m} = \sum_{j=1}^{k_n-1} g\left(\frac{j}{k_n}\right)\left(\sigma_{i\Delta_n}\Delta_{i+m+j}^n W + \Delta_{i+m+j}^n U\right),$$

and then we deduce the claim based on $\bar{Y}_{i,i+m}$, i.e.

$$\check{V}_{i\Delta_n} \equiv \frac{1}{h_n\sqrt{\Delta_n}}\sum_{m=0}^{h_n-1}\bar{Y}_{i,i+m}^2 - \theta\psi_2 V_{t_n} - \frac{1}{\theta}\psi_1\omega^2 \xrightarrow{\mathbb{P}} 0. \tag{A.1}$$

Now, it is readily seen that for each $m \geq 0$, $\Delta_n^{-1/4}\bar{Y}_{i,i+m}$ has (conditionally on $\mathcal{F}_{(i+m)\Delta_n}$) mean zero and variance equal to $k_n\sqrt{\Delta_n}\psi_2^n V_{t_n} + \dfrac{1}{k_n\sqrt{\Delta_n}}\psi_1^n\omega^2$. Furthermore,

$$k_n\sqrt{\Delta_n}\psi_2^n V_{t_n} + \frac{1}{k_n\sqrt{\Delta_n}}\psi_1^n\omega^2 - \theta\psi_2 V_{t_n} - \frac{1}{\theta}\psi_1\omega^2 \xrightarrow{\mathbb{P}} 0.$$

as $n \to \infty$. It follows that

$$\frac{1}{h_n\sqrt{\Delta_n}}\sum_{m=0}^{h_n-1}\mathbb{E}\left[\bar{Y}_{i,i+m}^2 \mid \mathcal{F}_{(i+m)\Delta_n}\right] - \theta\psi_2 V_{t_n} - \frac{1}{\theta}\psi_1\omega^2 \xrightarrow{\mathbb{P}} 0,$$

and therefore suffices to prove that

$$\frac{1}{h_n\sqrt{\Delta_n}}\sum_{m=0}^{h_n-1}\left(\bar{Y}_{i,i+m}^2 - \mathbb{E}\left[\bar{Y}_{i,i+m}^2 \mid \mathcal{F}_{(i+m)\Delta_n}\right]\right) \xrightarrow{\mathbb{P}} 0.$$

To this end, we note that

$$\Delta_n^{-1}\mathbb{E}\Big[\big(\bar{Y}_{i,i+m}^2 - \mathbb{E}\big[\bar{Y}_{i,i+m}^2 \mid \mathcal{F}_{(i+m)\Delta_n}\big]\big)\big(\bar{Y}_{i,i+l}^2 - \mathbb{E}\big[\bar{Y}_{i,i+l}^2 \mid \mathcal{F}_{(i+l)\Delta_n}\big]\big)\Big] \leq C, \qquad \text{for } |m-l| \leq k_n,$$

$$\Delta_n^{-1}\mathbb{E}\Big[\big(\bar{Y}_{i,i+m}^2 - \mathbb{E}\big[\bar{Y}_{i,i+m}^2 \mid \mathcal{F}_{(i+m)\Delta_n}\big]\big)\big(\bar{Y}_{i,i+l}^2 - \mathbb{E}\big[\bar{Y}_{i,i+l}^2 \mid \mathcal{F}_{(i+l)\Delta_n}\big]\big)\Big] = 0, \qquad \text{for } |m-l| > k_n.$$

Thus,

$$\mathbb{E}\left[\left(\frac{1}{h_n\sqrt{\Delta_n}}\sum_{m=0}^{h_n-1}\Big(\bar{Y}_{i,i+m}^2 - \mathbb{E}\big[\bar{Y}_{i,i+m}^2 \mid \mathcal{F}_{(i+m)\Delta_n}\big]\Big)\right)^2\right] \leq C\frac{k_n}{h_n} \to 0,$$

and (A.1) follows.

We are thus left with the assertion:

$$S_n = \tilde{V}_{i\Delta_n} - \check{V}_{i\Delta_n} \xrightarrow{\mathbb{P}} 0. \tag{A.2}$$

To prove this we define $X''(\kappa) = (\delta 1_{\{\Gamma\leq\kappa\}})\star(\mu-\nu)$ and $B''(\kappa) = B'' - (\delta 1_{\{\Gamma>\kappa\}}\star\nu)$ for each $\kappa \in (0,1)$, where $B''_t = \int_0^t b_s ds + \int_0^t \int_{\mathbb{R}} \delta(s,z) 1_{\{|\delta(s,z)|>1\}}\mu(d(s,z))$. Moreover, we let $\Omega_n(\kappa)$ denote the set on which the Poisson process $\mu\big([0,s]\times\{z : \Gamma > \kappa\}\big)$ has no jumps on the interval $(t, t+(k_n+h_n)\Delta_n]$. Note that $\Omega_n(\kappa) \nearrow \Omega$, as $n\to\infty$. On $\Omega_n(\kappa)$, $\Delta_j^n Z = \sigma_{i\Delta_n}\Delta_j^n W + \Delta_j^n U + \Delta_j^n B''(\kappa) + \Delta_j^n Y' + \Delta_j^n X''(\kappa)$, for $j = i+1,\ldots,i+h_n+k_n$, where $Y'_t = \int_{i\Delta_n\wedge t}^t (\sigma_s - \sigma_{i\Delta_n})dW_s$. Now, from the inequality (Jacod and Protter, 2012, p. 260):

$$|(x+y+z+w)^2 - x^2| \leq \epsilon x^2 + \frac{C}{\epsilon}(y^2+z^2+w^2)$$

with

$$x = \frac{\bar{Y}_{i,i+m}}{\Delta_n^{1/4}}, \quad y = \frac{\bar{X}''_{i+m}(\kappa)}{\Delta_n^{1/4}}, \quad z = \frac{\bar{Y}'_{i+m}}{\Delta_n^{1/4}}, \quad \text{and} \quad w = \frac{\bar{B}''_{i+m}(\kappa)}{\Delta_n^{1/4}},$$

we find that on $\Omega_n(\kappa)$:

$$|S_n| \leq \frac{1}{h_n}\sum_{m=0}^{h_n-1}\left[\epsilon\left|\frac{\bar{Y}_{i,i+m}}{\Delta_n^{1/4}}\right|^2 + \frac{C}{\epsilon}\left(\left|\frac{\bar{X}''_{i+m}(\kappa)}{\Delta_n^{1/4}}\right|^2 + \left|\frac{\bar{Y}'_{i+m}}{\Delta_n^{1/4}}\right|^2 + \left|\frac{\bar{B}''_{i+m}(\kappa)}{\Delta_n^{1/4}}\right|^2\right)\right].$$

As $|\Delta_i^n B''(\kappa)| \leq C\Delta_n$, uniformly in i, we see that

$$\left|\frac{\bar{B}''(\kappa)}{\Delta_n^{1/4}}\right| \leq C\Delta_n^{1/4}.$$

Then, by means of the Burkholder-Davis-Gundy inequality:

$$\mathbb{E}\Big[|\bar{Y}'_{i+m}|^2 \mid \mathcal{F}_{(i+m)\Delta_n}\Big] \leq Ck_n\Delta_n\mathbb{E}\Big[\sup_{s\in[i\Delta_n,(i+h_n+k_n)\Delta_n]}|\sigma_s - \sigma_{i\Delta_n}|^2 \mid \mathcal{F}_{(i+m)\Delta_n}\Big]. \tag{A.3}$$

As σ is càdlàg and can be assumed to be bounded, the last term converges to zero. Next, for the components involving $\bar{Y}_{i,i+m}$, it holds that (Podolskij and Vetter, 2009a, Lemma 1):

$$\mathbb{E}\Big[|\bar{Y}_{i,i+m}|^2 \mid \mathcal{F}_{(i+m)\Delta_n}\Big] \leq K\Delta_n^{1/2}. \tag{A.4}$$

In addition, from Lemma 2.1.5 in Jacod and Protter (2012):

$$\mathbb{E}\Big[\big|\bar{X}''_{i+m}(\kappa)\big|^2\Big] \leq C\Delta_n k_n \xi(\kappa),$$

where $\xi(\kappa) = \int_{\{z:\Gamma(z)\leq\kappa\}} \Gamma(z)^2 \lambda(dz)$. This, along with (A.3) – (A.4), implies that:

$$\mathbb{E}\big[|S_n| 1_{\{\Omega_n(\kappa)\}}\big] \leq C\epsilon + \frac{C}{\epsilon}\left(\xi(\kappa) + \mathbb{E}\bigg[\sup_{s\in[i\Delta_n,(ih_n+k_n)\Delta_n]} |\sigma_s - \sigma_{i\Delta_n}|^2\bigg] + \Delta_n^{1/4}\right).$$

Furthermore, for all $\eta > 0$, $\{|S_n| > \eta\} \subset \Omega_n(\kappa)^{\complement} \cup \big(\{|S_n| > \eta\} \cap \Omega_n(\kappa)\big)$. Markov's inequality then yields that

$$\mathbb{P}\big(|S_n| > \eta\big) \leq \mathbb{P}\Big(\Omega_n(\kappa)^{\complement}\Big) + \frac{1}{\eta}\left(C\epsilon + \frac{C}{\epsilon}\left(\xi(\kappa) + \mathbb{E}\bigg[\sup_{s\in[i\Delta_n,(ih_n+k_n)\Delta_n]} |\sigma_s - \sigma_{i\Delta_n}|^2\bigg] + \Delta_n^{1/4}\right)\right).$$

With $n \to \infty$, it follows that

$$\limsup_{n\to\infty} \mathbb{P}\big(|S_n| > \eta\big) \leq \frac{1}{\eta}\left(C\epsilon + \frac{C}{\epsilon}\xi(\kappa)\right). \tag{A.5}$$

As this is true for all $\epsilon, \kappa \in (0,1)$, we can pick $\epsilon = \xi(\kappa)^{1/2}$. Doing so and letting $\kappa \to 0$ verifies (A.2) and hence completes the proof of Lemma 2.1. ∎

Proof of Theorem 3.1

The proof follows the structure of Lemma 1 and Theorem 1 in Li, Todorov, and Tauchen (2013). We start by showing that $\mathbb{P}\big(V_t = x\big) = 0$. By assumption, F_T is continuous almost surely, so letting $F_T(x-) = \lim_{y\uparrow x} F_T(y)$, it holds that $\mathbb{P}\big(F_T(x) \neq F_T(x-)\big) = 0$. Hence,

$$\int_0^T \mathbb{P}\big(V_t = x\big)dt = \mathbb{E}\bigg[\int_0^T (1_{\{V_t\leq x\}} - 1_{\{V_t<x\}})dt\bigg] = T\mathbb{E}\big(F_T(x) - F_T(x-)\big) = 0.$$

Thus, $\mathbb{P}(V_t = x) = 0$ for a.e. $t \in [0,T]$. Next, we show the pointwise consistency of $F_{n,T}(x)$. We therefore fix $x \in \mathbb{R}_+$ and notice that

$$\begin{aligned}\mathbb{E}\Big[\big|F_{n,T}(x) - F_T(x)\big|\Big] &\leq \frac{1}{T}\int_0^T \mathbb{E}\Big[\big|1_{\{\hat{V}_t\leq x\}} - 1_{\{V_t\leq x\}}\big|\Big]dt \\ &= \frac{1}{T}\int_0^T \mathbb{E}\Big[\big|\big(1_{\{\hat{V}_t\leq x\}} - 1_{\{V_t\leq x\}}\big)1_{\{V_t\neq x\}}\big|\Big]dt.\end{aligned}$$

Applying Lemma 2.1 and Lebesgue's dominated convergence theorem (twice), we deduce that $\mathbb{E}\Big[\big|F_{n,T}(x) - F_T(x)\big|\Big] \to 0$, which implies the pointwise consistency, $F_{n,T}(x) \xrightarrow{\mathbb{P}} F_T(x)$. As $F_{n,T}$ and F_T are non-decreasing functions and F_T is continuous a.s., this convergence is locally uniform (see, e.g., Resnick, 1998). Furthermore, as the paths of σ_t are càdlàg, for any $\eta > 0$ there exists some constant M such that $\mathbb{P}(\sup_{t\in[0,T]} V_t > M) < \eta$, which means that $\mathbb{P}(F_T(M) \neq 1) < \eta$. In turn,

$$\begin{aligned}
\mathbb{P}\bigg(\sup_{x\in\mathbb{R}_+}\big|F_{n,T}(x) - F_T(x)\big| > \epsilon\bigg) &\leq \mathbb{P}\bigg(\sup_{x\in[0,M]}\big|F_{n,T}(x) - F_T(x)\big| > \epsilon\bigg) + \mathbb{P}\bigg(\sup_{x\geq M}\big|F_{n,T}(x) - F_T(x)\big| > \epsilon\bigg) \\
&\leq \mathbb{P}\bigg(\sup_{x\in[0,M]}\big|F_{n,T}(x) - F_T(x)\big| > \epsilon\bigg) + \mathbb{P}\bigg(\sup_{x\geq M}\big|F_{n,T}(x) - F_{n,T}(M)\big| > \frac{\epsilon}{3}\bigg) \\
&+ \mathbb{P}\bigg(\sup_{x\geq M}\big|F_{n,T}(M) - F_T(M)\big| > \frac{\epsilon}{3}\bigg) + \mathbb{P}\bigg(\sup_{x\geq M}\big|F_T(M) - F_T(x)\big| > \frac{\epsilon}{3}\bigg) \\
&\leq \mathbb{P}\bigg(\sup_{x\in[0,M]}\big|F_{n,T}(x) - F_T(x)\big| > \epsilon\bigg) + \mathbb{P}\bigg(1 - F_{n,T}(M) > \frac{\epsilon}{3}\bigg) \\
&+ \mathbb{P}\bigg(\big|F_{n,T}(M) - F_T(M)\big| > \frac{\epsilon}{3}\bigg) + \mathbb{P}\bigg(1 - F_T(M) > \frac{\epsilon}{3}\bigg).
\end{aligned}$$

Taking lim sup and using the pointwise and locally uniform convergence:

$$\limsup_{n\to\infty}\mathbb{P}\bigg(\sup_{x\in\mathbb{R}_+}\big|F_{n,T}(x) - F_T(x)\big| > \epsilon\bigg) \leq 2\eta,$$

and since η is arbitrary, this shows that

$$\mathbb{P}\bigg(\sup_{x\in\mathbb{R}_+}\big|F_{n,T}(x) - F_T(x)\big| > \epsilon\bigg) \to 0,$$

as $\Delta_n \to 0$. ■

Two auxiliary results

As in the paper, we will throughout this section assume that X is continuous.

Lemma A.1 *Suppose that Assumption 1*(i) *with $H \in (0,1)$ and 3 – 5 hold true, as does the rate condition $T^{1/2+\iota}\Delta_n^{\frac{H}{4H+2}-\iota} \to 0$, as $\Delta_n \to 0$ and $T \to \infty$, for some $\iota > 0$ and $h_n \asymp \Delta_n^{-\frac{4H+1}{4H+2}}$. Then,*

$$\sqrt{T}\sup_{x\in\mathbb{R}_+}\big|F_{n,T}(x) - F_T(x)\big| \xrightarrow{\mathbb{P}} 0. \tag{A.6}$$

Proof. The proof consists of three parts. First, we show that for each i and $p \geq 2$:

$$\mathbb{E}\Big[|\hat{V}_{i\Delta_n} - V_{i\Delta_n}|^p\Big] \leq C\Big[\big(k_n/h_n\big)^{p/2} + \big(h_n\Delta_n\big)^{pH} + \Delta_n^{p/4}\Big]. \tag{A.7}$$

Note that

$$\hat{V}_{i\Delta_n} - V_{i\Delta_n} = R_n^{(1)} + R_n^{(2)} + R_n^{(3)} + R_n^{(4)} + R_n^{(5)}, \tag{A.8}$$

where

$$R_n^{(1)} = \frac{1}{\theta\psi_2}\frac{1}{h_n\sqrt{\Delta_n}}\sum_{m=0}^{h_n-1}\bar{X}_{i+m}^2 - \sigma_{i\Delta_n}^2\bar{W}_{i+m}^2,$$

$$R_n^{(2)} = \frac{1}{\theta\psi_2}\frac{2}{h_n}\sum_{m=0}^{h_n-1}\Delta_n^{-1/2}\bar{X}_{i+m}\bar{U}_{i+m},$$

$$R_n^{(3)} = \frac{1}{\theta\psi_2}\frac{1}{h_n}\sum_{m=0}^{h_n-1}\Big(\Delta_n^{-1/2}\bar{U}_{i+m}^2 - \frac{\psi_1}{\theta}\omega^2\Big),$$

$$R_n^{(4)} = \frac{\psi_1}{\theta^2\psi_2}(\omega^2 - \hat{\omega}^2),$$

$$R_n^{(5)} = \sigma_{i\Delta_n}^2\left(\frac{k_n\Delta_n^{1/2}\psi_2^n}{\theta\psi_2} - 1\right) + \frac{1}{\theta\psi_2}\frac{1}{h_n}\sum_{m=0}^{h_n-1}\sigma_{i\Delta_n}^2\big(\Delta_n^{-1/2}\bar{W}_{i+m}^2 - k_n\Delta_n^{1/2}\psi_2^n\big).$$

Consider the random variable:

$$S_n = \sum_{m=0}^{l_nk_n-1} Y_m = \sum_{i=1}^{k_n} S_n^{(i)},$$

where $(Y_m)_{m\geq 1}$ are mean zero and (k_n-1)-dependent, such that for each i $S_n^{(i)} = \sum_{m=0}^{l_n} Y_{i-1+mk_n}$ is a sum of independent random variates and $\sup_{m\in\mathbb{N}}\mathbb{E}\big[|Y_m|^p\big] \leq C$. Applying the discrete Hölder and Burkholder inequality, we obtain for each $p \geq 2$:

$$\mathbb{E}\big[|S_n|^p\big] \leq k_n^{p-1}\sum_{i=1}^{k_n}\mathbb{E}\big[|S_n^{(i)}|^p\big] \leq k_n^p l_n^{p/2}.$$

This observation leads to

$$\mathbb{E}\Big[\big|R_n^{(2)}\big|^p\Big] + \mathbb{E}\Big[\big|R_n^{(3)}\big|^p\Big] + \mathbb{E}\Big[\big|R_n^{(5)}\big|^p\Big] \leq C\big(k_n/h_n\big)^{p/2} + C\Delta_n^{p/4},$$
$$\mathbb{E}\Big[\big|R_n^{(4)}\big|^p\Big] \leq C\big(\Delta_n/T\big)^{p/2}.$$

Turning next to $R_n^{(1)}$, we have for each i and m that

$$\bar{X}_{i+m} - \sigma_{i\Delta_n}\bar{W}_{i+m} = \sum_{j=1}^{k_n-1} g(j/k_n)A_{i+m+j},$$

where

$$A_{i+m+j} = \int_{(i+m+j-1)\Delta_n}^{(i+m+j)\Delta_n} b_s ds + \int_{(i+m+j-1)\Delta_n}^{(i+m+j)\Delta_n}(\sigma_s - \sigma_{i\Delta_n})dW_s.$$

The Burkholder inequality then implies that, for each $p \geq 2$,

$$\mathbb{E}\Big[\big|\bar{X}_{i+m} - \sigma_{i\Delta_n}\bar{W}_{i+m}\big|^p\Big] \leq C\Big(\big(k_n\Delta_n\big)^p + \big(k_n\Delta_n\big)^{p/2}\big(h_n\Delta_n\big)^{pH}\Big) \leq C\Big(\Delta_n^{p/2} + \Delta_n^{p/4}\big(h_n\Delta_n\big)^{pH}\Big),$$

uniformly in i and m. Using the above together with the identity $|a^2-b^2|^p = |a-b|^p|a+b|^p$, plus the discrete Hölder and Cauchy-Schwarz inequality, we deduce that:

$$\begin{aligned}\mathbb{E}\Big[\big|R_n^{(1)}\big|^p\Big] &\leq \frac{C}{h_n}\sum_{m=0}^{h_n-1}\mathbb{E}\Big[\big|\Delta_n^{-1/4}\big(\bar{X}_{i+m} - \sigma_{i\Delta_n}\bar{W}_{i+m}\big)\big|^p\big|\Delta_n^{-1/4}\big(\bar{X}_{i+m} + \sigma_{i\Delta_n}\bar{W}_{i+m}\big)\big|^p\Big] \\ &\leq C\Big(\Delta_n^{p/4} + \big(h_n\Delta_n\big)^{pH}\Big),\end{aligned}$$

which finishes the proof of (A.7).

We continue with the second part. We denote $\eta_{n,T} = \sup_{t\in[0,T]}\big|\hat{V}_t - V_t\big|$ and notice that

$$\eta_{n,T} \leq \sup_{i\in A_{n,T}}|\hat{V}_{i\Delta_n} - V_{i\Delta_n}| + \sup_{i\in A_{n,T}}\sup_{t\in((i-1)\Delta_n,i\Delta_n]}|V_t - V_{i\Delta_n}|.$$

where $A_{n,T} = \{i : 0 \leq i \leq [T/\Delta_n] - h_n - k_n\}$. Now, Assumption 1(i) implies:

$$\mathbb{E}\Big[\Big|\sup_{t\in((i-1)\Delta_n,i\Delta_n]}|V_t - V_{i\Delta_n}|\Big|^p\Big] \leq C\Delta_n^{pH}.$$

This result, combined with the maximal inequality and (A.7), means that

$$\mathbb{E}\Big[\big|\eta_{n,T}\big|^p\Big] \leq C\frac{T}{\Delta_n}\Big[\big(k_n/h_n\big)^{p/2} + \big(h_n\Delta_n\big)^{pH} + \Delta_n^{p/4} + \Delta_n^{pH}\Big] \leq C\frac{T}{\Delta_n}\Big[\big(k_n/h_n\big)^{p/2} + \big(h_n\Delta_n\big)^{pH}\Big],$$

which immediately delivers the bound

$$\mathbb{E}\Big[\big|\eta_{n,T}\big|^p\Big]^{1/p} \leq CT^{1/p}\Delta_n^{-1/p}\Big[\big(k_n/h_n\big)^{1/2} + \big(h_n\Delta_n\big)^H\Big].$$

As p can be arbitrarily large and optimizing $h_n \asymp \Delta_n^{-\frac{4H+1}{4H+2}}$, we deduce that

$$\eta_{n,T} = O_p\Big(T^\iota\Delta_n^{\frac{H}{4H+2}-\iota}\Big).$$

As for the final step in the proof, recall that by definition of $\eta_{n,T}$:

$$F_T(x-\eta_{n,T}) \leq F_{n,T}(x) \leq F_T(x+\eta_{n,T}).$$

Then,

$$\begin{aligned}\sqrt{T}\big|F_{n,T}(x) - F_T(x)\big| &\leq \sqrt{T}\big|F_T(x+\eta_{n,T}) - F_T(x-\eta_{n,T})\big| \\ &\leq \big|G_T(x+\eta_{n,T}) - G_T(x-\eta_{n,T})\big| + \sqrt{T}\big|F(x+\eta_{n,T}) - F(x-\eta_{n,T})\big|,\end{aligned} \tag{A.9}$$

where $G_T(x) = \sqrt{T}(F_T(x) - F(x))$.

As the density of F exists and is bounded:

$$\sqrt{T} \sup_{x \in \mathbb{R}_+} |F(x + \eta_{n,T}) - F(x - \eta_{n,T})| = \sqrt{T} \sup_{x \in \mathbb{R}_+} \int_{x-\eta_{n,T}}^{x+\eta_{n,T}} f(z)dz \leq 2K\sqrt{T}\eta_{n,T},$$

which—by the rate condition on $\eta_{n,T}$—implies that $\sqrt{T}\sup_{x\in\mathbb{R}_+} |F(x+\eta_{n,T}) - F(x-\eta_{n,T})| \xrightarrow{\mathbb{P}} 0$. And since $\eta_{n,T} \xrightarrow{\mathbb{P}} 0$, it follows that for any $\epsilon, \delta > 0$ there exists (n', T') such that $\mathbb{P}(\eta_{n,T} > \delta) \leq \epsilon/2$ for all pairs (n, T) with $n \geq n'$ and $T \geq T'$, which adheres to the rate condition. Let $\gamma > 0$ be given. Then,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{x\in\mathbb{R}_+} \big|G_T(x+\eta_{n,T}) - G_T(x-\eta_{n,T})\big| > \gamma \Big) &\leq \mathbb{P}\Big(\{\sup_{x\in\mathbb{R}_+} \big|G_T(x+\eta_{n,T}) - G_T(x-\eta_{n,T})\big| > \gamma\} \cap \{\eta_{n,T} < \delta\} \Big) \\
&\quad + \mathbb{P}\Big(\eta_{n,T} \geq \delta\Big) \\
&\leq \mathbb{P}\Big(\{ \sup_{|x-y|\leq 2\delta} \big|G_T(x) - G_T(y)\big| > \gamma\} \Big) + \mathbb{P}\Big(\eta_{n,T} \geq \delta\Big),
\end{aligned} \tag{A.10}$$

where $G_T(x) = \sqrt{T}(F_T(x) - F(x))$. It follows from Theorem 5.2 in Dehay (2005) that for any $\epsilon, \gamma > 0$ there exists $\delta > 0$ such that the first term in (A.10) is smaller than $\epsilon/2$. Hence,

$$\mathbb{P}\Big(\sup_{x\in\mathbb{R}_+} \big|G_T(x+\eta_{n,T}) - G_T(x-\eta_{n,T})\big| > \gamma \Big) \leq \epsilon.$$

As ϵ, γ are arbitrary, this verifies that $\sup_{x\in\mathbb{R}_+} |G_T(x+\eta_{n,T}) - G_T(x-\eta_{n,T})| \xrightarrow{\mathbb{P}} 0$, concluding the proof of Lemma A.1. ■

Lemma A.2 *Suppose that Assumption 1*(ii) *and 3 – 5 hold true, as does the rate condition* $T^{1/2+\iota}\Delta_n^{1/8-\iota} \to 0$*, as* $\Delta_n \to 0$ *and* $T \to \infty$*, for some* $\iota > 0$ *and* $h_n \asymp \Delta_n^{-3/4}$*. Then,*

$$\sqrt{T} \sup_{x\in\mathbb{R}_+} \big|F_{n,T}(x) - F_T(x)\big| \xrightarrow{\mathbb{P}} 0. \tag{A.11}$$

Proof. Recall that if σ_t satisfies Assumption 1(ii), the form of $V_t = \sigma_t^2$ is representable via (the general version of) Itô's Lemma:

$$V_t = V_0 + \int_0^t \bar{b}_s ds + \int_0^t \bar{\sigma}_s dW_s + \int_0^t \bar{\sigma}'_s dB_s + \int_0^t \int_{\mathbb{R}} \bar{\delta}(s,z)(\mu - \nu)(d(s,z)).$$

We split the variance in two: $V_t = V'_t(\epsilon) + V''_t(\epsilon)$, for each $\epsilon > 0$, where

$$V_t^c = V_0 + \int_0^t \bar{b}_s ds + \int_0^t \bar{\sigma}_s dW_s + \int_0^t \bar{\sigma}'_s dB_s,$$

$$V_t'(\epsilon) = V_t^c + \int_0^t \int_{\{z:\tilde{\Gamma}(z)\le\epsilon\}} \bar{\delta}(s,z)(\mu-\nu)(d(s,z)) - \int_0^t \int_{\{z:\tilde{\Gamma}(z)>\epsilon\}} \bar{\delta}(s,z)\nu(d(s,z)),$$
$$V_t''(\epsilon) = \int_0^t \int_{\{z:\tilde{\Gamma}(z)>\epsilon\}} \bar{\delta}(s,z)\mu(d(s,z)).$$

Moreover, we denote

$$J_n(\epsilon) = \{i \in \mathbb{Z} : 0 \le i \le [T/\Delta_n] - h_n - k_n, \mu([i\Delta_n,(i+1)\Delta_n) \times \{z : \tilde{\Gamma}(z) > \epsilon\}) = 0\},$$
$$S_n(\epsilon) = \cup_{i\in J_n(\epsilon)}[i\Delta_n,(i+1)\Delta_n),$$
$$A_{n,T} = \{i : 0 \le i \le [T/\Delta_n] - h_n - k_n\}.$$

In words, $J_n(\epsilon)$ is a set of indices that excludes the big jumps and $S_n(\epsilon)$ is the union of these. It leads to a decomposition $F_{n,T}(x) = F_{n,T}(x,\epsilon) + R_{n,T}(x,\epsilon)$ and $F_T(x) = F_T(x,\epsilon) + R_T(x,\epsilon)$ with

$$F_{n,T}(x,\epsilon) = \frac{1}{T}\int_{S_n(\epsilon)} 1_{\{\hat{V}_t\le x\}}dt, \quad R_{n,T}(x,\epsilon) = \frac{1}{T}\int_{[0,T]\setminus S_n(\epsilon)} 1_{\{\hat{V}_t\le x\}}dt,$$
$$F_T(x,\epsilon) = \frac{1}{T}\int_{S_n(\epsilon)} 1_{\{V_t\le x\}}dt, \quad R_T(x,\epsilon) = \frac{1}{T}\int_{[0,T]\setminus S_n(\epsilon)} 1_{\{V_t\le x\}}dt.$$

Here, F is the main term, while R is a remainder. A final piece of notation:

$$\eta_{n,T}(\epsilon) = \sup_{i\in J_n(\epsilon)} |\hat{V}_{i\Delta_n} - V_{i\Delta_n}| + \sup_{i\in A_{n,T}} \sup_{t\in((i-1)\Delta_n,i\Delta_n]} |V_t'(\epsilon) - V_{i\Delta_n}'(\epsilon)|.$$

Let $\epsilon_n \asymp \Delta_n^{1/8}$. The aim is to prove that

$$\eta_{n,T}(\epsilon_n) = O_p(T^\iota \Delta_n^{1/8-\iota}). \tag{A.12}$$

We deal first with the last term of $\eta_{n,T}(\epsilon_n)$, which we call

$$\tilde{\eta}_{n,T}(\epsilon_n) = \sup_{i\in A_{n,T}} \sup_{t\in((i-1)\Delta_n,i\Delta_n]} |V_t'(\epsilon_n) - V_{i\Delta_n}'(\epsilon_n)|.$$

Observe that

$$\sup_{t\in((i-1)\Delta_n,i\Delta_n]} |V_t'(\epsilon_n) - V_{i\Delta_n}'(\epsilon_n)| \le \gamma_{i,n}^{(1)} + \gamma_{i,n}^{(2)} + \gamma_{i,n}^{(3)},$$

where

$$\gamma_{i,n}^{(1)} = \sup_{t\in((i-1)\Delta_n,i\Delta_n]} |V_t^c - V_{i\Delta_n}^c|,$$
$$\gamma_{i,n}^{(2)} = \sup_{t\in((i-1)\Delta_n,i\Delta_n]} \Big|\int_{(i-1)\Delta_n}^t \int_{\{z:\tilde{\Gamma}(z)\le\epsilon_n\}} \bar{\delta}(s,z)(\mu-\nu)(ds,dz)\Big|,$$
$$\gamma_{i,n}^{(3)} = \sup_{t\in((i-1)\Delta_n,i\Delta_n]} \Big|\int_{(i-1)\Delta_n}^t \int_{\{z:\tilde{\Gamma}(z)>\epsilon_n\}} \bar{\delta}(s,z)\nu(ds,dz)\Big|.$$

Application of the Burkholder and maximal inequality yield that, for any $p \geq 2$,

$$\mathbb{E}\Big[\big|\sup_{i\in A_{n,T}} \gamma_{i,n}^{(1)}\big|^p\Big] \leq CT\Delta_n^{p/2-1}. \tag{A.13}$$

Moreover, for each $p \geq 2$ and i, Lemma 2.1.5 in Jacod and Protter (2012) implies that

$$\begin{aligned}\mathbb{E}\Big[\big|\gamma_{i,n}^{(2)}\big|^p\Big] &\leq C\Delta_n \int_{\{z:\tilde{\Gamma}(z)\leq\epsilon_n\}} \tilde{\Gamma}(z)^p\lambda(dz) + C\Delta_n^{p/2}\Big(\int_{\{z:\tilde{\Gamma}(z)\leq\epsilon_n\}} \tilde{\Gamma}(z)^2\lambda(dz)\Big)^{p/2} \\ &\leq C\big(\Delta_n\epsilon_n^{p-2} + \Delta_n^{p/2}\big).\end{aligned}$$

In view of the maximal inequality:

$$\mathbb{E}\Big[\big|\sup_{i\in A_{n,T}} \gamma_{i,n}^{(2)}\big|^p\Big] \leq CT\big(\epsilon_n^{p-2} + \Delta_n^{p/2-1}\big). \tag{A.14}$$

Finally,

$$\big|\sup_{i\in A_{n,T}} \gamma_{i,n}^{(3)}\big| \leq \sup_{i\in A_{n,T}} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\{z:\tilde{\Gamma}(z)>\epsilon_n\}} \tilde{\Gamma}(z)\nu(ds,dz) \leq C\Delta_n\epsilon_n^{-1}. \tag{A.15}$$

As a result, for any $p \geq 2$, a combination of (A.13) - (A.15) shows that

$$\mathbb{E}\Big[\big|\tilde{\eta}_{n,T}(m_n)\big|^p\Big]^{1/p} \leq C\big(T^{1/p}\Delta_n^{1/2-1/p} + T^{1/p}\epsilon_n^{1-2/p} + \Delta_n\epsilon_n^{-1}\big). \tag{A.16}$$

Now, turn to the first term of $\eta_{n,T}(\epsilon_n)$, which is $\hat{\eta}_{n,T}(\epsilon_n) \equiv \sup_{i\in J_n(\epsilon_n)} |\hat{V}_{i\Delta_n} - V_{i\Delta_n}|$. As in (A.8) in the proof of Lemma A.1, we disentangle, for each $i \in J_n(\epsilon_n)$,

$$\hat{V}_{i\Delta_n} - V_{i\Delta_n} = R_n^{(1)} + R_n^{(2)} + R_n^{(3)} + R_n^{(4)} + R_n^{(5)}.$$

It suffices to look at $R_n^{(1)}$, as the rates of the other four terms are equal. As above, utilizing Burkholder's inequality, we see that for each $p \geq 2$, $i \in J_n(\epsilon_n)$ and $0 \leq m \leq h_n - 1$:

$$\begin{aligned}\mathbb{E}\Big[\big|\bar{X}_{i+m} - \sigma_{i\Delta_n}\bar{W}_{i+m}\big|^p\Big] &\leq C\Big(\big(k_n\Delta_n\big)^p + \big(k_n\Delta_n\big)^{p/2}\mathbb{E}\Big[\sup_{t\in[i\Delta_n,(i+h_n)\Delta_n]} |\sigma_t - \sigma_{i\Delta_n}|^p\Big]\Big) \\ &\leq C\Big(\big(k_n\Delta_n\big)^p + \big(k_n\Delta_n\big)^{p/2}\big[(h_n\Delta_n)^{p/2} + h_n\Delta_n\epsilon_n^{p-2} + \big(h_n\Delta_n\big)^p\epsilon_n^{-p}\big]\Big) \\ &\leq C\Delta_n^{p/4}\Big[\big(h_n\Delta_n\big)^{p/2} + h_n\Delta_n\epsilon_n^{p-2} + \big(h_n\Delta_n\big)^p\epsilon_n^{-p}\Big],\end{aligned}$$

uniformly in i and m. Proceeding as in the proof of Lemma A.1, it follows that

$$\begin{aligned}\mathbb{E}\Big[\big|R_n^{(1)}\big|^p\Big] &\leq \frac{C}{h_n}\sum_{m=0}^{h_n-1} \mathbb{E}\Big[\big|\Delta_n^{-1/4}(\bar{X}_{i+m} - \sigma_{i\Delta_n}\bar{W}_{i+m})\big|^p\big|\Delta_n^{-1/4}(\bar{X}_{i+m} + \sigma_{i\Delta_n}\bar{W}_{i+m})\big|^p\Big] \\ &\leq C\Big[\big(h_n\Delta_n\big)^{p/2} + h_n\Delta_n\epsilon_n^{p-2} + \big(h_n\Delta_n\big)^p\epsilon_n^{-p}\Big].\end{aligned}$$

In conclusion, for each $p \geq 2$ and $i \in J_n(\epsilon_n)$, we get

$$\mathbb{E}\Big[\big|\hat{V}_{i\Delta_n} - V_{i\Delta_n}\big|^p\Big] \leq C\Big[(k_n/h_n)^{p/2} + (h_n\Delta_n)^{p/2} + h_n\Delta_n\epsilon_n^{p-2} + (h_n\Delta_n)^p\epsilon_n^{-p}\Big].$$

Hence, for any $p \geq 2$, the maximal inequality along with $k_n \asymp \Delta_n^{-1/2}$ and $h_n \asymp \Delta_n^{-3/4}$ yield

$$\mathbb{E}\Big[\big|\tilde{\eta}_{n,T}(\epsilon_n)\big|^p\Big]^{1/p} \leq CT^{1/p}\Delta_n^{-1/p}\big[\Delta_n^{1/8} + \epsilon_n^{1-2/p} + \Delta_n^{1/4}\epsilon_n^{-1}\big].$$

Merging this with (A.16):

$$\begin{aligned}\mathbb{E}\Big[\big|\eta_{n,T}(\epsilon_n)\big|^p\Big]^{1/p} &\leq CT^{1/p}\Delta_n^{-1/p}\big[\Delta_n^{1/8} + \epsilon_n^{1-2/p} + \Delta_n^{1/4}\epsilon_n^{-1}\big] \\ &\leq CT^{1/p}\Delta_n^{-1/p}\big[\Delta_n^{1/8} + \Delta_n^{1/8-1/4p}\big],\end{aligned}$$

where $\epsilon_n \asymp \Delta_n^{1/8}$ was used in the last line. Now, choosing large enough p implies (A.12).

Passing back to the main proof, it holds that

$$\mathbb{E}\Big[T^{-1}\int_{[0,T]\setminus S_n(\epsilon_n)} dt\Big] \leq C\Delta_n\epsilon_n^{-2}.$$

This delivers a bound for the remainder term(s):

$$R_{n,T}(x,\epsilon_n) + R_T(x,\epsilon_n) = O_p(\Delta_n\epsilon_n^{-2}). \tag{A.17}$$

As $V_t''(\epsilon_n)$ vanishes on $S_n(\epsilon_n)$, $\sup_{t\in S_n(\epsilon_n)}|\hat{V}_t - V_t| \leq \eta_{n,T}(\epsilon_n)$. Then, proceeding as in (A.9) we deduce that

$$\begin{aligned}\sqrt{T}|F_{n,T}(x,\epsilon_n) - F_T(x,\epsilon_n)| &\leq \sqrt{T}\big|F_T(x+\eta_{n,T}(\epsilon_n),\epsilon_n) - F_T(x-\eta_{n,T}(\epsilon_n),\epsilon_n)\big| \\ &\leq \sqrt{T}\big|F_T(x+\eta_{n,T}(\epsilon_n)) - F_T(x-\eta_{n,T}(\epsilon_n))\big| \\ &\leq \big|G_T(x+\eta_{n,T}(\epsilon_n)) - G_T(x-\eta_{n,T}(\epsilon_n))\big| + \sqrt{T}\big|F(x+\eta_{n,T}(\epsilon_n)) - F(x-\eta_{n,T}(\epsilon_n))\end{aligned}$$

where $G_T(x) = \sqrt{T}(F_T(x) - F(x))$. Here, the second inequality is due to integrating over a larger set. Now, arguing as in the end of the proof of Lemma A.1 and using (A.17), we conclude that

$$\sqrt{T}\sup_{x\in\mathbb{R}_+}|F_{n,T}(x) - F_T(x)| = o_p(1) + O_p(T^{1/2-\iota}\Delta_n^{1/8-\iota}) + O_p(T^{1/2}\Delta_n\epsilon_n^{-2}).$$

As $\epsilon_n \asymp \Delta_n^{1/8}$, our work is done. ■

Proof of Theorem 3.3

Notice that

$$\sqrt{T}\big(F_{n,T}(x) - F(x)\big) = \sqrt{T}\big(F_{n,T}(x) - F_T(x)\big) + \sqrt{T}\big(F_T(x) - F(x)\big). \tag{A.18}$$

Appealing to Lemma A.1-A.2, it holds that $\sup_{x\in\mathbb{R}_+}\sqrt{T}\big(F_{n,T}(x) - F_T(x)\big) \xrightarrow{\mathbb{P}} 0$. Furthermore, by Theorem 4.1 in Dehay (2005) it follows that $\sqrt{T}\big(F_T(x) - F(x)\big) \xrightarrow{d} N(0, \Sigma(x))$, where

$$\Sigma(x) = 2\int_0^\infty \big(F_t(x,x) - F(x)^2\big)dt$$

and $F_t(x,y) = \mathbb{P}\big(\sigma_0^2 \le x, V_t \le y\big)$. We next prove the proposed estimator of the asymptotic variance $\Sigma_{n,T}(x) \xrightarrow{\mathbb{P}} \Sigma(x)$, for all $x \in \mathbb{R}_+$. Thus, let $\xi \in (0, 1/3)$ be fixed and observe that:

$$\begin{aligned}\Sigma_{n,T}(x) &= 2\int_0^{T^\xi} \big(F_{t,n,T}(x) - F_{n,T}(x)^2\big)dt \\ &= 2\int_0^{T^\xi} \big(F_{t,n,T}(x) - F_{t,T}(x)\big)dt + 2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt + 2T^\xi\big(F_T(x)^2 - F_{n,T}(x)^2\big),\end{aligned} \tag{A.19}$$

where $F_{t,T}(x) = \dfrac{1}{T}\displaystyle\int_0^{T-T^\xi} 1_{\{V_{s+t}\le x, V_s \le x\}}ds$. The last term in the above display is asymptotically negligible by Lemma A.1 – A.2:

$$\begin{aligned}T^\xi\big|F_T(x)^2 - F_{n,T}(x)^2\big| &= T^\xi\big|F_T(x) - F_{n,T}(x)\big|\big|F_T(x) + F_{n,T}(x)\big| \\ &\le 2T^\xi\big|F_T(x) - F_{n,T}(x)\big| \xrightarrow{\mathbb{P}} 0.\end{aligned}$$

Next,

$$F_{t,T}(x - \eta_{n,T}) \le F_{t,n,T}(x) \le F_{t,T}(x + \eta_{n,T}),$$

hence $\big|F_{t,n,T}(x) - F_{t,T}(x)\big| \le F_{t,T}(x+\eta_{n,T}) - F_{t,T}(x-\eta_{n,T})$. By definition of $F_{t,T}$, it follows that for any $z, y \in \mathbb{R}_+$ with $z > y$:

$$\begin{aligned}F_{t,T}(z) - F_{t,T}(y) &= \frac{1}{T}\int_0^{T-T^\xi} 1_{\{y<\sigma_{s+t}^2\le z, y<\sigma_s^2\le z\}}ds \\ &\le \frac{1}{2T}\int_0^{T-T^\xi} 1_{\{y<\sigma_{s+t}^2\le z\}} + \frac{1}{2T}\int_0^{T-T^\xi} 1_{\{y<\sigma_s^2\le z\}}ds \\ &= F_T(z) - F_T(y) + O\big(T^{\xi-1}\big).\end{aligned}$$

Taking $z = x + \eta_{n,T}$ and $y = x - \eta_{n,T}$:

$$\int_0^{T^\xi} \big|F_{t,n,T}(x) - F_{t,T}(x)\big|dt = T^\xi\big(F_T(x+\eta_{n,T}) - F_T(x-\eta_{n,T})\big) + O\big(T^{2\xi-1}\big).$$

As in the proof of Lemma A.1-A.2, we can show that

$$\int_0^{T^\xi} \left|F_{t,n,T}(x) - F_{t,T}(x)\right| dt \xrightarrow{\mathbb{P}} 0.$$

Finally, for the remaining term in (A.19), we first note that stationarity of $(V_t)_{t\geq 0}$ implies that $\mathbb{E}\big[F_{t,T}(x)\big] = \mathbb{E}\big[F_t(x,x)\big] + O\big(T^{\xi-1}\big)$ and $\mathbb{E}\big[F_T(x)\big] = F(x)$. Hence,

$$\mathbb{E}\left[2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt\right] = 2\int_0^{T^\xi} \big(F_t(x,x) - F(x)^2\big)dt - 2T^\xi \mathrm{var}\big(F_T(x)\big) + O\big(T^{2\xi-1}\big).$$

Since $\alpha(t) = O(t^{-\gamma})$ and $\xi \in (0, 1/3)$, Theorem 3.3 in Dehay (2005) implies that $T^\xi \mathrm{var}\big(F_T(x)\big) \to 0$ as $T \to \infty$. Therefore,

$$\mathbb{E}\left[2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt\right] \to \Sigma(x),$$

as $T \to \infty$. It thus suffices to show that $2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt - \mathbb{E}\Big[2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt\Big] \to 0$ in L^2, as $\sqrt{T}a_{n,T} \to 0$, $\Delta_n \to 0$ and $T \to \infty$, in order to conclude that $\Sigma_{n,T}(x) \xrightarrow{\mathbb{P}} \Sigma(x)$. To this end we decompose the difference as follows

$$\begin{aligned} 2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt - \mathbb{E}\left[2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt\right] &= 2\int_0^{T^\xi} \big(F_{t,T}(x) - F_t(x,x)\big)dt \\ &\quad + T^\xi\big(F_T(x)^2 - F(x)^2\big) \\ &\quad + 2T^\xi \mathrm{var}\big(F_T(x)\big) + O\big(T^{2\xi-1}\big). \end{aligned}$$

By Minkowski's Inequality, it is enough to look at each term on the right-hand side separately. We define $\tilde{F}_{t,T}(x) = \frac{1}{T}\int_0^T 1_{\{V_{s+t}\leq x, V_s \leq x\}} ds$ and note that $\tilde{F}_{t,T}(x) = F_{t,T}(x) + O\big(T^{\xi-1}\big)$, $\mathbb{E}\big[\tilde{F}_{t,T}(x)\big] = F_t(x,x)$ and

$$2\int_0^{T^\xi} \big(F_{t,T}(x) - F_t(x,x)\big)dt = 2\int_0^{T^\xi} \big(\tilde{F}_{t,T}(x) - F_t(x,x)\big)dt + O\big(T^{2\xi-1}\big).$$

The Cauchy-Schwarz Inequality yields that

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^{T^\xi} \big(\tilde{F}_{t,T}(x) - F_t(x,x)\big)dt\right)^2\right] &\leq T^\xi \int_0^{T^\xi} \mathrm{var}\big(\tilde{F}_{t,T}(x)\big)dt \\ &= O\big(T^{2\xi-1}\big), \end{aligned}$$

where the last inequality follows from the calculations on p. 941 in Dehay (2005). Therefore,

$$2\int_0^{T^\xi} \big(F_{t,T}(x) - F_t(x,x)\big)dt \to 0$$

in L^2. As both F_T and F are bounded by 1, the bounded convergence theorem and Theorem 3.3 in Dehay (2005) implies that

$$T^{\xi}\left(F_T(x)^2 - F(x)^2\right) \to 0 \tag{A.20}$$

in L^2, as $T \to \infty$. Thus,

$$2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt - \mathbb{E}\Big[2\int_0^{T^\xi} \big(F_{t,T}(x) - F_T(x)^2\big)dt\Big] \to 0$$

in L^2, given the rate condition from Theorem 3.3. This implies that $\Sigma_{n,T}(x) \xrightarrow{\mathbb{P}} \Sigma(x)$. ■

We now state a multivariate version of Theorem 3.3.

Lemma A.3 *Suppose that the conditions of Theorem 3.3 are fulfilled. Let $p \in \mathbb{N}$ be given. Then, for any $x_1, \dots, x_p \in \mathbb{R}_+$,*

$$\big(G_{n,T}(x_1), \dots, G_{n,T}(x_p)\big) \xrightarrow{d} N\big(0, \Sigma(x_1, \dots, x_p)\big),$$

where $\Sigma(x_1, \dots, x_p)$ is the $p \times p$ covariance matrix with elements of the form

$$\Sigma(x_1, \dots, x_p)_{ij} = \int_0^\infty \big(F_t(x_i, x_j) + F_t(x_i, x_j) - 2F(x_i)F(x_j)\big)dt,$$

for $i, j = 1, \dots, p$. Furthermore, let $\xi \in (0, 1/3)$ be given. Then,

$$\Sigma_{n,T}(x_i, x_j) = \int_0^{T^\xi} \big(F_{t,n,T}(x_i, x_j) + F_{t,n,T}(x_j, x_i) - 2F_{n,T}(x_i)F_{n,T}(x_j)\big)dt \xrightarrow{\mathbb{P}} \Sigma(x_1, \dots, x_p)_{ij}.$$

Proof. The first part follows by a Crámer-Rao device. Also, proving consistency of the covariance estimator goes along the lines of the univariate case. We therefore skip it. ■

Proof of Theorem 3.5

In view of Lemma A.1-A.2, we obtain that $\sup_{x\in\mathbb{R}_+} \sqrt{T}\big(F_{n,T}(x) - F_T(x)\big) \xrightarrow{\mathbb{P}} 0$. On the other hand, Theorem 5.2 in Dehay (2005) implies a functional central limit theorem for the process $G_T = \big\{\sqrt{T}\big(F_T(x) - F(x)\big), x \in \mathbb{R}_+\big\}$. ■

Proof of Corollary 3.6

This follows from Theorem 3.5 by noting that the sup and weighted L^2 norm are both continuous mappings from $C_b(\mathbb{R}_+) \to \mathbb{R}$. ■

Proof of Proposition 3.7

From Corollary 3.6 the limiting distribution of rL^2 under $\mathcal{H}_0$ is $||G_F||_w^2$, where G_F is a Gaussian process with covariance operator Σ. The distribution can therefore be written as $||G_F||_w^2 = \sum_{i=1}^{\infty} \lambda_i \Phi_i^2$, where Φ_i are independent standard normal random variables, and λ_i are the eigenvalues of Σ (see, e.g., Kuo, 1975). Moreover, from Bosq (2000, chapter 1) we deduce that $\tilde{\Sigma} = \sum_{i=1}^{\infty} \lambda_i < \infty$, so that $\mathbb{E}\big[||G_F||_w^2\big] = \tilde{\Sigma}$. Combining this result with the fact that $T(F)$ is a quadratic form of mean zero Gaussian random variables, it follows that for all $\alpha \in (0, 0.215)$:

$$\mathbb{P}\big(T(F) > \alpha\big) = 1 - \mathbb{P}\big(T(F) \leq \alpha\big) \leq 1 - \mathbb{P}\big(\chi^2 \leq \alpha\big) = \mathbb{P}\big(\chi^2 > \alpha\big),$$

where the inequality is due to Székely and Bakirov (2003). This concludes the proof of the first statement, while the second part follows upon observing that $||G_F||_w^2 \xrightarrow{\mathbb{P}} \infty$ under $\mathcal{H}_a$. ■

Proof of Corollary 3.8

The proof follows from Theorem 19.23 in van der Vaart (1998). ■

B Simulation of critical values for goodness-of-fit testing

To describe how critical values for our goodness-of-fit tests are generated, we again fix the sampling frequency at Δ_n and the time horizon at $[0, T]$.

The stochastic volatility model is assumed to be indexed by a parameter vector υ. We discriminate between the case where υ is known (I), and the more realistic setup where υ is unknown, but a preliminary estimate $\hat{\upsilon}$ is available along with an estimate of the noise variance $\hat{\omega}$ (II). As a biproduct, we emphasize that the latter approach also accounts for the estimation error induced by the infill step, which is negligible in the limit but can be sizable in small samples.

We proceed as follows:

I. Without parameter estimation

1. Simulate a volatility path of length T based on υ.
2. Sample the path at frequency Δ_n.
3. Construct the EDF, F_T.
4. Calculate the t-statistic with F based on υ.

5. Repeat step 1 – 4 B times.

II. With parameter estimation

1. Simulate a volatility path of length T based on $\hat{\upsilon}$. Call it $(\sigma_t)_{t\in[0,T]}$.
2. Construct noisy log-returns at sampling frequency Δ_n according to: $\Delta_i^n Z = \sigma_{(i-1)\Delta_n}\sqrt{\Delta_n}\Phi_1 + \hat{\omega}\Phi_2$, where Φ_1 and Φ_2 are independent standard normal random variables.
3. Construct the REDF, $F_{n,T}$.
4. Retrieve a new estimate of the parameter vector, denoted by $\tilde{\upsilon}$, from the artificial data.
5. Calculate the t-statistic with F based on $\tilde{\upsilon}$.
6. Repeat step 1 – 5 B times.

In both I. and II., the critical values are subsequently found as the α-quantile of the distribution of t-statistics based on the above B repetitions of the simulation experiment.

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